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TOPOLOGY OF VARIETIES WITH NON-ISOLATED SINGULARITIES

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Topology of Varieties with Non-Isolated Singularities

Aurélio Menegon Neto

À minha amada mãe,
Léia.

"Viver e não ter a vergonha de ser feliz,
Cantar, a beleza de ser um eterno aprendiz
Eu sei que a vida devia ser bem melhor e será,
Mas isso não impede que eu repita:
É bonita, é bonita e é bonita!
E a vida? E a vida o que é, diga lá, meu irmão?
Ela é a batida de um coração?
Ela é uma doce ilusão?
Mas e a vida? Ela é maravilha ou é sofrimento?
Ela é alegria ou lamento?
O que é? O que é, meu irmão?
Há quem fale que a vida da gente é um nada no mundo,
É uma gota, é um tempo. Que nem dá um segundo,
Há quem fale que é um divino mistério profundo,
É o sopro do criador numa atitude repleta de amor.
Você diz que é luta e prazer,
Ele diz que a vida é viver,
Ela diz que melhor é morrer
Pois amada não é, e o verbo é sofrer.
Eu só sei que confio na moça
E na moça eu ponho a força da fé,
Somos nós que fazemos a vida
Como der, ou puder, ou quiser,
Sempre desejada por mais que esteja errada,
Ninguém quer a morte, só saúde e sorte,
E a pergunta roda, e a cabeça agita.
Fico com a pureza das respostas das crianças:
É a vida! É bonita e é bonita!"

Gonzaguinha

Ao meu saudoso pai,
in memorian.

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Abstract

In this thesis, we study the topology of real and complex varieties with non-isolated singularity, by means of the study of the topology of its Milnor fibre and its degeneration to the singular variety. When the variety is a *line singularity*, we describe such degeneration in terms of the *Lê Polyhedra*. When the singularity is given by a real analytic map-germ of the type $f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, with some hypothesis, we describe the degeneration of the boundary of its Milnor fibre to the *link* of the singularity. Moreover, when n is 3, we prove that this boundary, which is a 3-manifold, is actually a Waldhausen manifold.

Resumen

En la presente tesis, estudiamos la topología de variedades reales y complejas con singularidad no-aislada, por medio del estudio de la topología de su fibra de Milnor y su degeneración a la variedad singular. Cuando la variedad es una *line singularity*, describimos tal degeneración en terminos de los *Poliedros de Lê*. Cuando la singularidad es dada por un germen real analítico del tipo $f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, con ciertas hipótesis, describimos la degeneración de la frontera de su fibra de Milnor al *link* de la singularidad. Además, cuando n es 3, probamos que esta frontera, que es una 3-variedad, es en realidad una variedad de Waldhausen.

Resumo

Na presente tese, estudamos a topologia de variedades reais e complexas com singularidade não-isolada, por meio do estudo da topologia de sua fibra de Milnor e sua degeneração à variedade singular. Quando a variedade é uma *line singularity*, descrevemos tal degeneração em termos dos *Poliedros de Lê*. Quando a singularidade é dada por um germe real analítico do tipo $f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, com certas hipóteses, descrevemos a degeneração da fronteira de sua fibra de Milnor ao *link* da singularidade. Além disso, quando n é 3, provamos que essa fronteira, que é uma 3-variedade, em realidade é uma variedade de Waldhausen.

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Introduction

The topology of complex varieties with an isolated singularity has been widely studied by many important mathematicians, inspired mainly by the work [28] of Milnor, in 1968. Since then much progress has been obtained in this field. However, the topology of real and complex varieties with non-isolated singularity is still not well understood, and recently it has been object of study of important mathematicians like Fernandez de Bobadilla, Kato, Lê, Mastumoto, Massey, Nemethi, Pichon, Seade, Siersma, Teissier and many others.

A classical point of view in Singularity Theory is the study of the topology of the Milnor fibre of a hypersurface. The idea of studying the critical level of a complex function by studying the non-critical level was used by many authors like Milnor, Hirzebruch, Brieskorn, Pham and others. This lead to the classical Fibration Theorem of Milnor and the study of the degeneration of the Milnor fibre to the singular one.

