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Markovian bridges, Brownian excursions, and stochastic fragmentation and coalescence

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To Abi, who never lacked faith

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Introduction

Brownian motion is an universal and ubiquitous mathematical object in both theoretical and applied studies of probability theory. It is universal in the sense physicists have given to the word: it describes common features emerging in the study of all members of a class of different models. It is ubiquitous, since it describes other universal objects. This work stems from the desire to understand how Brownian motion can be conditioned to act in a way that it doesn't, giving rise to Brownian bridges and excursions, and how these conditionings occur in studies of Brownian motion arising in models of stochastic fragmentation and coalescence.

One of Paul Lévy's fundamental remarks about Brownian motion is that if we know that its values at times $t_1 < t_2$ are x_1, x_2 and $t \in (t_1, t_2)$, then its value at time t given the former is Gaussian with mean μ_t and variance σ_t^2 where

$$\mu_t = \frac{t_2 - t}{t_2 - t_1} x_1 + \frac{t - t_1}{t_2 - t_1} x_2 \text{ and } \sigma_t^2 = \frac{(t - t_1)(t_2 - t)}{t_2 - t_1}.$$

This enabled him to construct Brownian motion by an interpolation procedure. However, it also led him to the construction of Brownian motion on $[0, t]$ conditioned on its values at times 0 and t; since this process bridges the two values, in the sense of going from one to the other, it has been termed the Brownian bridge. If B is a Brownian motion (starting at zero), then the Brownian bridge $b^{x,y,t}$ that goes from x to y in t units of time can be represented as a linear transformation of B: for $s \in [0, t]$

$$b_s^{x,y,t} = x + B_s - \frac{s}{t} B_t + (y - x) \frac{s}{t}.$$

The fundamental property of the Brownian bridge is that it provides us with a disintegration of the law of Brownian motion, in the sense that

$$\mathbb{E}\Big(F\Big((B_s)_{s\in[0,t]}\Big)\, f(B_t)\Big) = \int \mathbb{P}(B_t \in dx)\, \mathbb{E}\big(F\big(b^{0,x,t}\big)\big)\, f(x)\,.$$

(Here, F is a nonnegative measurable functional whose arguments are continuous trajectories...) There is a general existence theorem for disintegrations of probability measures, but unfortunately, disintegrations are not unique. Suppose we could find, as in several examples, a function g such that

$$\mathbb{E}\Big(F\Big((B_s)_{s\in[0,t]}\Big)\, f(B_t)\Big) = \int \mathbb{P}(B_t \in dx)\, g(x)\, f(x)$$

for a given class of functions f, like continuous and bounded or measurable and bounded. Then we could assert the almost sure equality (with respect to the law of B_t, that is, with respect to Lebesgue measure):

$$\mathbb{E}\big(F\big(b^{0,x,t}\big)\big) = g(x)\,.$$

But if we are interested in the value of the left-hand side for, say, $x = 1$, we are in no position to assert that it is equal to $g(1)$. If both sides were continuous, this would be possible. Paul Lévy's construction of the Brownian bridge gives us a weakly continuous disintegration in x and y, and so for a big class of functionals F, we can assert the continuity of $\mathbb{E}\big(F\big(b^{0,x,t}\big)\big)$. However, Paul Lévy's construction of the Brownian bridge uses fundamental properties of Gaussian random variables. Therefore, the first question we will address is about the possibility of constructing continuous disintegrations to other types of processes, for example, Lévy processes or more generally other Markov processes. An answer will be given for a subclass of Feller processes in Chapter 1. Feller processes are strong Markov processes; in discrete time, the strong Markov property is a straightforward extension of the Markov property, but to go to continuous time, continuity properties satisfied by Feller processes are useful. Since a part of the Markov property can be recast in a way that highlights its symmetry with respect to the direction of the flow of time, a second question to be addressed in Chapter 1 is about the possibility of extending the strong Markov property when this direction is reversed.

Our extension will be done in terms of Markovian bridges, through continuity considerations. We will also provide examples and applications of Markovian bridges and our backward strong Markov property.

Paul Lévy also used his interpolation procedure to study Brownian motion between two consecutive zeros. (One has to be careful with the preceding phrase since the zero set of Brownian motion has no isolated points; what he actually studied was the behaviour of Brownian motion between the last zero preceding a deterministic time, and the first zero after it; because of the Gaussian character of B_t, $\mathbb{P}(B_t = 0) = 0$ for all $t > 0$.) He obtained the law of a Brownian excursion of a given length v; they can all be expressed in terms of the law of the normalized Brownian excursion, which is the case $v = 1$. In Chapter 2, we will review some known facts about these processes, emphasizing their interpretation as Markovian bridges.

We have mentioned the universality of Brownian motion. It can be instantiated in Donker's invariance principle, which we now state. For a given probability measure ν on $\mathbb{R}$, with finite variance σ^2 and zero mean, let $(S_n)_{n\in\mathbb{N}}$ be a random walk with step distribution ν, that is, the sequence of partial sums associated to a sequence of independent random variables with common distribution ν. Then the sequence of stochastic processes S^n given by $S^n_t = S_{[nt]}/\sqrt{\sigma^2 n}$ converges in law to Brownian motion. The fact to stress is that the limit process B does not depend on the step distribution ν of the random walk. We have also mentioned the ubiquitous character of Brownian motion; let us exemplify with the recent example of plane trees. Plane trees are combinatorial trees (connected graphs with no cycles) on which a total order that respects branches has been given. They can be thought of as a collection of vertices and edges and can drawn in the plane using the total order; they can also be coded by a continuous function on $\mathbb{R}$ (called the contour) on which the leaves correspond to locations of the maxima and the inner vertices correspond to the local minima. Both representations can be seen in Figure 1. There are other codings of plane trees such as the depth-first walk, which will have a rôle in what follows. There is a collection of random trees, called Galton-Watson trees, which are parametrized by a probability law μ on $\{-1, 0, 1, \ldots\}$, called the offspring distribution, and for which an invariance principle was given by Aldous during the early nineties. It specifies that, under some conditions on μ, the depth-first

FIGURE 1. A plane tree and its contour.

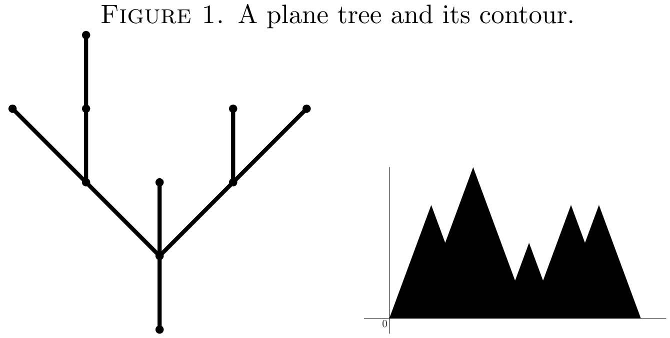
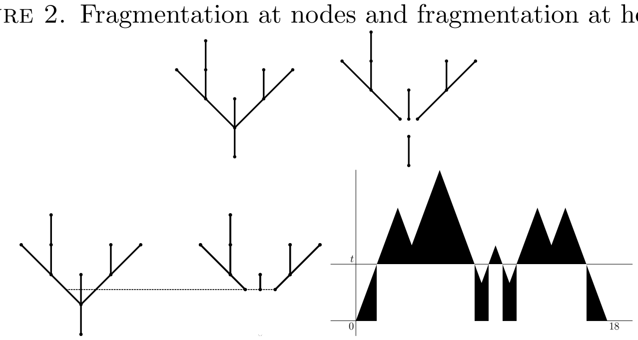


FIGURE 2. Fragmentation at nodes and fragmentation at heights.



walk associated to a Galton-Watson tree can be rescaled to converge to the normalized Brownian excursion, independently of μ. In this way, Brownian motion lurks behind the asymptotics for random trees: the normalized Brownian excursion can be thought to code a tree-like object termed the continuum random tree.

There is nothing simpler than fragmenting a tree, just choose a way of disconnecting it. For example, one could remove a node, or cut down everything less than a given distance to the root. The latter case will be called the fragmentation at heights. Both fragmentations can be visualized in Figure 2. When the tree is coded by its contour, denoted f (or by its depth-first walk) one can define the height framentation in terms of the set $\{s \geq 0 : f(s) > t\}$, where t is the height at which we cut down the tree. As t grows, the associated set has less elements. Chapter 3 is devoted to the study of the height fragmentation of the normalized Brownian excursion, or equivalently, the height fragmentation of the continuum random tree. Note that the associated fragmentation process becomes empty when the height is above the maximum of the excursion. We say that the fragmentation process disintegrates, and one of our tasks will be to see how this happens. We will also study another aspect of this fragmentation: how pieces fall off a randomly selected fragment called the tagged fragment.

Finally, the subject of Chapter 4 is contrary to fragmentation: in it, a model for stochastic coalescence, called the multiplicative coalescent, will be considered. The appearance of excursions in this chapter is quite surprising. The classical random graph model of Erdős and Rényi can be thought to be part of an growing family of graphs with n vertices which starts at the trivial graph and grows until it becomes the complete graph. One can index this family by $[0, \infty)$ to obtain a continuous time Markov chain on graphs with n vertices whose jumps add edges one at a time to build up the graph. Spanning trees of its connected components are, conditionally on their sizes, uniform (and independent). Hence, concatenation of their depth-first walks might lead, as $n \to \infty$, to concatenation of independent Brownian excursions. The way they are concatenated can be specified by the analysis of the vector of lengths of the components in the evolving random graph process; this turns out to be a continuous time Markov chain called the multiplicative coalescent. A recent representation of it will be used to reappraise some known results on random graphs in this last chapter; in particular, we will comment on why excursions of stochastic processes are inherent to the study of multiplicative coalescence.

Mathematics is a collective enterprise. As such, the results of this thesis are built upon numerous related studies. While the precise references belong to the main body of this work, the author would like to collect in the introduction the new mathematics the reader might encounter:

- A weak limit construction of bridges of Feller processes under verifiable assumptions, as well as an analysis of their weak continuity with respect to the parameters involved. (Theorem 1 in p. 12.) Numerous examples where the conditions are met; their existence can be established through other techniques, but the analysis of their weak continuity and their approximation is new in most cases. (Section 3 of Chapter 1, p.25.) As a new application of weakly continuous Markov bridges, an extension of Jeulin's limit theorem for the normalized Brownian excursion to bridges of transient Bessel processes is provided. (Application 2, p. 36.)

- A framework and a proof of a backward strong Markov property. (Theorem 2, p. 23.) It is applied throughout this work to gain insight into several path transformations connecting Brownian motion and related Markov processes. A new application is the construction of a stable subordinator conditioned to die at a given point by a path transformation of the trajectories of stable subordinators. (Application 3, p. 40.)
- A further study of the height fragmentation of the continuum random tree, with some related computations for the height fragmentation of other stable trees is the subject of Chapter 3. A relationship is established between the tagged fragment of such fragmentations, the two-parameter Poisson-Dirichlet distributions, and stable subordinators conditioned to die at a given point (Theorems 7 in p. 69 and 8 in p. 70). Also, explicit limiting objects for the asymptotic disintegration of the height fragmentation are found in the Brownian case (Theorem 9, p. 71).
- Advances of a study pertaining to the classical Erdős-Rényi model of random graphs, the multiplicative coalescent and excursions of stochastic processes is the subject of Chapter 4. Here, a recent representation of the multiplicative coalescent is reexamined to explain the appeareance of Gumbel's law in asymptotic computations of the probability that a given random graph is connected (Section 2, p. 110), to gain some insight into the emergence of the giant component (Sections 3 and 4, p. 112 and p. 114) and to propose an answer to the following question: why are excursions of stochastic processes relevant to the study of the multiplicative coalescent? (Section 2, p. 110)

CHAPTER 1

Markovian Bridges

In this chapter we will give a construction of Markovian bridges where weak continuity is emphasized. A Markovian bridge is a probability measure taken from a disintegration of the law of an initial part (of non-random length) of the trajectory of a Markov process given its future behaviour. As such, Markovian bridges admit a natural parametrization in terms of the state space of the process and of the length of the trajectory. The emphasis on weak convergence is explained by the fact that the parametrized collection of probability measures is not uniquely determined unless an additional restriction on it, such as weak continuity with respect to the spatial index, is imposed. It turns out that weak convergence considerations are also useful to the construction of Markovian bridges and lead to approximation theorems for them by means of conditionings of the original Markov process. Once such a weakly continuous collection of Markovian bridges is constructed, each element satisfies an inhomogeneous Markov property. The construction of Markovian bridges can be further generalized to other types of disintegrations of Markovian laws. An example of this would be the construction of first-passage bridges which disintegrate the law of a Markov process stopped at its first hitting-time of a set given the value of the hitting time and the terminal value of the process.

Markovian bridges also appear when disintegrating the initial part of a trajectory of random length of a Markov process (under some restrictions on the random length) and this allows for a statement dual to the usual Strong Markov Property when the direction of time is reversed. We will state and prove such a property, which is termed here the backward strong Markov property. The use of this theorem leads to conceptual proofs of properties of Brownian motion including a construction of the normalized Brownian excursion from Brownian motion. As will be exemplified, the backward strong Markov property is very useful when tied to additional self-similarity assumptions.

All these notions will be applied in the construction and analysis of Brownian related Markov processes, with special regards towards the normalized Brownian excursion to be introduced in Chapter 2. Much of the material presented is classical although many of the proofs are original and so chosen because they share a common technique: the use of Markovian bridges. In the final section we point out how the selected material is related to the published literature.

Considerations of weak continuity usually require spaces to be Polish, that is, separable and complete, so that individual laws are tight. Also, weak continuity of a Markovian family of probability laws usually requires the Feller property and metric spaces that are locally compact and with a countable base (LCCB) so that in particular they are Polish. Our setting will therefore be that of Feller processes on an LCCB metric space where we will impose conditions under which we can speak of Markovian bridges: the existence of transition densities.

We will assume that the reader is familiar with weak convergence theory and Feller processes. This chapter was written with [**Kal02**] in mind as a reference for a succint exposition of the basic facts of both theories.

1. Construction and Weak Continuity of Markovian Bridge Laws

In this chapter, we will work on an arbitrary locally compact metric space with a countable base (or LCCB for short) denoted (S,ρ). More particularly, we will consider a **Markovian family** of probability measures on this space which satisfy the Feller property, by which the following is meant. Let D_∞ stand for the Skorohod space of càdlàg functions from $[0,\infty)$ into S and consider on it the shift operators $\theta_t : D_\infty \to D_\infty$ given by $\theta_s f : s' \to f(s+s')$. Let $X = (X_s)_{s\geq 0}$ denote the canonical process, and write $\mathscr{F}$ and $(\mathscr{F}_s)_{s\geq 0}$ for the σ-field and the canonical filtration generated by X.

Definition. A **Markovian family** on (S,ρ) is a collection of probability measures $(\mathbb{P}_x)_{x\in S}$ on D_∞ indexed by the elements of S which satisfies

Starting Point Property: For all $x \in S$:

$$\mathbb{P}_x(X_0 = x) = 1.$$

Measurability Property: For all $F \in b\mathscr{F}$,

$$x \mapsto \mathbb{E}_x(F)$$

is measurable.

Markov Property: For every $F \in b\mathscr{F}_s$ and every $G \in b\mathscr{F}$:

$$\mathbb{E}_x(F \cdot G \circ \theta_s) = \mathbb{E}_x(F \cdot \mathbb{E}_{X_s}(G))\,.$$

A Markovian family $(\mathbb{P}_x)_{x\in S}$ is said to satisfy the Feller property (and we will therefore speak of a **Feller-Markov** family) if the operators $(P_s)_{s\geq 0}$ defined on $b\mathscr{B}_S$ by means of $P_s f(x) = \mathbb{E}_x(f(X_s))$ are an extension of a Fellerian semigroup.

Of course, Feller-Markov families are in bijection with (conservative) Feller semigroups.

Let us provide the notion of Markovian bridge and the heuristics associated to its construction. We seek to build a version of the conditional law of $(X_s)_{s\leq t}$ given $X_t = y$ under $\mathbb{P}_x$, which call Markovian bridge from x to y of length t. One could appeal to the general theorem on existence of regular conditional distributions (see for example [**Kal02**, Thm. 6.3, p.107]), but that result builds the whole family of conditional laws as y varies and does not give control over individual conditional laws. Since we are working on a Polish space, we might impose further regularity conditions on conditional laws such as their weak continuity as y varies; since there is at most one weakly continuous disintegration, this singles out specific conditional laws. This will be the strategy we will follow. To that end, fix $x \in S$ and consider a Feller-Markov family $(\mathbb{P}_x)_{x\in S}$ on (S, ρ) and its associated semigroup $P = (P_s)_{s\geq 0}$ and suppose that P_s admits a transition density $p_s(x, \cdot)$ with respect to a σ-finite measure μ on (S, ρ) in the sense that

$$P_s f(x) = \int f(y)\, p_s(x, y)\ \mu(dy)\,.$$

Let $0 < s < t$ and note that for every $F \in b\mathscr{F}_s$ and every $f \in b\mathscr{B}_S$ the Markov property and the Tonelli-Fubini theorem imply

$$\begin{aligned}\mathbb{E}_x(F \cdot f(X_t)) &= \mathbb{E}_x(F \cdot P_{t-s}f(X_s)) \\ &= \int f(y)\, \mathbb{E}_x(F \cdot p_{t-s}(X_s, y))\ \mu(dy)\,.\end{aligned}$$

By restricting the last integral to

$$\mathscr{P}_t = \{y \in S : p_t(x, y) > 0\}\,,$$

we obtain our base formula

$$\mathbb{E}_x(F \cdot f(X_t)) = \int_{\mathscr{P}_t} \mathbb{E}_x\left(F \cdot \frac{p_{t-s}(X_s, y)}{p_t(x, y)}\right) f(y)\, p_t(x, y)\ \mu(dy)\,.$$

To construct a version of the conditional law of $(X_s)_{s\leq t}$ given $X_t = y$ under $\mathbb{P}_x$, one could therefore seek to build a law $\mathbb{P}^t_{x,y}$ on the Skorohod space of càdlàg trajectories of $[0, t]$ into S, denoted D_t, such that for every $s < t$, $\mathbb{P}^t_{x,y}$ is absolutely continuous with respect to $\mathbb{P}_x$ with Radon-Nikodým density $M^s_{x,y}$ given by

$$M^s_{x,y} = \frac{d\mathbb{P}^t_{x,y}|_{\mathscr{F}_s}}{d\mathbb{P}_x|_{\mathscr{F}_s}} = \frac{p_{t-s}(X_s, y)}{p_t(x, y)}, \tag{1}$$

because for such measures the equality

$$\mathbb{E}_x(F \cdot f(X_t)) = \int_{\mathscr{P}_t} \mathbb{P}^t_{x,y}(F)\, f(y)\, p_t(x, y)\ \mu(dy) \tag{2}$$

would follow for $s < t$. Equation 2 contains a disintegration of the law of $(X_r)_{r<s}$ with respect to X_t under $\mathbb{P}_x$. The laws $\mathbb{P}^t_{x,y}$ are usually called bridges since under clearly stated hypotheses, the starting point condition

$$\mathbb{P}^t_{x,y}(X_0 = x) = 1$$

as well as the **ending point condition**

$$\mathbb{P}^t_{x,y}(X_{t-} = X_t = y) = 1 \tag{3}$$

are satisfied. This explains why, even if we succeed at constructing such a law $\mathbb{P}^t_{x,y}$, the local absolute continuity relationship (1) would not hold for $s = t$, unless of course the law of X_t under $\mathbb{P}_x$ charges y and for the examples we shall consider this is not the case. However, if we can build the laws $\mathbb{P}^t_{x,y}$ satisfying the local absolute continuity relationship (1) and the ending point condition (3) we can extend (2) to $s = t$ by the following argument. Let $\sigma_t : \cup_{s>t} D_s \to D_t$ be defined by

$$\sigma_t f(s) = \begin{cases} f(s) & \text{if } s < t \\ f(t-) & \text{if } s \geq t \end{cases}.$$

Then the ending point condition (3) implies that for every $F \in b\mathscr{F}_t$, $\mathbb{P}^t_{x,y}(F = F \circ \sigma_t) = 1$ and, since Feller processes do not jump at fixed times, $\mathbb{P}_x(F = F \circ \sigma_t) = 1$. The disintegration (2) can be extended to $\mathscr{F}_{t-} = \sigma(X_s : s < t)$ by a monotone class argument and if $F \in b\mathscr{F}_t$ then $F \circ \sigma_t \in b\mathscr{F}_{t-}$ so that:

$$\begin{aligned} \mathbb{P}_x(F f(X_t)) &= \mathbb{P}_x(F \circ \sigma_t f(X_t)) \\ &= \int_{\mathscr{P}_t} \mathbb{P}^t_{x,y}(F \circ \sigma_t)\, f(y)\, p_t(x,y)\ \mu(dy) \\ &= \int_{\mathscr{P}_t} \mathbb{P}^t_{x,y}(F)\, f(y)\, p_t(x,y)\ \mu(dy)\,. \end{aligned}$$

To continue our discussion of bridges, recall that weak continuity of the bridge laws is implied by tightness and weak continuity of one-dimensional distributions. Weak continuity of one-dimensional distributions is implied by continuity in variation, which is implied by continuity of the densities by Scheffe's lemma. Hence, the following hypotheses are not far fetched:

(H1): $y \mapsto p_s(x,y)$ is continuous for all $s \in (0,t]$.
(H2): The Chapman-Kolmogorov equations

$$p_t(x,y) = \int p_{t-s}(x,z)\, p_s(z,y)\ \mu(dz)$$

hold for each $y \in \mathscr{P}_t$, and for $0 < s < t$.

Together, they imply the weak-continuity of finite dimensional distributions, at least for times $s < t$, since the first implies the almost sure convergence

$$M^s_{x,z} \to M^s_{x,y}$$

as $z \to y$ under $\mathbb{P}_x$, and the second one implies the applicability of Scheffe's lemma, since it implies that the integral of $M^s_{x,y}$ with respect to $\mathbb{P}_x$ is equal to 1.

Another technical hypothesis, needed to deduce tightness of bridge laws, is the following:

(H3): $s \mapsto p_s(x, y)$ is continuous for all $x, y \in S$.

Under the set of hypotheses **H1-H3** we will prove our basic existence result.

THEOREM 1. *On D_t, the laws $\mathbb{P}_x(\cdot \mid X_t \in B_\delta(y))$ converge weakly as $\delta \to 0$ to a law $\mathbb{P}^t_{x,y}$ which satisfies the following three conditions*

(A) the local absolute continuity relationship (1),

(B) the ending point condition (3)*, and*

(C) $y \mapsto \mathbb{P}^t_{x,y}$ is weakly continuous.

PROOF. Property B is immediate once we prove weak convergence, so let us focus on the latter, in the usual way, by establishing tightness and the convergence of the finite-dimensional distributions. Some technical preliminaries are needed.

Let us first see that the support of μ is S: let $y \in S$ and consider $\delta > 0$. Then, there exists $t > 0$ such that

$$\mathbb{P}_y(X_t \in B_\delta(y)) > 0$$

since X_t converges in probability to y as $t \to 0$ under $\mathbb{P}_y$, because of the Feller property. Since

$$\mathbb{P}_y(X_t \in B_\delta(y)) = \int_{B_\delta(y)} p_t(y, z)\ \mu(dz)\,,$$

it follows that $\mu(B_\delta(y)) > 0$.

Now we will obtain the approximation

$$\lim_{\delta \to 0, z \to y} \frac{\mathbb{P}_x(X_s \in B_\delta(z))}{\mu(B_\delta(z))} = p_s(x, y)\,. \tag{4}$$

of the transition density p_s. Since $p_s(x, \cdot)$ is continuous at y, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|p_s(x, y) - p_s(x, z)| < \varepsilon$ for all $z \in B_\delta(y)$. Therefore, for all $\delta' < \delta/2$ and all $z \in B_{\delta/2}(y)$:

$$\left| p_s(x, y) - \frac{1}{\mu(B_{\delta'}(z))} \int_{B_{\delta'}(z)} p_s(x, z')\ \mu(dz) \right| < \varepsilon,$$

so that (4) holds.

The next step is to note that if $y \in \mathscr{P}_t$ then for all $\delta > 0$,

$$\mathbb{P}_x(X_t \in B_\delta(z)) > 0.$$

This is because, by hypothesis **H1**, there exists δ_0 such that $p_t(x,z) > 0$ for all $z \in B_{\delta_0}(y)$. Therefore, for all $\delta \leq \delta_0$,

$$\mathbb{P}_x(X_t \in B_\delta(y)) = \int_{B_\delta(y)} p_t(x,z)\ \mu(dz) > 0$$

since otherwise, $\mu(B_\delta(y)) = 0$.

We will now take care of Property A. For any $F \in b\mathscr{F}_s$ where $s < t$, the Markov property implies the equality

$$\mathbb{P}_x(\,F \mid X_t \in B_\delta(y)) = \mathbb{E}_y\left(F \cdot \frac{\mathbb{P}_{X_s}(X_{t-s} \in B_\delta(y))}{\mathbb{P}_x(X_t \in B_\delta(y))}\right),$$

the right-hand side of which converges to

$$\mathbb{P}_x\left(F \cdot \frac{p_{t-s}(X_s,y)}{p_t(x,y)}\right)$$

because of (4), and Scheffe's lemma. The latter is applicable because of the Chapman-Kolmogorov equations. From this, we conclude something quite a bit stronger than the convergence of finite-dimensional distributions: for any $s < t$, the law of $(X_r)_{r \leq s}$ converges in variation (hence weakly) to a law $\mathbb{P}^{t,s}_{x,y}$ on D_s such that

$$\mathbb{P}^{t,s}_{x,y}(A) = \mathbb{E}_x\left(\mathbf{1}_A \cdot \frac{p_{t-s}(X_s,y)}{\mathbb{P}_t(x,y)}\right).$$

In particular, if $\tilde{\omega}(f,t,h)$ denotes the so-called modified modulus of continuity on D_t given by

$$\tilde{\omega}(f,t,h) = \inf_{\{t_i\}} \max_i \max_{s,s' \in [t_{i-1},t_i)} \rho(f(s)\,, f(s'))$$

where the infimum extends over all partitions

$$0 = t_0 < t_1 < \cdots < t_n = t$$

such that $t_i - t_{i-1} > h$, then the above functional weak convergence implies the following condition: for all $\varepsilon > 0$ and $s < t$

$$\lim_{h \to 0} \limsup_{\delta \to 0} \mathbb{P}_x(\,\tilde{\omega}(X,s,h) > \varepsilon \mid X_t \in B_\delta(y)) = 0. \tag{5}$$

We will use (5) to study the tightness of our approximations

$$\mathbb{P}_x(\cdot \mid X_t \in B_\delta(y))$$

as $\delta \to 0$. Let $Z_h = \sup_{s,s' \in [0,h]} \rho(X_s, X_{s'})$. It suffices, in view of the convergence of finite-dimensional distributions on $[0,s)$ and the fact that the law of X_t under the approximating law converges weakly to unit mass at y so that all finite-dimensional distributions converge, to verify the following for all $\varepsilon > 0$:

$$\lim_{h\to 0} \lim_{\delta \to 0} \mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon \mid X_t \in B_\delta(y)) = 0.$$

To that end, we will now prove a technical result displayed in (6). By the Feller property, for any compact set $K \subset S$, the laws $(\mathbb{P}_z)_{z\in K}$ are weakly continuous on D_h with respect to z. Since for each individual law

$$\lim_{h\to 0} \mathbb{P}_z(Z_h > \varepsilon) \to 0$$

and $z \mapsto \mathbb{P}(Z_h > \varepsilon)$ is continuous (because Feller processes do not jump at fixed times and Z_h seen as a functional on D_∞ is continuous at f if f is continuous at h) and increasing in h, then

$$\lim_{h\to 0} \sup_{z \in K} \mathbb{P}_z(Z_h > \varepsilon) \to 0. \tag{6}$$

Otherwise, there would be two sequences, (z_n) in K and (h_n) decreasing to zero, such that

$$\liminf_{n\to 0} \mathbb{P}_{z_n}(Z_{h_n} > \varepsilon) > 0.$$

However, since K is compact, there exists a subsequence (z_{n_k}) converging to $z \in K$ and because Feller processes do not admit fixed-time discontinuities and have càdlàg paths:

$$\begin{aligned} 0 < \liminf_{k\to\infty} \mathbb{P}_{z_{n_k}}\left(Z_{h_{n_k}} > \varepsilon\right) &\leq \liminf_{m\to\infty} \lim_{k\to\infty} \mathbb{P}_{z_{n_k}}(Z_{h_m} > \varepsilon) \\ &= \lim_{m\to\infty} \mathbb{P}_z(Z_{h_m} > \varepsilon) = 0. \end{aligned}$$

To continue our main line of argument, note that by local compactness, there exists a $\delta > 0$ such that $B_\delta(y)$ has compact closure. We will write

$$\begin{aligned}&\mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon \mid X_t \in B_\delta(y)) \\ &= \mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_{t-h} \in B_\delta(y) \mid X_t \in B_\delta(y)) \\ &+ \mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_{t-h} \notin B_\delta(y) \mid X_t \in B_\delta(y))\end{aligned}$$

and bound each one of the summands of the right-hand side. For the first one, use Bayes rule

$$\begin{aligned}&\mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_{t-h} \in B_\delta(y) \mid X_t \in B_\delta(y)) \\ &= \mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_t \in B_\delta(y) \mid X_{t-h} \in B_\delta(y)) \, \frac{\mathbb{P}_x(X_{t-h} \in B_\delta(y))}{\mathbb{P}_x(X_t \in B_\delta(y))}.\end{aligned}$$

However, in view of the Markov property, the technical result of the last paragraph, hypothesis **H3** and the transition density approximation (4):

$$\begin{aligned}&\mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_t \in B_\delta(y) \mid X_{t-h} \in B_\delta(y)) \\ &\le \frac{\mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_{t-h} \in B_\delta(y))}{\mathbb{P}_x(X_{t-h} \in B_\delta(y))} \\ &\le \sup_{z \in B_\delta(y)} \mathbb{P}_z(Z_h > \varepsilon)\end{aligned}$$

and

$$\lim_{h\to 0} \lim_{\delta \to 0} \frac{\mathbb{P}_x(X_{t-h} \in B_\delta(y))}{\mathbb{P}_x(X_t \in B_\delta(y))} = \lim_{h \to 0} \frac{p_{t-h}(x,y)}{p_t(x,y)} = 1,$$

so that

$$\lim_{h\to 0} \limsup_{\delta \to 0} \mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_{t-h} \in B_\delta(y) \mid X_t \in B_\delta(y)) = 0.$$

We will now obtain a second bound by means of

$$\begin{aligned}&\mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_{t-h} \notin B_\delta(y) \mid X_t \in B_\delta(y)) \\ &\le \mathbb{P}_x(X_{t-h} \notin B_\delta(y) \mid X_t \in B_\delta(y)) \\ &= 1 - \frac{\mathbb{P}_x(X_{t-h} \in B_\delta(y)\,, X_t \in B_\delta(y))}{\mathbb{P}_x(X_{t-h} \in B_\delta(y))} \frac{\mathbb{P}_x(X_{t-h} \in B_\delta(y))}{\mathbb{P}_x(X_t \in B_\delta(y))};\end{aligned}$$

we have already seen that if $\delta \to 0$ and then $h \to 0$, the second factor in the right-hand side of the last equality converges to 1. To study the first factor, write it as

$$1 - \frac{\mathbb{P}_x(X_{t-h} \in B_\delta(y)\,, X_t \notin B_\delta(y))}{\mathbb{P}_x(X_{t-h} \in B_\delta(y))}$$

and use the Feller property in the following manner: for δ small enough (so that $B_\delta(y)$ has compact closure) and $\delta' \in (0, \delta)$, let $\phi : S \to [0,1]$ be a continuous function which is equal to 1 on $B_{\delta'}(y)$ and vanishes outside $B_\delta(y)$; since ϕ is continuous and vanishes at infinity, the Feller property implies that for all $z \in B_{\delta'}(y)$

$$\mathbb{P}_z(X_h \notin B_\delta(y)) \leq \mathbb{E}_z(1 - \phi(X_h)) = |\mathbb{E}_z(\phi(X_h)) - \phi(z)| \leq \|P_h - \mathrm{Id}\,\|.$$

Since the previous estimation does not depend on $\delta' < \delta$, our conclusion is that it holds for all $z \in B_\delta(y)$ and so, by the Markov property:

$$\frac{\mathbb{P}_x(X_{t-h} \in B_\delta(y))\,, X_t \notin B_\delta(y)}{\mathbb{P}_x(X_{t-h} \in B_\delta(y))} \leq \|P_h - \mathrm{Id}\,\|.$$

We finally obtain

$$\lim_{h\to 0} \lim_{\delta \to 0} \mathbb{P}_x(\, Z_h \circ \theta_{t-h} > \varepsilon, X_{t-h} \notin B_\delta(y) \mid X_t \in B_\delta(y)) = 0,$$

which implies the existence of a law $\mathbb{P}^t_{x,y}$ on D_t to which $\mathbb{P}_x(\,\cdot \mid X_t \in B_\delta(y))$ converges weakly to as $\delta \to 0$. As we have already remarked, $\mathbb{P}^t_{x,y}$ satisfies the local absolute continuity relationship (1). It also satisfies the ending point condition since the law of X_t under $\mathbb{P}_x$ conditionally on $\{X_t \in B_\delta(y)\}$ is concentrated on $B_\delta(y)$.

To conclude the proof of the theorem, we must examine the weak continuity of $\mathbb{P}^t_{x,y}$ as y varies. To do it, we will prove that if $K \subset S$ is compact in $\mathscr{P}_t$ then $\left(\mathbb{P}_x(\,\cdot \mid X_t \in B_\delta(z))\right)_{z\in K, \delta>0}$ is tight in D_t. If this is true then $\left(\mathbb{P}^t_{x,z}\right)_{z\in K}$ will be tight and because as $z \to y \in \mathscr{P}_t$, $\mathbb{P}^t_{x,z}$ converges in variation to $\mathbb{P}^t_{x,y}$ on D_s and the ending point condition is satisfied, then the finite-dimensional distributions of $\mathbb{P}^t_{x,z}$ converge to those of $\mathbb{P}^t_{x,y}$ and therefore, there is also weak convergence. To analyze the tightness of $(\mathbb{P}_x(\,\cdot \mid X_t \in B_\delta(z)))_{z\in K, \delta>0}$, we note that tightness holds on D_s for each $s < t$, so that it suffices to prove, for all $\varepsilon > 0$:

$$\lim_{h\to 0} \lim_{\delta\to 0, z\to y} \mathbb{P}_x(\, Z_h \circ \theta_{t-h} > \varepsilon \mid X_t \in B_\delta(z)) = 0.$$

Our previous arguments can be extended to this case, since by the density approximation (4):

$$\lim_{h\to 0}\lim_{\delta\to 0, z\to y}\frac{\mathbb{P}_x(X_{t-h}\in B_\delta(z))}{\mathbb{P}_x(X_t\in B_\delta(z))}=1$$

and for sufficiently small δ (so that $B_{2\delta}(y)$ has compact closure) and $z\in B_\delta(y)$, we have that

$$\lim_{h\to 0}\sup_{z'\in B_\delta(z)}\mathbb{P}_{z'}(X_h\notin B_\delta(z))\le\lim_{h\to 0}\sup_{z'\in B_{2\delta}(y)}\mathbb{P}_{z'}(X_h\notin B_{2\delta}(y))=0$$

by (6) and

$$\mathbb{P}_x(\,X_t\notin B_\delta(z)\mid X_{t-h}\in B_\delta(z))\le\|P_h-\operatorname{Id}\|.$$

□

Finally, we will use a stronger hypothesis to study joint weak continuity in the ending point and the length. However, since bridge laws associated to different lengths are defined on different Skorohod spaces, we need to specify the interpretation of weak continuity we will use. For every $f\in D_t$, we can associate the function $f^t\in D_\infty$ given by $f^t(s)=f(s\wedge t)$. This mesurable mapping will be denoted by i_t and we will say that the sequence of measures $\mathbb{P}_n^{t_n}$ on D_{t_n} converge weakly if $\mathbb{P}_n^{t_n}\circ i_{t_n}^{-1}$ converges weakly in D_∞. To simplify notations, from this point on, we will think of bridge measures as defined on D_∞ by identifying $\mathbb{P}^t_{x,y}$ with $\mathbb{P}^t_{x,y}\circ i_t^{-1}$.

A technical hypothesis, which supersedes **H1** and **H3** and is related to the joint weak continuity of bridge laws with respect to the ending point and the length, is the following:

(H1'): $(s,y)\mapsto p_s(x,y)$ is continuous for all $x\in S$.

Lemma 1. *Under* $\mathbf{H1'}$ *and* $\mathbf{H2}$*: the bridge laws* $\left(\mathbb{P}^t_{x,y}\right)$ *are jointly continuous in* y *and* t*.*

Proof. Let us prove that as $t'\to t$ and $z\to y$ (in $\mathscr{P}_t$), $\mathbb{P}^{t'}_{x,z}$ converges in law to $\mathbb{P}^t_{x,y}$. As in Theorem 1, under **H1'** we have convergence in variation of $\mathbb{P}^{t'}_{x,z}|_{\mathscr{F}_s}$ to $\mathbb{P}^t_{x,y}|_{\mathscr{F}_s}$ if $s<t$ and, because of the ending point condition, this implies not only the convergence of the finite-dimensional distributions but also a tightness criterion on the compact intervals of $[0,\infty)\setminus\{t\}$. Hence, we must only prove the following for all $\varepsilon>0$:

$$\lim_{h\to 0}\lim_{\delta\to 0, z\to y, t'\to t}\mathbb{P}_x(\,Z_h\circ\theta_{t'-h}>\varepsilon\mid X_{t'}\in B_\delta(z))=0.$$

Again, we can use the same arguments as in Theorem 1 since under **H1**$'$, (4) can be generalized to:

$$\lim_{\delta \to 0, z \to y, s' \to s} \frac{\mathbb{P}_x(X_{s'} \in B_\delta(z))}{\mu(B_\delta(z))} = p_s(x, y) \,.$$

The other bounds needed did not depend on the length parameter t'. □

We could, by the same techniques and **H1'** to joint continuity of transition densities in all variables, handle joint continuity in (t, x, y). This will be used in Chapter 2.

Let us note that if $F : D_\infty \to \mathbb{R}$ is measurable then the function which is equal to $\mathbb{P}^t_{x,y}(F)$ if $p_t(x, y) > 0$ and equal to zero otherwise is measurable. First, let us note that the set $\{(t, x, y) : p_t(x, y) > 0\}$ is measurable because $(t, x, y) \mapsto p_t(x, y)$ is measurable, since it is jointly continuous in (t, y) for fixed x and measurable in x for fixed (t, y), by the measurability property of Markovian families and the density approximation (4) implied by hypothesis **H1'**. For the rest of the argument, we will work on the set $\{(t, x, y) : p_t(x, y) > 0\}$. Let us note that if $F \in b\mathscr{F}_s$ and $s < t$, then the local absolute continuity relationship (1) implies that $x \mapsto \mathbb{P}^t_{x,y}(F)$ is measurable and by the monotone class theorem, we see that the measurability extends first to any $F \in b\mathscr{F}_t$ and then to any measurable F. Since by Lemma 1, $(t, y) \mapsto \mathbb{P}^t_{x,y}(F)$ is continuous if F is, we see that $(t, x, y) \mapsto \mathbb{P}^t_{x,y}(F)$ is measurable whenever F is continuous. By a monotone class argument, the preceding measurability extends to measurable F. This will be used in Section 2.

As a final comment, let us go back to equation (1), we note that the bridge law $\mathbb{P}^t_{x,y}$ is a space-time harmonic transform (also called h-transform) of $\mathbb{P}_x$; let us be more precise.

Definition. If the funcion $h : S \to \mathbb{R}$ is excessive (for Feller-Markov families this is equivalent to the supermartingale character of $(f(X_s))_{s \geq 0}$ under $\mathbb{P}_x$ for every $x \in S$) we define a new sub-Markovian semigroup $P^h = \left(P^h_t\right)_{t \geq 0}$ on $\left(S^h, d\right)$ where $S^h = \{0 < h < \infty\}$ and

$$P^h_t f = P_t(hf)\,/h.$$

This semigroup is called the h**-transform** of P.

The terminology is also used when a Markov process or a Markov family are associated to the semigroup P^h. A general discussion of h transforms is found in [**DM87**], see paragraphs XII.4.75 (p. 59) and XIV.2.28-34 (spanning pages 329-335). Another discussion is presented in [**RW00**, III.45, p.296]. The link with equation (1) is that $(\mathbb{P}_{t-s}(X_s, y))_{s\in[0,t)}$ is a supermartingale under $\mathbb{P}_x$; since (X_s, s) is called the space-time process associated to X, we might think of the space-time process under $\mathbb{P}^t_{x,y}$ as an h-transform of the space-time process under $\mathbb{P}_x$. Note however that the supermartingale we are using is indexed by $[0,t)$ instead of $[0,\infty)$.

2. The Backward Strong Markov Property

In this section, we will analyze a generalization of the usual Strong Markov Property for Feller processes in which Markovian bridges play a prominent role. This is possible since the Markov property can be expressed in a way that highlights its symmetry with respect to the direction in which time flows. The key concept in this expression is that of conditional independence; it leads naturally to the introduction of a temporal window in the usual Markov property in which Markovian bridges first appear. When used with the weak continuity of Markovian bridges, the deterministic temporal window can be substituted for a random one, leading to the concept of a Backward Optional Time and then to the Backward Strong Markov Property.

Let us first discuss the concept of conditional independence of σ-fields.

Definition. In a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, we will say that **two subσ-fields** (of $\mathscr{F}$) $\mathscr{G}_1$ and $\mathscr{G}_2$ **are conditionally independent given a third one** $\mathscr{H}$, denoted by

$$\mathscr{G}_1 \underset{\mathscr{H}}{\perp} \mathscr{G}_2,$$

if for every $G_1 \in b\mathscr{G}_1$ and $G_2 \in b\mathscr{G}_2$:

$$\mathbb{E}(G_1 \cdot G_2|\mathscr{H}) = \mathbb{E}(G_1|\mathscr{H}) \cdot \mathbb{E}(G_2|\mathscr{H}) .$$

As for traditional independence (to which the conditional variety particularizes to when the conditioning σ-field is trivial), we can define conditional independence relating to collections of sets or random variables (or both) as that of the generated σ-fields. Note that the smaller the conditioning class, the closer we are to independence.

A basic summary of the properties of conditional independence which we will need is the following:

Proposition 1. *The σ-fields $\mathscr{G}_1$ and $\mathscr{G}_2$ are conditionally independent given $\mathscr{H}$ if and only if for all $G \in b\mathscr{G}_1$:*

$$\mathbb{E}(G|\mathscr{G}_2, \mathscr{H}) = \mathbb{E}(G|\mathscr{H})\,.$$

Furthermore for any σ-fields $\mathscr{H}, \mathscr{G}, \mathscr{G}_1, \mathscr{G}_2, \ldots$, the following conditions are equivalent:

(i) $\mathscr{G} \underset{\mathscr{H}}{\perp} \mathscr{G}_1, \mathscr{G}_2, \ldots$.

(ii) For any $n \geq 1$*:* $\mathscr{G} \underset{\mathscr{H}, \mathscr{G}_1, \ldots, \mathscr{G}_n}{\perp} \mathscr{G}_{n+1}$.

Finally, if $\mathscr{G}_1 \perp_{\mathscr{H}} \mathscr{G}_2$ *and* $\mathscr{G}_1' \subset \mathscr{G}_1$ *then*

$$\mathscr{G}_1' \underset{\mathscr{H}}{\perp} \mathscr{G}_2 \text{ and } \mathscr{G}_1 \underset{\mathscr{H}, \mathscr{G}_1'}{\perp} \mathscr{G}_2$$

The first property is the **asymmetric expression of conditional independence** and is the link between conditional independence and the Markov property, as will be expanded upon. The second of the above properties will be refered to as the **chain rule for conditional independence**. The third property consists of the **downwards monotone character of conditional independence** in the non-conditioning σ-fields and a **partial upwards monotone character in the conditioning σ-field**. It is a trivial application of the chain rule since under the conditions stated, $\sigma(\mathscr{G}_1, \mathscr{G}_1') = \sigma(\mathscr{G}_1)$. We cannot expect a general upwards monotone character to hold: for example, if X and Y are two independent random variables on $\{-1, 1\}$ which take the two values with equal probability, and $Z = XY$, then X and Y are independent but they are not conditionally independent given Z, since the conditional law of Y given X, Z is concentrated at XZ and the conditional law of Y given Z is the same as that of Y since Y and Z are independent. The following formulation of the preceding example might be more impressive. Let $\mathscr{H}_1 \subset \mathscr{H}_2 \subset \mathscr{H}_3$ then

$$\mathscr{G}_1 \underset{\mathscr{H}_1}{\perp} \mathscr{G}_2 \text{ and } \mathscr{G}_1 \underset{\mathscr{H}_3}{\perp} \mathscr{G}_2 \text{ do not imply } \mathscr{G}_1 \underset{\mathscr{H}_2}{\perp} \mathscr{G}_2.$$

To relate this section with the last one, let us consider a Markovian family $(\mathbb{P}_x)_{x \in S}$ on an LCCB space (S, ρ) associated with the conservative Feller semigroup $(T_t)_{t \geq 0}$. We recall the notation for the canonical space

D_∞ on which $\mathbb{P}_x$ is defined, as well as for the canonical process X, the canonical filtration $(\mathscr{F}_t)_{t\geq 0}$ and the shift operators $(\theta_t)_{t\geq 0}$ on D_∞. Let us define, for a fixed time t, the σ-fields associated to the past before time t, $\mathscr{F}_t$, and to the future after time t, $\mathscr{F}^t = \sigma(X_s : s \geq t)$. Since $\mathscr{F}^t = \sigma(X \circ \theta_t)$, the Markov property implies that for any $G \in b\mathscr{F}^t$,

$$\mathbb{P}_x(G \mid \mathscr{F}_t) = \mathbb{P}_x(G \mid X_t)\,.$$

This is precisely the asymmetric expression of the conditional independence

$$\mathscr{F}_t \underset{X_t}{\perp} \mathscr{F}^t$$

which amounts to saying that the past and the future are conditionally independent given the present. This expression is invariant under change in the direction of the flow of time. Now we will introduce a temporal window in this conditional independence by defining, for $s < t$, the σ-field $\mathscr{F}_t^s = \sigma(X_r : r \in [s,t])$.

Lemma 2. *Under $\mathbb{P}_x$, the σ-fields $\mathscr{F}_t^s$ and $\mathscr{F}_s \vee \mathscr{F}^t$ are conditionally independent given X_s, X_t.*

Proof. According to the chain rule for conditional independence, we need to verify the following:

$$\mathscr{F}_t^s \underset{X_s,X_t}{\perp} \mathscr{F}^t \text{ and } \mathscr{F}_t^s \underset{\mathscr{F}^t,X_s,X_t}{\perp} \mathscr{F}_s. \tag{7}$$

The first conditional independence in (7) holds because $\mathscr{F}_t \perp_{X_t} \mathscr{F}^t$, so that the inclusion $\mathscr{F}_t^s \subset \mathscr{F}_t$ implies $\mathscr{F}_t^s \perp_{X_t} \mathscr{F}^t$, which implies the desired conclusion since X_s is $\mathscr{F}_t^s$-measurable.

The second conditional independence in (7) is obtained as follows: for $s < t$, write $\mathscr{F}^s = \mathscr{F}^t \vee \mathscr{F}_t^s$, so that the chain rule, applied to $\mathscr{F}_s \perp_{X_s} \mathscr{F}^s$, allows us to obtain $\mathscr{F}_s \perp_{X_s,\mathscr{F}^t} \mathscr{F}_t^s$; to conclude, it suffices to apply the partial upwards character of conditional independence since X_t is $\mathscr{F}_t^s$-measurable. □

Not only does the Markov property imply conditional independence, as we have already argued, it also gives a precise description of the conditional law of events in $\mathscr{F}^t$ given X_t. To complement the preceding lemma in the same direction, let us place ourselves under **H1-H3**, and note that thanks to the Markov property, we obtain:

The conditional law of $X^{s,t} = \big(X_{(r+s)\wedge t}\big)_{r\geq 0}$ given X_s, X_t under $\mathbb{P}_x$ is $\mathbb{P}^{t-s}_{X_s,X_t}$.

This is because for any $F \in b\mathscr{F}_{t-s}$ and any $f, g \in b\mathscr{B}_S$:

$$\begin{aligned}\mathbb{E}_x\big(F\big(X^{s,t}\big)\, f(X_s)\, g(X_t)\big) &= \mathbb{E}_x\big(f(X_s)\, \mathbb{E}_{X_s}\big(F\big(X^{0,t-s}\big)\, g(X_{t-s})\big)\big)\\ &= \mathbb{E}_x\Big(f(X_s)\, \mathbb{E}_{X_s}\Big(\mathbb{P}^{t-s}_{X_0,X_{t-s}}(F)\, g(X_{t-s})\Big)\Big)\\ &= \mathbb{E}_x\Big(f(X_s)\, \mathbb{P}^{t-s}_{X_s,X_t}(F)\, g(X_t)\Big).\end{aligned}$$

We shall generalize the preceding conditional description to a *strong Markov property with respect to future events*. Actually the method of proof will be analogous to a known one for the Strong Markov Property: we will discretize the problem, then we shall use the local property of conditional expectation (to be stated shortly), and finally, continuity considerations will be used to transport conclusions of the discrete setup to the continuous one. The target result needs the following:

DEFINITION. A **backward optional time** is a random variable $L : D_\infty \to [0,\infty]$ such that $\{L > t\} \in \mathscr{F}^t$ for all $t > 0$.

For a backward optional time L, the **σ-field of events ocurring after** L, denoted $\mathscr{F}^L$, is defined to be $\sigma(X \circ \theta_L)$.

As a first example, let us note that if $U \subset S$ is open, then the **last visit to** U equal to zero if X is never in U and equal to

$$\sup\{s \geq 0 : X_s \in U\}$$

otherwise is a backward optional time. A second example would be the last visit to an open set (just) before a fixed time t given by

$$L^t_U = \begin{cases} 0 & \text{if } \{s < t : X_s \in U\} = \emptyset\\ \sup\{s < t : X_s \in U\} & \text{otherwise}\end{cases}.$$

They are backward optional times since by the right-continuous character of the trajectories,

$$\{L_U > t\} = \bigcup_{s>t}\{X_s \in U\} = \bigcup_{s>t, s\in\mathbb{Q}}\{X_s \in U\} \in \mathscr{F}^t$$

and for $s < t$

$$\left\{L_U^t > s\right\} = \bigcup_{s' \in (s,t)} \{X_{s'} \in U\} = \bigcup_{s' \in (s,t), s' \in \mathbb{Q}} \{X_{s'} \notin U\} \in \mathscr{F}^s.$$

The first example belongs to the following class of random times, which are all backward optional times:

Definition. A **cooptional time** is a random variable $L : D_\infty \to [0, \infty]$ such that $L \circ \theta_t = (L - t)^+$.

Cooptional times are backward optional times since, by definition they are random variables, and then

$$\{L > t\} = \left\{(L - t)^+ > 0\right\} = \theta_t^{-1}(\{L > 0\}) \in \mathscr{F}^t.$$

This provides another proof that the last visit to an open set is a backward optional time. However, our second example, that of the last visit to an open set before a fixed time, is an example of a backward optional time which is not cooptional.

Backward optional times are the key to opening random temporal windows in the Markov property. However, to provide a statement closer to the usual expression of the Strong Markov property, we will use the **shift and stop operators** $\sigma_t^s : D_\infty \to D_\infty$ given by

$$\sigma_t^s f(r) = \begin{cases} f(r+s) & \text{if } r+s < t \\ f(t-) & \text{if } r+s \geq t \end{cases}.$$

They are continuous on D_∞ (or on $D_{t'}$ if $t' > t$) because they were defined by means of $f(t-)$ instead of $f(t)$.

To make sense of the following result, let us recall that we have identified bridge laws on D_t with their image on D_∞ under the embedding $i_t : (f(s))_{s \in [0,t]} \mapsto (f(s \wedge t))_{s \geq 0}$. We had argued after the proof of Lemma 1 that $(t, x, y) \mapsto \mathbb{P}_{x,y}^t(F)$ is measurable for any measurable $F : D_\infty \to \mathbb{R}$.

Theorem 2 (The Backward Strong Markov Property). *Under* **H1'** *and* **H2**, *let S and L be a stopping and a backward time respectively. Then for any initial distribution ν on S and any $F \in b\mathscr{F}$,*

$$\mathbb{E}_\nu\big(F \circ \sigma_L^S \,\big|\, \mathscr{F}_S, \mathscr{F}^L, X_{L-}\big) = \mathbb{P}_{X_S, X_{L-}}^{L-S}(F)$$

almost surely on $\{S < L < \infty\}$.

During the course of the proof of the backward strong Markov property, we will use:

Definition. Given **two σ-fields** $\mathscr{G}$ and $\mathscr{G}'$, we say that they **agree on a set** A, written $\mathscr{G} = \mathscr{G}'$ on A, if $A \in \mathscr{G} \cap \mathscr{G}'$ and $A \cap \mathscr{G} = A \cap \mathscr{G}'$.

Lemma 3 (Local property of conditional expectation). *On a probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *, let* $\mathscr{G}$ *and* $\mathscr{G}'$ *be sub-*σ*-fields of* $\mathscr{F}$ *and consider two integrable random variables* ξ, ξ'*. Suppose that* $\mathscr{G} = \mathscr{G}'$ *on* A *and that* $\xi = \xi'$ *almost surely on* A*. Then*

$$\mathbb{E}(\xi | \mathscr{G}) = \mathbb{E}(\xi' | \mathscr{G}') \ \textit{ almost surely on } A.$$

Proof of the backward strong Markov property.

Because of the strong Markov property, it suffices to prove the theorem when $S = 0$; we will simplify the notation for σ_L^0 to σ_L.

Let

$$L^n = \sum_{k=0}^{\infty} \frac{k}{2^n} \mathbf{1}_{(\frac{k}{2^n}, \frac{k+1}{2^n}]}(L) .$$

Then L^n is a random time strictly smaller than L which increases with n towards L. Since L is a backward optional time:

$$\left\{ L^n = \frac{k}{2^n} \right\} = \left\{ \frac{k}{2^n} < L \leq \frac{k+1}{2^n} \right\} \in \mathscr{F}^{k/2^n} .$$

Furthermore, the σ-fields $\mathscr{F}^{k/2^n}$ and $\mathscr{F}^{L^n}$ agree on the set $\{L^n = k/2^n\}$ since θ_{L^n} coincides with $\theta_{k/2^n}$ on that set. For every bounded and measurable $H : D_\infty \to \mathbb{R}$

$$\mathbb{E}_\nu \Big(H \circ \sigma_{k/2^n} \mathbf{1}_{(L^n = k/2^n)} \ \Big| \ \mathscr{F}^{k/2^n} \Big) = \mathbb{P}^{k/2^n}_{X_0, X_{k/2^n}} (H) \, \mathbf{1}_{(L^n = k/2^n)},$$

so that by the local property of conditional expectation:

$$\mathbb{E}_\nu \Big(H \circ \sigma_{L^n} \ \Big| \ \mathscr{F}^{L^n} \Big) = \mathbb{P}^{L^n}_{X_0, X_{L^n}} (H) \ \text{ a.s. on } \{L^n > 0\}. \tag{8}$$

If H is actually continuous and bounded then $H \circ \sigma_{L^n} \to H \circ \sigma_L$. If $A \in \mathscr{F}^L$ and $B \in \mathscr{B}_S$ then $A \cap \{X_{L-} \in B\} \cap \{L_n > 0\} \in \mathscr{F}^{L^n}$, and so (8) implies

$$\begin{aligned} &\mathbb{E}_\nu \Big(H \circ \sigma^{L^n} \mathbf{1}_A \mathbf{1}_{X_{L-} \in B} \mathbf{1}_{L^n > 0} \Big) \\ &= \mathbb{E}_\nu \Big(\mathbf{1}_A \mathbf{1}_{X_{L-} \in B} \mathbb{P}^{L^n}_{X_0, X_{L^n-}} (H) \, \mathbf{1}_{L^n > 0} \Big) . \end{aligned}$$

The left-hand side of the preceding expression converges to

$$\mathbb{E}_\nu\big(H\circ\sigma^L \mathbf{1}_A \mathbf{1}_{X_{L-}\in B}\mathbf{1}_{L>0}\big)$$

as $n\to\infty$, while the right-hand side converges to

$$\mathbb{E}_\nu\Big(\mathbf{1}_A\mathbf{1}_{X_{L-}\in B}\mathbb{P}^L_{X_0,X_{L-}}(H)\,\mathbf{1}_{L>0}\Big)$$

by Lemma 1, so that

$$\mathbb{E}_\nu\big(H\circ\sigma_L\;\big|\;\mathscr{F}^L, X_{L-}\big)=\mathbb{P}^L_{X_0,X_{L-}}(H)\ \text{ a.s. on } \{L>0\}.$$

□

3. Examples of Markovian bridges

In this section we will meet some examples of Feller processes for which bridges can be built using Theorem 1. We will start with a description of the probabilistic objects to consider: Brownian motion, Bessel processes, Brownian motion killed upon reaching zero, and stable subordinators. With this, we will introduce the associated bridges. At some points, we will need facts concerning Lévy processes and Bessel processes. Although more precise information will follow, our main references will be [**Ber96a**] and [**RY99**]. In the examples that follow, the LCCB space (S,ρ) will be either $\mathbb{R}$, $\mathbb{R}^n$ or $\mathbb{R}_+=[0,\infty)$, endowed with the usual metrics. First, let us recall the definition of linear Brownian motion, in which $S=\mathbb{R}$.

Definition. A **Lévy process** is a stochastic process Y with trajectories on D_∞ which satisfies the following:

(i) $Y_0=0$ and
(ii) Y has stationary and independent increments.

A **linear Brownian motion** is a Lévy process $B=(B_t)_{t\geq 0}$ which satisfies the following:

(i) B has continuous trajectories,
(ii) B_t has a centered Gaussian distribution with variance t.

For $\delta\in\mathbb{Z}_+$, a **Brownian motion on** $\mathbb{R}^\delta$ is the vector $\big(B^1,\dots,B^\delta\big)$ where $B^1,\dots B^\delta$ are independent linear Brownian motions.

Example 1. For $x\in\mathbb{R}$, let us define $\mathbb{P}_x$ to be the law of $B+x$ on D_∞; it is concentrated on the set C_∞ of continuous functions on $[0,\infty)$ with values on $\mathbb{R}$ which is a Borel subset of D_∞. From the independence and stationarity of the increments, it follows that $(\mathbb{P}_x)_{x\in\mathbb{R}}$ is a Feller-Markov family (cf. [**Ber96a**, I.2, Prop. 4, p.18]). Because of the Gaussian character of the one-dimensional distributions of Brownian motion, its semigroup $(P_t)_{t\geq 0}$ is given explicitly by:

$$P_t f(x) = \mathbb{E}(f(B_t + x)) = \int f(y)\, \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}\, dy$$

where $f \in b\mathscr{B}_S$. It follows that hypothesis **H1'** holds for the transition density p_t with respect to Lebesgue measure given by

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}.$$

Let us see why hypothesis **H2**, which is the validity of the Chapman-Kolmogorov equations, holds for p_t. We had remarked in Section 1 that, generally, hypothesis **H2** follows from the Markov property and specific boundedness properties of the transition densities; this represents a first example. By the Markov property for Brownian motion, for $s < t$ and $f \in b\mathscr{B}_{\mathbb{R}}$:

$$\int f(y)\, p_t(x,y)\ dy = \int f(y) \int p_s(x,z)\, p_{t-s}(z,y)\ dy\, dz$$

so that for almost all y with respect to Lebesgue measure:

$$p_t(x,y) = \int p_s(x,z)\, p_{t-s}(z,y)\ dz.$$

However, the left-hand side is continuous in y, and since $p_{t-s}(z,y) \leq 1/\sqrt{2\pi\,(t-s)}$, we can use the bounded convergence theorem in the right-hand side to verify its continuity with respect to y, so that the equality between them holds identically and not only almost surely.

Example 2. On the canonical space of continuous functions, we define

$$T_a = \inf\{t \geq 0 : X_t = a\}.$$

Under $\mathbb{P}_0$, $(T_a)_{a\geq 0}$ is a stochastically continuous process with independent increments, which admits a càdlàg modification characterized by the fact that

$$\mathbb{P}_0\big(e^{-qT_a}\big) = e^{-a\sqrt{2q}}.$$

(cf. [**RY99**], II.3.7 in p.71 and III.3.9 in p.107.) This modification is a Lévy process which gives rise to a Feller-Markov family $\left\{\tilde{\mathbb{P}}^{1/2}_x : x \geq 0\right\}$

on $[0,\infty)$, where $\tilde{\mathbb{P}}_x^{1/2}$ is the law of the càdlàg modification plus x under $\mathbb{P}_0$. Since the law of T_a admits the (continuous) density

$$f_a^{1/2}(s) = \frac{a}{\sqrt{2\pi s^3}} e^{-a^2/2s} \mathbf{1}_{s>0}$$

under $\mathbb{P}_x$, the independence and homogeneity of the increments imply that $\tilde{\mathbb{P}}_x^{1/2}$ admits a transition density with respect to Lebesgue measure given by

$$\tilde{p}_t^{1/2}(x,y) = f_t^{1/2}(y-x)\,.$$

We note that hypotheses **H1'** and **H2** are in place, the latter as for Brownian motion since $\tilde{p}_t^{1/2}$ is bounded. Note that $f_t^{1/2}(x)$ is positive if and only if $x>0$.

Example 3. The preceding examples are particular cases of Lévy processes for which one can build bridges; the unproved facts can be consulted in [**Ber96b**] or [**Sat99a**]. If ξ is a real Lévy process, the characteristic function of ξ_t has no zeros, and so there exists a continuous function $\Psi : \mathbb{R} \to \mathbb{C}$, called the **characteristic exponent**, such that

$$\mathbb{E}\big(e^{iu\xi_t}\big) = e^{-t\Psi(u)}.$$

It characterizes the law of ξ. When the Lévy process is increasing, one talks instead of a **subordinator**, and one defines the **Laplace exponent** Φ by means of

$$\mathbb{E}\big(e^{-q\xi_t}\big) = e^{-t\Phi(q)}.$$

If $\mathbb{P}_x^\Psi$ denotes the law of $\xi + x$, then $\big(\mathbb{P}_x^\Psi\big)_{x\in\mathbb{R}}$ is a Feller Markov family; in the case of subordinators, $\big(\mathbb{P}_x^\Psi\big)_{x\geq 0}$ is Feller-Markov. Suppose now that Φ is such that $\exp(-t\Phi)$ is integrable for any $t>0$; by Fourier inversion, one can prove that the law of ξ_t is absolutely continuous and admits a jointly continuous density f_t^Φ bounded on $[t,\infty)\times\mathbb{R}$ (the second factor is $\mathbb{R}_+$ in the subordinator case) for each $t>0$. By independence and homogeneity of the increments, the transition density for X_t under $\mathbb{P}_x$ can be taken equal to $f_t^\Phi(\cdot - x)$, which implies the validity of hypotheses **H1'** and **H2**, where the latter holds as for linear Brownian motion by the bounded character of the density. In [**Sha69**], it is proven that f_t^Φ is positive on the interior of the support of the law of ξ_t, which is of the form (dt,∞) for all $t>0$ or $(-\infty,dt)$ for all $t>0$, where $d\in[-\infty,\infty)$;

$d = -\infty$ if the absolute value of the Lévy process is not a subordinator and it is finite otherwise.

In particular, we can apply the preceding reasoning to **stable subordinators of index** $\beta \in (0,1)$ since the characteristic exponent satisfies

$$\left|e^{-t\Psi(u)}\right| = e^{-tC\,|u|^{\beta}}$$

for some $C > 0$ and $\beta \in (0,1)$; in this case the Laplace exponent is given by $\Phi^{\beta}(q) = Cq^{\beta}$. (Cf. [**Sat99b**, Remark 14.18, p.87].) In this case, we will simplify the notation $\mathbb{P}_a^{\Psi^\beta}$ to $\tilde{\mathbb{P}}_a^{\beta}$. This example generalizes the one involving the hitting-times of Brownian motion T_a, $a \geq 0$ which corresponds to the case $C = \sqrt{2}$ and $\beta = 1/2$. We will use the laws $\tilde{\mathbb{P}}_a^{\beta}$ to give an application of Markovian bridges in the next section.

Example 4. In complete analogy to the linear case, for $\vec{x} \in \mathbb{R}^n$, we define $\mathbb{P}_{\vec{x}}$ to be the law of $\vec{x} + \vec{B}$, where $\vec{B}$ is a Brownian motion on $\mathbb{R}^{\delta}$. Then $(\mathbb{P}_{\vec{x}})_{\vec{x} \in \mathbb{R}^{\delta}}$ is a Feller-Markov family with a transition density with respect to Lebesgue measure given by

$$p_t(\vec{x}, \vec{y}) = \frac{1}{(2\pi t)^{\delta/2}} e^{-\|\vec{x}-\vec{y}\|^2/2t}.$$

Again, hypotheses **H1'** and **H2** hold for this transition density, where the Chapman-Kolmogorov equations hold as for linear Brownian motion by an application of the bounded convergence theorem. Here, we can be surprised by the fact that since the transition density is positive, we can obtain the bridge from $\vec{0}$ to $\vec{0}$ of length t by our approximations, even though for $\delta \geq 2$, Brownian motion does not visit $\vec{0}$ at positive times almost surely. (This is because of the polarity of one-point sets for $\delta \geq 2$, cf. [**RY99**, V.2.7, p. 191].)

Example 5. The next family of processes we shall consider is that of Bessel processes of dimension $\delta \in [0, \infty)$. When $\delta \in \mathbb{Z}_+$, they can be related in a simple manner to Brownian motion. The case $\delta \notin \mathbb{Z}_+$ is handled via stochastic differential equations, which might obscure the Brownian relationship, so that when possible, different arguments will be given for each case. When $\delta \in \mathbb{Z}_+$, the **Bessel process of dimension** δ is defined as follows: Let $\vec{X}$ be the canonical process on the space of continuous functions from $[0, \infty)$ to $\mathbb{R}^{\delta}$ and consider $Y_t = \|\vec{X}_t\|$. The law of the Bessel process of dimension $\delta \in \mathbb{Z}_+$ starting at x, denoted $\mathbb{P}_x^{\delta}$, is the law of Y under $\mathbb{P}_{\vec{x}}$, for any $\vec{x} \in \mathbb{R}^\delta$ such that $\|\vec{x}\| = x$. One proves that the law $\mathbb{P}^\delta_x$ depends only on x and not on the vector $(x_1, \dots, x_n)$ whose norm is x. This is because the law $\mathbb{P}_{\vec{x}}$ depends on $\vec{x}$ only through its norm. In [**RY99**, VI.3.1, p.251] it is argued that $\left(\mathbb{P}^\delta_x\right)_{x\in[0,\infty)}$ is a Markovian family; its Feller property is immediate from that of $(\mathbb{P}_{\vec{x}})_{\vec{x}}$. For $\delta \in [0,\infty)\setminus\mathbb{Z}_+$, one proceeds by constructing first the **square Bessel process of dimension** δ which starts at $x \geq 0$, whose law is denoted $\mathbb{Q}^\delta_x$, as the unique strong solution of the stochastic differential equation

$$Z_t = x + 2\int_0^t \sqrt{|Z_s|}\, dB_s + \delta t$$

where B is a linear Brownian motion. One proves that Z is actually nonnegative, so that the absolute value in the stochastic integral is unnecessary. Of course, when $\delta \in \mathbb{Z}_+$, the two definitions coincide. By careful use of Lévy's characterization of Brownian motion (found in [**RY99**, IV.3.6, p.150], it is proved that for $\delta, \delta', x, x' \geq 0$, the sum of the two coordinates of the canonical process on the space of continuous functions with values in $[0,\infty)^2$ under $\mathbb{Q}^\delta_x \otimes \mathbb{Q}^{\delta'}_{x'}$ has law $\mathbb{Q}^{\delta+\delta'}_{x+x'}$ (cf. [**RY99**, XI.1.2, p.440]), a fact abridged as $\mathbb{Q}^\delta_x * \mathbb{Q}^{\delta'}_{x'} = \mathbb{Q}^{\delta+\delta'}_{x+x'}$, where $*$ represents an extension of the usual convolution to the canonical spaces we are working on. When $\delta \in \mathbb{Z}_+$, this additivity property can be understood in terms of Brownian motion: we had defined multidimensional Brownian motion as built from independent linear Brownian motions. The additivity property implies

$$\mathbb{Q}^\delta_x\left(e^{-\lambda X_t}\right) = A(\lambda)^x\, B(\lambda)^\delta\,,$$

a simplification of [**RY99**, XI.1.3, p.440], and the calculation of constants can be performed by setting $\delta = 1$:

$$\mathbb{Q}^\delta_x\left(e^{-\lambda X_t}\right) = \frac{1}{(1+2\lambda t)^{\delta/2}} e^{-x\frac{\lambda}{1+2\lambda t}}.$$

Inverting this expression, one obtains the transition density associated with square Bessel processes; it features the **modified Bessel function of the first kind**, denoted I_ν, given by

$$I_\nu(x) = \sum_{k=0}^\infty \left(\frac{x}{2}\right)^{\nu+2k} \frac{1}{k!\Gamma(1+\nu+k)}. \tag{9}$$

We shall not need it but in a transformed form stated next.

The **Bessel process of dimension** δ which starts at x is defined as $\sqrt{X}$ under $\mathbb{Q}^{\delta}_{x^2}$; its law will be denoted by $\mathbb{P}^{\delta}_{x}$ and it constitutes, a Feller-Markov family on $[0,\infty)$ whose transition density with respect to Lebesgue measure, which is expressed in a simpler fashion in terms of the **index** $\nu = \delta/2 - 1$ associated to the dimension $\delta > 0$, is given by

$$p^{\delta}_t(x,y) = \frac{1}{t}\left(\frac{y}{x}\right)^{\nu} y e^{-\left(x^2+y^2\right)/2t} I_{\nu}\left(\frac{xy}{t}\right)$$

for $x > 0$ and $t > 0$. For $x = 0$, we have the expression

$$p^{\delta}_t(0,y) = \frac{y^{2\nu+1}}{2^{\nu} t^{\nu+1}\Gamma(\nu+1)} e^{-y^2/2t}.$$

This transition density satisfies hypotheses $\mathbf{H1}'$ and **H2**. The latter holds because the transition density satisfies

$$\sup_{x\in[0,\infty)} \sup_{y\leq M} p^{\delta}_t(x,y) < \infty$$

for any $M > 0$. Let us verify the preceding assertion. First of all, from the computation of the Laplace transform of the square Bessel process, the quantity

$$\mathbb{P}^{\delta}_{x}(X^s_t)$$

is finite for all $s \geq 0$, so that

$$\lim_{y\to\infty} p^{\delta}_t(x,y)\, y^s = 0. \tag{10}$$

Since I_{ν} is increasing and $1 + \nu = \delta/2 > 0$, then

$$\sup_{y\leq M} p^{\delta}_t(x,y) \leq p^{\delta}_t(x,M)\, e^{M^2/2t},$$

also, note that

$$p^{\delta}_t(x,M) = \left(\frac{M}{x}\right)^{2\nu+1} p^{\delta}_t(M,x)\,.$$

By continuity,

$$\sup_{x\leq 1} p^{\delta}_t(x,M) < \infty$$

and then

$$\sup_{x>1} p^{\delta}_t(x,M) = \sup_{x>1} p^{\delta}_t(M,x)\left(\frac{M}{x}\right)^{2\nu+1} < \infty;$$

this follows by considering the sign of $2\nu+1$ making use of (10). Another way to verify the validity of **H2** is to bound the transition density using the asymptotic equality

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}} e^x$$

valid as $x \to \infty$ (cf. [**Leb65**, 5.11.10, p. 123]), which implies

$$\sup_{x\in\mathbb{R}_+, y\leq M} p^\delta_t(x,y) < \infty$$

for any $M > 0$. We can therefore construct Bessel bridges from x to y for any $x \geq 0$ and $y > 0$. It is possible to consider $y = 0$ for a bridge law by the following heuristic argument; recall that for $\delta \in \mathbb{Z}_+$, the Bessel process is the norm of a δ-dimensional Brownian motion, so that it is natural to suppose that the Bessel bridge is the norm of the Brownian bridge. We had argued that it is possible to construct a Brownian bridge from $\vec{0}$ to $\vec{0}$, so that a $0 \to 0$ Bessel bridge makes sense. This can be formalized using Theorem 1 if instead of using Lebesgue measure λ, we use the σ-finite measure with density $y \mapsto y^{2\nu+1}$ with respect to Lebesgue measure, which would imply the fact that the transition density of $\left\{\mathbb{P}^\delta_x : x \in [0,\infty)\right\}$ with respect to it assigns a positive value to 0 starting from any $x \in [0,\infty)$, and satisfies hypotheses **H1'** and **H2**. By using the identity

$$p^\delta_t(x,y) = \left(\frac{y}{x}\right)^{1+2\nu} p^\delta_t(y,x)\,,$$

we note that the time-reversal mapping on D_t, which sends f to $s \mapsto f(t-s)$, interchanges the starting and the ending point when applied to a Bessel bridge.