For a complex hypersurface in $\mathbb{C}^n$ with an isolated singularity, Lê Dung Trang described in [22] such degeneration through the *Lê Polyhedron*. In this thesis, we extend these concepts to the non-isolated singularities, describing the degeneration of the Milnor fibre of a *line singularity* to the singular fibre. We also describe the degeneration of the boundary of the Milnor fibre to the link of the singularity, for certain classes of real and complex non-isolated singularities.

We are going to consider mainly two kinds of map-germs; the first one is a holomorphic germ of function defined on the n-dimensional complex affine space, that is,

$$f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$$

and the second one is a real analytic map-germ given by the product of a a complex function f as above and the complex conjugated of another complex function g, that is,

$$f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$$

supposing that such real analytic map-germ has an isolated critical value at $0 \in \mathbb{C}$. A. Pichon and J. Seade proved in [36] that such map-germ $f\bar{g}$ has the Thom a_f-property, and therefore it has a Milnor fibration

$$f : N(\mathbf{B}_\epsilon, \mathbf{D}^*_\eta) \to \mathbf{D}^*_\eta,$$

where $\mathbf{B}_\epsilon$ is a sufficiently small closed ball around 0 in $\mathbb{C}^n$, $\mathbf{D}_\eta$ is a disk in $\mathbb{C}$ centered at 0 and with a sufficiently small radius η with respect to ϵ, and $\mathbf{D}^*_\eta$ is the punctured disk $\mathbf{D}_\eta \backslash \{0\}$.

In the first chapter, we fix our notation and recall some basic results which are used through the thesis.

In the second chapter, we study the degeneration of the Milnor fibre of a holomorphic germ of function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ to the singular one. In the case of an isolated singularity, Lê Dung Trang refined in [22] the idea of the vanishing homology and proved that there exists a polyhedron in the Milnor fibre such that the Milnor fibre deformation retracts to such polyhedron. He also proved that there is a continuous map from the Milnor fibre to the singular one, which restricts to a homeomorphism outside the polyhedron and takes the polyhedron to the singular point.

It is unlikely that there is a natural extension of this result to holomorphic functions with arbitrary critical locus. Yet, we show that there is such theorem for an important class of singularities called *line singularities*, defined by D. Siersma in [40]. These are complex singularity germs with critical locus a complex smooth curve. The theorem we prove is the following:

Theorem 2.2.2 *If $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is a line singularity, then there exist ϵ and η sufficiently small, with $0 < \eta << \epsilon << 1$, such that for any $t \in \boldsymbol{D}^*_\eta$:*

- (i) *There exists a polyhedron P_t, of real dimension 3, in the Milnor fibre F_t such that F_t deformation retracts to P_t;*
- (ii) *If f does not admit a good Milnor radius, there exists a (contractible) polyhedron P_0 of real dimension 3 in the singular fibre F_0 such that F_0 deformation retracts to P_0; and there is a continuous map $\Psi_t : \mathring{F}_t \to \mathring{F}_0$ which sends P_t to P_0 and such that Ψ_t restricts to a homeomorphism from $\mathring{F}_t \backslash P_t$ to $\mathring{F}_0 \backslash P_0$;*
- (iii) *If f admits a good Milnor radius, there exists a collapsing map $\Psi_t : \mathring{F}_t \to \mathring{F}_0$ sending P_t to $\Sigma \cap \boldsymbol{B}_\epsilon$ and such that Ψ_t restricts to a homeomorphism from $\mathring{F}_t \backslash P_t$ to $\mathring{F}_0 \backslash (\Sigma \cap \boldsymbol{B}_\epsilon)$.*

In chapter 3, we study the topology of real analytic map-germs of the type $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. The understanding of the topology of its Milnor fibre and its degeneration to the singular one is essential to the two last chapters. The main new results of chapter 3 are:

Theorem 3.3.5 *Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two holomorphic functions such that the real analytic map-germ $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ has an isolated critical value. Let $\pi : \tilde{M} \to \mathbb{C}^2$ be an embedded resolution of the curve fg at the origin and let $E = \cup_{i=1}^{s} E_i$ be a decomposition of the exceptional divisor of π in irreducible components. Let a_i and b_i denote the multiplicity of E_i in the total transform of f and g, respectively. Set $d_i := |a_i - b_i|$ and let r_i be the number of double points of the total transform of fg in E_i. Then the zeta function of the monodromy of the Milnor fibration of $f\bar{g}$ is given by:*

$$Z(t) = \prod_{i=1}^{s} (1 - t^{d_i})^{r_i - 2}.$$

Theorem 3.4.1 *Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two holomorphic germs of function with no common irreducible components and such that the real analytic germ given by $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ has an isolated critical point at $0 \in \mathbb{C}^2$. Suppose that $g(x, y) = g(y)$. Then there exist ϵ and η sufficiently small, with $0 < \eta << \epsilon << 1$, such that for any $t \in \boldsymbol{D}_\eta^*$ there exists a polyhedron P_t, of real dimension 1, in the Milnor fibre F_t of f such that F_t deformation retracts to P_t. Moreover, there exists a continuous map $\Psi_t : F_t \to F_0$ which sends P_t to $\{0\}$ and such that Ψ_t restricts to a homeomorphism from $F_t \backslash P_t$ to $F_0 \backslash \{0\}$.*

In chapter 4, we study the boundary of the Milnor fibre of complex and real analytic map-germs by means of the so called *vanishing zone*. The concept of vanishing zone is well known for holomorphic germs of functions, and we extend it for real analytic map-germs. The goal is to construct a neighbourhood of the link of the critical locus inside the Milnor sphere, whose properties are given by the following theorem, which combines the results of chapter 4:

Theorem 4.3.5 *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$, with $n \geq m$, be a real analytic map-germ such that $0 \in \mathbb{R}^m$ is an isolated critical value with the Thom a_f-property. Suppose that the singular set Σ of f has at most an isolated singularity. Let ϵ be a Milnor radius for f and set $L(\Sigma) := \Sigma \cap \boldsymbol{B}_\epsilon$ and $L_t := f^{-1}(t) \cap \boldsymbol{S}_\epsilon$, for t sufficiently small. Then:*

(*i*) *There exists a neighbourhood* W *of* $L(\Sigma)$ *in* $\boldsymbol{S}_\epsilon$*, which is a fibre bundle over* $L(\Sigma)$ *with fibre a disk, such that* $L_t\backslash\mathring{W}$ *is homeomorphic to* $L_0\backslash\mathring{W}$*;*

(*ii*) *The intersection* $W_t := L_t \cap W_t$ *is a fibre bundle over* $L(\Sigma)$ *if, and only if, the intersection* $W_0 := L_0 \cap W_t$ *is a fibre bundle over* $L(\Sigma)$*, which happens if, and only if, either* Σ *or* $\Sigma\backslash\{0\}$ *is a stratum of a Whitney stratification of* f*.*

In chapter 5, we study the topology of the 3-manifold given by the boundary of the Milnor fibre of a real analytic map-germ of the type $f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ with an isolated critical value. This study was motivated by the problem of finding new classes of 3-manifolds, besides the links of isolated complex surfaces singularities, that have a rich geometric structure. In this direction, F. Michel and A. Pichon announced in [23] an interesting theorem stating that if f is a holomorphic map-germ $(\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ with a 1-dimensional critical locus, then the boundary of the Milnor fiber is a Waldhausen manifold. The original proof contained a gap, and then in [26] F. Michel, A. Pichon and C. Weber provided a proof valid for some classes of singularities. A. Nemethi and A. Szilard provided in [32] a complete proof of the general case of the theorem. F. Michel and A. Pichon also provided in [25] a complete proof, which is more in the spirit of the original method they proposed. In a joint work with J. Fernandez de Bobadilla [3], we proved the following theorem, which is the main result of chapter 5:

Theorem 5.2.3 *Let* $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ *be two holomorphic functions such that the real analytic germ given by* $f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ *has an isolated critical value at* $0 \in \mathbb{C}$*. Then the boundary of the Milnor fibre of* $f\bar{g}$ *is a Waldhausen manifold.*

Although our proof has some inspiration from the method of Nemethi and Szilard, and has some points in common with that of Michel and Pichon, it provides a shorter proof of the theorem for the holomorphic case which generalizes to non-holomorphic real analytic germs of the type $f\bar{g}$. The content of [3] is explained in chapter 5 bellow.