The particular case $\delta = 3$ deserves particular attention: thanks to the explicit expression

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh(x)\,,$$

the transition density of the 3-dimensional Bessel process can be expressed in terms of the transition density of linear Brownian motion as:

$$\begin{aligned} p^3_t(x,y) &= \frac{1}{t}\sqrt{\frac{y}{x}}\, y e^{-\left(x^2+y^2\right)/2t}\sqrt{\frac{2t}{\pi x y}}\sinh(xy/t) \\ &= \frac{y}{x}\left(p_t(x,y) - p_t(x,-y)\right) \end{aligned}$$

for $x, y > 0$. For $x = 0$, we have

$$p_t^3(0,y) = 2y\frac{\partial p_t}{\partial x}(0,y) = \sqrt{\frac{2}{\pi t^3}}y^2 e^{-y^2/2t}.$$

EXAMPLE 6. Bessel processes are particular instances of Bessel processes in the wide sense, introduced in [**Wat75**], which will provide the next example of stochastic processes for which one can build bridges by weak continuity. Let $\delta > 0$, $c \geq 0$ and consider $\nu = \delta/2 - 1$ and

$$\rho_c(x) = 2^\nu \Gamma(1+\nu)\left(\sqrt{2c}x\right)^{-\nu} I_\nu\left(\sqrt{2c}x\right)$$

where I_ν is the modified Bessel function of the first kind given in (9). A **Bessel process in the wide sense** with index (δ, c) is a diffusion process on $[0,\infty)$ determined by the local generator

$$L^{\delta,c} = \frac{1}{2}\frac{\partial}{\partial x^2} + \left(\frac{\delta-1}{2x} + \frac{\rho_c'(x)}{\rho_c(x)}\right)\frac{\partial}{\partial x};$$

the point 0 is a reflecting boundary when $0 < \alpha < 2$ and an entrance boundary for $\alpha \geq 2$. When $c = 0$, this is just an ordinary Bessel process. Their law starting at x will be denoted $\mathbb{P}_x^{\delta,c}$. Bessel processes in the wide sense can also be interpreted as Bessel processes with drift: for integer $\delta \geq 1$, $\mathbb{P}_0^{\delta,c}$ is the law of the modulus of δ-dimensional Brownian motion with a drift vector $\vec{c}$ of length c that starts at zero (cf. [**PY81**, Remark 5.4.iii, p.319]). The last result is actually proved through a third description of Bessel processes in the wide sense contained in [**PY81**, Sect. 3 & 4]: the law $\mathbb{P}_x^{\delta,c}$ is locally absolutely continuous with respect to $\mathbb{P}_x^\delta$. To describe this relationship, we introduce $\alpha = \sqrt{2c}$, the hitting-time T_y of y by the canonical process, and the functions

$$\phi_\alpha(x,y) = \mathbb{P}_x^\delta\left(e^{-\alpha T_y}\right) \quad \text{and} \quad \phi_{\alpha\uparrow}(y) = \begin{cases} \phi_\alpha(x_0,y) & y \leq x_0 \\ 1/\phi_\alpha(x_0,y) & y > x_0 \end{cases}$$

where x_0 is any element of $(0,\infty)$; the choice of x_0 affects the definition of $\phi_{\alpha+}$ by a constant factor, as can be seen from [**IM74**]. For any t, the restriction of $\mathbb{P}_x^{\delta,c}$ to $\mathscr{F}_t$ is absolutely continuous with respect the restriction of $\mathbb{P}_x^\delta$ to $\mathscr{F}_t$ and the Radon-Nikodým derivative is given by

$$\frac{d\mathbb{P}_x^{\delta,c}|_{\mathscr{F}_t}}{d\mathbb{P}_x^\delta|_{\mathscr{F}_t}} = e^{-\alpha t}\frac{\phi_{\alpha\uparrow}(X_t)}{\phi_{\alpha\uparrow}(x)}.$$

From the form of the Radon-Nikodým derivative, we see that the finite-dimensional distributions of the bridges of $\mathbb{P}_x^{\delta,c}$ do not depend on c, they are just Bessel bridges. (The law $\mathbb{P}_x^{\delta,c}$ is a kind of space-time h-transform of $\mathbb{P}_x$, just discussed for bridges in p. 18.) Therefore, we get not only the existence of bridge laws but also their weak continuity with respect to the parameters involved, because this is the case for $c = 0$. From the weak continuity of the bridge laws, it follows also that, even if we have not checked the validity of the hypotheses ensuring the applicability of Theorem 1, we can obtain the bridges of $\mathbb{P}_x^{\delta,c}$ by approximation as if the hypotheses held: for any continuous and bounded $\mathscr{F}_t$-measurable functional F we get

$$\mathbb{P}_x^{\delta,c}(F \mid X_t \in B_\delta(y)) = \frac{1}{\mathbb{P}_x^{\delta,c}(X_t \in B_\delta(y))} \int_{B_\delta(y)} \mathbb{P}_{x,z}^{\delta}(H)\, \mathbb{P}_x^{\delta,c}(X_t \in dz)\,;$$

since the integrand on the right-hand side is continuous, the left-hand side converges to $\mathbb{P}_{x,y}^{\delta}(H)$ as $\delta \to 0$. However, we might as well use Theorem 1 since the transition densities, denoted $p_t^{\delta,c}$ are continuous in all three variables and we can bound $1/\phi_{\alpha\uparrow}(x)$ by $1/\mathbb{P}_0^{\delta}\big(e^{-\alpha T_{x_0}}\big)$ (the latter quantity is finite since otherwise $\mathbb{P}_0^{\delta}(T_{x_0} = 0) = 1$ which implies by scaling that under $\mathbb{P}_0^{\delta}$ the process never leaves zero which contradicts the existence of a transition density) and so

$$\sup_{x\in\mathbb{R}} \sup_{y\le M} p_t^{\delta,c}(x,y) < \infty$$

for all $M > 0$ which implies **H2**.

To end this example, let us mention a trajectorial construction of Bessel bridges from the trajectories of Bessel processes, contained in Theorem (5.8) of [**PY81**, p. 324]. It states that the law of the bridge of a Bessel process (with or without drift) from x to y of length t can be obtained as:

- the law of $\big(uX_{1/u-1/t}\big)_{u\in[0,t]}$ under $\mathbb{P}_{y/t}^{\delta,\sqrt{2x}}$ or as
- the law of $\big(\big(\frac{t-u}{t}\big)\, X_{tu/(t-u)}\big)_{u\in[0,t]}$ under $\mathbb{P}_x^{\delta,\sqrt{y/t}}$.

Example 7. The next example, very related to example 5 with dimension $\delta = 3$, is that of Brownian motion killed upon reaching zero. The state space of this process is $(0,\infty) \cup \{\Delta\}$, where we think of Δ as an isolated point. We will define it first in terms of Brownian motion: on the canonical space of continuous functions on $[0,\infty)$ with values in $\mathbb{R}$, we let T_0 be the (possibly infinite) hitting time of $\{0\}$ by the canonical process given by

$$T_0 = \begin{cases} \infty & \text{if } \{t \geq 0 : X_s = 0\} = \emptyset \\ \inf\{t \geq 0 : X_s = 0\} & \text{otherwise} \end{cases}.$$

We will define a new trajectory, which will be càdlàg and with values in $(0,\infty) \cup \{\Delta\}$ by setting

$$X_t^\dagger = \begin{cases} X_t & \text{if } t < T_0 \\ \Delta & \text{otherwise} \end{cases};$$

under $\mathbb{P}_x$, this will be Brownian motion started at x and killed when it reaches zero. We will denote its law by $\mathbb{P}_x^\dagger$ and define $\mathbb{P}_\Delta^\dagger$ as the probability measure concentrated on the function identically equal to Δ; the collection $\left(\mathbb{P}_y^\dagger\right)_{x \in (0,\infty) \cup \{\Delta\}}$ is a Markovian family and if μ is the σ-finite measure on $(0,\infty)$ equal to Lebesgue measure on $(0,\infty)$ plus a point mass at Δ, then the law of X_t under $\mathbb{P}_x^\dagger$ is absolutely continuous with respect to μ and the transition density can be expressed in terms of that of linear Brownian motion as

$$p_t^\dagger(x,y) = \begin{cases} p_t(x,y) - p_t(x,-y) & \text{if } x,y \in (0,\infty) \\ \mathbb{P}_x(T_0 \leq t) & \text{if } x \in (0,\infty) \text{ and } y = \Delta \\ 1 & \text{if } x = y = \Delta \end{cases}.$$

In [**RY99**, III.3.29, p. 114], some indications of why the preceding is true are analyzed and the explicit expression of the positive quantity $\mathbb{P}_x(T_0 \leq t)$ are given in terms of the Gaussian distributions. The reader might ask why we have chosen to kill Brownian motion at zero and not only to stop it. The answer is that the transition density for stopped Brownian motion is not continuous, so that we would not be able to apply the results of Sections 1 and 2; by adding the isolated point, we force continuity and in particular, hypotheses **H1'** and **H2** are satisfied, the latter because $p_t^\dagger$ is bounded by $1 \vee \sqrt{2/\pi t}$. Because of the approximation of bridge laws, when the endpoints belong $(0,\infty)$, the corresponding bridge does not pass through Δ and has continuous trajectories. Let us note that for $x, y \in (0,\infty)$, we have

$$p_t^3(x,y) = \frac{y}{x} p_t^\dagger(x,y);$$

if $h(x) = x$, the preceding relationship between transition densities implies that the 3-dimensional Bessel process is the Doob h-transform (see the definition in p. 18) of Brownian motion killed upon reaching zero, which implies that the bridge of the Bessel process of dimension 3 coincides with that of Brownian motion killed upon reaching zero. This is seen by writing down the finite-dimensional distributions of the corresponding bridge laws with starting point and ending point in $(0, \infty)$. However, we have seen that the 3-dimensional Bessel bridge has a limit as the endpoint tends to zero, and that it makes sense when the starting point is equal to zero.

4. Applications related to Brownian motion

Application 1. For Brownian motion B and its associated Feller-Markov family $(\mathbb{P}_x)\, x \in \mathbb{R}$ of probability measures, we have seen that hypotheses **H1′** and **H2** hold. We can therefore apply Theorem 1 to construct the associated bridge measures $\mathbb{P}^t_{x,y}$ which are jointly weakly continuous with respect to $y \in \mathbb{R}$ and $t > 0$. For $v > 0$, let us introduce the **Brownian scaling operators** S_v on D_∞ by means of

$$S_v f(t) = \sqrt{v} f(t/v)\,; \tag{11}$$

they can be thought of as defined also on D_t and their image would then belong to $D_{t \cdot v}$. Using the explicit form of the transition densities of Brownian motion, the scaling relationship

$$\mathbb{P}_x \circ S_v^{-1} = \mathbb{P}_{x\sqrt{v}}$$

follows. In the same manner, using the transition densities of the associated bridge laws, we arrive at

$$\mathbb{P}^t_{x,y} \circ S_v^{-1} = \mathbb{P}^{t \cdot v}_{x\sqrt{v}, y\sqrt{v}}.$$

Let us provide with this an application of the backward strong Markov property. Suppose that we are working on the canonical space of continuous trajectories and let $g = \sup\{s \leq 1 : X_s = 0\}$; note that g is then a backward optional time. This happens because for every $t \leq 1$

$$\{g \geq t\} = \bigcup_{s \in [t,1]} \{X_s = 0\} = \bigcap_{\varepsilon > 0} \bigcup_{s \in [t,1]} \{|X_s| < \varepsilon\}$$

where the second equality stems from the fact that the trajectories are continuous and that $[t, 1]$ is compact. The equality

$$\{g \geq t\} = \bigcap_{\varepsilon>0, \varepsilon\in\mathbb{Q}} \bigcup_{s\in[t,1]\cap\mathbb{Q}} \{|X_s| < \varepsilon\} \in \mathscr{F}^t$$

follows. Since $\{g > t\} = \cap_{\varepsilon>0,\varepsilon\in\mathbb{Q}} \{g > t + \varepsilon\}$, it follows that $\{g > t\} \in \mathscr{F}^t$ for all $t \leq 1$. Under $\mathbb{P}_0$, it is known that $g > 0$ because the zero set of Brownian motion does not have isolated points (cf. [**RY99**, III.3.12, p.109]). Thanks to the backward strong Markov property, it follows that a conditional law of σ_0^g given $\mathscr{F}^g$ under $\mathbb{P}_0$ is $\mathbb{P}_{0,0}^g$. However, by the scaling relationship between the bridge laws, it follows that a conditional law of $S_{1/g} \circ \sigma_0^g$ given $\mathscr{F}^g$ under $\mathbb{P}_0$ is equal to $\mathbb{P}_{0,0}^1$ from which we deduce the independence of

$$\left(\frac{1}{\sqrt{g}} B_{s\cdot g}\right)_{s\in[0,1]} \quad \text{and} \quad (B_{s+g})_{s\geq 0}$$

as well as the fact that the former is a Brownian bridge between zero and zero of length one.

Note that the preceding computations rely solely on the scaling property of Brownian motion, hence, the same conclusion (with an appropriate bridge) can be arrived at for Bessel processes of dimension $\delta \in (0, 2)$. (Their last zero before time occurs at a positive time.) This fact will be used in Section 4 of Chapter 2.

Application 2. Our next application is related to Bessel bridges and in particular to their behaviour as the length goes to infinity. The Bessel bridge of length t and dimension δ between x and y will be denoted $\mathbb{P}_{x,y}^{t,\delta}$. Let $\tilde{}$ stand for the time-reversal operation on D_∞, which sends f to $s \mapsto f((t-s) \vee 0)$ and $\tilde{X}$ for the time-reversal of the canonical process: $\tilde{X}_s = X_{(t-s)\vee 0}$. Although the notation does not reflect the variable t, it will be used with bridge laws of a given length and so it will be deduced from the context. For any $a > 0$ and $s \geq 0$, let $L_a^s = \sup\{r \leq s : X_r \in [0, a)\}$ and let $\tilde{L}_a^s = L_a^s \circ \tilde{X}$. Let L_a be the last visit to $[0, a)$ by a given process on $[0, \infty)$; it could be defined as L_a^∞. Under $\mathbb{P}_{0,0}^{t,\delta}$, we will prove that for any $\alpha \in (0, 1)$,

$$\left(\sigma_{L_a^{\alpha t}} \circ X, \sigma_{\tilde{L}_a^{(1-\alpha)t}} \circ \tilde{X}\right) \text{ converges in variation as } t \to \infty \tag{12}$$

towards two independent Bessel processes of dimension δ which start at zero and are stopped upon their last visit to a for every $\delta > 2$. The key to our study will be the following scaling identity satisfied by the Bessel transition density:

$$p_t^\delta(x,y) = \frac{1}{\sqrt{t}} p_1^\delta\left(x/\sqrt{t}, y/\sqrt{t}\right).$$

This readily implies the following scaling relationship between Bessel bridge laws:

$$\mathbb{P}_{x,y}^{t,\delta} \circ S_v^{-1} = \mathbb{P}_{x\sqrt{v},y\sqrt{v}}^{tv,\delta}.$$

where S_v is the scaling operator introduced in the preceding application (see equation (11)).

We will start with a related result for fixed times instead of the random ones of last visit: we will prove that if we stop X and $\tilde{X}$ at the fixed times s_1 and s_2, the resulting processes converge in variation to two independent Bessel processes stopped at times s_1 and s_2 respectively. On the two-fold product of D_∞ with itself, let Φ be a bounded measurable functional and consider such that $0 < s_1 < \alpha t$ and $0 < t - s_2 < (1-\alpha)\, t$ to define $\Phi_{s_1,s_2} : D_t \times D_t \to \mathbb{R}$ by means of

$$\Phi_{s_1,s_2}(f,g) = \Phi(\sigma_{s_1}(f)\,, \sigma_{s_2}(g)).$$

We can then write

$$\mathbb{P}_{0,0}^{t,\delta}\left(\Phi_{s_1,s_2}\left(X,\tilde{X}\right)\right) = \mathbb{P}_{0,0}^{1,\delta}\left(\Phi_{s_1,s_2}\left(X,\tilde{X}\right) \circ S_t\right).$$

Using the fact that $\Phi_{s_1,s_2}\left(X,\tilde{X}\right) \circ S_t$ is $\mathscr{F}_{s_1/t} \otimes \mathscr{F}^{1-s_2/t}$, the Markov property coupled with time-reversibility give

$$\mathbb{P}_{0,0}^{1,\delta}\left(\Phi_{s_1,s_2}\left(X,\tilde{X}\right) \circ S_t\right) = \mathbb{P}_{0,0}^{1,\delta}\left(\mathbb{P}_{0,X_\alpha}^{\alpha,\delta} \otimes \mathbb{P}_{0,\hat{X}_{1-\alpha}}^{1-\alpha,\delta}(\Phi_{s_1,s_2} \circ S_t)\right),$$

a fact that can be verified by using functionals Φ which factorize and extending it to general functionals by a monotone class argument. We obtain

$$\mathbb{P}_{0,0}^{t,\delta}\left(\Phi_{s_1,s_2}\left(X,\tilde{X}\right)\right) = \mathbb{P}_{0,0}^{1,\delta}\left(\mathbb{P}_{0,\sqrt{t}X_\alpha}^{\alpha t,\delta} \otimes \mathbb{P}_{0,\sqrt{t}\hat{X}_{1-\alpha}}^{(1-\alpha)t,\delta}(\Phi_{s_1,s_2})\right)$$

by the scaling property bounding Bessel bridges. We are now ready to take the limit in the preceding quantity as $t \to \infty$. Let

$$M_x^{v,s,\delta}(X) = \frac{d\mathbb{P}_{0,x}^{v,\delta}|_{\mathscr{F}_s}}{d\mathbb{P}_0^{\delta}|_{\mathscr{F}_s}};$$

by the local absolute continuity relationship (1) for bridge laws:

$$\text{(13)} \qquad M_{\sqrt{t}x}^{\alpha t,s_1,\delta} = \frac{p^{\delta}_{\alpha t - s_1}\left(X_{s_1}, \sqrt{t}x\right)}{p^{\delta}_{\alpha t}\left(0, \sqrt{t}x\right)} = \frac{p^{\delta}_{\alpha - s_1/t}\left(X_{s_1}/\sqrt{t}, x\right)}{p^{\delta}_{\alpha}(0, x)} \overset{a.s.}{\to} 1$$

as $t \to \infty$ by continuity of the Bessel densities. Since each $M_x^{s,v,\delta}$ integrates 1, Scheffé's lemma implies the L^1 character of the preceding convergence. To apply the argument to the product of bridge laws instead of to each factor, we need to work on the three-fold product of D_t with itself and denote by X, Y and Z the first, second and third coordinate processes; X will be used when integrating against the bridge laws and Y and Z against the laws of Bessel proceses. We may then write:

$$\left|\mathbb{P}_{0,0}^{t,\delta}\left(\Phi_{s_1,s_2}\left(X, \tilde{X}\right)\right) - \mathbb{P}_0^{\delta} \otimes \mathbb{P}_0^{\delta}(\Phi_{s_1,s_2})\right| \leq \|\Phi\|_{\infty} \times$$
$$\mathbb{P}_{0,0}^{1,\delta}\left(\mathbb{P}_0^{\delta} \otimes \mathbb{P}_0^{\delta}\left(\left|1 - M_{\sqrt{t}X_\alpha}^{\alpha t,s_1,\delta}(Y)\, M_{\sqrt{t}\tilde{X}_{1-\alpha}}^{(1-\alpha)t,s_2,\delta}(Z)\right|\right)\right)$$

where the second factor of the right-hand side goes to zero by dominated convergence since

$$\mathbb{P}_0^{\delta} \otimes \mathbb{P}_0^{\delta}\left(\left|1 - M_{\sqrt{t}X_\alpha}^{\alpha t,s_1,\delta}(Y)\, M_{\sqrt{t}\tilde{X}_{1-\alpha}}^{(1-\alpha)t,s_2,\delta}(Z)\right|\right) \leq 2.$$

We will now tackle the asymptotic behaviour of the two ends of the Bessel bridge up to the random times $L_a^{\alpha t}$ and $\tilde{L}_a^{(1-\alpha)t}$ when $\delta > 2$ or equivalently, $\nu > 0$. This will be achieved by extending what we have just proved to the case where s_1 and s_2 go to infinity. Let us be more precise: note that (13) is valid in the L^1 sense if we let $s_1 \to \infty$ with the restriction $s_1 = o(t)$. This is because if $s_1 = o(t)$ then, by scaling,

$$\mathbb{P}_0^{\delta}\left(X_{s_1}/\sqrt{t}\right) = \frac{\sqrt{s_1}}{\sqrt{t}}\mathbb{P}_0^{\delta}(X_1) \to 0$$

as $t \to \infty$, which implies $X_{s_1}/\sqrt{t} \to 0$ in probability. By continuity of the densities, $M^{\alpha t, s_1, \delta}_{\sqrt{t}x}$ converges in probability to 1, and because this density integrates 1, we conclude its convergence to 1 in L^1 by the following argument:for any sequence $(t_n, s_n) \to \infty$ such that $s_n = o(t_n)$, and any subsequence (t'_n, s'_n) of it, there exists a further subsequence (t''_n, s''_n) such that $M^{\alpha t''_n, s''_n, \delta}_{\sqrt{t''_n}x} \to 1$ almost surely, and by Scheffé's lemma, the convergence holds in L^1 as well. Since the subsequence (t'_n, s'_n) is arbitrary, we conclude that $M^{\alpha t, s_1, v}_{\sqrt{v}x}$ converges to 1 in L^1 as $t \to \infty$ if $s_1 = o(t)$. With this and dominated convergence, we see that if s_1 and s_2 vary with t in such a way that $s_i = o(t)$ then

$$\lim_{t\to\infty} \sup_{\|\Phi\|_\infty \leq 1} \left| \mathbb{P}^{t,\delta}_{0,0}(\Phi_{s_1,s_2}) - \mathbb{P}^\delta_0 \otimes \mathbb{P}^\delta_0(\Phi_{s_1,s_2}) \right| = 0.$$

This will be relevant once we prove that

$$\mathbb{P}^{t,\delta}_{0,0}\left(L^{\alpha t}_a = L^s_a\right), \mathbb{P}^\delta_0(L^\infty_a = L^s_a) \to 1 \tag{14}$$

as $s \to \infty$ with $s \leq \alpha t$ when $\delta > 2$. The limits in (14) will be used in the following way: for every measurable and bounded functional Φ on $D_\infty \times D_\infty$, define Φ_{a,s_1,s_2} by means of

$$\Phi_{a,s_1,s_2}(f,g) = \Phi\Big(\sigma_{L^{s_1}_a(f)}(f)\,, \sigma_{L^{s_2}_a(g)}(g)\Big)$$

and note that

$$\Phi_{a,\alpha t,(1-\alpha)t}\Big(X, \tilde{X}\Big) = \Phi_{a,s_1,s_2}\Big(X, \tilde{X}\Big)$$

whenever

$$L^{\alpha t}_a = L^{s_1}_a \text{ and } \tilde{L}^{(1-\alpha)t}_a = L^{s_2}_a\Big(\tilde{X}\Big).$$

By the triangular inequality

$$\begin{aligned}
&\left|\mathbb{P}^{t,\delta}_{0,0}\Big(\Phi_{a,\alpha t,(1-\alpha)t}\Big(X,\tilde{X}\Big)\Big) - \mathbb{P}^\delta_0 \otimes \mathbb{P}^\delta_0(\Phi_{a,\infty,\infty}(Y,Z))\right| \\
\leq &\|\Phi\|_\infty \mathbb{P}^{t,\delta}_{0,0}\Big(L^{\alpha t}_a = L^{s_1}_a, \tilde{L}^{s_2}_a = \tilde{L}^{(1-\alpha)t}_a\Big) \\
+&\|\Phi\|_\infty \mathbb{P}^{t,\delta}_{0,0}\Big(\mathbb{P}^\delta_0 \otimes \mathbb{P}^\delta_0\Big(\Big|1 - M^{\alpha t,s_1,\delta}_{\sqrt{t}X_{s_1}}(Y)\, M^{(1-\alpha)t,s_2,\delta}_{\sqrt{t}X_{s_2}}(Z)\Big|\Big)\Big) \\
+&\|\Phi\|_\infty \mathbb{P}^\delta_0(L^\infty_a \neq L^{s_1}_a)\, \mathbb{P}^\delta_0(L^\infty_a \neq L^{s_2}_a)\,,
\end{aligned}$$

so that to obtain (12), it suffices to verify (14). Let $E_{a,s}$ denote the event that the canonical process does not visit $[0,a)$ after s; we have $\{L_a^\infty = L_a^s\} = E_{a,s}$. When $\delta > 2$, it is known that $X_t \to \infty$ under $\mathbb{P}_0^\delta$, thanks to scale function computations (cf. [**RY99**, XI.1, p. 442]), so that

$$\mathbb{P}_0^\delta\big(L_a^{\alpha t} = L_a^s\big) \geq \mathbb{P}_0^\delta(E_{a,s}) \to 1$$

as $s,t \to \infty$. By scaling, we have also

$$\mathbb{P}_0^\delta\Big(L^\alpha_{a/\sqrt{t}} = L^{s/t}_{a/\sqrt{t}}\Big) \to 1$$

as $s,t \to \infty$. By dominated convergence, we conclude

$$\begin{aligned}\mathbb{P}_{0,0}^{t,\delta}\big(L_a^{\alpha t} = L_a^s\big) &= \mathbb{P}_{0,0}^{1,\delta}\Big(L^\alpha_{a/\sqrt{t}} = L^{s/t}_{a/\sqrt{t}}\Big)\\ &= \mathbb{P}_0^\delta\left(\mathbf{1}_{L^\alpha_{a/\sqrt{t}} = L^{s/t}_{a/\sqrt{t}}} M^{1,\alpha,\delta}_{X_\alpha}\right) \to 1,\end{aligned}$$

so that (14) holds.

Application 3. Our next application is related to stable subordinators and is obtained by a Doob transformation, recall the definition of page 18. As we have seen in the Example 7, Doob's transformation leaves bridge laws invariant, since the function used to perform it cancels out. Associated to the stable subordinator σ^β of index β, we can define the Markovian family $\tilde{\mathbb{P}}^\beta_a$, $a \geq 0$ such that $\tilde{\mathbb{P}}^\beta_a$ is the law of $a + \sigma^\beta$. We can also construct the Markovian family associated to the negative of σ^β: $\hat{\mathbb{P}}^\beta_a$ will be the law of $a - \sigma^\beta$ and $\hat{\mathbb{P}}^\beta_a$, $a \in \mathbb{R}$ is a Feller-Markov family. Let f^β_t be the density of σ^β_t so that $f^\beta_t(\cdot - a)$ is the transition density associated to $\tilde{\mathbb{P}}^\beta_a$ and $f^\beta_t(\cdot + a)$ is the transition denstity associated to $\hat{\mathbb{P}}^\beta_a$. Recall from Example 3 that f^β is positive on $(0,\infty)$ and zero elsewhere.

It is possible to calculate the potential density u^β given by

$$u_\beta(a) = \int_0^\infty f^\beta_t(a)\ dt$$

as shown in [**Sat99b**, Example 37.19, p. 261]. The explicit expression is

$$u_\beta(a) = \frac{1}{C\Gamma(\beta)\, a^{1-\beta}} \mathbf{1}_{a>0}.$$

For any $0 < b$ we define $h_\beta : [0,\infty) \to [0,\infty)$ equal to $u_\beta(b-a)$ if $a \leq b$ and equal to zero otherwise. With it, we will consider the Doob h_β-transform of $\tilde{\mathbb{P}}^\beta_a$, denoted $\tilde{\mathbb{P}}^{h_\beta}_a$; it is a measure on the Skorohod space D_∞

on $[0, \infty) \cup \{\Delta\}$ concentrated on trajectories with values on $[0, b) \cup \{\Delta\}$. Each measure $\tilde{\mathbb{P}}_a^{h_\beta}$ is determined by local absolute continuity and the fact that trajectories are absorbed at Δ at the possibly infinite random time $\zeta = \inf\{s \geq 0 : X_s = \Delta\}$: it as the (only) probability measure such that for all $A \in \mathscr{F}_s$

$$\tilde{\mathbb{P}}_a^{h_\beta}(A \cap \{s < \zeta\}) = \tilde{\mathbb{P}}_a^{\beta}\left(\frac{h_\beta(X_s)}{h_\beta(x)} \mathbf{1}_A\right).$$

See [**DM87**] for additional details and references on the construction of such a measure by means of projective limits of probability measures. The family $\tilde{\mathbb{P}}_a^{h_\beta}$, $a \in [0, b) \cup \{\Delta\}$ is Markovian and is associated to the Markov process termed the **stable subordinator conditioned to die at** b. We will dwell on this interpretation using Markovian bridges. Firstly, note the following:

$$\tilde{\mathbb{P}}_a^{h_\beta}(s < \zeta) = \tilde{\mathbb{P}}_a^{\beta}\left(\frac{h_\beta(X_s)}{h_\beta(a)}\right).$$

By the Chapman-Kolmogorov equations and the Tonelli-Fubini theorem, it follows that for $a < b < c$:

$$\tilde{\mathbb{P}}_a^{\beta}\left(\frac{h_\beta(X_s)}{h_\beta(a)}\right) = \int_{\mathbb{R}} u_\beta(b - c)\, f_s^\beta(c - a)\; db = \int_s^\infty f_r^\beta(b - a)\; dr.$$

The preceding quantity is differentiable in s and tends to zero as $t \to \infty$, and so

$$\tilde{\mathbb{P}}_a^{h_\beta}(\zeta \in ds) = \frac{f_s^\beta(b - a)}{u_\beta(b - a)}\, ds \text{ and } \mathbb{P}_a^{h_\beta}(\zeta < \infty) = 1.$$

The denomination for $\mathbb{P}_a^{h_\beta}$ is justified because

$$\tilde{\mathbb{P}}_a^{h_\beta}(X_{\zeta -} = b) = 1$$

as we will now see. To this end, let us first calculate a version of the conditional law of $(X_s)_{s \leq t}$ given $\zeta = t$; this could be done by calculating the finite-dimensional distributions or using compact notation as follows: for any $F \in b\mathscr{F}_s$, we have the relationship

$$\tilde{\mathbb{P}}_a^{h_\beta}(F \mathbf{1}_{\zeta > t}) = \tilde{\mathbb{P}}_a^{\beta}\left(F \frac{h_\beta(X_t)}{h_\beta(a)}\right) = \tilde{\mathbb{P}}_a^{\beta}\left(F \mathbb{P}_{X_s}^{\beta}\left(\frac{h_\beta(X_{t-s})}{h_\beta(a)}\right)\right)$$

valid by the local absolute continuity relationship and the Markov property. If we let $\tilde{\mathbb{P}}_{a,b}^{t,\beta}$ denote the law of the bridge of $\tilde{\mathbb{P}}_a^{\beta}$ of endpoint b and length t, we conclude from the preceding expression that

$$-\frac{d}{dt}\tilde{\mathbb{P}}_a^{h_\beta}(F\mathbf{1}_{\zeta>t}) = \tilde{\mathbb{P}}_a^{\beta}\left(F\frac{f_{t-s}^{\beta}(b-X_s)}{h_\beta(a)}\right)$$
$$= \tilde{\mathbb{P}}_{a,b}^{t,\beta}(F)\,\frac{f_t(b-a)}{h_\beta(a)} = \tilde{\mathbb{P}}_{a,b}^{t,\beta}(F)\,\frac{d}{dt} - \tilde{\mathbb{P}}_a^{h_\beta}(\zeta>t)$$

and so it follows that one version of the conditional law of $(X_s)_{s\leq t}$ given $\zeta = t$ is $\tilde{\mathbb{P}}_{a,b}^{t,\beta}$. (It is a good choice since it is weakly continuous with respect to t as we have already argued.) However, from Theorem 1 we have

$$\tilde{\mathbb{P}}_{a,b}^{t,\beta}(X_{t-}=b)=1 \text{ which implies } \tilde{\mathbb{P}}_a^{h_\beta}(X_{\zeta-}=b)=1$$

by integration.

As another application, we will now construct the stable subordinator conditioned to die at b by means of the stable subordinator. More specifically, consider the backward optional time $L=\sup\{t\geq 0 : X_s<1\}$ and construct $Y=(Y_t)_{t\geq 0}$ by means of

$$Y_t = \begin{cases} \frac{b}{g}X_{t(g/b)^\beta} & \text{if } t\,(g/b)^\beta < L \\ \Delta & \text{otherwise} \end{cases}.$$

Then the law of Y under $\tilde{\mathbb{P}}_a^{\beta}$ is $\tilde{\mathbb{P}}_a^{h_\beta}$. The proof is an application of the backward strong Markov property and the fact that the conditional law of L given $g=x$ under $\tilde{\mathbb{P}}_0^{\beta}$ admits the density

$$\frac{f_s^\beta(x)}{u_\beta(x)},$$

a fact that will be proven afterwards. The latter law is equal to that of ζ under $\tilde{\mathbb{P}}_0^{h_\beta}$ when $b=1$. Since the stable densities satisfy the scaling property

$$f_t^\beta(x) = t^{-1/\beta}f_1^\beta\left(t^{-1/\beta}x\right),$$

which can be proven by computing Laplace transforms in the almost everywhere sense and then in the pointwise sense by continuity of the densities, the conditional law of $L\,(g/b)^\beta$ given $g=x$ under $\tilde{\mathbb{P}}_0^{\beta}$ is equal to the law of ζ for the subordinator conditioned to die at b started at 0. From this:

$$\text{the absorption time of } Y \text{ is distributed like } \zeta \text{ under } \tilde{\mathbb{P}}_0^{h_\beta}.$$

We will now compute the conditional law of Y given (L, g): note that $Y = S^{\beta}_{(g/b)^{\beta}} \circ \sigma_L$, where the scaling operator $S^{\beta}_v : D_\infty \to D_\infty$ is defined by

$$S^{\beta}_v f(s) = f(s/v)\, v^{1/\beta}. \tag{15}$$

By the backward strong Markov property, the conditional law of Y given (L, g) under $\tilde{\mathbb{P}}^{\beta}_0$ is $\tilde{\mathbb{P}}^{L,\beta}_{0,g} \circ S^{\beta}_{(g/b)^{\beta}}$, and by the scaling properties of the stable density, the latter is equal to $\tilde{\mathbb{P}}^{L(b/g)^{\beta},\beta}_{0,b}$; since it only depends on $L\,(b/g)^{\beta}$, it follows that

$$\text{the conditional law of } Y \text{ given } L\,(b/g)^{\beta} \text{ under } \tilde{\mathbb{P}}^{\beta}_0 \text{ is } \tilde{\mathbb{P}}^{L(b/g)^{\beta},\beta}_{0,b}.$$

Summarizing, the law of the absorption time of Y is equal to the law of the absorption time of the stable subordinator conditioned to die at b started at zero, and, conditionally on the absorption times, Y and the conditioned subordinator are bridges of the stable subordinator which start at 0, end at b, and whose length is the corresponding absorption time. We conclude that Y is a stable subordinator, of index β, conditioned to die at y and started at zero.

It remains to compute the law of L given g. To this end, we will need certain facts about subordinators, their jumps, and Poisson processes; this will be done in Chapter 2.

5. Bibliographical Notes

In this section, we will relate our results on Markovian bridges to the published literature on the subject. We will concentrate on the notion of Markovian bridge, its construction, and on the backward strong Markov property, since giving an overview of its applications is beyond the current capabilities of the author.

We find the notion of Markovian bridge as far back as 1939, in Paul Lévy's famous paper [**Lev39**]. In Section 6 (p. 304), entitled *L'interpolation dans les processus stochastiques*, he discusses the finite-dimensional distributions of the bridge of stable processes and constructs the Brownian bridge rigorously, thanks to the continuous character of its trajectories. As a matter of fact, from the Gaussian character of the finite-dimensional distributions of Brownian motion, the following representation of the Brownian bridge is given: if $x, y \in \mathbb{R}$, $t > 0$ and B is a Brownian motion started at x, then the stochastic process b on $[0,t]$ defined by

$$b_s = B_s + (y - B_t)\,(s/t)$$

has law $\mathbb{P}^t_{x,y}$. More formally, what one can do with Paul Lévy's representation is to prove that the constructed processes disintegrate $(B_s)_{s\in[0,t]}$ with respect to B_t, and since their weak continuity is obvious by construction, we deduce that they coincide with our bridge laws constructed in Theorem 1. This way of reasoning is used in [**RY99**] to introduce the Brownian bridge in page 37 and to interpret is as conditioned Brownian motion in Excercise 3.16 of page 41. As explained there, the construction extends to other dimensions and would enable us to construct bridges of Bessel processes of integer dimensions in a trajectorial manner. Paul Lévy continues to use his interpolation procedure in [**Lev44b**] and [**Lev44a**] to relate properties of Brownian bridges and of Brownian motion.

We can find many examples of generalizations of the strong Markov property to random times; such generalizations consist of two parts: a statement of conditional independence of past and future with respect to the present at a given random time τ, and a description of the conditional law of the pre-τ and post-τ parts of the process given some notion of the present, which can be the σ-field generated by τ and X_τ, or only X_τ, or even more exotic ones. The first such example is [**MSW72**], in which the authors study cooptional times and the particular case of coterminal times. Pittenger and Shih retake and extend this study in [**PS73**]. Jacobsen introduces in [**Jac74**] a class of random times called splitting times and states an unpublished result of D. Williams: in discrete time, the past and the present are conditionally independent given the value of a splitting time and of the process at it. Jacobsen treats the case of *positively inclined diffusions* and their future minimum times, and describes the diffusion after a future minimum time. He gives evidence of the difficulties in passing from discrete to continuous time. Another particular case of a splitting time τ is studied by Millar in [**Mil77**], where a description of a post-τ process is given. In the same year, Jacobsen and Pitman give in [**JP77**] a characterization of random times τ in discrete time for which either the pre-τ or the post-τ fragments of a Markov chain are Markovian and such that the two fragments are conditionally independent given "the position of the inner endpoint of the Markovian fragment at τ or $\tau - 1$". Two other studies of generalizations of Markov properties are [**GS79b**] and [**GS81**]. In the first one, one proves that conditional independence is valid at cooptional times and in the second one, they introduce diverse notions of present with which one can prove Markov type properties.

We now proceed with the main references for the material presented in this chapter. First comes Kallenberg's 1981 article [**Kal81**], more precisely pages 785 and 786, in which the author presents the general notion of the bridge of Lévy processes, by means of the absolute continuity condition that we used, and states the backward strong Markov property for them. He introduces bridge laws through the local absolute continuity relationship, although no details are given as to the existence of such a measure, and weak continuity is confronted through use of the convergence criteria of processes with exchangeable increments of [**Kal73**]. Temporal weak continuity is defined as weak continuity on D_1 of rescaled processes $(X_{st})_{s\in[0,1]}$, which is different to our notion of it; they coincide whenever lengths are finite, but when considering lengths which go to infinity, we would have problems rescaling. We have to wait 11 years until the appereance of [**FPY93**] in which a framework for the existence of measures satisfying the local absolute continuity, and the starting and ending point condition characterizing bridge laws, is given. It involves two right Markov processes (with values in a Lusin space) in duality and the existence of densities with respect to the measure for which duality holds. The bridge laws are constructed from Parthasarathy's theorem on the existence of probability measures on inverse limits (found in [**Par67**]), but only on the space of càdlàg functions on $[0, t)$. The duality hypothesis then comes into play to prove the ending point property, hence allowing the extensions of the laws to D_t. Their framework does not require the Feller property so that it is more general than ours. If the state space were Polish, and the transition densities continuous, in their context we would in general only be able to prove weak continuity (with respect to the ending point) of the images under σ_s $(s < t)$ of bridge laws of length t. (This would be enough to prove versions of the backward strong Markov property. Further conditions seem to be necessary to prove weak continuity up to time t, which is precisely where the Feller property was useful to us.) A general backward strong Markov property is stated and the envisioned proof would imply the use of results of [**GS79b**] and [**GS81**] with space time processes. In our context, the backward strong Markov property is in a certain sense very simple in discrete time and weak continuity takes care of the extension to continuous time. Therefore, it seems necessary to study other types of Markovian disintegrations and if weak continuity considerations lead to more general Markov type properties like those in the preceding paragraph.

The examples have already been considered by a variety of authors. We have discussed the genesis of the Brownian bridge and of bridges of Lévy processes. In [**FPY93**], they construct diffusion bridges, and although weak continuity is not considered, this gives in particular a construction of bridges of Bessel processes and killed Brownian motion. Application 1 has been introduced in several places, in particular in [**Yor95**] it is seen as a part of a path decomposition of Brownian motion and proved by the time inversion transformation $(B_t)_{t>0} \mapsto t\left(B_{1/t}\right)_{t>0}$ which does not alter the law of the Brownian motion B started at zero but which has the effect of turning backward optional times into optional times. Work in progress with Loïc Chaumont is aimed at extending this pathwise construction of bridges, not only when going from zero to zero, to other self-similar Markov processes (which might not satisfy the time inversion property). Application 2 is an extension of a result of Jeulin (which corresponds to $\delta = 3$) found in [**Jeu80**, Thm. 6.41, p.127]. The original proof depended on explicit calculations that can be performed with Brownian motion and estimates whose extension to other cases seem difficult. As explained by Jeulin, the application was aimed at understanding results of Getoor and Sharpe (found in [**GS79a**]) in which the two scaled ends of an excursion acted as if independently. Application 3 was conceived out of the results concerning conditioned subordinators of Chapter 3 and the author has not found references concerning this particular trajectorial construction.

CHAPTER 2

The Normalized Brownian Excursion

In this chapter, we will introduce the Normalized Brownian Excursion. It will be defined as a limit of conditioned Brownian bridges, so that a first problem consists in explaining why the weak limit exists. For this, we will go back to the relationship between the bridges of Brownian motion killed upon reaching zero and of the three-dimensional Bessel process. The weak limit construction of the normalized Brownian excursion will explain its appearance in the study of Brownian motion. In particular, it will be seen to arise as a renormalized part of Brownian motion by means of the backward strong Markov property. As a further application of the former, we will also see how and why it arises in the study of the excursion process of Brownian motion. For this last application, we will need to review some facts about Poisson point processes, which already appeared in the last chapter.

1. The Normalized Brownian excursion as a weak limit of conditioned Brownian bridges

Let us define the infimum functional on Skorohod space, denoted I, as $I(f) = \inf\{f(s) : s \geq 0\}$. Under the probability measure governing a Brownian bridge from zero to zero of length one, which was denoted by $\mathbb{P}^1_{0,0}$ in Chapter 1, I is almost surely finite, and our goal in this section will be to study the probability measure $\mathbb{P}^1_{0,0}(\cdot \mid I > -\varepsilon)$ governing the Brownian bridge conditioned on remaining above $-\varepsilon$, as $\varepsilon \to 0$. Not only does a weak limit exist as $\varepsilon \to 0$, but we can identify it with the law of a three-dimensional Bessel bridge between zero and zero of length one. For the record, we state the following result:

Theorem 3. *The family of probability measures*

$$\left(\mathbb{P}^1_{0,0}(\cdot \mid I > -\varepsilon)\right)_{\varepsilon > 0}$$

converges weakly as $\varepsilon \to 0$ *towards* $\mathbb{P}^{3,1}_{0,0}$.

Thanks to this theorem, we can explain why $\mathbb{P}^{3,1}_{0,0}$ is called the law of the **normalized Brownian excursion**. Its approximation by conditioned Brownian bridges ensures that it starts at zero, is absorbed at zero at time one, and is nonnegative. Proving that, actually, normalized Brownian excursions are positive on $(0,1)$ will have to wait.

Proof of Theorem 3. We will start by proving an equality in law between two bridges. Let $\mathbb{P}^{t,\dagger}_{x,y}$ stand for the bridge between x and y of length t of Brownian motion killed upon reaching zero. We had already remarked that when working with $\mathbb{P}^{t,\dagger}_{x,y}$, the trajectories do not pass through Δ and so we will assume that the bridge laws are defined on the Skorohod space over $(0,\infty)$ and concentrated on continuous trajectories. In this sense, the equality between $\mathbb{P}^{t}_{x,y}(\,\cdot \mid I > 0)$ and $\mathbb{P}^{t,\dagger}_{x,y}$ for $x, y \in (0,\infty)$ will be verified. By Theorem 1 and the relationship between Brownian motion and killed Brownian motion, $\mathbb{P}^{t,\dagger}_{x,y}$ is the weak limit as $\delta \to 0$ of

$$\mathbb{P}_x \circ \sigma_t^{-1}(\,\cdot \mid T_0 > t, X_t \in B_\delta(y))\,. \tag{16}$$

Since $\{T_0 > t\} = \{I \circ \sigma_t > 0\}$, the Brownian bridge $\mathbb{P}^{t}_{x,y}$ conditioned on $I > 0$ can also be approximated by (16) as $\delta \to 0$, proving the desired equality in law.

Secondly, we will verify the equality in law between $\mathbb{P}^{1}_{0,0}(\,\cdot \mid I > -\varepsilon)$ and the image of $\mathbb{P}^{1}_{\varepsilon,\varepsilon}(\,\cdot \mid I > -\varepsilon)$ under the mapping $f \mapsto f - \varepsilon$. More generally, since $\mathbb{P}_x$ is the image of $\mathbb{P}_{x+z}$ under the mapping $f \mapsto f - z$, it follows by the weak approximation of Theorem 1 that $\mathbb{P}^{t}_{x,y}$ is the image of $\mathbb{P}^{t}_{x+z,y+z}$ under the same mapping.

For the proof of Theorem 3, it suffices to remember that $\mathbb{P}^{t,\dagger}_{x,y}$ is equal to $\mathbb{P}^{t,3}_{x,y}$ as noted in Example 7. Until now, we have proved weak continuity with respect to the ending point of the bridge; however, just as the proof of weak continuity extends to the temporal parameter under appropriate conditions, the existence of a tri-continuous transition density for δ-dimensional Bessel process which is positive when the starting point and ending point are 0 suffices to verify the joint weak continuity of the laws $\mathbb{P}^{t,\delta}_{x,y}$ with respect to $(x,y,t) \in [0,\infty)^2 \times (0,\infty)$, so that

$$\mathbb{P}^{1}_{0,0}(\,\cdot \mid I > -\varepsilon) = \mathbb{P}^{3,t}_{\varepsilon,\varepsilon} \circ (f \mapsto f - \varepsilon)^{-1} \to \mathbb{P}^{3,t}_{0,0}$$

as $\varepsilon \to 1$. □

Of course, if we start with a bridge of length t instead of with one of length one in the preceding reasoning, one obtains a limiting process which can be called the **Brownian excursion of length** t.

2. A pathwise construction of the normalized Brownian excursion from Brownian motion

Our present objective will be to see how the normalized Brownian excursion arises as a renormalized part of Brownian motion, in a similar way to the Brownian bridge construction of Application 1. Let B be a Brownian motion and consider g and d to be the last zero (resp. first zero) before (resp. after) time 1, thought of as defined on canonical space but applied here to B. We will show that conditionally on $\mathscr{F}_g$ and $\mathscr{F}^d$, the stochastic process defined by

$$Y_s = \left|B_{(g+s)\wedge d}\right|$$

is a Brownian excursion of length $d-g$; it is called the **excursion of** B **straddling** 1. However, in Application 2, we had seen how Bessel bridges are affected by the Brownian scaling operator S_v; in our case, applying $S_{1/d-g}$ to Y results in the normalized Brownian excursion $\mathbf{e}$ defined as follows:

$$\mathbf{e}_s = \frac{B_{g+(s\wedge 1)(d-g)}}{\sqrt{d-g}};$$

it is independent of $\mathscr{F}_g$ and $\mathscr{F}^d$. Let $\mathfrak{s}$ stand for the sign of the excursion straddling one, equal to the sign of B_1. Since $-B$ has the same law as B, $\mathfrak{s}$ is uniform on $\{-1,1\}$.

The objective of this section is to prove the following pathwise construction of the normalized Brownian excursion.

THEOREM 4. *The law of* $\mathbf{e}$ *given* $\mathscr{F}_g \vee \mathscr{F}^d \vee \sigma(\mathfrak{s})$ *is* $\mathbb{P}^{1,3}_{0,0}$.

It suffices to see that a version of the conditional law of Y given $\mathscr{F}_g \vee \mathscr{F}^d$ is $\mathbb{P}^{3,d-g}_{0,0}$ and for this we will use the equality between the Brownian bridge conditioned on remaining nonnegative and the bridge of Brownian motion killed upon reaching zero. The underlying heuristic is that Y behaves, before its hitting time of zero, like Brownian motion conditioned on remaining positive, and so the backward strong Markov property would explain the appearance of its bridge; however, we will be calculating with the law of B, and for this process, g is not stopping time and d is not backward optional, impeding the application of our Markov type property. The proof is therefore a circumvention of this point based on repeating several steps in the proof of the backward strong Markov property.

As a remark, note that since $-B$ has the same law as B, then for $x, y < 0$, the absolute value of the Brownian bridge between x and y of any length conditioned on remaining negative has the same law as the Brownian bridge between $|x|$ and $|y|$ of the same length conditioned on remaining positive.

PROOF. Consider $0 < s < 1 \wedge t$, $F \in b(\mathscr{F}_s \vee \mathscr{F}^t)$ and $G \in b\mathscr{F}_\infty$. Denote by R the operator which sends f to $|f|$. Using the equality of the sets $\{g < s, t < d\} = \{T_0 \circ \theta_s > t - s\}$, valid because $s < 1 \wedge t$, a temporal window in the Markov property gives

$$\begin{aligned}\mathbb{E}(FG \circ R \circ \sigma_t^s \mathbf{1}_{g<s,t<d} \mathbf{1}_{\mathfrak{s}=i}) &= \mathbb{E}(FG \circ R \circ \sigma_t^s \mathbf{1}_{T_0 \circ \theta_s > t-s} \mathbf{1}_{\mathfrak{s}=i}) \\ &= \mathbb{E}\Big(F \mathbb{P}^{t-s}_{X_s, X_t}(G \circ R \mathbf{1}_{T_0 > t-s})\, \mathbf{1}_{\mathfrak{s}=i}\Big).\end{aligned}$$

(Notice that $\mathfrak{s} = \operatorname{sgn}(X_s)$ when $g < s < d$.) In particular, when G is equal to one, we obtain

$$\mathbb{E}(F \mathbf{1}_{g<s,t<d} \mathbf{1}_{\mathfrak{s}=i}) = \mathbb{E}\Big(F \mathbb{P}^{t-s}_{X_s, X_t}(T_0 > t-s)\, \mathbf{1}_{\mathfrak{s}=i}\Big)$$

and so, using the equality between the Brownian bridge conditioned on remaining nonnegative and the Bridge of Brownian motion killed when it reaches zero proved in Theorem 3 gives

$$\begin{aligned}&\mathbb{E}(FG \circ R \circ \sigma_t^s \mathbf{1}_{g<s,t<d \mathbf{1}_{\mathfrak{s}=i}}) \\ &= \mathbb{E}\Big(F \mathbb{P}^{t-s}_{X_s, X_t}(G \circ R \mathbf{1}_{T_0 > t-s})\, \mathbf{1}_{\mathfrak{s}=i}\Big) \\ &= \mathbb{E}\Big(F \mathbb{P}^{t-s,\dagger}_{|X_s|, |X_t|}(G)\, \mathbb{P}^{t-s}_{X_s, X_t}(T_0 > t-s)\, \mathbf{1}_{\mathfrak{s}=i}\Big) \\ &= \mathbb{E}\Big(F \mathbf{1}_{g<s,t<d} \mathbb{P}^{t-s,\dagger}_{|X_s|, |X_t|}(G)\, \mathbf{1}_{\mathfrak{s}=i}\Big).\end{aligned}$$

We now have the ingredients to apply the approximations on which the proof of the backward strong Markov property is based. As in that proof, let us construct an increasing approximation g^n to g written in terms of g (resp. a decreasing approximation d^n to d) such that they both take at most a countable number of values, such that $g < g^n < d^n < d$ whenever $d < g$ (which occurs almost surely since $\mathbb{P}(B_1 = 0) =$

0). Note the inclusion $\mathscr{F}_g \vee \mathscr{F}^d \cap \{g^n = s, d^n = t\} \subset \mathscr{F}_s \vee \mathscr{F}^t$ since on $\{g^n = s, d^n = t\}$, $g < s$ and $t < d$. If $A \in \mathscr{F}_g \vee \mathscr{F}^d$ and $G \in b\mathscr{F}_\infty$, then by summing over the values of g^n and d^n, we get

$$\begin{aligned}
\mathbb{E}\big(\mathbf{1}_A G \circ R \circ \sigma_{d_n}^{g_n} \mathbf{1}_{\mathfrak{s}=i}\big) &= \sum_{s<t} \mathbb{E}(\mathbf{1}_A \mathbf{1}_{g^n=s,d^n=t} G \circ R \circ \sigma_t^s \mathbf{1}_{\mathfrak{s}=i}) \\
&= \sum_{s<t} \mathbb{E}\Big(\mathbf{1}_A \mathbf{1}_{g^n=s,d^n=t} \mathbb{P}^{t-s,\dagger}_{|X_s|,|X_t|}(G)\, \mathbf{1}_{\mathfrak{s}=i}\Big) \\
&= \sum_{s<t} \mathbb{E}\left(\mathbf{1}_A \mathbf{1}_{g^n=s,d^n=t} \mathbb{P}^{d^n-g^n,\dagger}_{|X_{g^n}|,|X_{d^n}|}(G)\, \mathbf{1}_{\mathfrak{s}=i}\right) \\
&= \mathbb{E}\left(\mathbf{1}_A \mathbb{P}^{d^n-g^n,\dagger}_{|X_{g^n}|,|X_{d^n}|}(G)\, \mathbf{1}_{\mathfrak{s}=i}\right).
\end{aligned}$$

If we now consider continuous functionals G on D_∞, and let $n \to \infty$, we obtain

$$\mathbb{E}(\mathbf{1}_A G \circ \sigma_d^g \mathbf{1}_{\mathfrak{s}=i}) = \mathbb{E}\Big(\mathbf{1}_A \mathbb{P}^{d-g,3}_{0,0}(G)\, \mathbf{1}_{\mathfrak{s}=i}\Big)$$

which proves our theorem. □

As a corollary, we obtain the following fact:

$$X_s > 0 \text{ for all } s \in (0,t)\ \mathbb{P}^{t,3}_{0,0}\text{-almost surely.}$$

For $t = 1$, it is a consequence of the fact that $\mathbf{e}$ has that property, and for general t, we deduce it from the previous case by scaling.

In the pathwise construction of the Brownian bridge (Application 1 of Chapter 1), we noted that the process b defined by

$$b_t = \frac{B_{g\cdot(t\wedge 1)}}{\sqrt{g}}$$

is independent of $\mathscr{F}^g$; however, $\mathscr{F}^g = \sigma\big(g, \mathbf{e}, \mathfrak{s}, \mathscr{F}^d\big)$ and we now know that $\mathbf{e}$ is independent of $\sigma\big(g, \mathfrak{s}, \mathscr{F}^d\big)$. Also, by the Markov property, the triple $(g, d-g, \mathfrak{s})$ is independent of $\mathscr{F}^d$, and by invariance of the law of Brownian motion under $f \mapsto -f$, $\mathfrak{s}$ is independent of $(g, d-g)$. We can therefore reconstruct a Brownian motion by combining 5 independent elements: a Brownian bridge between 0 and 0 of length, say b, a couple $\big(\tilde{g}, \tilde{d}\big)$ distributed as (g,d), a normalized Brownian excursion $\mathbf{e}$, a random variable $\mathfrak{s}$ uniform on $\{-1,1\}$,and a Brownian motion B. The pathwise construction is the following: put

$$\tilde{B}_t = \begin{cases} b_{t/\tilde{g}}\sqrt{\tilde{g}} & \text{if } t \leq \tilde{g} \\ \mathfrak{s}\mathbf{e}_{(t-\tilde{g})/(\tilde{d}-\tilde{g})}\sqrt{\tilde{d}-\tilde{g}} & \text{if } t \in (\tilde{g}, \tilde{d}) \\ B_{t-\tilde{d}} & \text{if } t \geq \tilde{d} \end{cases}.$$

As a final remark, remember the independence of $\mathbf{e}$ and $\mathscr{F}_g \vee \mathscr{F}^d$; by considering excursions *straddled* by times different than 1, we could think that the excursions of Brownian motion away from zero, renormalized to have length one, are independent and that Brownian motion can be reconstructed by putting normalized Brownian excursions over the complement of the zero set of Brownian motion, scaled in a Brownian way to fit correctly. This is essentially true, although a complication arises because a same excursion could straddle two different deterministic times; we should therefore select a random sequence $(t_n)_{n\in\mathbb{N}}$ such that the excursion straddling t_n differs from the one straddling t_m if $n \neq m$, and consider the corresponding straddling excursions renormalized to have length one. It is not clear however, if this random choice has a repercussion in their joint law. A precise way of choosing a random sequence $(t_n)_{n\in\mathbb{N}}$ is described in [**RY99**, Exercise XII.2.18, p. 487]. In Section 5, we will give sense to an abstract formulation of our conjecture without labelling and to prove it with a particular labelling. However, the method presented in this section will be extended (in a slightly different context) in Subsection 2.1 of Chapter 3.

3. The excursion process of Brownian motion and the normalized Brownian excursion

In this section we will introduce the excursion process of Brownian motion and explain the appearance of the normalized Brownian excursion in its study. For this, it will be necessary to review some facts about Poisson point process which are not only necessary to the fulfillment of our present aims but also imposed by previous and posterior needs. In particular, we will verify the unproved statement crucial to our pathwise construction of killed stable subordinators found in Application 3. We will also pave the way for a conditional independence lemma to be stated in the following chapter.

Let B be a Brownian motion and denote by $\mathscr{Z}$ its zero set equal to $\{s \geq 0 : B_s = 0\}$. By the strong Markov property, Brownian motion is **regenerative at zero** in the following sense: if T is a stopping time for Brownian motion then the conditional law of $B \circ \theta_T$ given $\mathscr{F}_T$ equals that of B on the set $\{T < \infty, B_T = 0\}$. By Theorem 22.11 in [**Kal02**, p. 436], there exists a nondecreasing, adapted process L, called the **regenerative local time of** B whose support is $\mathscr{Z}$. Since $\mathscr{Z}$ is a closed set, its complement can be expressed as a countable union of intervals (u, v) which will be termed **excursion intervals**. Introduce the right-continuous inverse τ of L given by

$$\tau_l = \inf\{t \geq 0 : L_t > l\}.$$

Then $\mathscr{Z}$ is the closure of the image of τ and so the excursion intervals are in one-to-one correspondence with the set of jump times of τ , $J = \{l > 0 : \Delta\tau_l > 0\}$ (we define the **jump of X at time** t, denoted ΔX_t, to be the quantity $X_t - X_{t-}$): excursion intervals are precisely (τ_{l-}, τ_l) with $l \in J$. With this correspondence, we will label the excursions of B away from zero: for every $l \in J$, define the continuous trajectory e^l by means of

$$e^l_t = B_{(\tau_{l-}+t)\wedge\tau_l}.$$

Each e^l is an excursion of Brownian motion away from zero, but they are indexed by the random set J. To start working with the excursions without having to refer to their labels, we introduce excursion space E consisting of elements e of D_∞ which are continuous and for which there exists $L = L(e) \in [0, \infty]$ (called the length of the excursion e) such that $e_t \neq 0$ if and only if $t \in (0, L)$; it will be given the σ-field generated by the valuations, just as for D_∞. Although the notation L is used with two different objects, namely the local time process and the length functional on excursion space, the precise meaning will be clear from the context. The value $L = 0$ is considered to allow for the trivial excursion which never leaves zero and $L = \infty$ gives way to infinite excursions. Consider also the random point measure Ξ on $[0, \infty) \times E$ by

$$\Xi(A) = \sum_{l \in J} \mathbf{1}_A\big((l, e^l)\big).$$

The family of random measures $\Xi_t, t \geq 0$ defined as the restriction of Ξ to $[0, t] \times E$ give rise to what is called the **excursion process** of Brownian motion. The excursion process allows for explicit calculations for Brownian motion, thanks to the following definitions and theorem; a more detailed account is found in Sections 1 and 2 of Chapter XII in [**RY99**] although, for the notion of local time, we have used Chapter 22 of [**Kal02**], and to reconcile it with the one used in the former reference, Proposition 22.14 of the latter. The complication that arises from using local time for regenerative sets stems from its unicity up to a contant multiple. We will make a choice of the constant so that the two choices of local time really coincide.

Definition. Consider an arbitrary measurable space $(S, \mathscr{S})$ and write $M(S)$ for the set of σ-finite measures on S together with the σ-field generated by the maps $\mu \mapsto \mu(A)$ where $A \in \mathscr{S}$. A **random measure** is a random variable with values on $M(S)$. The **intensity** of a random measure ξ is the measure on $\mathscr{S}$ given by $A \mapsto \mathbb{E}(\xi(A))$.

A **Poisson random measure with intensity** ν is a random measure ξ with the following properties:

- if $A_1, \ldots, A_n \in \mathscr{S}$ are pairwise disjoint, then $\xi(A_1), \ldots, \xi(A_n)$ are independent, and
- for every $A \in \mathscr{S}$ satisfying $\nu(A) < \infty$, $\xi(A)$ has a Poisson distribution with mean $\nu(A)$.

A **Poisson point process with characteristic measure** ν is a Poisson random measure on $[0, \infty) \times S$ whose intensity is expressed in terms of Lebesgue measure λ on $[0, \infty)$ as $\lambda \otimes \nu$.

Theorem 5. *The excursion process Ξ is a Poisson point process.*

The characteristic measure of Ξ is called Itô's measure and will be denoted by n. Its restriction to positive excursions is denoted n_+. Thanks to the invariance of the law of B under $B \mapsto -B$, it follows that the restriction of n to negative excursions, denoted n_-, is the image of n_+ under $e \mapsto -e$. The characteristic measure is the key to explicit calculations for Brownian motion, as will be exemplified in this and the following chapter, so we will attempt a description of it, known as Itô's description of Itô's measure; it corresponds to a disintegration of n_+ with respect to the length functional. It is not only conceptually useful, but also for practical reasons because the image of n_+ under L is known. The two results combine as follows.

Proposition 2. *The family of measures* $\mathbb{P}^{t,3}_{0,0}, t > 0$ *disintegrate* n_+ *with respect to* L. *Furthermore, there exists a contant* C *such that*

$$n_+(L > v) = C/\sqrt{v}$$

and if the constant is taken equal to $\sqrt{2/\pi}$, *calculations agree with the ones obtained through semimartingale local time.*

The proof of Proposition 2 relies on a property of Poisson point processes which will be described after some preliminaries; its proof is found in [**RY99**, XII.1.13, p. 477].

Let ξ be a Poisson point process with characteristic measure ν on $(S, \mathscr{S})$ and let $U \in \mathscr{S}$ satisfy $\nu(U) \in (0, \infty)$. If

$$T = \inf\{t \geq 0 : \xi([0,t] \times U) > 0\},$$

there exists a unique (albeit random) $e^T \in S$ such that $\xi|_{\{T\}\times S} = \delta_{(T, e^T)}$. We have:

Lemma 4. T *and* e^T *are independent and*

$$\mathbb{P}\big(e^T \in A\big) = \nu(A \cap U)\,/\nu(U)\,.$$

Proof of Proposition 2. The reasoning is very similar to the one used in the proof of the strong Markov property under n_+ of [**RY99**, XII.4.1, p. 495]. First, we define the height functional $H : E \to \mathbb{R}_+$ by $H(e) = \sup_{s\geq 0} e_s$ and use Lemma 4 with

$$T = \inf\left\{l \geq 0 : H\big(e^l\big) > x\right\}$$

for any $x > 0$. To do so, we are forced to verify the membership of $n_+(H > x)$ to $(0, \infty)$: since $\tau_l < \infty$, on any compact interval $[0, l]$ there is at most a finite quantity of positive excursions of B away from zero which have height greater than x before time τ_l and so $l n_+(H > x) < \infty$; since $\limsup_{t\to\infty} B_t = \infty$, there is almost surely an excursion of B of height greater than x on $[0, \infty)$ and so $\infty n_+(H > x) > 0$ so that $n_+(H > x) > 0$. Using Lemma 4 we note that $n_+(\cdot \mid H > x)$ is the law of e^T; however, $e^T \circ \theta_{T_x}$ coincides with $\sigma^{T_x}_{T_0\circ\theta_{T_x}}(B)$ or, said otherwise, the law of the portion of e^T after it reaches level x is that of a Brownian motion started at x and stopped upon reaching zero. The following symbolic translation ensues: for any measurable and bounded functional Φ on E

$$n_+(\mathbf{1}_{H>x}\Phi \circ \theta_{T_x}) = n_+(H > x)\,\mathbb{P}_x(\Phi \circ \sigma_{T_0})\,. \tag{17}$$

The two parts of Proposition 2 will follow from the application of this result.

We will now carry out the disintegration of n_+ with respect to the length using the backward strong Markov property and the preceding paragraph. Consider $x > y > 0$ and let L_y be the last visit to y before its first visit to zero performed by the canonical process; then $X \circ \sigma_{L_y}$ has the same law under $\mathbb{P}_x$ and under $\mathbb{P}^{\dagger}_x$. However, under $\mathbb{P}^{\dagger}_x$ we can use the backward strong Markov property to deduce that $X \circ \sigma_{L_y}$ has conditional law $\mathbb{P}^{L_y,\dagger}_{x,y}$ given L_y. As $y \to 0$ $L_y \to T_0$ and so, using the identification of the bridges of killed Brownian motion and the three-dimensional Bessel process, we conclude that given T_0, the conditional law of $X \circ \sigma_{T_0}$ given T_0 under $\mathbb{P}_x$ is $\mathbb{P}^{T_0,3}_{x,0}$. Equation (17) translates to

$$\begin{aligned} n_+(\mathbf{1}_{H>x}\Phi \circ \theta_{T_x} f(T_0 \circ \theta_{T_x})) &= n_+(H > x)\,\mathbb{P}_x(\Phi \circ \sigma_{T_0} f(T_0)) \\ &= n_+(H > x)\,\mathbb{P}_x\left(\mathbb{P}^{T_0,3}_{x,0}(\Phi)\, f(T_0)\right) \\ &= n_+\left(\mathbf{1}_{H>x}\mathbb{P}^{T_0\circ\theta_{T_x},3}_{x,0}(\Phi)\, f(T_0 \circ \theta_{T_x})\right). \end{aligned}$$

We will take the limit of the preceding quantity as $x \to 0$ using the dominated convergence theorem justifying its use because $n_+(L > v) < \infty$; this is true as for $\{H > x\}$ because for Brownian motion there is at most a finite number of excursions exceeding length v on a compact time interval. By considering continuous and bounded f and Φ, with f vanishing on a neighborhood of zero, we obtain as $x \to 0$

$$n_+(\Phi f(L)) = n_+\big(\mathbb{P}^{L}_{0,0}(\Phi)\, f(L)\big)$$

since n_+ is concentrated on $H > 0$.

Finally, we will characterize the law of L under n_+ up to a constant; this is the best we can do when using regenerative local time. We will prove

$$n_+\big(1 - e^{-qL}\big) = C\sqrt{2q/\pi},$$

since this implies $n_+(L > v) = C/\sqrt{v}$. By monotone convergence,

$$n_+\big(1 - e^{-qL}\big) = \lim_{x\to 0} n_+\big(\mathbf{1}_{H>x}\left(1 - e^{-qT_0\circ\theta_{T_x}}\right)\big);$$

the right-hand side can be calculated with equation (17):

$$n_+\big(\mathbf{1}_{H>x}\left(1 - e^{-qT_0\circ\theta_{T_x}}\right)\big) = n_+(\mathbf{1}_{H>x})\,\mathbb{P}_x\big(1 - e^{-qT_0}\big).$$

The law of T_0 under $\mathbb{P}_x$ admits $q \mapsto \exp\left(x\sqrt{2q}\right)$ as a Laplace transform (as in Example 2 of Chapter 1). Now we will calculate $n_+(H > x)$ up to a multiplicative constant: if $y > x$, we get by (17),

$$n_+(T_y < \infty) = n_+(T_x < \infty, T_y \circ \theta_{T_x} < \infty) = n_+(\mathbf{1}_{T_x < \infty}) \, \mathbb{P}_x(T_y < T_0) \, .$$

By scale function computations $\mathbb{P}_x(T_y < T_0) = x/y$ (cf. [**RY99**, VII.3, p.300]) and so there exists $C' > 0$ such that $n_+(H > x) = C'/x$. We finally get

$$\begin{aligned} n_+\left(1 - e^{-qL}\right) &= \lim_{x \to 0} n_+\left(\mathbf{1}_{H > x} \left(1 - e^{-qT_0 \circ \theta_{T_x}}\right)\right) \\ &= \lim_{x \to 0} \frac{C}{x} \left(1 - e^{-x\sqrt{2q}}\right) = C' \sqrt{2q}. \end{aligned}$$

If we let $C = \sqrt{2/\pi}$ (or $C' = \sqrt{2}$), then the law of L under n_+ coincides with the one obtained by computations with semimartingale local time performed in [**RY99**, XII.2.8, p. 484]. □

4. Inverse local times of recurrent Bessel processes.

In this section we will consider an application of the preceding computations in the context of Bessel processes of dimension $\delta \in (0, 2)$. This will provide a further example for the application of the backward strong Markov property. Let $\mathscr{Z}$ stand for the zero set of the canonical process, whose structure we have studied under $\mathbb{P}_0$ and shall undertake under $\mathbb{P}_0^\delta$ for $\delta \in (0, 2)$. By the strong Markov property of $\mathbb{P}_0^\delta$, $\mathscr{Z}$ is a regenerative set and as such, it admits a local time process L which starts at zero, is continuous, and has $\mathscr{Z}$ as a support. Furthermore, this local time process is an **additive functional** in the sense that

$$L_{t+s} = L_s + L_t \circ \theta_s.$$

This implies that its generalized inverse τ given by

$$\tau_l = \inf\left\{s \geq 0 : L_s > l\right\} \text{ satisfies } \tau_{l+\cdot} - \tau_l = \tau \circ \theta_{\tau_l}.$$

Since τ_l is a stopping time for X and $X_{\tau_l} = 0$, $\tau_{\cdot + l} - \tau_l$ is independent of $(\tau_{l'})_{l' \in [0,l]}$ and has the same law as τ; it is therefore a subordinator. We will characterize it as a stable subordinator of index $1 - \delta/2 \in (0, 1)$ by computing its Laplace exponent. This will consist of two parts: the separate calculation of its drift and of its Lévy measure. The drift d of τ is zero because from [**Ber96a**, IV, Corollary 6, p. 112] one knows that

$$\int_0^t \mathbf{1}_{(X_s=0)}\,ds = dL_t$$

and the law of X_t under $\mathbb{P}_0^\delta$ is absolutely continuous with respect to Lebesgue measure for $t > 0$ giving

$$\mathbb{P}_0^\delta\left(\int_0^t \mathbf{1}_{X_s=0}\right) = 0;$$

since the local time process L is not identically equal to zero, we must have $d = 0$. It follows that τ is the sum of its jumps, meaning

$$\tau_l = \sum_{l' \le l} \Delta\tau_{l'}$$

and this will be the key to the identification of its Lévy measure by lifting the discussion in the preceding section to this case. Just as before, we can define the excursion process $(\Xi_t)_{t\ge 0}$ under $\mathbb{P}_0^\delta$ and it is a Poisson point process, whose characteristic measure shall be called n^δ; it is concentrated on positive excursions since Bessel processes are nonnegative. The preceding display translates to

$$\tau_l = \sum_{l' \in J, l' \le l} L\left(e^{l'}\right),$$

where we remember that L is used here as the length functional on excursion space. This will be put to use by means of the exponential formula for Poisson point processes which we now state. For the proof, see [**RY99**, XII.12, p.476].