Finally, in chapter 6 we describe the degeneration of the boundary of the Milnor fibre to the link of a real analytic map-germ of the type $f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, with certain hypothesis. The result is the following theorem:

Theorem 6.0.4 *Let* $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be two holomorphic germs of function such that* $f^{-1}(0)$ *intersects* $g^{-1}(0)$ *transversally at* $0 \in \mathbb{C}^n$*, and such that the real analytic map-germ* $f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *has an isolated critical value. Suppose that either* Σ *or* $\Sigma\backslash\{0\}$

is a stratum of a Whitney stratification of fg, with complex dimension k. Also suppose that $\frac{\partial g}{\partial z_1} = 0$. Let W be a vanishing zone for $f\bar{g}$. Then, for any $t \neq 0$ sufficiently small, there exist:

(i) *a polyhedron P_t in $W_t = L_t \cap W$, of real dimension $n - k$, such that W_t deformation retracts to P_t;*

(ii) *a continuous map $\Psi_t : W_t \to W_0 = L_0 \cap W$ which restricts to a homeomorphism from $W_t \backslash P_t$ to $W_0 \backslash L(\Sigma)$ and sends P_t to $L(\Sigma)$.*

Chapter 1

Preliminaries

In this chapter, we establish our notation and present some concepts, definitions and classical results that will be used in the next chapters. To see more details about the subjects presented here, the reader should review the literature listed in the bibliography.

1.1 Basic notation

- If A and B are two sets, we denote by $A \backslash B$ the complement of B in A;
- If ϵ is a real positive number and x is a point in a metric space X, then $\mathbf{B}_\epsilon(x)$ denotes the closed ball in X centered at x and with radius ϵ;
- When X is the n-dimensional real or complex affine space, $\mathbf{B}_\epsilon$ denotes the ball in X centered at the origin and with radius ϵ;
- $\mathbf{B}^*_\epsilon(x)$ denotes the punctured ball $\mathbf{B}_\epsilon(x)\backslash\{x\}$, and $\mathbf{B}^*_\epsilon := \mathbf{B}_\epsilon\backslash\{0\}$;
- The sphere $\mathbf{S}_\epsilon(x)$ is the boundary of $\mathbf{B}_\epsilon(x)$;
- When $\mathbf{B}_\epsilon(x)$ is a 2-dimensional disk, we denote it by $\mathbf{D}_\epsilon(x)$.

1.2 Geometric Simplicial Complex

A set of points in $\mathbb{R}^n$ is *affinely independent* if it is not contained in a hyperplane. An affinely independent set in $\mathbb{R}^n$ contains at most $n+1$ points. The *convex hull* of a finite set $X \subset \mathbb{R}^n$ is the set of all weighted averages of points in X:

$$conv(X) := \left\{ \sum_{x\in X} \lambda_x x \;\middle|\; \sum_{x\in X} \lambda_x = 1 \text{ and } 0 \le \lambda_x \le 1 \text{ for all } x \in X \right\}.$$

The convex hull of a finite set of points is called a *polytope*. A hyperplane *supports* a polytope if it does not intersect its interior. The intersection of a polytope and any supporting hyperplane is a proper face of the polytope. It is easy to check that $conv(X) \cap h = conv(X \cap h)$ for any point set X and supporting hyperplane h; thus, proper faces of polytopes are also polytopes. In particular, the empty set is a polytope.

A *k-simplex* is the convex hull of a set of $k+1$ affinely independent points, called its *vertices*; a *simplex* is a k-simplex for some integer k. For example, a tetrahedron is a 3-simplex, a triangle is a 2-simplex, a line segment is a 1-simplex, a point is a 0-simplex, and the empty set is the unique (-1)-simplex. A *face* of a simplex is the convex hull of a subset of its vertices; in particular, a *facet* of a k-simplex is the convex hull of all but one of its vertices. Thus, every k-simplex has exactly $k+1$ facets and exactly 2^k faces. Every face of a simplex is a simplex.