Proposition 3. *Let ξ be a Poisson random measure with intensity ν on $(S, \mathscr{S})$. If $f : S \to [0,\infty)$ is measurable, then*

$$\mathbb{E}\left(e^{\int f\,d\xi}\right) = e^{\int \left(1-e^{-f}\right)d\nu}.$$

As an application, we note that

$$\mathbb{P}_0^\delta\left(e^{-q\tau_l}\right) = \mathbb{P}_0^\delta\left(e^{-\sum_{l'\in J, l'\le l} q\Delta\tau_{l'}}\right) = e^{-ln^\delta\left(1-e^{-qL}\right)}.$$

The quantity $n^\delta\big(1-e^{-qL}\big)$ will be calculated as in the preceding section; to do so, we needed the following two quantities:

$$\mathbb{P}_x^\delta\big(e^{-qT_0}\big) \text{ and } n^\delta(T_x<\infty)$$

as well as an extension of (17) to n^δ. The latter can be dealt with by the same methods use to establish it for the Brownian case. By scale function computations (the scale function of squared Bessel processes is found in [**RY99**, XI.1, p. 442]) we conclude conclude that for $x<y$

$$n^\delta(T_y<\infty)=n^\delta(T_x<\infty)\,\mathbb{P}_x(T_y<T_0)=n^\delta(T_x<\infty)\left(\frac{x}{y}\right)^{2\beta}$$

where $\beta=1-\delta/2$, and so there exists $C'>0$ such that

$$n^\delta(T_x<\infty)=C'x^{-2\beta}.$$

An explicit expression for $\mathbb{P}_x^\delta\big(e^{-qT_0}\big)$ is found in [**Ken78**]. It ultimately involves the Bessel functions I_β and $I_{-\beta}$ (whose explicit expression is found in Example 5 of Chapter 1 by introducing the modified Bessel function of the second kind K_β defined by

$$K_\beta=\frac{I_{-\beta}-I_\beta}{2\sin(\pi\beta)}.$$

It is given by

$$\mathbb{P}_x^\delta\big(e^{-qT_0}\big)=\frac{2}{\Gamma(\beta)}\left(\frac{\sqrt{2q}x}{2}\right)^\beta K_\beta\Big(\sqrt{2q}x\Big).$$

By means of the explicit expression of I_ν, we see that

$$I_\nu(x)\sim_{x\to 0}\left(\frac{x}{2}\right)^\nu\frac{1}{\Gamma(1+\nu)}.$$

Since $\mathbb{P}_x^\delta$ has a weak limit as $x\to 0$, it follows that

$$\lim_{x\to\infty}\mathbb{P}_x\big(1-e^{-qT_0}\big)=0$$

and so then, from the asymptotic behaviour of I_β and $I_{-\beta}$ near zero, it is found that

$$\mathbb{P}_x\big(1-e^{-qT_0}\big)\sim_{x\to 0}C''\left(\sqrt{q}x\right)^\beta I_\beta\Big(\sqrt{2q}x\Big).$$

Hence:

$$n^\delta\big(1-e^{-qL}\big)=C'''\lim_{x\to 0}x^{-2\beta}\left(\sqrt{q}x\right)^\beta I_\beta\Big(\sqrt{2q}x\Big)=Cq^\beta.$$

As announced, for $\delta \in (0,2)$ and under $\mathbb{P}_0^\delta$, inverse local time is a stable subordinator of index $\beta = (1-\delta/2) \in (0,1)$.

We will conclude this section with another application of the backward strong Markov property to the inverse local time of Bessel bridges.

Application 4. Within the framework of this section, let us consider the inverse local time τ under $\mathbb{P}_{0,0}^{1,\delta}$. First of all, let us note that the existence of local time (let alone inverse local time) is not obvious. In [**Kal02**, Prop. 22.12, p. 437], the following approximation is given for regenerative local time: let $\eta_t(A)$ be the quantity of excursions that belong to A and are completed before time t; if $A_1, A_2, \dots$ are Borel subsets of excursion space such that $n^\delta(A_i)$ belongs to $(0,\infty)$ and tends to ∞ as $i \to \infty$ then

$$\sup_{s \leq t} \left| \frac{\eta_t(A_i)}{n^\delta(A_i)} - L_s \right| \to 0$$

almost surely as $i \to \infty$ under $\mathbb{P}_0^\delta$. By the local absolute continuity of bridge laws (see (1)), the same convergence holds under $\mathbb{P}_{0,0}^{1,\delta}$ if $t < 1$ and it is that almost sure limit which we call local time. We shall use the sets $A_i = \{L > \varepsilon_i\}$, where $\varepsilon_i \to 0$, because we have calculated $n^\delta(A_i) = D\varepsilon_i^{-\beta}$.

By the backward strong Markov property, if we let g denote the last zero of the canonical process before time 1, then as in Application 1, under $\mathbb{P}_x^\delta$,the stochastic process b given by

$$b_t = \frac{1}{\sqrt{g}} X_{(t \wedge 1)g}$$

has law $\mathbb{P}_{0,0}^{1,\delta}$. The local time of b, denoted $L(b)$, obtained by the previous approximations, is expressed in terms of that of X as follows:

$$L_t(b) = g^{-\beta} L_{t \cdot g}.$$

It follows that the inverse local time of b, denoted $\tau(b)$, is given by

$$\tau_l(b) = \frac{1}{g} \tau_{l \cdot g^\beta}.$$

If we note that $g = \tau_{L_1-}$, application 3 implies that the inverse local time of a Bessel bridge of dimension δ is a stable subordinator of index $\beta = 1 - \delta/2$ started at zero and conditioned to die at one. Another (perhaps more direct) proof of this fact will be given in Chapter 3.

Having introduced Poisson point processes, we will now conclude application 3. It remains to prove that if $L_1 = \sup\{s \geq 0 : X_s < 1\}$ and $g = X_{L_1}$, then under the law of stable subordinator of index β started at zero, $\tilde{\mathbb{P}}_0^{\beta}$, the conditional law of L_1 given g has density $s \mapsto f_s^{\beta}(x)/u_{\beta}(x)$, where f_s^{β} is the density of X_s under $\tilde{\mathbb{P}}_0^{\beta}$ and u_{β} is the potential density associated to $\tilde{\mathbb{P}}_x^{\beta}, x \geq 0$. Thanks to the Lévy-Itô decomposition of Lévy processes (see [**Ber96a**, I.1,Thm. 1]), a stable subordinator increases only by jumps, so that under $\tilde{\mathbb{P}}_0^{\beta}$, $X_t = \sum_{s \leq t} \Delta X_s$. (In the preceding sum, there is at most a countable quantity of non-zero terms.) Note that if $f : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$ is measurable, then

$$f(L_1, g) = \sum_s f(s, X_{s-})\, \mathbf{1}_{X_{s-} < 1 < X_{s-} + \Delta X_s}$$

since only one term is positive. Since under $\tilde{\mathbb{P}}_0^{\beta}$ the jump process of X, given by $(\Delta X_t)_{t \geq 0}$ is a Poisson point process whose characteristic measure π^{β} is absolutely continuous with respect to Lebesgue measure with a density given by

$$x \mapsto \frac{\beta C}{\Gamma(1-\beta)\, x^{1+\beta}} \mathbf{1}_{x>0},$$

we can use the additive formula (cf. [**RY99**, XII.1.10, p.475] to compute

$$\begin{aligned}
\tilde{\mathbb{P}}_0^{\beta}(f(L_1, g)) &= \tilde{\mathbb{P}}_0^{\beta}\left(\sum_s f(s, X_{s-})\, \mathbf{1}_{X_{s-} < 1 < X_{s-} + \Delta X_s}\right) \\
&= \int_0^{\infty} \tilde{\mathbb{P}}_0^{\beta}\big(f(s, X_{s-})\, \mathbf{1}_{X_{s-}<1} \pi^{\beta}([1 - X_{s-}, \infty))\big)\, ds \\
&= \int_0^{\infty} \tilde{\mathbb{P}}_0^{\beta}\left(f(s, X_{s-})\, \mathbf{1}_{X_{s-}<1} \frac{C}{\Gamma(1-\beta)\,(1 - X_{s-})^{\beta}}\right) ds.
\end{aligned}$$

We can substitute X_{s-} with X_s in the preceding computation, since $\tilde{\mathbb{P}}_0^{\beta}(X_{s-} = X_s) = 1$, to obtain

$$\tilde{\mathbb{P}}_0^{\beta}(f(L_1, g)) = \int_0^{\infty} \int_0^1 f(s, x)\, f_s^{\beta}(x)\, \frac{C}{\Gamma(1-\beta)\,(1-x)^{\beta}}\, dx\, ds.$$

We therefore see that the joint law of (L_1, g) under $\tilde{\mathbb{P}}_0^{\beta}$ is absolutely continuous with respect to Lebesgue measure, with a version of the density given by

$$(s,x) \mapsto f_s^{\beta}(x)\,\frac{C}{\Gamma(\beta)\,(1-x)^{\beta}}\mathbf{1}_{0<x<1}.$$

Using the explicit value of the potential density u_β, we see that the law of g under $\mathbb{P}_0^{\delta}$ has the density

$$\frac{1}{\Gamma(\beta)\,\Gamma(1-\beta)\,x^{1-\beta}\,(1-x)^{\beta}},$$

so that g has the generalized arc-sine law with parameter β. We see then that the conditional density of L_1 given $g=x$ can be taken equal to

$$s \mapsto f_s^{\beta}(x)\,C\Gamma(1-\beta)\,x^{1-\beta} = \frac{f_s^{\beta}(x)}{u_\beta(x)}$$

as announced.

5. Conditional independence of excursions given their lengths

In this section, we will formalize the heuristic put forth at the end of Section 2, namely, that if we consider the excursions of Brownian motion that start before inverse local time l, ordering them by their length in a decreasing manner, and scale them in a Brownian way to have length one, one obtains a sequence of independent normalized Brownian excursions. This will be seen to be a consequence of more general considerations involving Poisson point processes.

Let $(S,\mathscr{S})$ and $(R,\mathscr{R})$ be a arbitrary measurable spaces. The following definitions and property come from [**Kal02**, Ch. 12]

Definition. A **point process** on S is an integer valued random measure. A **probability kernel** from S to T is a function $\nu : S\times\mathscr{R}\to[0,1]$ such that for each $x\in\mathscr{S}$, $A\mapsto\nu(x,A)$ is a probability measure on $\mathscr{R}$ and for each $A\in\mathscr{R}$, $x\mapsto\nu(x,A)$ is measurable. Let ν be a probability kernel from S to T; for each $\overline{\mathbb{Z}}_+$-valued measure $\mu=\sum_k\delta_{a_k}$ on $\mathscr{S}$, we define the law $\mathbb{Q}_\mu$ as that of $\zeta=\sum_k\delta_{(a_k,b_k)}$, where the b_k are independent and b_k has law $\nu(a_x,\cdot)$. A **ν-randomization** of a point process ξ on S is a point process ζ on $S\times T$ whose conditional law given ξ is $\mathbb{Q}_\xi$. The **Laplace functional** of a random measure ξ on $\mathscr{S}$ is the map defined on measurable nonnegative functions on S and is given by

$$f\mapsto\mathbb{E}\Big(e^{-\int f\,d\xi}\Big)$$

Laplace functionals characterize laws of random measures. In the context of the excursion process of Brownian motion, we could think that the Poisson point process consisting of triples consisting of local times of occurrence of excursions, lengths of excursions, and the excursions themselves is a randomization of the Poisson point process consisting only of the pairs formed by the first two coordinates. This would formalize the statement that excursions are conditionally independent given their lengths, although no labelling is required. Of course, the randomizing kernel would be $(l, v, A) \mapsto \mathbb{P}^{v,3}_{0,0}(A)$, as we shall see.

In the following proposition, consider the kernel $\hat{\nu}$ from S to $S \times T$ defined in terms of the kernel ν from S to T by means of $\hat{\nu}(s, \cdot) = \delta_s \otimes \nu(s, \cdot)$. Also, write $\langle \xi, f \rangle$ for the integral of f with respect to ξ.

PROPOSITION 4. *ζ is a ν-randomization of ξ if and only if*

$$\mathbb{E}\Big(e^{-\langle \zeta, f\rangle}\Big) = \mathbb{E}\Big(e^{\langle \xi, \log\left(\langle \hat{\nu}, e^{-f}\rangle\right)\rangle}\Big).$$

THEOREM 6. *Let S be a Borel space and Ξ a Poisson random measure on S with intensity measure μ. Let $h : S \to (0, \infty)$ be such that $\mu \circ h^{-1}$ has no atoms and is finite on (ε, ∞) for all $\varepsilon > 0$. Define the random measure Ξ^h on $(0, \infty) \times S$ by $\Xi^h(A \times B) = \Xi\big(h^{-1}(A) \times B\big)$. Then*

(1) There exists a probability kernel ν from $(0, \infty)$ to S such that Ξ^h is a ν-randomization of $\Xi \circ h^{-1}$.

(2) Si $((l_i, s_i))_{i\in\mathbb{N}}$ are the atoms of Ξ^h ordered by their decreasingly with respect to their first coordinate, then (l_i, s_i) are random variables, $(s_i)_{i\in\mathbb{N}}$ are conditionally independent given $(l_i)_{i\in\mathbb{N}}$ and a version of the conditional law of s_i given $(l_i)_{i\in\mathbb{N}}$ is $\nu(l_i, \cdot)$.

PROOF. To construct the kernel ν, consider the measure μ^h on the set $(0, \infty) \times S$ given by $\mu^h(A \times B) = \mu\big(h^{-1}(A) \cap B\big)$. We can therefore disintegrate μ^h (an extension of Theorem 6.3 in [**Kal02**, p.107]) by means of a kernel ν from $(0, \infty)$ to S as follows:

$$\mu^h(C) = \int\int \mathbf{1}_C(r, s)\, \mu \circ h^{-1}(dr)\, \nu(r, ds).$$

With the kernel so constructed, the next step is to verify that Ξ^h is a Poisson random measure with intensity μ^h. To do this, consider nonnegative measurable functions g on $(0, \infty) \times S$ of the form $g(r, s) = f_1(r)\, f_2(s)$, where f_1 and f_2 are nonnegative measurable functions defined on $(0, \infty)$ and S respectively. By the exponential formula for Poisson random measures, we get

$$\mathbb{E}\Big(e^{-\langle \Xi^h, g\rangle}\Big) = \mathbb{E}\Big(e^{-\langle \Xi, f_1\circ h^{-1} f_2\rangle}\Big) = e^{-\langle \mu, 1-e^{-f_1\circ h^{-1} f_2}\rangle} = e^{-\langle \mu^h, 1-e^{-g}\rangle},$$

which by monotone class arguments and the fact that the Laplace functional characterizes laws of random measures, implies our assertion.

We will now verify that Ξ^h is a ν-randomization of $\Xi \circ h^{-1}$. To do this, we are to verify

$$\mathbb{E}\Big(e^{-\langle \Xi^h, f\rangle}\Big) = \mathbb{E}\Big(e^{\langle \Xi\circ h^{-1}, \log\langle \hat{\nu}, e^{-f}\rangle\rangle}\Big).$$

Using the exponential formula for $\Xi \circ h^{-1}$ (which is a Poisson random measure with intensity $\mu \circ h^{-1}$), we get

$$\mathbb{E}\Big(e^{\langle \Xi\circ h^{-1}, \log\langle \hat{\nu}, e^{-f}\rangle\rangle}\Big) = e^{-\langle \mu\circ h^{-1}, \langle \hat{\nu}, e^{-f}\rangle\rangle}.$$

The right-hand side is just a re-expression of $\exp\big(-\langle \mu^h, f\rangle\big)$ by the definition of ν and the fact that it is a probability kernel.

Now we will tackle the second part of the theorem, which is when the Borel hypothesis is important because it implies that (l_i, s_i) are random variables; the hypothesis about the lack of atoms of $\mu \circ h^{-1}$ tells us that $l_i \neq l_j$ if $i \neq j$ almost surely. To verify the conditional independence of $(s_i)_{i\in\mathbb{N}}$ given $(l_i)_{i\in\mathbb{N}}$ and perform a computation of their law, we will compute the conditional expectation of $\sum_{i=1}^n \lambda_i s_i$ given $(l_i)_{i\in\mathbb{N}}$. Recall that a regular conditional distribution of Ξ^h given $\sigma\big(\Xi \circ h^{-1}\big) = \sigma(l_i : i \in \mathbb{N})$ is $\mathbb{Q}_{\Xi\circ h^{-1}}$, and that the latter is characterized by its Laplace functional:

$$\int e^{-\langle m, f\rangle} \mathbb{Q}_{\Xi\circ h^{-1}}(dm) = e^{-\langle \Xi\circ h^{-1}, \log \hat{\nu} e^{-f}\rangle}. \tag{18}$$

By a monotone class argument, equality (18) can be extended to functions $f : \Omega \times (0, \infty) \times S \to [0, \infty)$ which are $\sigma(l_i : i \in \mathbb{N}) \otimes \mathscr{B}_{(0,\infty)} \otimes \mathscr{S}$ measurable, with the understanding that we will integrate with $\omega \in \Omega$ fixed on the left hand side. When using the function f given by $f(\omega, r, s) = \sum_{i=1}^n \mathbf{1}_{r=l_i(\omega)} \lambda_i s$, for which

$$\langle \Xi^h, f\rangle = \sum_{i=1}^n \lambda_i s_i \text{ and } \langle \Xi \circ h^{-1}, \log \hat{\nu} e^{-f}\rangle = \sum_{i=1}^n \int e^{-\lambda_i s}\, \nu(l_i, ds),$$

we obtain

$$\mathbb{E}\Big(e^{-\sum_{i=1}^n \lambda_i s_i} \,\Big|\, l_i, i \in \mathbb{N}\Big) = \prod_{i=1}^n \int e^{-\lambda_i s}\, \nu(l_i, ds)\,.$$

□

As an example, consider the absolute values of the excursions of Brownian motion that occur before inverse local time l; by Itô's representation of the Itô measure, the kernel that disintegrates n_+ with respect to the length is $\nu(v,\cdot) = \mathbb{P}^{v,3}_{0,0}$. We conclude that, when ranked in decreasing length and scaled in a Brownian way to have length one, their conditional law given the sequence of their lengths is that of independent normalized Brownian excursions.

An extension of the previous ideas is needed in Subsection 2.2 of Chapter 3.

6. Bibliographical notes

Theorem 3 is a classical result by Durret, Iglehart and Miller proved in [**DIM77**]. Convergence of the finite-dimensional distributions is not that difficult, but as usual, one has to do some delicate estimates to prove tightness. In the aforementioned reference, they use the explicit expressions of the joint and marginal laws of the cumulative minimum and maximum processes of Brownian motion. Probably the easiest way to obtain this theorem is to use Lévy's explicit construction of the Brownian bridge and the three-dimensional Bessel bridge (see the bibliographical notes to Chapter 1) and the h-transform relationship between the corresponding Markov processes. This would imply the desired weak convergence. For us, the tightness argument is ultimately handled by the general considerations of Theorem 1.

The pathwise construction of the normalized Brownian excursion of Theorem 4 is basically found in [**Chu76**], although the author does not scale the process Y to obtain $\mathbf{e}$. Earlier references are [**Lev48**, p.233], where, as for the Brownian bridge, Lévy uses his interpolation method to construct Brownian excursions, and Section 2.9 of [**IM74**], where they prove not only our conditional independence assertion but also the link with the three dimensional Bessel bridge. Arguments are given to generalize the result to the independence of renormalized excursions labelled in a particular way. The technique we have introduced to deal with it is explored and extended in Subsection 2.1 of Chapter 3. A further path decomposition, which splits our excursion **e** into two other parts and relates them to Brownian meanders, can be found in Lecture 4 of [**Yor95**].

Itô's description of the Itô measure is the subject of XII.4 of [**RY99**]. The authors deal with it by means of explicit computations. Our objective was to show that a conceptual proof of the relationship between n_+ and $\mathbb{P}^{1,3}_{0,0}$ could be given, thanks to the backward strong Markov property, independently of the value of $n_+(L > v)$.

Local times of Bessel processes were found to have a distribution related to the Mittag-Leffler one by Molčanov and Ostrovskiĭ in [**MO69**] by means of occupation time calculations. Since the law of the inverse of a stable subordinator is equally related to that distribution, their result is expected from our point of view; however, we have not seen if the two notions of local time coincide. Exercise XI.1.25 of [**RY99**] offers an alternative, not for regenerative local time, but for local time in the sense of occupation densities.

CHAPTER 3

The Height Fragmentation of the Normalized Brownian Excursion

In this chapter, a further analysis of the fragmentation at heights of the normalized Brownian excursion is presented. Specifically we study a representation for the mass of a tagged fragment in terms of a Doob transformation of the 1/2-stable subordinator and use it to study its jumps; this accounts for a description of how a typical fragment falls apart. These results carry over to the height fragmentation of the stable tree. Also, the sizes of the fragments in the Brownian height fragmentation when it is about to reduce to dust are described in a limit theorem.

1. Introduction and statement of the results

A new class of stochastic processes, that of self-similar fragmentations, has been introduced by Bertoin in [**Ber01**] and [**Ber02**]. This class bears a close relationship with that of positive self-similar Markov processes, for which there has been renewed interest in recent years. Also, as Aldous points out in his survey [**Ald99**], fragmentation processes might be of use in the study of coalescence. This idea has been exemplified in a construction of the standard additive coalescent, as shown in [**AP98**]. In that paper, the authors provide a relationship between a fragmentation process, constructed from the Continuum Random Tree (CRT), and a stable subordinator. In their own words, the relationship should be obtainable directly from the one between the CRT and the normalized Brownian excursion (they use a combinatorial method). This remark is one of the motivations for the following paragraphs since we settle the question of providing an analogous relationship between a fragmentation process, intimately related to the one introduced by Aldous and Pitman, and a stable subordinator. This is done mainly by analysis of continuous-time stochastic processes. The reader can consult [**Ber06**] for a primer on stochastic fragmentation and coalescence.

We shall study the height fragmentation of the normalized Brownian excursion, which was introduced by Bertoin as the second example illustrating the theory of self-similar fragmentations developed in [**Ber02**]. We shall also deal with its generalization to normalized excursions of spectrally positive α-stable Lévy processes reflected at their minima with $1 < \alpha < 2$, introduced by Miermont in [**Mie03**]. Although both processes can be thought to belong to the same parametric family $(F^\alpha)_{\alpha\in(1,2]}$ of fragmentations, their irreconcilable difference lies in the fact that, following Miermont, the first one is binary while the second one is infinitary. This means that fragments separate into two pieces in the $\alpha = 2$ case and into infinitely many pieces when $\alpha \in (1, 2)$. This difference is just the reflection of the fact that while Brownian trajectories have continuous sample paths, other stable Lévy processes feature jumps. It affects the sophistication of the arguments needed to study them. As we hope to make apparent in this note, past an initial threshold, aspects of both fragmentations can be studied without recourse to different arguments. However, the Brownian case admits a simpler representation in terms of the normalized Brownian excursion which makes a more visual analysis feasible; this has not been shown to be true for $\alpha \in (1, 2)$. Since some of our results are valid for all $\alpha \in (1, 2]$, we choose to present both the Brownian and the general proofs when possible.

The Brownian height fragmentation can be defined as follows: let $\mathbf{e}$ be a normalized Brownian excursion, and for nonnegative t, define F^2_t by

$$F^2_t = \{s \in (0, 1) : \mathbf{e}_s > t\}\,; \tag{19}$$

then the Brownian height fragmentation is the decreasing family of sets given by $F^2 = \left(F^2_t\right)_{t\geq 0}$. In Figure 1, a visualization of F^2_t is proposed, with some other quantities of interest that shall be introduced in the following paragraphs. The process F^2 takes values in the space $\mathscr{V}$ of open subsets of $(0, 1)$, where a suitable metric exists which turns it into a compact space. The first thing to discuss about such an object is why it constitutes a self-similar fragmentation. This was only mentioned by Bertoin, and we shall deduce it by direct analysis of the law of the normalized

Brownian excursion and by an indirect one which exploits the relationship with Itô's measure. This shall be the content of Section 2. This point has been discussed by Miermont when proving that his fragmentation processes are self-similar, using the concept of height process associated to a spectrally positive Lévy processes (introduced by Duquesne and Le Gall in [**DLG02**]). We refer to [**Mie03**] for the definition of F^α when $\alpha \in (1, 2)$.

A simple process tied to any self-similar interval fragmentation is the mass of its tagged fragment. Instead of tracking down the behavior of the whole fragmentation process, we select the interval that contains an independent uniform random variable, with the objective of recording how its mass is lost. As Bertoin proves in [**Ber02**] using the results of [**Ber01**], the mass of the tagged fragment is a decreasing and positive self-similar Markov process which can be therefore represented in terms of a Lévy process using the Lamperti transformation. A self-similar fragmentation is characterized by three parameters: the index of self-similarity, the erosion rate and the dislocation measure. Performing a random time-change in the self-similar fragmentation can alter its index without affecting the other two parameters. For example, the Brownian fragmentation at heights and the one introduced by Aldous and Pitman are fragmentations which share the same erosion coefficient and dislocation measure, though the first one has index equal to $-1/2$ and the second one equal to $1/2$. This is the intimate relationship between the two fragmentations alluded to above. Information about the erosion coefficient and the dislocation measure can sometimes be inferred from the study of the tagged fragment. However, as the next theorem shows, there is another representation of which additional use can be made. To state the result, let us introduce a uniform random variable U independent of the fragmentation process F^α and define the mass of the tagged fragment at time t, denoted by χ_t, to be the size of the connected component of F^α_t which contains U. This process is absorbed at zero in finite time; when $\alpha = 2$ this stems from the continuity of $\mathbf{e}$ and for general α, from the continuity of the height process. The next result is a relationship between the tagged fragment of F^α and a stable subordinator of index $\beta = 1 - 1/\alpha \in (0, 1/2]$. The process referred to in the statement is a Doob transform of a stable subordinator via its potential density. Recall Application 3 in page 40.

THEOREM 7. *The mass of the tagged fragment of F^α has the same law as the opposite of a stable subordinator of index β starting at one and conditioned to die at zero.*

We will prove this theorem in the Brownian case by analysis under the Itô measure of positive excursions, while the general case will be the consequence of the duality considerations involving positive self-similar Markov processes of [**BY02**]. From this, we can use a formula by Perman, Pitman and Yor pertaining the law of jumps of subordinators (found in [**PPY92**]) in conjunction with a description of the conditional law the tagged fragment given its death time (a consequence of the general considerations of [**Kal81**] and [**FPY93**]) to establish the following theorem:

THEOREM 8. *The law of the decreasing rearrangement of the absolute values of the jumps of the mass of a tagged fragment of F^α is the two-parameter Poisson-Dirichlet distribution with parameters (β, β).*

Further information regarding the two-parameter Poisson-Dirichlet distribution is found in the survey paper [**PY97**]. The preceding theorem is analyzed in Section 4. While the general strategy has been outlined, the Brownian case is proved by a visual argument relying on a path transformation between the normalized Brownian excursion and the Brownian bridge introduced in [**BP94**]. Its connection to a fragmentation obtained by obliteration of ancestral lines in Lévy trees and the coagulation and fragmentation operators of Dong-Goldschmidt-Martin (cf. [**DGM06**]) is also mentioned. Since the Poisson-Dirichlet distribution with parameters (β, β) arises as the distribution of the ranked lengths of the excursions of a Bessel bridge of dimension $2(1-\beta)$ starting and ending at zero, we shall link Theorems 7 and 8 by commenting on the relationship between the aforementioned bridge and the conditioned subordinator.

Our last results concern limit theorems for the Brownian height fragmentation at the moment where it reduces to dust. To be more specific, let us note that, because of the continuity of $\mathbf{e}$, the first level t at which $F^2_t = \emptyset$ exists, is finite and equal to the maximum of $\mathbf{e}$, denoted by M. Secondly, when we replace the deterministic level t by the random one $M - t$ in (19), we obtain a random variable with values in $\mathscr{V}$ which will be denoted $\hat{F}^2_t$. If R and R' are two independent realizations of the Bessel process of dimension three starting at zero and we set $Z_t = R_t$ if $t > 0$

and R'_{-t} if $t < 0$. Finally, let $S \in (0,1)$ be the almost surely unique location of the maximum M, then the following holds:

THEOREM 9. *As $t \to 0+$, the random set*

$$\frac{\hat{F}_t^2 - S}{t^2}$$

converges in distribution to the random open set of $\mathbb{R}$ given by

$$\{s \in \mathbb{R} : Z_s < 1\}.$$

Apart from studying the proper topology on the family of open sets of $\mathbb{R}$ in Section 5, the preceding theorem will be proved. This time, a path transformation leaving the law of **e** invariant suggested by B. Haas will be used to state the problem into one concerning deterministic levels in lieu of the random ones, while a limit theorem for the normalized Brownian excursion similar to the one given by Jeulin in [**Jeu80**] will allow us to conclude. Invariance of the law of **e** by Haas' path transformation can be deduced from the Vervaat transformation introduced in [**Ver79**], but since a treatment using continuous-time stochastic processes was promised, we prove it by means of William's description of the Itô measure and his reversibility theorems for the three-dimensional Bessel processes. The original proof of invariance under Vervaat's transformation ([**Ver79**]) was combinatorial in nature and Biane provides in [**Bia86**] an explanation by continuous-time methods similar to the one we shall give.

As a consequence of the preceding result we find:

COROLLARY 1. *Let M_t be the Lebesgue measure of $\hat{F}_t^2$ and L_t be the Lebesgue measure of the interval of $\hat{F}_t^2$ that contains S. Then M_t/t^2 and L_t/t^2 converge in distribution as $t \to 0+$ to laws with Laplace transform $q \mapsto \left(1/\cosh\left(\sqrt{2q}\right)\right)^2$ and $q \mapsto \left(1/\sinh\left(\sqrt{2q}\right)\right)^2$ respectively.*

The laws encountered in the corollary belong to the two infinitely divisible families studied in [**BPY01**] and [**PY03**].

2. The fragmentation property

Informally, the notion of self-similar fragmentation on the set $\mathscr{V}$, also called a self-similar interval fragmentation, states that such a process is Markovian and that the evolution of the process on the different connected components is independent of the rest and mimics the whole, perhaps on a

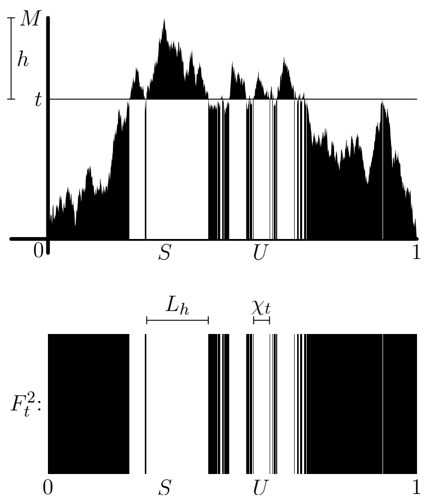


FIGURE 1. Visualization of F_t^2

different time scale. To state it formally, consider first the following metric on $\mathscr{V}$ introduced in [**Ber02**]: for any $V \in \mathscr{V}$, let χ_V be the continuous function on $[0,1]$ given by $\chi_V(x) = d(x, [0,1] \setminus V)$, and for $V_1, V_2 \in \mathscr{V}$, set $d_{\mathscr{V}}(V_1, V_2) = \|\chi_{V_1} - \chi_{V_2}\|_\infty$. The distance between V_1 and V_2 is equal to the Hausdorff distance between V_1^c and V_2^c (where complementation is with respect to $[0,1]$) and it turns $\mathscr{V}$ into a separable compact metric space; we shall therefore speak of Hausdorff's topology on $\mathscr{V}$. Next, suppose we are given probability measures $p_t((0,1), \cdot)$ on $\mathscr{V}$, and for every interval $I \in \mathscr{V}$, consider the unique affine function $g_I : \mathbb{R} \to \mathbb{R}$ that maps $(0,1)$ into I and preserves the order; define

$$p_t^\alpha(I, \cdot) = p_r((0,1), \cdot) \circ g_I^{-1}$$

to be the image of $p_r((0,1), \cdot)$ by g_I, where $r = |I|^\alpha$ and finally, for any $V \in \mathscr{V}$, decompose V as the disjoint union of intervals $(I_i)_{i\in\mathbb{N}}$ and define $p_t^\alpha(V, \cdot)$ to be the distribution of the union of $\{X_i : i \in \mathbb{N}\}$ where $(X_i)_{i\in\mathbb{N}}$ are independent and X_i has law $p_t^\alpha(I_i, \cdot)$. A *self-similar interval fragmentation with index* α is a Markov process F with values in $\mathscr{V}$, which starts at $(0,1)$, is continuous in probability and has $(p_t^\alpha)_{t\geq 0}$ as a semigroup, where $p_t((0,1), \cdot)$ is the law of F_t.

We will be concerned first with the following result.

PROPOSITION 5. *The $\mathscr{V}$-valued process $\left(F_t^2\right)_{t\geq 0}$ is a self-similar interval fragmentation with index $-1/2$.*

We will prove Proposition 5 in two ways, the first one will be a direct analysis of the normalized Brownian excursion, possible since it is a limit of Brownian bridges conditioned to remain positive. This first study will be done in subsection 2.1. However, classical excursion theory provides another way of understanding the fragmentation property, mainly through the Markov property and Itô's description of the Itô measure, and so we will present this method in subsection 2.2. It is termed an indirect method since one obtains a conclusion about the normalized Brownian excursion by inheriting it from the σ-finite intensity measure of positive excursions of Brownian motion thoroughly presented in [**RY99**]. This process of inheritance has been exemplified in [**Bia86**] and [**LG93**], and alluded to by Miemont in [**Mie03**] when speaking of the fragmentation property for his height fragmentation. It poses applicability problems because of the need to establish a weak continuity, which we will deal with. Informally, the fragmentation property for the Brownian height fragmentation can be thought of as follows: if instead of the normalized Brownian excursion $\mathbf{e}$ we are given $\mathbf{e}\wedge t$, then $\mathbf{e}$ can be reconstructed by placing, on top of each connected component of $\{s\in(0,1):\mathbf{e}_s>t\}$, independent normalized Brownian excursions, where independence is between themselves and $\mathbf{e}$, scaled to fit the corresponding component in a Brownian way. Verifying the preceding phrase will be the key to proving Proposition 5 in our direct analysis; it is however not evident how to do so, since we have to find a way to label the excursions in order to prove their independence. Following [**Ber00**], which deals with a similar problem but in the framework of Brownian motion instead of the normalized Brownian excursion, this could be done by ranking the excursions of $\mathbf{e}$ above t by their lengths and then scaling them to obtain normalized ones. This would induce a labeling of the excursions with which one can speak of independence. In our direct analysis, we will circumvent this problem by focusing on a finite quantity of excursions. In the indirect analysis, the heuristic will be a consequence of excursion theory and of the process of inheritance. However both proofs ultimately rely on the following fact: there exists a multiplicative system of functions on $\mathscr{V}$ generating its Borel

σ-algebra. The family of functions is $\mathscr{M} = \{e^{-f} : f \in \mathscr{D}\}$ where $\mathscr{D}$ is the set of functions $f : \mathscr{V} \to \mathbb{R}_+$ for which there exists a positive measure μ on $(0,1)$, absolutely continuous with respect to Lebesgue measure λ, such that $f(V) = \mu(V)$. To use it in conjunction with the multiplicative systems lemma (cf. [**RY99**, p.3] and [**Dyn04**, p. 209] for a proof) we need the following result.

Lemma 5. *The classes $\mathscr{D}$ and $\mathscr{M}$ both generate $\mathscr{B}_{\mathscr{V}}$.*

From the preceding lemma, it follows that a probability measure on $\mathscr{V}$ is determined by its values on $\mathscr{M}$. For example, a fragmentation semigroup (p_t^α) is characterized by the fact that for every $f \in \mathscr{D}$, if we denote by $C(V)$ the set of connected components of $V \in \mathscr{V}$:

$$p_t^\alpha\big(V, e^{-f}\big) = \prod_{I \in C(V)} p_{|I|^\alpha t}\big((0,1), e^{-f\circ g_I}\big).$$

Before turning to the fragmentation property, let us prove Lemma 5 and settle the first two technical points of the definition of self-similar fragmentation: that F_t^2 constitutes a random variable, and that $t \mapsto F_t^2$ is continuous in probability. In fact, we shall see first that if T is a random variable with values in $[0,\infty)$, then

$$F_T^2 = \{s \in (0,1) : \mathbf{e}_s > T\}$$

is a $\mathscr{V}$-valued random variable. (This implies $\hat{F}_t^2$ is also a random variable.) To do that, we note that thanks to display (2) in [**Ber02**], $t \mapsto F_t^2$ is right-continuous on $[0,\infty)$, so that it suffices to prove that F_t^2 is a random variable for every deterministic $t \in [0,\infty)$. By Lemma 5 it suffices to prove that for every measure μ on the Borel sets of $(0,1)$, $\mu\big(F_t^2\big)$ is measurable. Since

$$\mu\big(F_t^2\big) = \int_0^1 \mathbf{1}_{\mathbf{e}_s > t}\, \mu(ds)$$

and the trajectories of $\mathbf{e}$ are continuous, we obtain the measurability of $\mu\big(F_t^2\big)$ and as a consequence, that of F_t^2. For the fact that $t \mapsto F_t^2$ is continuous in probability, we mention that display (2) in [**Ber02**] implies

$$F_{t+}^2 = F_t^2 \quad \text{and} \quad F_{t-}^2 = F_t^2 \cup \operatorname{Int}\{s \in (0,1) : \mathbf{e}_s = t\},$$

where $\operatorname{Int} A$ is the interior of the set A. For every $s \in (0,1)$ and every $t > 0$, the law of $\mathbf{e}_s$ assigns no mass to $\{t\}$; it follows by Tonelli's theorem that the Lebesgue measure of $\{s \in (0,1) : \mathbf{e}_s = t\}$ is zero so that it has, almost surely, no interior. This implies the desired continuity.

Proof of Lemma 5. It suffices to see that $\sigma(\mathscr{D}) = \mathscr{B}_{\mathscr{V}}$.

As a consequence of Lemma 2 in [**Ber02**], we see that for every measure μ on $\mathscr{B}_{(0,1)}$ absolutely continuous with respect to Lebesgue measure, the function $V \mapsto \mu(V)$ is continuous, hence $\mathscr{B}_{\mathscr{V}}$-measurable, implying $\sigma(\mathscr{D}) \subset \mathscr{B}_{\mathscr{V}}$. To verify the converse inclusion, we note that the definition of $d_{\mathscr{V}}$ implies $\mathscr{B}_{\mathscr{V}} = \sigma(\chi)$ where χ is the function given by $V \mapsto \chi_V$. We will finish the proof by verifying that χ is $\sigma(\mathscr{D})$-measurable. As the Borel subsets of the space of continuous functions on $[0,1]$ equipped with the uniform norm are generated by the projections $f \mapsto f(t)$ for $t \in (0,1)$, the asserted measurability for χ will follow if we verify that for every $t \in (0,1)$, the function $V \mapsto \chi_V(t) = d(t,[0,1] \setminus V)$ is $\sigma(\mathscr{D})$-measurable. However, for every $t \in (0,1)$, $\chi_V(t) \leq t \wedge (1-t)$ and for every $\varepsilon \in (0, t \wedge (1-t))$ we can define the measure μ on $(0,1)$ as Lebesgue measure concentrated on $(t-\varepsilon, t+\varepsilon)$ for which the following holds:

$$\{d(t,V^c) \geq \varepsilon\} = \{\mu(V) = 2\varepsilon\} \in \sigma(\mathscr{D})\,.$$

□

2.1. Direct Analysis of the fragmentation property. The content of this section is a proof of Proposition 5 by means of direct analysis of the law of the normalized Brownian excursion. To this end, we will work on the canonical spaces C_v of continuous real valued functions on $[0,v]$, where (abusing notation) the canonical process X is defined. For positive x, y and v, let us consider the law $\pi^v_{x,y}$ of a Brownian bridge between x and y of length v conditioned to remain positive. In this subsection, we shall think of the law of the Brownian excursion of length v as the weak limit of $\pi^v_{x,y}$ as x and y tend to 0; it will be denoted π^v. The adjective normalized is used when the length v is equal to 1, and we will simplify notation in this case by denoting π^1 by π. The Brownian excursion of length v is a stochastic process which starts at zero, ends at zero at time v and remains positive on $(0,v)$. It satisfies the following Markov property: if $\mathscr{F}_s = \sigma(X_u : u \leq s)$, $\mathscr{F}^s = \sigma(X_u : u \geq s)$ and θ_s is the usual shift operator then for $0 < s_1 < s_2 < v$ and bounded $\mathscr{F}_{s_2-s_1}$

and $\mathscr{F}_{s_1} \vee \mathscr{F}^{s_2}$-measurable functionals Φ and Ψ we have that

$$\pi(\Phi \circ \theta_{s_1} \Psi) = \pi\left(\pi^{s_2 - s_1}_{X_{s_1}, X_{s_2}}(\Phi)\, \Psi\right). \tag{20}$$

Also, by the self-similarity properties of Brownian motion, the following scaling relationship between π^v and π^1 follows: the law of $\left(\sqrt{v} X_{t/v}\right)_{t \in [0,v]}$ under π is π^v.

Let ν be the kernel for which $\nu(a, v, \cdot)$ is the law of $\left(X_{(s-a)^+}\right)_{s \geq 0}$ under π^v. We shall see that the fragmentation property of F^2 is the consequence of the following conditional independence result:

LEMMA 6. *Let $\varepsilon > 0$ and let (g_i, d_i) be the excursion interval of the i-th excursion of length greater than ϵ of the canonical process X above level t ($g_i = d_i = 0$ if there are less than i excursions whose length exceeds ε); defining*

$$X^i_s = \begin{cases} X_s & \text{if } s \in [0,1] \setminus (g_i, d_i) \\ t & \text{otherwise} \end{cases}, \quad Y^i_s = \begin{cases} X_s - t & \text{if } s \in (g_i, d_i) \\ 0 & \text{otherwise} \end{cases}$$

the conditional law under π of Y^i given $\sigma\left(X^i\right)$ is $\nu(g_i, d_i - g_i, \cdot)$.

The reader might wish to consult Figure 2 for a visual reference to the processes introduced in the preceding lemma. To apply the lemma, we need to define the process of excursions of X above t and its relation to the process below t. To do that, consider the σ-algebra $\mathscr{F}^t$ given by $\sigma(X_s \wedge t : s \geq 0)$. Also, let A be the additive functional given by

$$A_s = \int_0^s \mathbf{1}_{(X_s \leq t)}\, ds$$

with right-continuous inverse α. Defining

$$X^s_r = \begin{cases} X_r - t & \text{if } r \in (\alpha_{s-}, \alpha_s) \\ 0 & \text{otherwise} \end{cases},$$

and using Lemma 6, we will see that for every $\mathscr{F}^t$-measurable and bounded functional Φ and every measurable and nonnegative f defined on $[0, \infty) \times$

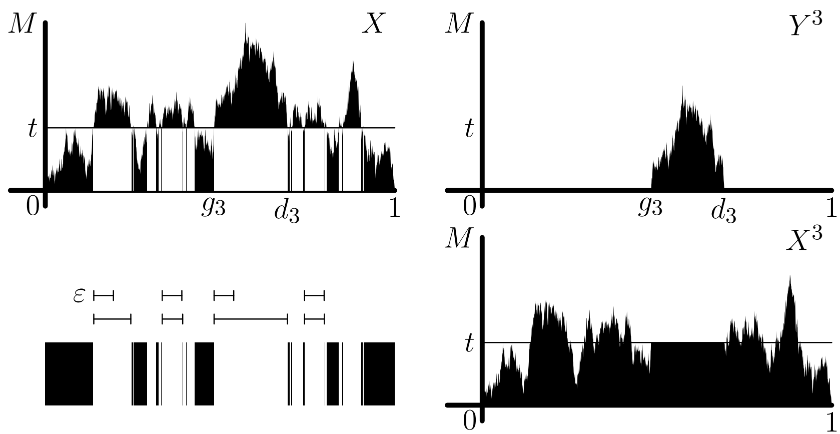


FIGURE 2. The processes of Lemma 6.

C_1:

(21)

$$\pi\left(\Phi \exp\left(-\sum_{\{r:\alpha_r<\infty,\Delta\alpha_r\neq 0\}} f(\alpha_{r-},\Delta\alpha_r,X^r)\right)\right) =$$

$$\pi\left(\Phi \exp\left(\sum_{\{r:\alpha_r<\infty,\Delta\alpha_r\neq 0\}} \log\left(\int e^{-f(\alpha_{r-},\Delta\alpha_r,x)}\,\nu(\alpha_{r-},\Delta\alpha_r,dx)\right)\right)\right).$$

The last equality should be confronted with the characterization of randomized point processes in [**Kal02**, Lemma 12.2, p.227]. Its intuitive content is that under π, if instead of X we are given $X \wedge t$, then X can be reconstructed by placing, on top of each connected component of $\{s \in (0,1) : X_s > t\}$, independent normalized Brownian excursions, where independence is between themselves and $X \wedge t$, scaled in a Brownian way and shifted to fit the corresponding component. To obtain (21), notice that by monotone convergence, it is enough to consider the case where $f(a,v,x)$ is zero if $v \le \varepsilon$. In that case, using the notation of Lemma 6 we see that

$$\pi\left(\Phi \exp\left(-\sum_{\{r:\alpha_r<\infty,\Delta\alpha_r\neq 0\}} f\left(\alpha^u_{r-},\Delta\alpha_r,X^r\right)\right)\right) = \pi\left(\Phi\prod_i e^{-f\left(g_i,d_i-g_i,Y^i\right)}\right)$$

and so the fact that Y^j is $\sigma\left(X^i\right)$-measurable if $i \neq j$ and the description of the conditional law of Y^i given $\sigma\left(X^i\right)$ imply:

$$\pi\left(\Phi\prod_i e^{-f\left(g_i,d_i-g_i,Y^i\right)}\right) = \pi\left(\Phi\prod_i \pi\left(e^{-f\left(g_i,d_i-g_i,Y^i\right)}\,\middle|\,\sigma\left(X^i\right)\right)\right) = \pi\left(\Phi\prod_i \int e^{-f(g_i,d_i-g_i,x)}\,\nu(g_i,d_i-g_i,dx)\right).$$

The last expression in the preceding display reduces to the right-hand side of (21).

The scaling relationship bonding the π^v laws and display (21) allow us to deal with the fragmentation property of F^2, by calculating the conditional expectation of $g\left(F^2_{t_1+t_2}\right)$ given $\mathscr{F}^{t_1}$, where g belongs to the multiplicative system studied in the introduction to this section. If $\psi_t(f) = \{s \in (0,1) : f(s) > t\}$, we have seen why ψ_t is a measurable function from C_1 into $\mathscr{V}$. We shall consider the inverse α of A defined above with respect to level t_1 instead of t. Then, by σ-additivity:

$$f\left(F^2_{t_1+t_2}\right) = \sum_{\{s:\alpha_s<\infty,\Delta\alpha_s\neq 0\}} f(\psi_{t_2}(X^s))$$

and so, writing $C(V)$ for the set of connected components of $V \in \mathscr{V}$, (21) implies

$$\pi\left(e^{-f\left(F^2_{t_1+t_2}\right)}\,\middle|\,\mathscr{F}^{t_1}\right) = \exp\left(\sum_{(a,b)\in C\left(F^2_{t_1}\right)} \log\left(\int e^{-f\left(\psi_{t_2}(x)\right)}\,\nu(a,b-a,dx)\right)\right).$$

If we let $p_t((0,1))$ be the law of ψ_t under π and $\alpha = -1/2$, then by the scaling properties bonding together the laws $(\pi^v)_{v>0}$, we see that

$\nu(a, b-a, \cdot) \circ \psi_t^{-1}$ is equal to $p_r((0,1)) \circ g_{(a,b)}^{-1}$, where $r = t\,(b-a)^\alpha$. The multiplicative system lemma and Lemma 5 allow us to conclude that $\left(F_t^2\right)_{t\geq 0}$ is Markovian (since $\sigma\left(F_u^2 : u \leq t\right) \subset \mathscr{F}^t$) and that its semi-group is that of a self-similar fragmentation built from the family of laws $(p_t((0,1)))_{t\geq 0}$ and the index $-1/2$. To finish the proof of Proposition 5 we shall now examine the validity of Lemma 6.

Lemma 6 is an assertion of conditional independence that will obtained by use of the interpretation of Brownian excursions as a limit of conditioned Brownian bridges and of the Markov property we considered. This will be possible by exemplifying and extending the *ansatz* put forth by Jacobsen in [**Jac74**] during his study of splitting-times, although g_i above is not one of them.

Proof of Lemma 6. Let us introduce $\mathscr{F}^i = \sigma\left(X^i\right)$ and note that if $s_2 - s_1 > \varepsilon$ then

$$\mathscr{F}^i \bigcap \{g_i < s_1, s_2 < d_i\} \subset \mathscr{F}_{s_1} \vee \mathscr{F}^{s_2} \bigcap \left\{\inf_{s\in[s_1,s_2]} X_s > t\right\}.$$

Although the above inclusion is not valid if $0 \leq s_2 - s_1 \leq \varepsilon$, it is enough for our needs since the family $\left\{\mathscr{F}^i \bigcap \{g_i < s_1, s_2 < d_i\} : s_2 - s_1 > \varepsilon\right\}$ generates $\mathscr{F}^i$. Actually, the set $\{g_i < s_1, s_2 < d_i\}$ is written as

$$A \cap \left\{\inf_{s\in[s_1,s_2]} X_s > t\right\}$$

with $A \in \mathscr{F}_{s_1}$, so that if Φ is any bounded $\mathscr{F}^i$-measurable functional and Ψ is bounded and $\mathscr{F}_{s_2-s_1}$-measurable, then $\Phi \mathbf{1}_{g_i<s_1,s_2<d_i}$ is $\mathscr{F}_{s_1} \vee \mathscr{F}^{s_2}$-measurable and an application of (20) gives

$$\begin{aligned}&\pi(\Phi\mathbf{1}_{g_i<s_1,s_2<d_i}\Psi\circ\theta_{s_1}(X-t)) = \\ &\pi\left(\Phi\mathbf{1}_A \pi_{X_{s_1},X_{s_2}}\left(\Psi(X-t)\,\mathbf{1}_{\inf_{s\in[0,s_2-s_1]} X_s>t}\right)\right),\end{aligned}$$

so that by applying (20) again, we find

$$\begin{aligned}&\pi(\Phi\mathbf{1}_{g_i<s_1,s_2<d_i}\Psi\circ\theta_{s_1}(X-t)) = \\ &\pi\left(\Phi\mathbf{1}_{g_i<s_1,s_2<d_i}\pi_{X_{s_1},X_{s_2}}\left(\Psi(X-t)\,\middle|\,\inf_{s\in[0,s_2-s_1]} X_s > t\right)\right).\end{aligned}$$

By spatial homogeneity of the laws of Brownian bridges, we see that for $x, y > t$, the law of $X - t$ under $\pi^r_{x,y}$ conditioned on remaining above level t is $\pi^r_{x-t,y-t}$, so that we can transform the right-hand side of the preceding display to obtain the equality

$$\pi(\Phi \mathbf{1}_{g_i<s_1,s_2<d_i} \Psi \circ \theta_{s_1}(X-t)) = \pi\Big(\Phi \mathbf{1}_{g_i<s_1,s_2<d_i} \pi^{s_2-s_1}_{X_{s_1}-t,X_{s_2}-t}(\Psi(X))\Big).$$

Let us define the following approximations $g^k_i = j/2^n$ if $(j-1)/2^n \leq g^i < j/2^n$ and $d^i_k = j/2^k$ if $j/2^n < d_i \leq (j+1)/2^n$. Applying the equality in the last display, this time with Ψ a bounded $\mathscr{F}_s$-measurable functional where $s > \varepsilon$, we obtain:

$$\pi\Big(\Phi \mathbf{1}_{d^k_i - g^k_i > s} \Psi \circ \theta_{g^k_i}\Big) = \pi\left(\Phi \mathbf{1}_{d^k_i - g^k_i > s} \pi^{d^k_i - g^k_i}_{X_{g^k_i}, X_{d^k_i}}(\Psi)\right).$$

(To see that, one could decompose over sets of the form

$$\left\{g^k_i = l/2^k, d^k_i = m/2^k\right\}.)$$

Performing a limit when $k \to \infty$, but restricting our attention to continuous (instead of measurable) functionals Ψ we obtain

$$\pi(\Phi \mathbf{1}_{d_i-g_i>s} \Psi \circ \theta_{g_i}) = \pi\Big(\Phi \mathbf{1}_{d_i-g_i>s} \pi^{d_i-g_i}_{0,0}(\Psi)\Big)$$

by weak convergence of the laws of Brownian bridges conditioned on remaining positive to the Brownian excursion. It follows that $\pi^{d_i-g_i}_{0,0}$ can be considered as a version of the conditional law of $(X_{g_i+s})_{s\in[0,d_i-g_i]}$ given $\mathscr{F}^i$. □

Note. As a consequence of Lemma 6 we have that if

$$Z^i = \Big(X_{g_i+s(d_i-g_i)}/\sqrt{d_i-g_i}\Big)_{s\in[0,1]},$$

then under π, X^i and Z^i are independent and Z^i has law π.

2.2. Indirect analysis: the fragmentation property through excursion theory. We shall now examine a second proof of Proposition 5 based on arguments of excursion theory and the notion of randomization of point processes; we refer to [**RY99**] for the former while the latter may be found in [**Kal02**, p.226].

There is a link between Itô's measure of positive excursions of Brownian motion, denoted n_+, and the law of the normalized Brownian excursion. It is called Itô's description of n_+. To recall it, let $(E, \mathscr{E})$ denote excursion space consisting of continuous functions $e : [0,\infty) \to [0,\infty)$ for which there exists $L = L(e) \geq 0$, called the length of the excursion, such that $e(t) \neq 0$ iff $0 < t < L$, together with the σ-algebra generated by the canonical process X. This σ-algebra is also the one generated by the topology of E when we use a metric for uniform convergence on compact sets. Itô's description of n_+ is that the law of L under n_+ admits a density given by $v \mapsto 1/2\sqrt{2\pi v^3}\mathbf{1}_{v\geq 0}$ and that the conditional law of $(e_t)_{t\leq L}$ given $L = v$ is that of a Brownian excursion of length v, denoted π^v, so that it has the law of $\left(\sqrt{v}X_{s/v}\right)_{t\in[0,v]}$ under π (simplified notation for π^1). This means that for every bounded and measurable functional Φ on E and measurable $g : [0,\infty) \to [0,\infty)$, the following equality holds:

$$n_+(g(L)\,\Phi) = \int_0^\infty \frac{dv}{2\sqrt{2\pi v^3}} g(v)\,\pi^v(\Phi)\,.$$

Hence, a possibility for proving a statement concerning the normalized Brownian excursion is to see that a similar statement which involves the length holds true for n_+ followed by a conditioning by the length. In other words, if we find that

$$n_+(g(L)\,\Phi) = \int_0^\infty \frac{dv}{2\sqrt{2\pi v^3}} g(v)\,f(v)\,, \tag{22}$$

for every positive and measurable g, then it is tempting to conclude that $f(v) = \pi^v(\Phi)$. Unfortunately, from an integral equality such as (22), we can only conclude an almost everywhere equality (with respect to Lebesgue measure). However, if we prove that f and $v \mapsto \pi^v(\Phi)$ are continuous, then we will be able to conclude that $f(v) = \pi^v(\Phi)$ for every positive v, in particular for $v = 1$, and then we will have made a calculation for the normalized Brownian excursion by studying n_+. We will follow this strategy to give a second proof of Proposition 5.

Let us note that under n_+, the set $\{s > 0 : X_s > t\}$ is an open subset of $\mathbb{R}$; a proper topology, similar to Hausdorff's topology on $\mathscr{V}$, is then needed. For an open subset $V \subset \mathbb{R}$, let $\mathscr{V}^V$ be the set of open subsets of V. Bertoin's metric on the $\mathscr{V}^{(0,1)}$ discussed in the introduction to this section can be immediately extended to a metric d_V for $\mathscr{V}^V$, when V is a bounded open set; it turns this space into a compact and separable metric space, hence a Polish one. If V is an unbounded (open) set, we can define

$$d_V = \sum_{n\in\mathbb{Z}} \frac{d_{V\cap(n,n+1)}}{2^n}$$

so that $\mathscr{V}^V$ is again a Polish space. It is in this sense that we will consider random open subsets of $\mathbb{R}$; choosing a bounded metric giving the same topology of $\mathbb{R}$ in the definition of the Hausdorff distance would have the same effect. The domain of the function $\psi_t : f \mapsto \{s : f(s) > t\}$, introduced in the last subsection, can be extended to excursion space E, and it is measurable. Lemma 5 can also be extended to each of the spaces $\mathscr{V}^V$, by considering measures on V instead of $(0,1)$ in the definition of the multiplicative system $\mathscr{M}$. We will need the extension of $p_t^{-1/2}$, defined in the last subsection, to bounded open subsets V of $\mathbb{R}$: if $\{(a_i, b_i)\}_{i\in\mathbb{N}}$ is an interval decomposition of V and $(V_i)_{i\in\mathbb{N}}$ are independent random variables with values in $\mathscr{V}^{\mathbb{R}}$, V_i distributed as $\psi_{t(b_i-a_i)^{-1/2}}\left(\left(X_{(t-a_i)^+}\right)\right)$ under $\pi^{b_i-a_i}$, $p_t^{-1/2}(V,\cdot)$ will be the law of the union of the V_i. For any $g \in \mathscr{M}$ and any $\sigma(X \wedge t_1)$-measurable and positive functional Φ, we will prove that

$$n_+(\Phi g(\psi_{t_1+t_2}(X))) = n_+\left(\Phi p_{t_2}^{-1/2}(\psi_{t_1}(X)\,, g)\right). \tag{23}$$

Thanks to Itô's description of n_+ and because the length L of excursions is measurable with respect to $\sigma(X \wedge t_1)$ for any positive t_1, the last equation implies that for any nonnegative measurable function h on $\mathbb{R}$:

$$\int_0^\infty \frac{dv}{2\sqrt{2\pi v^3}} h(v)\, \pi^v(\Phi g(\psi_{t_1+t_2}(X))) = \int_0^\infty \frac{dv}{2\sqrt{2\pi v^3}} h(v)\, \pi^v\left(\Phi p_{t_2}^{-1/2}(\psi_{t_1}(X)\,, g)\right).$$

By focusing on continuous and bounded Ψ and g, we can use the scaling relationship between the laws π^v, the continuity in probability of $t \mapsto \psi_t$ under π^v (proved in the introduction to this section) and the fact that $V \mapsto aV$ $(a \neq 0)$ is continuous from $\mathscr{V}^{\mathbb{R}}$ into itself to conclude that the expression containing π^v on the left-hand side of the preceding display is continuous as a function of v. The same conclusion for the right-hand side is obtained by reasoning as in the proof of the Feller property of self-similar interval fragmentations in [**Ber02**]. We conclude the equality

$$\pi^v(\Phi g(\psi_{t_1+t_2}(X))) = \pi^v\left(\Phi p_{t_2}^{-1/2}(\psi_{t_1}(X)\,, g)\right),$$

which implies the fragmentation property for $(\psi_t(X))_{t\geq 0}$ under π^v, since $\mathscr{M}$ is multiplicative and generates the topology of $\mathscr{V}^{\mathbb{R}}$ and it is enough to prove the equality in the last display for continuous functionals Φ in order to deduce it for measurable and bounded ones.

It remains to prove equation (23). We shall reduce the study of the σ-finite measure n_+ to the study of Brownian motion: recall that under n_+, the canonical process is Markovian and has the semigroup of Brownian motion killed when it reaches zero. We can formalize the preceding phrase with the following setup: let $\mathbb{P}^\dagger_x$ denote the law of Brownian motion started at x and killed when it reaches zero, T_t denote the hitting time of t by the canonical process X, consider a nonnegative measurable functional Φ of $(X_{s\wedge T_t}, s \geq 0)$, a nonnegative and measurable functional Ψ and the usual shift operators (θ_t). The Markovian character of n_+ is then stated as follows

$$n_+(\Phi\Psi \circ \theta_{T_t}) = n_+\left(\Phi\mathbb{P}^\dagger_t(\Psi)\right).$$

Hence, if we prove that for any positive $\sigma(X \wedge t_1)$-measurable functional Ψ and $g \in \mathscr{M}$:

$$\mathbb{P}^\dagger_{t_1}(\Psi g(\psi_{t_1+t_2}(X))) = \mathbb{P}^\dagger_{t_1}\left(\Psi p_{t_2}^{-1/2}(\psi_{t_1}(X)\,, g)\right), \tag{24}$$

we will obtain equality (23) in the case where Φ is of the form $\Phi'\Psi \circ \theta_{T_t}$ where Φ' and Ψ are nonnegative and measurable with respect to $\sigma(X_{s\wedge T_t} : s \geq 0)$ and $\sigma(X \wedge t)$. This is enough to conclude the validity of (23), and so we will next focus on (24). To study the excursions of killed Brownian motion, one can start by studying those of Brownian motion and perform a selection of them. To this end, we shall use the notion of randomization of point processes and a lemma concerning it.

Let $\mathbb{P}_t$ be the law of Brownian motion starting at t. Under it, consider the point process Ξ^+ (Ξ^-) of excursions and their lengths of the canonical process X above (below) level t (shifted to start and end at zero) in the (semimartingale) local time scale. If L is the local time at t for X and τ is its inverse, $\Xi^\pm$ admits the representation

$$\Xi^\pm = \sum_{\substack{\Delta\tau_l \neq 0 \\ \operatorname{sgn}\left(e^l\right) = \pm 1}} \delta_{(l, \Delta\tau_l, e^l)}$$

where $\operatorname{sgn}(e)$ is 1 if the excursion e is positive and -1 if it is negative, $\Delta\tau_- = \tau_l - \tau_{l-}$ is the jump of τ at l and e^l is the excursion of X away from t that starts at τ_{l-} and ends at τ_l:

$$e_s^l = \begin{cases} X_{\tau_{l-}+s} - t & \text{if } \tau_{l-} < s < \tau_l \\ 0 & \text{otherwise} \end{cases}.$$

Note that in [**RY99**], the point process equal to $\Xi^+(\cdot, \mathbb{R}_+, \cdot) + \Xi^-(\cdot, \mathbb{R}_+, \cdot)$ is proved to be a Poisson point process. However, Itô's description of the Itô measure allows the following conclusion: Ξ^+ and Ξ^- are independent Poisson point processes and the characteristic measure of Ξ^+ is

$$A \mapsto n_+(A) = \int_0^\infty \frac{dv}{2\sqrt{2\pi v^3}} \pi^v(A)\,.$$

We recall that π^v is the law of the Brownian excursion of length v. If in the last expression one uses the image of π^v under $X \mapsto -X$ instead of π^v, one obtains the characteristic measure of Ξ^-.

Associated to $\Xi^\pm$ is the Poisson point process $\eta^\pm$ of lengths of the excursions of X above (+) or below (−) level t:

$$\eta^\pm = \sum_{\substack{\Delta\tau_l \neq 0 \\ \operatorname{sgn}(e^l) = \pm 1}} \delta_{(l, \Delta\tau_l)}$$

By the reverse implication to Lemma 12.2.iii in [**Kal02**, p.227] (which is not proved there,) the expressions of the characteristic measures of $\Xi^\pm$ and $\eta^\pm$ imply that the latter is a randomization of the former, where the randomizing kernel is $(l, v, A) \mapsto \pi^v(A)$. This follows by use of the exponential formula for Poisson point processes since one first application implies

$$\begin{aligned} &\mathbb{E}\left(\exp\left(-\sum_{\substack{\Delta\tau_l \neq 0 \\ \operatorname{sgn}(e^l) = \pm 1}} f\left(l, \Delta\tau_l, e^l\right)\right)\right) \\ &= \exp\left(-\int_0^\infty \int_0^\infty \int_E \left(1 - e^{-f(l,v,\pm e)}\right) \pi^v(de)\, \frac{dv}{2\sqrt{2\pi v^3}} \lambda(dl)\right), \end{aligned}$$

where λ is Lebesgue measure; by use of

$$g(l, v) = \log \int_E e^{-f(l,v,\pm e)}\, \pi^v(de)$$

the right hand side in the preceding display can be expressed as

$$\exp\left(-\int_0^\infty \int_0^\infty \left(1-e^{-g(l,v)}\right) \frac{dv}{2\sqrt{2\pi v^3}}\lambda(dl)\right)$$

and a second application of the exponential formula transforms the preceding expression into

$$\mathbb{E}\left(\exp\left(-\sum_{\substack{\Delta\tau_l\neq 0\\ \operatorname{sgn}\left(e^l\right)=\pm 1}} g(l,\Delta\tau_l)\right)\right).$$

This suffices to prove that $\Xi^\pm$ is a randomization of $\eta^\pm$.

Since $\sigma(\Xi^+)$ and $\sigma(\Xi^-)$ are independent and $\eta^\pm$ is $\sigma(\Xi^\pm)$-measurable, it follows that Ξ^+ and Ξ^- are conditionally independent given $\sigma(\eta^+,\eta^-)$. Next, we will define two random measures using both Ξ^+ and Ξ^-: let

$$\Xi = \sum_{\substack{\Delta\tau_l\neq 0\\ \operatorname{sgn}\left(e^l\right)=1}} \delta_{(l,\tau_{l-},\Delta\tau_l,X^l)} \quad \text{and} \quad \eta = \sum_{\substack{\Delta\tau_l\neq 0\\ \operatorname{sgn}\left(e^l\right)=1}} \delta_{(l,\tau_{l-},\Delta\tau_l)}.$$

where $X^l_t = e^l_{(t-\tau_{l-})^+}$. Since η^+ is $\sigma(\eta)$-measurable and $\sigma(\eta,\Xi^-) = \sigma(\eta^+,\Xi^-)$, the independence between Ξ^+ and Ξ^- implies that Ξ^+ and Ξ^- are conditionally independent given η. To relate the preceding discussion to the canonical process, let us note that $\sigma(\eta,\Xi^-)$ coincides with $\sigma(X\wedge t)$; that η and Ξ^- are measurable with respect to $X\wedge t$ follows from the fact that L, and hence τ are measurable with respect to $X\wedge t$ as are the negative excursions. In the other direction, we can use the equality

$$X_s\wedge t = \sum_{\substack{\Delta\tau_l\neq 0\\ \operatorname{sgn}\left(e^l\right)=-1}} \mathbf{1}_{\tau_{l-}<s\leq\tau_l} e^l_{s-\tau_{l-}}.$$

The next lemma will allow us to conclude the proof of (24). To state it, introduce the kernel ν' (appealing to the kernel ν defined in the last subsection) where $\nu'(l,a,v,\cdot)$ is the law of $X_{(\cdot-a)^+}$ under π^v. Also, in order to perform the selection of the excursions of a killed Brownian motion from those of Brownian motion, consider the (random) set $A = [0,T_0]\times\mathbb{R}^2_+\times E$, which we will use to restrict both Ξ and η.

Lemma 7. *The conditional law of* $\Xi|_A$ *given* $\sigma(X \wedge t)$ *is that of a* ν'*-randomization of* $\eta|_A$.

To use it, note that thanks to it, it for every nonnegative $F : \mathbb{R}_+^3 \times E \to \mathbb{R}_+$ we let

$$G(l,a,v) = \log \int e^{-F(l,a,v,e)}\, \nu'(l,a,v,de)$$

then

$$\mathbb{P}_t\left(\exp\left(- \sum_{\substack{l<L_{T_0},\Delta\tau_l\neq 0\\ \operatorname{sgn}(e^l)=1}} F\big(l,\tau_{l-},\Delta\tau_l,X^l\big) \right) \middle| X \wedge t \right) =$$

$$\exp\left(\sum_{\substack{l<L_{T_0},\Delta\tau_l\neq 0\\ \operatorname{sgn}(e^l)=1}} G(l,a,v,de) \right).$$

Since the functional whose conditional expectation we are calculating depends only on $X_{\cdot\wedge T_0} \wedge t$, as does the right hand side, we can substitute $\mathbb{P}_t$ by $\mathbb{P}_t^\dagger$ in the preceding display. Additionally, if $g = e^{-f} \in \mathscr{M}$ in the space $\mathscr{V}^{\mathbb{R}}$, then

$$\mathbb{P}_t^\dagger(g \circ \psi_{t+h} \mid X \wedge t) = p_h^{-1/2}(\psi_t, g)\,,$$

since

$$g \circ \psi_{t+h} = \exp\left(- \sum_{\substack{l<L_{T_0},\Delta\tau_l\neq 0\\ \operatorname{sgn}(e^l)=1}} f \circ \psi_h\big(X^l\big) \right)$$

and

$$p_h^{-1/2}(\psi_t, g) = \exp\left(- \sum_{\substack{l<L_{T_0},\Delta\tau_l\neq 0\\ \operatorname{sgn}(e^l)=1}} \int f \circ \psi_h(e)\ \nu'(l,\tau_{l-},\Delta\tau_l,de) \right).$$

This proves (24) assuming Lemma 7. Let us finish this section by analyzing the latter.

Proof of Lemma 7. We know that Ξ^+ is a randomization of η_+ and by the conditional independence between Ξ^+ and Ξ^- given η, we have that for any measurable $f : \mathbb{R}_+^2 \times E \to \mathbb{R}_+$

$$\mathbb{P}_t\left(\exp\left(-\sum_{\substack{\Delta\tau_l\neq 0\\ \operatorname{sgn}(e^l)=1}} f\big(l,\Delta\tau_l,e^l\big)\right)\middle|\,\eta,\Xi^-\right) = \exp\left(\sum_{\substack{\Delta\tau_l\neq 0\\ \operatorname{sgn}(e^l)=1}} \log\int e^{-f(l,\Delta\tau_l,e)}\right)\pi^{\Delta\tau_l}(de)\,. \tag{25}$$

This implies the existence of a regular disintegration of Ξ^+ given η,Ξ^-: the conditional law of Ξ^+ given η,Ξ^- is that of a randomization of η where the randomization kernel is $(l,v)\mapsto\pi^v$. This enables us to extend the equality of the preceding display to $\mathscr{B}_{\mathbb{R}_+^2\times E}\otimes\sigma(\eta,\Xi^+)$-measurable functions $f:\mathbb{R}_+^2\times E\times\Omega\to\mathbb{R}_+$. If $g:\mathbb{R}_+^3\times E\to\mathbb{R}_+$ is measurable, let $h^{a,v}e(t)=e_{(t-a)^+/v}\sqrt{v}$ and use (25) with

$$f(l,v,e,\omega)=g(l,\tau_{l-},v,h^{\tau_{l-},v}(e))\,\mathbf{1}_{t<L_{T_0}(\omega)}$$

(note that L_{T_0} is $\sigma(\eta,\Xi^+)$-measurable) to obtain the equality

$$\mathbb{P}_t\left(\exp\left(-\sum_{\substack{l<L_{T_0},\Delta\tau_l\neq 0\\ \operatorname{sgn}(e^l)=1}} g\big(l,\tau_{l-},\Delta\tau_l,X^l\big)\right)\middle|\,\eta,\Xi^-\right)$$
$$=\exp\left(\sum_{\substack{l<L_{T_0},\Delta\tau_l\neq 0\\ \operatorname{sgn}(e^l)=1}}\log\int e^{-g(l,\tau_{l-},\Delta\tau_l,x)}\,\nu'(\tau_{l-},\Delta\tau_l,de)\right).$$

□

3. The representation of the tagged fragment

In this section, we shall prove Theorem 7. This will be done, in the Brownian case in subsection 3.1, by means of Bismut's decomposition of Itô's measure, while it will rely on considerations involving positive self-similar Markov processes, like the mass of a tagged fragment of a self-similar fragmentation, in the general case of the fragmentations F^α in subsection 3.2.

We start by describing the opposite of a stable subordinator conditioned to die at zero. This process is a Doob transformation of the opposite of a stable subordinator via its potential density that was introduced, in a more general context, in [**Cha96**]. There, the author proves that this process dies at a finite time and that it tends to zero at its death-time, justifying the name given to it. His proof relies on the classification of coharmonic and coinvariant functions for Lévy processes of [**Sil80**] and therefore, we shall present a way of obtaining it by an approach closer to the techniques used in this paper, namely, the use of the Markovian bridges introduced in [**FPY93**]. For an explicit expression of the infinitesimal generator and other aspects of this process, see [**CC06**].

Let σ be a stable subordinator of index β (the construction of the conditioned subordinator allows for $\beta \in (0,1)$, but the reader should keep in mind that for us, $\beta = 1/\alpha \in (0, 1/2]$), with Laplace exponent ψ given by $q \mapsto Cq^\beta$ for nonnegative q. It is known that σ_t admits a density f_t for positive t, for which there is no simple explicit expression except in the case $\beta = 1/2$, and that the potential operator of σ admits a density $u(x,y) = u(0, y-x)$ given explicitly by $u(0,y) = \mathbf{1}_{y\geq 0}/C\Gamma(\beta)\, y^{1-\beta}$. Since σ and $-\sigma$ are in duality with respect to Lebesgue measure, it follows that the potential density of $-\sigma$ is $\hat{u}(y,x) = u(x,y)$. Also, $\hat{h}(x) = \hat{u}(x,0)$ is a potential for the semigroup of $-\sigma$, since if $\left\{\hat{P}_t : t \geq 0\right\}$ denotes the semigroup of $-\sigma$ then

$$\hat{P}_t h(x) = \int_t^\infty f_s(x)\, ds \to 0 \quad (t \to \infty)\,. \tag{26}$$

So, we might consider the Doob transformation of $-\sigma$ by h, $-\sigma^h$, and we shall denote its sub-Markovian family of distributions by $\left\{\hat{\mathbb{P}}^h_x : x > 0\right\}$. Under $\hat{\mathbb{P}}^h_x$, the process dies almost surely in finite time because of the potential character of h: the left-hand side of (26) divided by $h(x)$ is the probability that, starting at x, the death-time ζ^h of $-\sigma^h$ is greater than t. To complete the interpretation of $-\sigma^h$, let us see that $\hat{\mathbb{P}}^h_x$-almost surely, $-\sigma^h_{\zeta^h-} = 0$. To this end, let us determine the conditional law of $-\sigma^h_t, t < \zeta$ given $\zeta = a$: since

$$\hat{\mathbb{P}}^h_x\big(\zeta^h \in da\big) = \frac{f_a(x)}{h(x)} da$$

the Markov property implies that for decreasing x_i and increasing t_i:

$$\hat{\mathbb{P}}^h_x\big(-\sigma^h_{t_1} \in dx_1, \ldots, -\sigma^h_{t_n} \in dx_n, \zeta^h \in da\big) \,/\, dx_1 \cdots dx_n \, da$$
$$= \frac{f_{t_1}(x_1 - x)\, f_{t_2}(x_2 - x_1) \cdots f_{t_n}(x_n - x_{n-1})\, f_{a-t_n}(x_n)}{h(x)}$$

so that a version of the conditional law of $-\sigma^h_t, t < \zeta$ given $\zeta = a$ under $\hat{\mathbb{P}}^h_x$ is that of a bridge of $-\sigma$ between x and 0 of length a. Thanks to Proposition 1 in [**FPY93**], we know that the left-hand limit at a of such a bridge is equal to 0 almost surely, and this implies that $-\sigma^h_{\zeta-} = 0$ $\mathbb{P}_x$-almost surely.

Using the self-similarity of σ, it follows that $-\sigma^h$ is a positive self-similar Markov process; this fact will be crucial to establishing Theorem 7 for all $\alpha \in (1, 2]$. However, in the Brownian case, we only need to calculate the finite-dimensional distributions of the tagged fragment and compare them to those of $-\sigma^h$, as we will do in subsection 3.1 armed with Bismut's representation of the Itô measure (found in [**RY99**]) followed by a conditioning by the length.

3.1. An analysis under Itô's measure. As in subsection 2.2, we shall work with Itô's measure of the positive excursions and we will use the same notation. However, since the notion of tagged fragment involves an independent uniform random variable, we are forced to introduce the measure $\tilde{n}_+$ over $\tilde{E} = [0, \infty) \times E$ given by

$$\tilde{n}_+(dt, de) = \frac{1}{L} \mathbf{1}_{t \in (0,L)} \, dt \, n_+(de) \, .$$

If we define the functions X, U and ζ on $\tilde{E}$ by $X(t, e) = e$, $U(t, e) = t$ and $\zeta(t, e) = e_t$, X will take the place of the excursion $\mathbf{e}$, U will take the place of our independent uniform random variable and ζ will be the death time of the tagged fragment once we extend the definitions of F

and χ over to $\tilde{E}$ by taking into account the length L in their definitions and use the process X instead of $\mathbf{e}$ as follows: $F_t = \{s \in (0, L) : X_s > t\}$ and χ_t is the length of the connected component of F_t that contains U. As a final preliminary before commencing the proof of Theorem 7 in the Brownian case, let us recall Bismut's description of the Itô measure (cf. [**RY99**, XII.4.7, p.502]): under $L \cdot \tilde{n}_+$, the law of ζ is Lebesgue measure on $[0, \infty)$ and conditionally on $\zeta = a$,

$$(X_{s\wedge U})_{s\geq 0} \quad \text{and} \quad \left(X_{(L-s)^+\wedge(1-U)}\right)_{s\geq 0}$$

are two independent Bessel processes of dimension 3 processes stopped at their last visit to a. Therefore, under $L \cdot \tilde{n}_+$ and conditionally on $\zeta = a$, the tagged fragment behaves like the process obtained by subtracting the last visit process of the sum of two independent Bessel processes of dimension three on $[0, a]$ its final value; by one of William's time reversal theorems (cf. [**RY99**, VII.4.6, p.317]), it behaves like the process obtained by subtracting 1/2-stable subordinator with Laplace exponent $q \mapsto 2\sqrt{2q}$ on the time interval $[0, a]$ its final value, which corresponds to the length L of the excursion. If T is such a subordinator, conditionally on $\zeta = a$, χ would be equal in law to $(T_a - T_t)_{t\in[0,a]}$ and L would be just T_a.

Brownian proof of Theorem 7. Let T be as above and f_t denote the density of T_t, given by

$$f_t(x) = \frac{\sqrt{2}t}{\sqrt{\pi x^3}} e^{-2t^2/x}.$$

The considerations of the preceding paragraph allow us to write

$$\begin{aligned}&\tilde{n}_+(\zeta \in da, \chi_{t_1} \in dx_1, \ldots, \chi_{t_n} \in dx_n, L \in dv) \,/\, da\, dx_1 \cdots dx_n\, dv \\ &= \frac{1}{v}\, f_{t_1}(v - x_1)\, f_{(t_2-t_1)}(x_1 - x_2) \cdots f_{(t_n-t_{n-1})}(x_{n-1} - x_n)\, f_{(a-t_n)}(x_n)\end{aligned}$$

for decreasing $x_1, \ldots, x_n$ in $[0, v]$ (the tagged fragment decreases in size) and increasing $t_1, \ldots, t_n$ in $[0, a]$; note that the factor $1/v$ comes from the fact that we are not working with $L \cdot \tilde{n}_+$ (as in Bismut's description of n_+) but with $\tilde{n}_+$.

Integrating a out of the right hand side of the last display over the interval (t_n,∞) gives

$$\begin{aligned}&\tilde{n}_+(\chi_{t_1}\in dx_1,\dots,\chi_{t_n}\in dx_n, L\in dv)\,/\,dx_1\cdots dx_n\,dv\\&=\frac{1}{v}\,f_{t_1}(v-x_1)\,f_{(t_2-t_1)}(x_1-x_2)\cdots f_{(t_n-t_{n-1})}(x_{n-1}-x_n)\,\frac{1}{2\sqrt{2\pi x_n}}.\end{aligned}$$

Conditioning by length, using $\tilde{n}_+(\zeta\in dv)=1/2\sqrt{2\pi v^3}$, allows the following conclusion:

$$\begin{aligned}&\pi(\chi_{t_1}\in dx_1,\dots,\chi_{t_n}\in dx_n)\,/\,dx_1\cdots dx_n\\&=f_{t_1}(v-x_1)\,f_{(t_2-t_1)}(x_1-x_2)\cdots f_{(t_n-t_{n-1})}(x_{n-1}-x_n)\,\frac{\sqrt{v}}{\sqrt{x_n}}.\end{aligned}$$

The right-hand side of the preceding display portrays the density of the finite-dimensional distributions of the opposite of a 1/2-stable subordinator with Laplace exponent $q\mapsto 2\sqrt{2q}$ conditioned to die at zero started at v. □

3.2. An analysis through positive self-similar Markov processes. The definition of the fragmentation F^α is not as simple when $\alpha\in(1,2)$ as in the $\alpha=2$ case previously introduced (recall that $-1/\alpha$ stands for the index of the self-similar fragmentation) because its construction depends on the so-called height process; see [**DLG02**] for the definition of the height process and [**Mie03**] for the definition of F^α when $\alpha\in(1,2)$. For Brownian motion the height process is a scaled version of reflected Brownian motion, as argued in [**DLG05**, p.566], and that justifies the representation and analysis of F^2 we have given, up to a constant that relates to the speed of the fragmentation and whose influence will be apparent later on. However, for our needs, concentrating on a tagged fragment of F^α will be enough.

The tagged fragment associated to F^α, denoted by χ^α, is a self-similar Markov process that is absorbed continuously at zero in finite time ζ^α. Thanks to the Lamperti transformation[1] it is associated to a subordinator ξ^α whose Lévy measure has been explicitly calculated, in [**Ber02**] and

[1]For information and further references regarding self-similar Markov process, the Lamperti transformation and its relationship to exponential functionals of Lévy processes, see the recent survey [**BY05**].

[**Mie03**], and is given by

$$x \mapsto \sqrt{\frac{2}{\pi}} \frac{e^x}{(e^x - 1)^{3/2}} \tag{27}$$

for the Brownian case (a multiple of (11) in [**Ber02**], as explained there) and, recalling that $\beta = 1 - 1/\alpha$,

$$x \mapsto \frac{\beta}{\Gamma(2-\beta)} \frac{e^x}{(e^x - 1)^{1+\beta}} \tag{28}$$

for $\alpha \in (1, 2)$ (the display after (12) in [**Mie03**]). The difference in the constant appearing is due to the fact that one uses the normalized Brownian excursion and not a scaled one corresponding to the height process of a Brownian excursion in Bertoin's construction of the fragmentation. We will now proceed with the proof of Theorem 7.

Proof of Theorem 7. The Lamperti transformation takes a Lévy process ξ and a real number a into the self-similar Markov process that starts at one given implicitly by

$$T_{\int_0^t \exp(a\xi_s)\, ds} = e^{\xi_t}.$$

The index of self-similarity of T, as defined in [**Lam72**], is then $1/a$. When applied to a subordinator ξ and successively with a and $-a$ for a positive a, it gives rise to two different processes, denoted by T and $\hat{T}$ respectively, which are nevertheless related to each other by duality of their resolvent operators with respect to Lebesgue measure on $(0, \infty)$, as shown in [**BY02**]. When T is a β-stable subordinator with Laplace exponent $q \mapsto Cq^\beta$, the associated subordinator ξ has Lévy measure

$$x \mapsto \frac{\beta C}{\Gamma(1-\beta)} \frac{e^x}{(e^x - 1)^{1+\beta}}$$

which coincides with (27) when $\beta = 1/2$ and $C = 2\sqrt{2}$ and with (28) when $\beta \in (1/2, 1)$ and $C = \Gamma(1-\beta) / \Gamma(2-\beta)$; the Lamperti transformation should be applied to ξ with $a = \beta$ to obtain T. It follows that the tagged fragment of F^α (we had denoted it by χ^α) is in resolvent duality with a β-stable subordinator. Since T_t admits a density f_t then, as argued in [**BY02**], when we view χ^α time-reversed from its death time, it behaves as T started at zero and conditioned to die at one via Doob's transformation with the excessive function $h(x) = \int_0^\infty f_t(1-x)\, dt$. Nagasawa's theorem on time-reversal then allows us to conclude that F^α has the same law as $-T$ started at one and conditioned to die at zero via $\hat{h}(x) = \int_0^\infty f_t(x)\,dt$. □

The calculations of the Lévy measure of the subordinator ξ associated to the death time of the tagged fragment of F^α performed in [**Ber02**] and [**Mie03**] were based on the fact that one can express the density of the death time of the tagged fragment in terms the density of a β-stable subordinator. With the notation introduced in the preceding proof, the density of the death time of the tagged fragment is $t \mapsto f_t(x)/h(x)$, where the constant C chosen in terms of β as mentioned during the course of the proof. This last expression should suffice to convince oneself of the validity of Theorem 7 because of the following result:

LEMMA 8. *The distribution of a decreasing and positive self-similar Markov process that is absorbed at zero in finite time is determined by its index and the law of its absorption time.*

PROOF. By self-similarity, it suffices to consider the case when the given process starts at one. If ζ denotes the absorption time of a decreasing and positive self-similar Markov process starting at one which is absorbed at zero in finite time obtained by applying the Lamperti transformation to a subordinator ξ, then there exists $\delta < 0$ (one over the self-similarity index) such that ζ has the same law as the exponential functional

$$A_\infty = \int_0^\infty \exp(\delta \xi_s)\,ds.$$

On the other hand, if ϕ is the Laplace exponent of ξ, then formula (4) in [**BY05**] used with $q = 0$ gives us

$$\mathbb{E}\big(\zeta^k\big) = \mathbb{E}\big(A_\infty^k\big) = \frac{k!}{\prod_{i=1}^k \phi(-\delta i)}.$$

It follows that the sequence of moments $\big(\mathbb{E}\big(\zeta^k\big)\big)_{k\in\mathbb{N}}$ determines the sequence $(\phi(-\delta i))_{i\in\mathbb{N}}$. However, the second sequence determines the moments of the bounded random variable $\exp(\delta\xi_t)$, so that it determines its law, hence that of ξ_t. Finally, it suffices to note that the distribution of ξ_t and the self-similarity index determine the distribution of the self-similar Markov process we started with. □

4. The falling apart of the tagged fragment and fragmentation by ancestral line obliteration

In this section, we shall prove Theorem 8. First, the Brownian case will be considered in subsection 4.1 using a path transformation relating the normalized Brownian excursion and the Brownian bridge, and known results on the distribution of the ranked length of excursions of the Brownian bridge away from zero. Then, we shall see how these results tie up in the construction of another self-similar fragmentation from the normalized Brownian excursion. Finally, the proof for the general case will be shown to be the consequence of our representation of the tagged fragment of F^α contained in Theorem 7, which allows us to calculate its conditional distribution given death-time and relate it to a stable subordinator, and known results on size-biased sampling of the jumps of subordinators. Also, a generalization of the fragmentation of subsection 4.1 will be sketched.

4.1. A visual argument for the Brownian case. The Brownian interpretation of Theorem 8 (that is, using the fragmentation F^2) is quite visual and depends on a path transformation, introduced by Bertoin and Pitman, between the normalized Brownian excursion and the reflected Brownian bridge which can be stated as follows (cf. [**BP94**, Theorem 3.2]): define $K^U = \left(K^U_s\right)_{s\in[0,1]}$ by

$$K^U_s = \begin{cases} \min_{s\leq u\leq U} \mathbf{e}_u & \text{for } s\in[0,U] \\ \min_{U\leq u\leq s} \mathbf{e}_u & \text{for } s\in[U,1]. \end{cases}$$

Then the process $b = \mathbf{e} - K^U$ is the absolute value of a Brownian bridge between 0 and 0 of length 1. Let us note, however that the lengths of the excursions of $\mathbf{e}$ above K^U are in one to one correspondence with the jumps of χ. Since the excursions of $\mathbf{e}$ above K^U are precisely the excursions of b away from zero, we conclude that the decreasing sequence of the jumps of χ has the same law as the decreasing sequence of the lengths of excursions of a Brownian bridge away from zero. By Proposition 7 in [**PY97**], this is the Poisson-Dirichlet law with parameters $(1/2, 1/2)$. This proves Theorem 8 for $\alpha = 2$.

The same type of analysis can be put to use in the construction of another fragmentation process. Define $b^0 = \mathbf{e}$ and suppose that $(U_i)_{i\geq 1}$

are independent (between themselves and $\mathbf{e}$) and uniformly distributed random variables. For $n \geq 1$ construct b^n as follows, and set $V_n = \{s \in (0,1) : b^n_s > 0\}$: let (a_{n-1}, b_{n-1}) be the connected component of V_{n-1} that contains U_n , $(K^n_s)_{s\in[0,1]}$ be given by

$$K^n_s = \begin{cases} 0 & \text{if } s \notin (a_{n-1}, b_{n-1}) \\ \min_{a_i \leq s \leq U_n} b^{n-1}_s & \text{if } a_{n-1} \leq s \leq U_n \\ \min_{U_n \leq s \leq b_i} b^{n-1}_s & \text{if } U_n \leq s \leq b_i \end{cases} ,$$

and $b^n = b^{n-1} - K^n$. To construct a self-similar interval fragmentation, let N be a Poisson process independent of $\mathbf{e}$ and $(U_i)_{i\geq 1}$, and set $F^o_t = V_{N_t}$. This fragmentation, which has self-similarity index 1, erosion coefficient zero and dislocation measure equal to PD(1/2, 1/2), shall be termed by ancestral line obliteration and we shall dwell next on its interpretation and on a calculation that can be performed with it.

As explained in Section 2 of [**LG05**], given a nonnegative continuous function $f : [0,1] \to \mathbb{R}_+$ and such that $f(0) = 0$ (it will be referred to as the coding function,) a pointed metric space (τ_f, d_f, ρ_f) belonging to the space of compact real trees can be constructed as follows: define the pseudo-metric d_f and the equivalence relation $\overset{f}{\sim}$ on $[0,1]$ by

$$d_f(s_1, s_2) = f(s_1) + f(s_2) - 2m_f(s_1, s_2) ,$$

where

$$m_f(s_1, s_2) = \min_{r\in[s_1\wedge s_2, s_1\vee s_2]} f(r) ,$$

and

$$s_1 \overset{f}{\sim} s_2 \text{ if and only if } d_f(s_1, s_2) = 0.$$

Then the quotient space $\tau_f = [0,1]/\overset{f}{\sim}$, with the induced distance (which will keep the notation d_f), is a compact real tree, to be rooted at the equivalence class of 0, denoted ρ_f. The locations of the local maxima of f correspond to the leaves of the tree τ_f; the leaves of the tree are elements which differ from the root and that do not disconnect it upon removal. The locations of the local minima of f code the nodes or branching points of τ_f (branching points disconnect the tree when removed). The equivalence classes of s_1 and s_2 $(s_1 < s_2)$ will branch from a common node, say $[s]$ satisfying $s_1 < s < s_2$, exactly when f is greater than $f(s)$ on $[s_1, s_2]$.

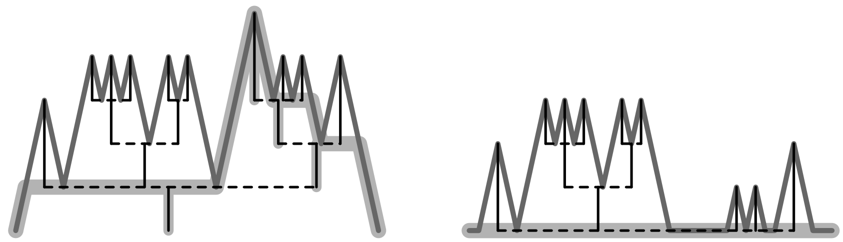


FIGURE 3. Trees coded by continuous functions and obliteration of ancestral lines.

A visualization of the trees coded by two functions is offered in Figure 3.

When the function f is replaced by a random continuous function, such as $\mathbf{e}$, it gives rise to a random tree; the random trees we shall be interested in are $(\tau_{b^n}, d_{b^n}, \rho_{b^n})$. The pointed metric space (τ_f, d_f, ρ_f) represents a genealogy coded by the function f. To continue the analogy presented in the last paragraph, let us note that the common ancestor of every element of τ_f is ρ_f, the most recent common ancestor of s_1 and s_2 is the equivalence class of any $r \in [s_1 \wedge s_2, s_1 \vee s_2]$ such that $m_f(s_1, s_2) = r$ and the line of descent traced from the ancestor ρ_f up to the equivalence class of s consists of equivalence classes of elements $r \in [0, 1]$ such that $f(r) = m_f(r, s)$. To concatenate with our fragmentation F^o, let us note that $K^n_s = m_{b^{n-1}}(U_n, s)$ and so $b^{n-1} - K^n$ represents the coding function for a tree that redefines the genealogy of $\tau_{b^{n-1}}$ by not taking into account the equivalence class of U^n (in b^{n-1}) and all its ancestors up to the root. The interpretation of this transformation between continuous functions and their associated trees does not appear to be reported elsewhere. Again, we refer to Figure 3 for a visual account of this procedure.

To end this subsection, let us describe the law of the decreasing sequence of masses of the components of V_n, which we shall denote m_n. To do this, let us note that m_n is obtained by taking a size-biased pick from m_{n-1} (the size of the component of V_{n-1} that contains U_n) and fragmentating it using a PD$(1/2, 1/2)$-distribution. So, m_n is obtained as the result of applying the fragmentation operator $\mathbf{Frag_{1/2}}$ of [**DGM06**] to m_{n-1}. Since m_1 has a PD$(1/2, 1/2)$ distribution, Theorem 3.1 in the last reference implies that for $n \geq 1$, m_n has a PD$(1/2, n - 1/2)$ distribution.

4.2. A computational argument for the general case. The aim of this subsection is to establish Theorem 8. Our strategy will be to analyze the implications of Theorem 7 by calculating the conditional law of the tagged fragment given its death time, and its relationship to stable subordinators. Then we shall use this conditional law in conjunction with formulae describing size-biased sampling of the jumps of subordinators to conclude.

We shall use the framework and notation considered in the introduction to section 3. One conclusion of the introduction is that under $\hat{\mathbb{P}}_1^{\hat{h}}$ and conditionally on $\zeta^h = a$, σ^h is a bridge of $-\sigma$ from 1 to 0 of length a, so the same result follows for the tagged fragment. Also, the bridge of $-\sigma$ from x to y of length v coincides with the opposite of that of σ between $-x$ to $-y$, so that the sizes of the jumps, in absolute value, are the same for both processes. We shall now prove Theorem 8.

Proof of Theorem 8. Let us recall that the Poisson-Dirichlet distribution with two parameters (see the survey [**PY97**] for further information and references), denoted by $\mathrm{PD}(\beta,\theta)$ (we will think of β as a fixed parameter) for $\theta > -\beta$, is a probability law on the space of decreasing sequences $v = v_1 > v_2 > \cdots > 0$ such that $\sum_i v_i = 1$, which is characterized by a property of their size-biased permutations: $V = (V_1, V_2, \ldots)$ has a $\mathrm{PD}(\beta,\theta)$ distribution iff for a size biased permutation $\tilde{V}$ of V, the random variables defined implicitly by $\tilde{V}_1 = Y_1$, $\tilde{V}_n = (1-Y_1)\cdots(1-Y_{n-1})\,Y_n$ are independent and Y_n has a Beta distribution with parameters $(1-\beta, \theta+n\beta)$. A size biased permutation of V is another sequence $\tilde{V}$ such that

$$\mathbb{P}\Big(\tilde{V}_1 = V_i \,|V\Big) = V_i$$

and

$$\mathbb{P}\Big(\tilde{V}_{n+1} = V_i \,|V, \tilde{V}_1, \ldots, \tilde{V}_n\Big) = \frac{V_i}{1-\tilde{V}_1-\cdots-\tilde{V}_n}\mathbf{1}_{V_i\neq\tilde{V}_j, 1\leq j\leq n}.$$

One of the objectives of [**PPY92**] is to construct the tools necessary for the analysis of size-biased permutations of the jumps of subordinators. Let us start the process $-\sigma^h$ at level one, simplifying notation by stipulating that $\mathbb{P}_1 = \mathbb{P}$, and consider the decreasing sequence V of the jumps of $-\sigma^h$ on $[0,\zeta]$, which will then sum up to one, and a size-biased permutation $\tilde{V}$ of V giving rise to the sequence Y as before. We shall use the density of the Lévy measure of σ, given by $\rho(x) = \beta C/\Gamma(1-\beta)\, x^{\beta+1}$ to define the function $\Theta(x) = x\rho(x)$. The discussion of the preceding paragraph and formula (2.d) of Theorem 2.1 in [**PPY92**], using the notation $\bar{x} = 1 - x$, allow the following:

$$\begin{aligned}&\mathbb{P}(Y_1 \in dx_1, \dots, Y_n \in dx_n, \zeta \in da) = \\ &v^n \Theta(x_1)\, \Theta(\bar{x}_1 x_2) \cdots \Theta(\bar{x}_1 \cdots \bar{x}_{n-1} x_n)\, f_a(\bar{x}_1 \cdots \bar{x}_n)\, \frac{1}{h(1)}.\end{aligned}$$

Now, we shall use the scaling identities of f_t to integrate a out of the last expression. Namely, since

$$f_t(y) = \frac{1}{y} f_{ty^{-\beta}}(1)\,,$$

we get

$$\begin{aligned}&\mathbb{P}(Y_1 \in dx_1, \dots, Y_n \in dx_n) \\ &= \frac{\beta^n}{\Gamma(1-\beta)^n} \mathbb{E}(\zeta^n)\, \Theta(x_1)\, \Theta(\bar{x}_1 x_2) \cdots \Theta(\bar{x}_1 \cdots \bar{x}_{n-1} x_n) \\ &= \frac{\beta^n}{\Gamma(1-\beta)^n} \mathbb{E}(\zeta^n)\, C^n\, (x_1 \cdots x^n)^{\beta-1}\, \bar{x}_1^{2\beta-1} \cdots \bar{x}_n^{(n+1)\beta-1}.\end{aligned}$$

Now, let us note that the last expression in the preceding display does not depend on C, which can be seen either by direct analysis of the law of ζ considering the scaling identity of f_t, or by the fact that the left-hand side of the first equality in the preceding display represents a probability density. The conclusion is that $Y_1, \dots, Y_n$ are independent and Y_n has a Beta distribution with parameters $1-\beta$ and $\beta + n\beta$, so that the sequence of jumps of χ in decreasing order has the PD(β, β) distribution. □

The Poisson-Dirichlet distribution of parameters (β, β) arises as the distribution of the ranked lengths of excursions of Bessel bridges of dimension $\delta = 2\,(1-\beta)$ starting and ending at zero (cf. [**PY97**, Proposition 7]). Since the inverse local time at zero of a Bessel process of dimension δ starting at zero is a stable subordinator of index β, it is natural to search for a similar representation for the inverse local time of our Bessel bridge; it turns out that inverse local time is a stable subordinator of index β

starting at zero and conditioned to die at 1 (through a Doob transformation via the potential density as described in Section 3). The proof is as follows: let $\mathbb{P}_0^\delta$ denote the law of a Bessel process of dimension δ starting at zero and $\mathbb{P}_{0,0}^{1,\delta}$ be the law of a Bessel bridge of dimension δ and length one starting and ending at zero. (For a general account of the theory of bridges of Markov processes, see [**FPY93**].) Using the explicit representations of the transition densities of Bessel processes (in terms of modified Bessel functions of the first kind) one can prove that, for $s < 1$, we have the following relationship between $\mathbb{P}_{0,0}^{1,\delta}$ and $\mathbb{P}_0^\delta$ (where X stands for the canonical process and $\mathscr{F}_s = \sigma(X_u : u \leq s)$):

$$\mathbb{P}_{0,0}^{1,\delta}|_{\mathscr{F}_s} = \left(\frac{1}{1-s}\right)^\beta e^{-\frac{X_s^2}{2(1-s)}} \cdot \mathbb{P}_0^\delta|\mathscr{F}_s.$$

Let τ denote the inverse local time at zero, where the local time is taken in the sense of regenerative sets (semimartingale local time vanishes as explained in [**RY99**, XI.1.5, p.442]). Just as in [**RY99**, VIII.1.3, p.326], we can extend the preceding equality to the stopping times τ_s on the set $\{\tau_s < \infty\}$ so that

$$\mathbb{P}_{0,0}^{1,\delta}|_{\mathscr{F}_{\tau_s}} = \left(\frac{1}{1-\tau_s}\right)^\beta \cdot \mathbb{P}_0^\delta|\mathscr{F}_{\tau_s}.$$

Since τ is a stable subordinator of index β under $\mathbb{P}_0^\delta$ (τ_s has density f_s), it follows that under $\mathbb{P}_{0,0}^{1,\delta}$, τ is Markovian and its transition density from x to y in s units of time is $f_s(y-x)\left((1-x)\,/\,(1-y)\right)^\beta$, so that it is a β-stable subordinator conditioned to die at 1.

5. Asymptotics at extinction

In this section we shall prove Theorem 9. The reader is asked to recall the framework introduced in the introduction in order to state it. A point that was not discussed there was the proper topology on the set of open subsets of $\mathbb{R}$ to be able to talk about weak convergence, which was introduced in Subsection 2.2. We also saw that that the $\mathscr{V}^{(0,1)}$-valued variable $\hat{F}_t^2$ is measurable, and since, for any open subset V of $\mathbb{R}$, the inclusion from $\mathscr{V}^V$ into $\mathscr{V}^\mathbb{R}$ is continuous, it is also a random variable with values in $\mathscr{V}^\mathbb{R}$. A similar argument implies that $\{t \in \mathbb{R} : Z_t < 1\}$ is a $\mathscr{V}^\mathbb{R}$-valued random variable. Let us note that a sequence $(\mu_n)_{n\in\mathbb{N}}$

of probability laws on $\mathscr{V}^{\mathbb{R}}$ converges in distribution to μ iff for every $i \in \mathbb{Z}$, $\mu_n \circ R^{-1}_{(-i,i)}$ converges in distribution to $\mu \circ R^{-1}_{(-i,i)}$. This is a direct consequence of the fact that, with the distance $d_{\mathbb{R}}$ defined on $\mathscr{V}^{\mathbb{R}}$, a subset A of $\mathscr{V}^{\mathbb{R}}$ is compact if and only if $\{V \cap (-i,i) : V \in A\}$ is compact for every $i \in \mathbb{Z}^+$.

Theorem 9 will be proved by the use of a path transformation to substitute the random time $M - t$ by t, and then we shall vary the length of the excursion instead of the parameter t, since a result concerning the Brownian excursion of length v when v tends to ∞ can be readily applied.

We shall now provide a path transformation of the normalized Brownian excursion that leaves its distribution invariant and which translates our problem into one involving initial times rather than the time M of extinction of F^2: if we let

$$\mathbf{e}^S_t = M - \mathbf{e}_{(S+t) \mod 1}$$

then we have:

Proposition 6. *$\mathbf{e}^S$ has the same law as $\mathbf{e}$.*

This path transformation of the normalized Brownian excursion was suggested by B. Haas in a private communication and it is illustrated in Figure 4.

Note that $\left(\hat{F}^2_t - S\right)/t^2$ can be obtained from the height fragmentation of $\mathbf{e}^S$ at level t and the random time S. The precise representation depends strongly on the value of S. However, on the event $nt^2 < S < 1-nt^2$, which is the whole space for fixed n as $t \to 0+$, we have the identity

$$R_{(-n,n)}\left(\frac{1}{t^2}\left(\hat{F}^2_t - S\right)\right) =$$
$$-\left\{s \geq 0 : \mathbf{e}^S_{st^2} < t, s < n\right\} \cup \left\{s \geq 0 : \mathbf{e}^S_{1-st^2} < t, s < n\right\}.$$

Therefore, we shall prove Theorem 9 by verifying that the right-hand side in the preceding display converges in law to the limit we have stipulated. Since $(\mathbf{e}_{st^2}/t)_{s\in(0,1/t^2)}$ has the law of a Brownian excursion of length $1/t^2$ (which was denoted π^{1/t^2}), denoting by X be the canonical process on excursion space, we have that

$$-\left\{s \geq 0 : \mathbf{e}^S_{st^2} < t, s < n\right\} \cup \left\{s \geq 0 : \mathbf{e}^S_{1-st^2} < t, s < n\right\}$$

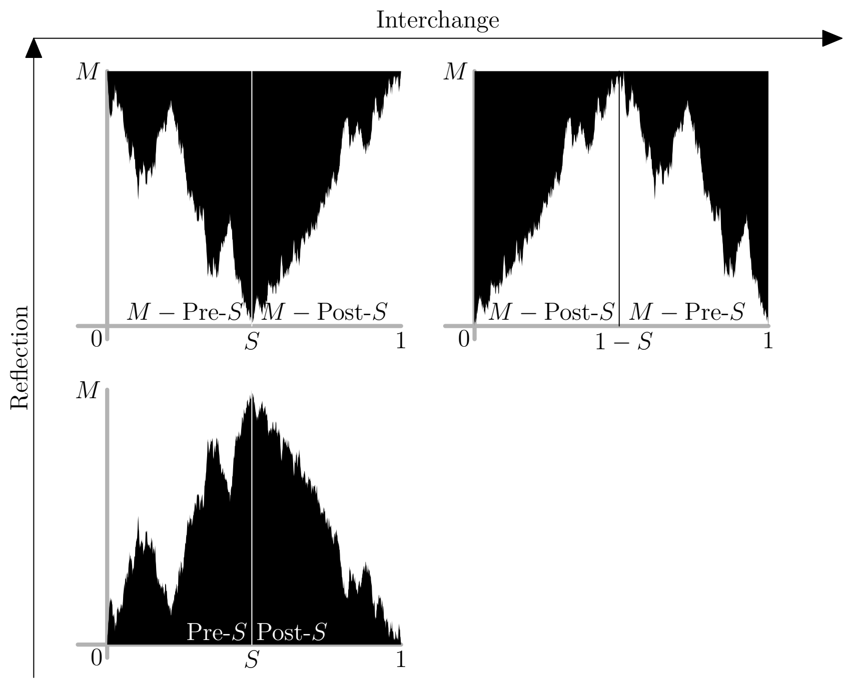


FIGURE 4. Haas' path transformation of the normalized Brownian excursion.

has the same law as

$$-\{s \in (0,n) : X_s < 1\} \cup \left\{s \in (0,n) : X_{1/t^2-s} < 1\right\}$$

under π^{1/t^2}, at least for $n < 1/t^2$. Appealing to a result of Jeulin (cf. [**Jeu80**, Th. 6.41 p. 127]) we will prove in subsection 5.2 that if F and G are bounded functionals depending on $(X_s)_{s\in(0,n)}$, and if we let $\tilde{X}_t = X_{v-t}$, then

$$\pi^v\Big(F(X)\,G\Big(\tilde{X}\Big)\Big) \to_{v\to\infty} \mathbb{P}_0^3(F)\,\mathbb{P}_0^3(G)\,. \tag{29}$$

This last result and the preceding discussion imply Theorem 9, as long as we can be convinced of the validity of Proposition 6.

Regarding Corollary 1, it suffices to remark that the size of the connected component of the set $\{s \in \mathbb{R} : Z_s < 1\}$ which contains zero is equal to the sum of the hitting times of level one by two independent three-dimensional Bessel processes starting at zero, which share $q \mapsto 1/\sinh\left(\sqrt{2q}\right)$ as a Laplace transform. On the other hand, the Lebesgue measure of $\{s \in \mathbb{R} : Z_s < 1\}$ is the sum of the occupation times of $(0, 1)$ of these independent Bessel processes and, thanks to the Ciesielski-Taylor identity for example (which equals their law to that of the occupation time of $(0, 1)$ by a Brownian motion starting at zero until it hits level one,) they have common Laplace transform given by $q \mapsto 1/\cosh\left(\sqrt{2q}\right)$.

5.1. Haas' path transformation. The aim of the following paragraphs is to show how Proposition 6 can be deduced from one of William's time reversal results relating Brownian motion killed when it reaches zero and the three-dimensional Bessel process on one hand and Itô's and William's descriptions of the Itô measure on the other.

Let n_+ be the Itô measure of positive excursions of Brownian motion introduced in subsection 3.1. We shall keep the notation. The reader is asked to recall Itô's description of the Itô measure since we shall perform a conditioning by the length on n_+. To carry out this program, we will also need William's description of the Itô measure, which is the following. Let $M : E \to \mathbb{R}_+$ denote the height of the excursion, given by $M = \sup_{s \geq 0} X_s$. Then the image law of M under n_+ admits the density $m \mapsto \mathbf{1}_{(0,\infty)}/2m^2$ and the conditional law of X under n_+ is that the pasting together of two independent three-dimensional Bessel processes started at zero and stopped when they reach level m, one of them concatenated in reverse time after the other. Now, let us recall the following time-reversal result, for which the reader is referred to [**RY99**, VII.4.8]: if R is a three-dimensional Bessel process starting at zero, $b > 0$ and T_b is the hitting time of b by R, then $(X_{T_b - t})_{0 \leq t \leq T_b}$ and $(b - X_t)_{0 \leq t \leq T_b}$ have the same law.

With these preliminaries, let us commence the proof of Proposition 6. Let S be the instant in which X attains its maximum and define, as for the normalized Brownian excursion, X^S by

$$X_t^S = M - X_{t+S \mod L}.$$

By using William's description of the Itô measure and his time-reversal result, we see that under n_+ conditionally on $M = m$, $\left(X^S, L\right)$ has the same law as (X, L), and so the same holds under n_+. If $g : \mathbb{R}^n \to \mathbb{R}$

is bounded and continuous, $f : (0, \infty) \to \mathbb{R}$ is a positive measurable function, and $0 \le t_1 \le \cdots \le t_n$ we obtain the equality

$$n_+\big(g\big(X^S_{t_1}, \dots, X^S_{t_n}\big)\, f(L)\big) = n_+(g(X_{t_1}, \dots, X_{t_n})\, f(L))\,. \tag{30}$$

However, by Itô's description of the Itô measure, the left hand side of the preceding display equals

$$\int_{t_n}^{\infty} dv\, \frac{1}{2\sqrt{2\pi v^3}} f(v)\, \pi^v(g(X_{t_1}, \dots, X_{t_n}))$$

while the right-hand side equals

$$\int_{t_n}^{\infty} dv\, \frac{1}{2\sqrt{2\pi v^3}} f(v)\, \pi^v\big(g\big(X^S_{t_1}, \dots, X^S_{t_n}\big)\big)\,.$$

Because the equality in (30) is valid for any positive measurable function f, we conclude from the weak continuity of $v \mapsto \pi^v$ that

$$\pi^v\big(g\big(X^S_{t_1}, \dots, X^S_{t_n}\big)\big) = \pi^v(g(X_{t_1}, \dots, X_{t_n}))\,,$$

for all $v > t_n$, so that π^v is invariant under the transformation $X \mapsto X^S$.

5.2. On Jeulin's limit theorem. In this subsection, we shall give a proof of (29). This result is analogous to Jeulin's limit theorem for the normalized Brownian excursion but it's verification will not rely on the delicate estimates used by the aforementioned author in [**Jeu80**]. This is because we stop our processes at fixed times instead of the random times of last visit.

We recall Jeulin's theorem, introduced and proved in [**Jeu80**]: if $\mathbf{e}$ is a normalized Brownian excursion and we define $X^\varepsilon = (X^\varepsilon_t)_{t \le r/\varepsilon^2}$ and $Y^\eta = (Y^\eta_t)_{t \le (1-r)/\eta^2}$ by

$$X^\varepsilon_t = \frac{1}{\varepsilon}\mathbf{e}_{\varepsilon^2 t} \quad \text{and} \quad Y^\eta_t = \frac{1}{\eta}\mathbf{e}_{1-\eta^2 t},$$

then the law of (X^ε, Y^η), both coordinates stopped when last visiting $a \ge 0$ before times r and $1-r$ respectively, converges in variation as $(\varepsilon, \eta) \to 0$ to the law of two independent Bessel processes of dimension three starting at zero and killed on their last visit to a. (The formulation in[**Jeu80**] does not mention convergence in variation; this is implied by the proof.)

Let us now discuss equation (29). We shall work on the canonical spaces C_v where the laws

$$\left\{\pi^v_{x,y} : x, y, v > 0\right\},$$

$\pi^v_{x,y}$ corresponding to a Brownian bridge from x to y of length v conditioned on remaining positive, are defined. We will denote by X and $(\mathscr{F}_t)_{t\geq 0}$ the canonical process and filtration.

As $y \to 0$, $\pi^v_{x,y}$ has a weak limit which shall be denoted π^v_y; this law satisfies a local absolute continuity with respect to the law of the three-dimensional Bessel process starting at zero, denoted $\mathbb{P}^3_0$, of the following form: $\pi^v_y|_{\mathscr{F}_s}$ is absolutely continuous with respect to $\mathbb{P}^3_0|_{\mathscr{F}_s}$ and the Radon-Nikodým derivative $M^{v,s}_y$ can be written in terms of the canonical process X, the transition density q_s of Brownian motion killed when it reaches zero and the density f_x of the hitting time of x by a Brownian motion started at zero as follows:

$$M^{v,s}_y(X) = \frac{q_{v-s}(X_s, y)}{2f_y(v)\, X_s}.$$

From this, one might infer an inhomogeneous Markov property for π^v_y.

We shall use the following facts: π^v is the weak limit of π^v_y as $y \to 0$ which satisfies the following result, combining its inhomogeneous Markov property with time-reversibility: if $\tilde{X}$ is the time-reversed process given by $\tilde{X}_s = X_{v-s}$ and Φ is functional on the two-fold product of canonical space with itself which is positive and $\mathscr{F}_{s_1} \otimes \mathscr{F}_{s_2}$-measurable, then for $0 < s_i < v - s_2 < v$

$$\pi^v\left(\Phi\left(X, \tilde{X}\right)\right) = \pi^v\left(\pi^{s_1}_{X_{s_1}} \otimes \pi^{s_2}_{\tilde{X}_{s_2}}(\Phi)\right). \tag{31}$$

We shall use this to establish a preliminary version of Jeulin's theorem with deterministic times in lieu of random ones. On the twofold-product of canonical space with itself, Φ will denote an arbitrary $\mathscr{F}_{s_1} \otimes \mathscr{F}_{s_2}$-measurable functional. Consider also the scaling operator $\mathscr{S}_u : C_v \to C_{v/u}$ defined as follows as follows: $\mathscr{S}_v f(s) = f(us)/\sqrt{u}$. Then, for $0 < s_1 < 1 - s_2 < 1$, as $(\varepsilon, \eta) \to (0,0)$:

$$\sup_{\|\Phi\|_\infty \leq 1} \left|\pi^1\left(\Phi\left(\mathscr{S}_{\varepsilon^2} \circ X, \mathscr{S}_{\eta^2} \circ \tilde{X}\right)\right) - \mathbb{P}^3_0 \otimes \mathbb{P}^3_0(\Phi)\right| \to 0. \tag{32}$$

In other words, Jeulin's theorem holds if we stop the processes at a fixed times instead of the random times of last visit. To see that (32) holds, we need only remark that, from the explicit expressions

$$q_s(x,y) = \frac{1}{\sqrt{2\pi s}}\left(e^{-(y-x)^2/2s} - e^{(x+y)^2/2s}\right)$$

and

$$f_x(s) = \frac{x}{\sqrt{2\pi s^3}} e^{-x^2/2s},$$

the convergence

$$M^{s\sqrt{v},s_1}_{y\sqrt{v}}(X) \to 1 \tag{33}$$

as $v \to \infty$ with the other arguments fixed follows. To use the preceding asymptotic equivalence, we shall work on the threefold product of canonical space with itself, with a measure constructed from π^1 and $\mathbb{P}^3_0 \otimes \mathbb{P}^3_0$, and denote by X, Y and Z the first, second and third coordinate processes; X will be used when integrating against π^1. By the bridge property (31) used with times s and $1-s$, the local absolute continuity between π^v_y and $\mathbb{P}^3_0$ and the scaling property $\pi^s_y \circ \mathscr{S}_{\varepsilon^2} = \pi^{s/\varepsilon^2}_{y/\varepsilon}$, we may write

$$\begin{aligned}
&\left|\pi^1\left(\Phi\left(\mathscr{S}_{\varepsilon^2}\circ X, \mathscr{S}_{\eta^2}\circ \tilde{X}\right)\right) - \mathbb{P}^3_0 \otimes \mathbb{P}^3_0(\Phi)\right| \\
&\leq \|\Phi\|_\infty \pi^1\left(\mathbb{P}^3_0 \otimes \mathbb{P}^3_0\left(\left|1 - M^{s/\varepsilon^2,s_1}_{X_s/\varepsilon}(Y)\, M^{(1-s)/\varepsilon^2,s_2}_{X_s/\varepsilon}(Z)\right|\right)\right)
\end{aligned}$$

for small ε and η. Since π^1-almost surely

$$x \mapsto \mathbb{P}^3_0 \otimes \mathbb{P}^3_0\left(M^{s/\varepsilon^2,s_1}_{x/\varepsilon}(Y)\, M^{(1-s)/\varepsilon^2,s_2}_{x/\varepsilon}(Z)\right)$$

converges to 1 by (33), and it integrates 1, it follows that the convergence holds also in L^1, proving (32), which is actually stronger than (29).

CHAPTER 4

Excursions and the Multiplicative Coalescent

In this chapter, we will analyze results from a work in progress concerning the relationship between a variation of the classical Erdős-Rényi model of random graphs, the multiplicative coalescent and the (Brownian) excursions of a random process related to Brownian motion. We will be concerned with three results: the appearance of Gumbel's law at the threshold for connectivity of random graphs, the threshold for the emergence of the giant component for random graphs and the critical window, together with its relationship to excursions of stochastic processes and the multiplicative coalescent, leading to a limit theorem concerning Brownian motion with time-dependent drift (with some heuristics for the drift term). We will start by introducing the probabilistic objects, and then pass on to the results. The arguments will be incomplete, emphasizing heuristics.

1. Random graphs and the multiplicative coalescent

Let us first introduce the **classical random graph model** $\mathbb{G}(n, M)$ ($n, M \in \mathbb{Z}_+$) whose deep study started in [**ER59**]: on the vertex set $V_n = \{1, \ldots, n\}$ consider the probability distribution on all graphs on V_n which is uniform on graphs with M edges. The variation we shall considered is known as the **binomial model** $\mathbb{G}(n, p)$ ($n \in \mathbb{Z}_+$ and $p \in [0, 1]$) which is a probability distribution on V_n which assigns the set consisting of a graph G with edge set $E(G)$ of cardinality $e(G)$ the probability

$$p^{e(G)} (1 - p)^{\binom{n}{2} - e(G)}.$$

This second model is extensively studied in [**Bol01**] and [**JŁR00**], although in both references, a link between the two models is established;

one can generally transfer asymptotic results (as $n \to \infty$) from one to the other.

When studying any one of these models, in particular as $n \to \infty$ (which would correspond to having more and more vertices), taking M or p as functions of n, it has been noticed that certain graph properties have a zero-one character, either the limiting probability for the occurrence of such a property tends to one or to zero. For example, it is known, and we shall see a reason why, that if $p_n = (\log(n) + c + o(1))/n$ then the probability that a $\mathbb{G}(n, p_n)$-distributed random graph is connected converges to $\exp(-\exp(-c))$ (which as c varies, conforms Gumbel's law, see for example [**KN00**]); a consequence is that $f : n \mapsto \log(n)/n$ is a **threshold function** for connectivity, in the sense that if $p_n = o(f(n))$, then under $\mathbb{G}(n, p_n)$, the probability of being connected tends to zero, while it tends to one if $f(n) = o(p_n)$. This is a result obtained in [**ER59**] for the classical random graph model by combinatorial methods, and in [**Bol01**] by means of structure theorems for the binomial model; based on the insights of this last method, one can compute the threshold for k-connectivity. (A graph G is k-connected if it has more than k vertices and by removing less than k vertices from G we obtain a connected graph.)

One can couple $\mathbb{G}(n, p)$-distributed random variables for example by using $\binom{n}{2}$ independent uniform random variables on $(0, 1)$ labelled by the posible edges on vertex set V_n and defining, for a fixed value of p, a random graph on V_n whose edges are those for which the corresponding uniform random variable is smaller than p. In this manner, one obtains, as p varies, an evolving random graph process. Bollobás' method for the determination of the threshold function for connectivity implies that, with probability tending to 1 as $n \to \infty$, the first instant the random graph process becomes connected coincides with the first instant the last isolated vertex disappears, that is, the last instant at which the minimum degree of the random graph process equals zero. We shall also comment on this result by linking it to the convergence in distribution of the random time the graph process becomes connected, conveniently shifted and renormalized to obtain Gumbel's law. Let us note that the threshold for connectivity is less precise than the original formulation of Erdős and Rényi, since it does not contemplate what happens as the random graph becomes connected, it does not account for the window for connectivity in which Gumbel's law appears. Similarly, one can verify, as we shall do, that

$n \mapsto 1/n$ is in a certain sense a threshold function for the emergence of the giant component: before $1/n$, all components are of size $o(n)$, and after $1/n$ there is one component whose size is comparable to n and all others have size $o(n)$. There is also a critical window to this threshold where an interesting limit theorem due to Aldous (cf. [**Ald97**]) holds: the vector that lists the sizes of the connected components of $\mathbb{G}\big(n, 1/n + t/n^{4/3}\big)$ in decreasing order ($t \in \mathbb{R}$), each entry multiplied by $n^{-2/3}$, converges in distribution to the law of the vector that lists the sizes (in decreasing order) of the excursions above its cumulative minimum of the stochastic process constructed from Brownian motion B by $s \mapsto B_s + ts - s^2/2$.

We will be concerned with three results: the threshold function for connectivity and the limit theorem concerning Gumbel's law, the threshold for the emergence of the giant component, and the limit theorem in the critical window. They will all be obtained by analyzing a stochastic process called the multiplicative coalescent. The **multiplicative coalescent** is a continuous time Markov chain on the set $S^{\downarrow}_f$ consisting of sequences $x = (x_1, x_2, \ldots)$ which are decreasing and equal to zero from a given index onwards. To describe its jump rates, consider for each $x \in S^{\downarrow}_f$ the element $x^{i\otimes j} \in S^{\downarrow}_f$ obtained by removing the elements indexed i and j from the sequence, adding $x_i + x_j$ to it, and reordering it; we say that $x^{i\otimes j}$ is obtained from x by coalescing coordinates i and j. The jump rate from x to x' is $x_i x_j$ if $x' = x^{i\otimes j}$ and zero otherwise. In [**Ald97**], Aldous gives the following graphical construction of the multiplicative coalescent: given $x \in S^{\downarrow}_f$ which has zeros after coordinate n, let $K_n = (V_n, E_n)$ be the complete graph on V_n and construct a random graph G_t as follows: the vertex set $V(G_t)$ is V_n while the edges $E(G_t)$ of G_t are the set of edges e of K_n for which $\xi_e \leq t$, where $(\xi_e)_{e\in E_n}$ are independent random variables and ξ_e has an exponential distribution with parameter $x_i x_j$ if e is the edge of K_n between i and j. As in the random graph process described above, one can couple all the random graphs by using the same exponential variables to construct them; in this way an increasing sequence of random graphs is obtained and as we vary the parameter t, the vector of the sizes (in decreasing order) of the connected components of G_t, say C_t, is an instance of the multiplicative coalescent which starts at x. The link between the graphical construction of the multiplicative coalescent and the $\mathbb{G}(n, p)$ model is as follows: if $x \in S^{\downarrow}_f$ has the first n coordinates equal to one and all others equal to zero, then G_t is an instance of the $\mathbb{G}(n, 1 - \exp(-t))$.

2. Connectedness and Gumbel's law

In terms of the graphical construction of the multiplicative coalescent, the random (stopping) time at which G_t first becomes connected coincides with the first instant the multiplicative coalescent is absorbed. However, the latter random variable is still far from simple: even though the waiting times between jumps of this Markov process are conditionally exponentially distributed, the parameters are dependent on its successive states. There is however, another construction of the multiplicative coalescent proposed by I. Armendáriz in [**Arm05**] which allows for a simple calculation of its absorption time. It runs as follows:

> Given $x \in S_f^{\downarrow}$ with zeros after coordinate n, consider n independent exponentials $\xi_1, \ldots, \xi_n$ of respective parameters $x_1, \ldots, x_n$. Let σ be the permutation that orders the exponentials increasingly and form n lines the i-th one of which passes through $(0, \xi_{\sigma_i})$ and has slope $-\sum_{j<i} x_{\sigma_j}$. If the first intersection between these lines is that of lines i and $i+1$ at time T_1, coalesce σ_i and σ_{i+1} to obtain a new vector which will represent the state at which the coalescent jumps to at time T_1. Delete line numbered $i+1$ and consider the first intersection of the remaining lines, say at time $T_2 > T_1$. Coalesce as before and continue until you have only the bottom line left, which will mark the moment at which the coalescent reaches an absorbing state.

These lines, and how they intersect, are depicted in Figure 1, together with a close-up on the most active part in terms of coalescence. Notice how the topmost line does not intersect any lines before the bottom one. This is interpreted as the fact that the first instant the graphical construction of the multiplicative coalescent becomes connected coincides with the first instant the coalescent is absorbed. It is not an artifact of the simulation, but quite common as we shall see. First of all, let us note that if $x \in$

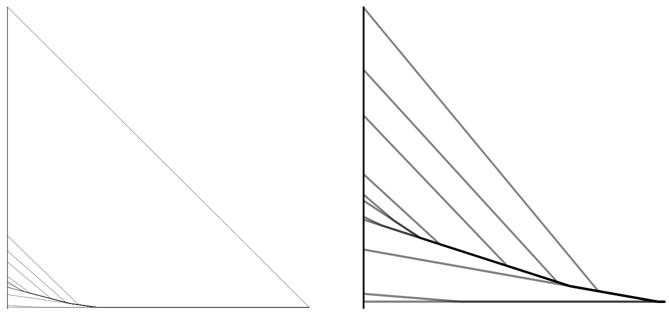


FIGURE 1. Armendáriz's representation of the multiplicative coalescent and a close-up.

$S_f^{\downarrow}$ and $\xi_1, \xi_2, \ldots$ are independent exponentials of respective parameters $x_1, x_2, \ldots$, then:

- If σ is the permutation that ranks $\xi_1, \xi_2, \ldots$ in increasing order, then σ is a **size-biased permutation** of x, meaning that if $x_1, \ldots, x_n \neq 0$ and $x_{n+1} = 0$ then

$$\mathbb{P}(\sigma_1 = i_1, \ldots, \sigma_n = i_n) = \frac{x_1}{\sum_{j\geq 1} x_{i_j}} \frac{x_2}{\sum_{j\geq 2} x_{i_j}} \cdots \frac{x_{n-1}}{\sum_{j\geq n-1} x_{i_j}}.$$

- Conditionally on σ, the random variables

$$\xi_{\sigma_1}, \qquad \xi_{\sigma_2} - \xi_{\sigma_1}, \qquad \ldots, \qquad \xi_{\sigma_n} - \xi_{\sigma_{n-1}}$$

  are independent, and they have exponential laws with respective parameters

$$x_{\sigma_1} + \cdots + x_{\sigma_n}, \qquad x_{\sigma_2} + \cdots + x_{\sigma_n}, \qquad \ldots, \qquad x_{\sigma_n}.$$

- In particular, if $x_1 = \cdots = x_n = \lambda$ then σ is uniform on the n-th symmetric group and so σ and $\xi^n \circ \sigma$ are independent. Also, $\xi_{\sigma_1}, \xi_{\sigma_2} - \xi_{\sigma_1}, \ldots, \xi_{\sigma_n} - \xi_{\sigma_{n-1}}$ are independent with exponential distributions of respective parameters $n\lambda$, $(n-1)\,\lambda$, …, λ.

When $x_1 = \cdots = x_n$, these results were already known to Gumbel, as can be seen in [**Gum58**, 2.1.7, p. 55]; the general case is simple to obtain.

Now, let us see that if $x_i = 1$ for $i = 1, \ldots, n$ and $x_{n+1} = 0$ then, with probability tending to 1, the topmost line intersects the bottom one before intersecting the $n-1$-th one:

$$\begin{aligned}
&\mathbb{P}(\text{Lines } n \text{ and } n-1 \text{ intersect before lines } n \text{ and } 1 \text{ do}) \\
&= \mathbb{P}\left(\xi_{\sigma_n} - \xi_{\sigma_{n-1}} < \left(\xi_{\sigma_{n-1}} - \xi_{\sigma_1}\right) \frac{1}{n-2}\right) \\
&= \mathbb{E}\left(e^{-\left(\xi_{\sigma_{n-1}} - \xi_{\sigma_1}\right)\frac{1}{n-2}}\right)
\end{aligned}$$

where the last equality stems from the independence of the increments of order statistics of equally distributed and independent exponential, as well as the fact that they are exponentially distributed. However, since

$$\mathbb{E}\left(\left|\left(\xi_{\sigma_{n-1}} - \xi_{\sigma_1}\right) \frac{1}{n-2}\right|\right) = \frac{1}{n-2} \sum_{i=1}^{n-1} \frac{1}{i},$$

we see that $\left(\xi_{\sigma_{n-1}} - \xi_{\sigma_1}\right) \frac{1}{n-2}$ converges to zero in probability, hence in distribution, implying that with probability tending to one, lines n and $n-1$ do not intersect before lines n and 1 do. Since $\xi_{\sigma_n} - \log(n)$ converges in law to a Gumbel random variable, and ξ_{σ_1} converges to zero in probability, then

$$\begin{aligned}
&\mathbb{P}\left(G_{(\log(n)+c+o(1))/n} \text{ is connected}\right) \\
&\sim \mathbb{P}(\xi_{\sigma_n} / (n-1) \leq (\log(n) + c + o(1)) / n) \to e^{-e^{-c}}.
\end{aligned}$$

The reader might wish to consult Chapter 7 of [**Bol01**] for other techniques to obtain this, and further elaborations and references.

3. Excursions, Glivenko-Cantelli, and the emergence of the giant component

Performing additional computations with Armendáriz's model representation of the multiplicative coalescent can become quite involved, since we have to keep track of which lines are intersecting. Let us now comment on a reinterpretation of it which allows the use of standard probabilistic techniques in order to get results. It makes use of the properties of the permutation that ranks independent exponential variables and on the differences between successive order statistics introduced in the last section. The representation runs as follows

Let $x \in S_f^{\downarrow}$ have exactly n positive coordinates, and consider n independent random variables $\xi_1, \ldots, \xi_n$

such that ξ_i has an exponential distribution of parameter x_i. Let σ be the permutation that ranks $\xi_1, \dots, \xi_n$, define $s_0, \dots, s_n$ as the partial sums associated to $x \circ \sigma$, given by

$$s_0 = 0, s_1 = x_{\sigma_1}, \dots, s_n = x_{\sigma_1} + \dots + x_{\sigma_n}.$$

Consider the stochastic process X on $[0, s_n)$ equal to 0 on $[s_0, s_1)$ and equal to $-\xi_{\sigma_i}$ on $[s_i, s_{i+1})$. Consider also $X^t = X + t\,\mathrm{Id}$ and, associated to it, form the vector S_t of sizes of the excursion intervals of X^t above its cumulative minimum $\underline{X}^t$ given by

$$\underline{X}^t_s = \min_{0 \le r \le s} X^t_r.$$

(The excursions intervals are the connected components of $\left\{s \in (0, s_n) : X^t_s - \underline{X}^t_s > 0\right\}$.) Finally, put $C^x = S \circ \sigma^{-1}$. The reader is asked to accept that C^x is a multiplicative coalescent which starts at x.

The preceding construction is essentially Armendáriz's representation of the multiplicative coalescent. Note that, conditionally on σ, X is an inhomogeneous random walk, which is amenable to computations, as seen in the next section. When $x_1 = \dots = x_n$, everything is simple, since there is no size-biased reordering, and we can actually omit the conditioning.

Let us move on to the emergence of the giant component. We will consider $x \in S^{\downarrow}_f$ defined by $x_i = 1$ for $1 \le i \le n$ and $x_{n+1} = 0$. Then X is a stochastic process on $[0, n)$ which is constant on each interval of the form $[k-1, k)$ ($0 \le k < n$, $k \in \mathbb{N}$) jumping at the end a negative quantity which is exponentially distributed with parameter $n - k + 1$. However, we might interpret $(X_{nt})_{t \in [0,1)}$ as follows: it is the negative inverse of the empirical distribution function F_n of the n independent exponentials of parameter 1, say $\xi_1, \dots, \xi_n$, given by

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\xi_i \le x}.$$

Then, if we set for $t \in [0, 1)$

$$Y_t = \inf\left\{s \ge 0 : F_n(s) \ge t\right\},$$

$-Y$ has the same law as $X_{n\cdot}$. By the Glivenko-Cantelli theorem, if F denotes the exponential distribution with parameter 1, then $\|F_n - F\|_\infty \to 0$ almost surely as $n \to \infty$ (when working on the same probability space). This is enough to prove that $-Y$ converges to $G : s \mapsto -\log(1/(1-s))$ uniformly on any compact subset of $[0, 1)$. Actually, we can do better, since $\xi_{(n)} = \max_{1 \le i \le n} \xi_i$ is of order $\log(n)$, meaning that

$$\text{if } \log(n) = o(f(n)) \text{ then } \xi_{(n)}/f(n) \to 0 \text{ in probability,}$$

and $\|F_n - F\|_\infty$ is of order $\sqrt{n}$, then $-Y$ converges to $G \vee -\xi_{(n)}$ uniformly on $[0, 1]$. If we let $Y^t = -Y + t\,\mathrm{Id}$ and define $\underline{Y}^t$ as its cumulative minimum process, and define G^t and $\underline{G}^t$ analogously, then $\left(Y^t, \underline{Y}^t\right)$ converges uniformly on $[0, 1)$ to $\left(G^t \vee -\xi_n, \underline{G}^t \vee -\xi_n\right)$. The excursions intervals of Y^t above $\underline{Y}^t$ are equal to $1/n$ times the state of a multiplicative coalescent started at x seen at time t/n; denote this last random variable by $C^x_{t/n}/n$. In general, uniform convergence would not imply convergence of lengths of excursions, especially when taken above zero. However, we are dealing with excursions above the cumulative minimum process; [**Ald97**] presents a version of why excursion lengths above the cumulative minimum converge, but the hypotheses do not apply here. However, from the simple form of our limit cumulative infimum we could make the following arguments rigorous. We distinguish two cases, $t > 1$ and $t < 1$, since for $t < 1$, G^t is strictly decreasing, while if $t > 1$, it is convex, and there exists $\rho_t \in (0, 1)$ such that it is strictly decreasing on $(\rho_t, 1)$, while it remains above its initial value of zero on $[0, \rho_t]$. If $t < 1$, then $G^t - \underline{G}^t$ is identically equal to zero and so $C^x_{t/n}/n$ converges to zero in distribution. If $t > 1$, then there exists a unique $\rho_t \in (0, 1)$ such that $G^t - \underline{G}^t > 0$ exactly on $(0, \rho_t)$. In this case, $C^x_{t/n}/n$ converges to $(\rho_t, 0, \ldots)$ in distribution. This is one way to obtain a threshold for the emergence of the giant component. Chapter 6 of [**Bol01**] discusses the emergence of the giant component in depth.

4. The critical window for the emergence of the giant component

We will proceed to the study of the critical window for the emergence of the giant component. We will prove the theorem of Aldous stated in the introduction, which for convenience is repeated here: the vector that lists the sizes of the connected components of $\mathbb{G}\left(n, 1/n + t/n^{4/3}\right)$ in decreasing order ($t \in \mathbb{R}$), each entry multiplied by $n^{-2/3}$, converges in distribution to the law of the vector that lists the sizes (in decreasing order) of the excursions above its cumulative minimum of the stochastic process constructed from Brownian motion B by $s \mapsto B_s + ts - s^2/2$.

As in the preceding section, the link between random graphs and excursions of stochastic processes is contained in our reinterpretation of Armendáriz's representation of the multiplicative coalescent. We will start by introducing, for $n \in \mathbb{Z}_+$ and $\lambda > 0$, the element $x^{\lambda,n} \in S_f^{\downarrow}$ which has n non-zero entries equal to λ, and study our representation of the multiplicative coalescent started at $X^{\lambda,n}$, denoted $C^{\lambda,n}$. $C^{\lambda,n}$ was constructed in the following manner: let $\xi_1, \ldots, \xi_n$ be independent and exponentially distributed random variables such that ξ_i has parameter $\lambda(n-i+1)$ and define $X^{t,\lambda,n} = \left(X_s^{t,\lambda,n}\right)_{s\geq 0}$ and $\left(\underline{X}^{t,\lambda,n}\right)$ by

$$X_s^{t,\lambda,n} = st + \sum_{i=1}^{[s]\wedge n} \xi_i \text{ and } \underline{X}_s^{t,\lambda,n} = \min_{r\leq s} X_r^{t,\lambda,n}.$$

Then the sequence of lengths of excursions of $X^{t,\lambda,n} - \underline{X}^{t,\lambda,n}$ in decreasing order, taken as a function of $t \geq 0$, has the same law as $C^{\lambda,n}$. Let $m_s^{t,n,\lambda}$ and $v_s^{n,\lambda}$ be defined as the mean and variance of $X_s^{t,\lambda,n}$, given explicitly by

$$m_s^{t,\lambda,n} = st + \sum_{i=1}^{[s/\lambda]\wedge n} \frac{1}{\lambda(n-i+1)}$$

and

$$v_s^{\lambda,n} = \sum_{i=1}^{[s/\lambda]\wedge n} \frac{1}{\lambda^2 (n-i+1)^2}.$$

Since $X^{t,\lambda,n}$ is an inhomogeneous random walk with drift, then

$$X^{t,\lambda,n} - m^{t,\lambda,n}$$

is a martingale, say $M^{\lambda,n}$, and

$$\left(M^{\lambda,n}\right)^2 - v^{\lambda,n}$$

is a martingale. Let us analyze the variance $v^{\lambda,n}$ as $n \to \infty$. We will do this by focusing on parameters λ (depending on n) such that this variance increases linearly; this would have the double purpose of making increments nearly interchangeable (so that excursion intervals would be comparable in size) and to materialize a Brownian motion as $n \to \infty$. (Results of [**JKŁP93**] imply that the critical window is characterized by the coexistence of complex components, i.e. those having at least one edge more than the quantity of vertices.) For example, if $1/\lambda = o(n)$, then

$$v_s^{\lambda,n} \sim \frac{s}{\lambda^3 n^2}.$$

To obtain a limit as $n \to \infty$ (which implies $\lambda \to 0$), we can take $\lambda = n^{-2/3}$. This checks with dividing component sizes by $n^{2/3}$ as implied by the theorem. A further analysis of the magnitudes of the jumps would allow us to use the martingale central limit theorem to conclude that $M^{n^{2/3},n}$ converges in law to Brownian motion. However, for us, $M^{\lambda,n}$ is not directly related to the multiplicative coalescent, but rather $X^{t,\lambda,n}$. In particular, it is preferable to take away a linearized drift term, than the non-linear one of $X^{t,\lambda,n}$. When $\lambda = n^{-2/3}$, then

$$\mu_s^{t,n^{-2/3},n} \sim n^{1/3}s \text{ and } \mu_s^{t,n^{-2/3},n} - n^{1/3}s \to s^2/2$$

uniformly on compact sets. Hence $X^{t+n^{1/3},n^{-2/3},n}$ converges in law to B^t given by $s \mapsto B_s + ts - s^2/2$. As in the preceding section, using Lemma 7 of [**Ald97**], this implies the convergence of $C_{t+n^{1/3}}^{n^{-2/3},n}$ to the vector of sizes (in decreasing order) of the excursions of B^t. The corresponding parameter for the binomial model is $p_n = 1 - e^{n^{-4/3}(t+n^{1/3})}$ which is asymptotically equivalent to $1/n + t/n^{4/3}$.

Notation guide

This appendix lists a selection by themes of the notation used throughout the work. Only notation that is used in different sections is included. Page references are given when the brief description is deemed insufficient.

Measure Theory and Probability
(All chapters)

Symbol	*Brief description*
$\mathscr{F},\mathscr{G},\mathscr{H}$	Calligraphic letters denote σ-fields.
$\sigma(\cdot)$	The σ-field generated by the argument.
$\mathscr{B}_\cdot$	Denotes the Borel σ-field of the subscript.
$b\mathscr{F}$	Bounded real-valued $\mathscr{F}$-measurable functions.
$\mathscr{F}_+$	Non-negative real-valued $\mathscr{F}$-measurable functions.
$\mathscr{F}\vee\mathscr{G}$	The σ-field $\sigma(\mathscr{F}\cup\mathscr{G})$.
$\mathrm{supp}(\cdot)$	The support of the given measure.
$\mathscr{G}_1\perp_{\mathscr{H}}\mathscr{G}_2$	Conditional independence of $\mathscr{G}_1$ and $\mathscr{G}_2$ given $\mathscr{H}$. p. 19

Canonical Space
(Chapters 1, 2 and 3)

Symbol	*Brief description*
(S,ρ)	Locally compact metric space with a countable base
x,y	Elements of S.
$\mathscr{B}_S$	Borel σ-field of (S,d).
D_∞	Skorohod space of càdlàg trajectories from $[0,\infty)$ to S.
D_t	Skorohod space of càdlàg trajectories from $[0,t]$ to S.

C_∞	Set of continuous functions from $[0,\infty)$ to S.
f, g	Elements of D_∞ or D_t.
$(X_r)_{r\geq 0}$	Canonical process on D_∞.
$(\mathscr{F}_r)_{r\geq 0}$	Canonical filtration: $\mathscr{F}_r = \sigma(X_u : u \leq r)$.
$\mathscr{F}^r$	Equal to $\sigma(X_u : u \geq r)$.
$\mathscr{F}_t^s$	Equal to $\sigma(X_u : u \in [s,t])$.
$X^{s,t}$	The stochastic process given by $(X_{r+s} \wedge t)_{r\geq 0}$.
$(\theta_t)_{t\geq 0}$	Shift operators on D_∞. p. 8

Note: the shift operators will also be used with random times.

$\mathscr{F}^L$	The σ-field $\sigma(X \circ \theta_L)$.
σ_t^s	Shift and stop operators on D_∞. p. 23
σ_t	Simplified notation for σ_t^0. p. 24

Note: the (canonical) notation will also be used on D_t.

S_v	Brownian scaling operators on D_∞. p. 35
S_v^β	Self-similar scaling operators to work with the index β. Reduces to S_v when $\beta = 2$. p. 43
T_y	Hitting time of the set y, p. 26

Markovian Bridges
(Chapter 1)

Symbol	*Brief description*
$(\mathbb{P}_x)_{x\in S}$	Feller-Markov family of probability meausres on D_∞.
$(P_s)_{s\geq 0}$	Semigroup associated to the Feller-Markov family.
μ	A σ-finite measure on S.
p_t	Transition density of P_t with respect to μ.
$\mathscr{P}_t$	The set $\{y \in S : p_t(x,y) > 0\} \cap \operatorname{supp}(\mu)$ for fixed x.
$\mathbb{P}_{x,y}^t$	Bridge law of length t between x and y. (Associated to the Markovian family $(\mathbb{P}_x)_{x\in S}$.)
$M_{x,y}^s$	Density of $\mathbb{P}_{x,y}^t\|_{\mathscr{F}_s}$ with respect to $\mathbb{P}_x\|_{\mathscr{F}_s}$.

Examples and Applications
(Chapter 1, Sections 3 and 4)

Symbol	*Brief description*

$(\mathbb{P}_x)_{x\in\mathbb{R}}$	Markovian family of Brownian motion. p. 25
$(P_t)_{t\geq 0}$	Semigroup associated to $\mathbb{P}_x$.
$p_t(x,\cdot)$	Density of X_t under $\mathbb{P}_x$.
$\left(\tilde{\mathbb{P}}^{\beta}_x\right)_{x\geq 0}$	Markovian family of a stable subordinator of index β. p. 28
f^{β}_t	Density of a stable subordinator of index β at time t.
$p^{\beta}_t(x,\cdot)$	Density of X_t under $\tilde{\mathbb{P}}^{\beta}_x$.
$(\mathbb{P}_{\vec{x}})_{\vec{x}\in\mathbb{R}^{\delta}}$	Markovian family of Brownian motion on $\mathbb{R}^{\delta}$. p. 28
$p_t(\vec{x},\cdot)$	Density of X_t under $\mathbb{P}_{\vec{x}}$.
Ψ	Characteristic exponent of a Lévy process. p. 27
Φ	Laplace exponent of a subordinator. p. 27
$\left(\mathbb{P}^{\Psi}_x\right)_{x\in\mathbb{R}}$	Markovian family of a Lévy process with characteristic exponent Ψ.
f^{Φ}_t	Density at time t of a Lévy process with characteristic exponent Ψ.
$\left(\mathbb{Q}^{\delta}_x\right)_{x\geq 0}$	Markovian family of a squared Bessel process of dimension δ. p. 29
$\left(\mathbb{P}^{\delta}_x\right)_{x\geq 0}$	Markovian family of a Bessel process of dimension δ. p. 30
$p^{\delta}_t(x,\cdot)$	Density of X_t under $\mathbb{P}^{\delta}_x$.
$\left(\mathbb{P}^{\dagger}_x\right)_{x\in(0,\infty)\cup\{\Delta\}}$	Markovian family of Brownian motion killed upon reaching zero. p.34
$p^{\dagger}(x,\cdot)$	Density of X_t under $\mathbb{P}^{\dagger}_x$.

Excursions
(Chapters 2 and 3)

Symbol	*Brief description*
$(E,\mathscr{E})$	Excursion space. p. 53
L	Depending on context:local time process p. 53 or length functional on E. p. 53
τ	Inverse local time process. p. 53
Ξ,Ξ^{+},Ξ^{-}	Point processes of excursions, p. 53

	positive excursions and negative excursions p.83.
n, n_+	Itô's measure and its restriction to positive excursions. p. 54

Self-similar fragmentations
(Chapter 3)

Symbol	*Brief description*
α	Self-similarity index.
β	Equal to $1 - 1/\alpha$
$\mathscr{V}$	Family of open sets of $(0, 1)$.
$d_{\mathscr{V}}$	Metric in $\mathscr{V}$. p. 72
$\mathscr{M}$	Multiplicative system generating $\mathscr{B}_{\mathscr{V}}$. p. 74
$\psi_t : C_1 \to \mathscr{V}$	Functional defining the height fragmentation associated to a continuous coding function. p. 78
F^α	Height fragmentation of excursions of spectrally positive α-stable processes. p. 68
χ	Tagged fragment of F^α. p. 69
M	Extinction time of F^α. p. 70
$\hat{F}^2$	F^2 time reversed from M. p. 70
$\mathbf{e}$	A normalized Brownian excursion.
π	Law of $\mathbf{e}$, shorthand for $\mathbb{P}^{1,3}_{0,0}$.
π^v	Law of a Brownian excursion of length v, shorthand for $\mathbb{P}^{v,3}_{0,0}$.
U	Uniform $(0, 1)$-valued random variable independent of $\mathbf{e}$.
$(p^\alpha_t)_{t \geq 0}$	Semigroup of an α-self similar fragmentation constructed from $(p_t)_{t \geq 0}$. p. 72

Multiplicative coalescence
(Chapter 4)

Symbol	*Brief description*
$\mathbb{G}(n, M)$	Erdős-Rényi classical random graph model. p. 107
$\mathbb{G}(n, p)$	Binomial random graph model. p. 107

V_n	The set $\{1, \ldots, n\}$.
K_n	Complete graph on V_n.
$S_f^{\downarrow}$	Set of decreasing sequences $x_1, x_2, \ldots$ with a finite quantity of non-zero entries.
$x^{i \otimes j}$	If $x \in S_f^{\downarrow}$: the sequence obtained by coagulation of x_i and x_j. $\mathbb{P}$. 109
C^x	Multiplicative coalescent which starts at $x \in S_f^{\downarrow}$. p. 113
$(G_t)_{t \geq 0}$	Graphical construction of the multiplicative coalescent. p. 109
X^t	The process given by $s \mapsto X_s + ts$.
$\underline{X}^t$	Cumulative minimum process of X^t. p. 113
B^t	Brownian motion with time-dependent drift. p. 116

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