A *geometric simplicial complex* is a set Δ of simplices in some Euclidean space $\mathbb{R}^n$, satisfying two conditions:

(1) every face of a simplex in Δ is also in Δ;

(2) the intersection of any two simplices in Δ is a face of both.

The simplices in Δ are called its *cells*. For example, the set of faces of any simplex define a simplicial complex. The underlying space of a simplicial complex Δ, denoted $|\Delta|$, is the union of its simplicies.

Definition 1.2.1 *Let X be a topological space. We say that X is a polyhedron (or that X is triangulable) if there exist a simplicial complex Δ and a homeomorphism $h : |\Delta| \to X$. In this case, Δ is said to be a triangulation of X.*

It is well known that topological manifolds of dimension 2 or 3, differentiable manifolds of any dimension, analytic and semi-analytic varieties are polyhedra.

1.3 The Euler characteristic

Originally, the *Euler characteristic* was defined by Euler for a 2-dimensional polyhedron P as the alternating sum

$$\chi(P) = n_0 - n_1 + n_2,$$

where n_0 is the number of vertices, n_1 is the number of edges and n_2 is the number of faces of P. He noticed that if P is homeomorphic to the sphere $\mathbf{S}^2$, then $\chi(P)$ is always 2.

Later, Poincaré generalized this result for any finite polyhedron P of any dimension, defining the Euler characteristic of P as the alternating sum

$$\chi(P) = \sum_i (-1)^i n_i,$$

where n_i is the number of i-dimensional simplices of P. We have the following important result due to Poincaré:

Theorem 1.3.1 *Let (Δ, h) and (Δ', h') be two triangulations of the same topological space X. Then $\chi(\Delta) = \chi(\Delta')$.*

Then we can define the Euler-Poincaré characteristic, or simply Euler characteristic to simplify notation, of a differentiable manifold M as the Euler characteristic of a triangulation K of M; it is a topological invariant. For example, the Euler characteristic of the sphere $\mathbf{S}^n$ is $\chi(\mathbf{S}^n) = 1 + (-1)^n$, and the Euler characteristic of the real projective space $\mathbb{P}^n$ is 0, if n is odd, and 1, if n even.

Equivalently, the Euler characteristic of a topological space X can be defined purely in terms of homology (and hence depends only on the homotopy type of X) by the formula

$$\chi(X) = \sum_n (-1)^n rank H_n(X).$$

Next we list some properties of the Euler characteristic, where X and Y are any two topological spaces:

(i) The Euler characteristic of their disjoint union is the sum of their Euler characteristics, since homology is additive under disjoint union. That is,

$$\chi(X \sqcup Y) = \chi(X) + \chi(Y);$$

(ii) If X and Y are sub-complexes of the simplicial complex $X \cup Y$, then

$$\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y);$$

(iii) For a k-sheeted covering space $X \to Y$ one has

$$\chi(X) = k \cdot \chi(Y).$$

1.4 Compact surfaces

A closed surface is a compact 2-dimensional real differentiable manifold without boundary. We should also admit compact surfaces with boundary, that is, compact 2-dimensional real manifolds with boundary.

The genus g of a connected, orientable, closed surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected. Equivalently, a connected, orientable, closed surface of genus g is the connected sum of g tori.

It can also be defined for compact orientable surfaces with boundary as the genus of the corresponding closed surface, that is, the closed surface obtained by gluing the smoothing of the cone over each boundary component (which are closed curves).

It is well known that closed orientable surfaces are determined, up to homeomorphism, by its genus, and similarly that connected compact orientable surfaces with boundary are classified by the number of boundary components and the class of the corresponding closed surface.

Now, since the sphere and the torus have Euler characteristics 2 and 0, respectively, it follows that the Euler characteristic of the connected sum of g tori is $2-2g$. Then if $\mathcal{S}$ is a connected, orientable, closed surface of genus g, we get the relation

$$\chi(\mathcal{S}) = 2 - 2g.$$

Analogous, if $\mathcal{S}$ is a connected, orientable, compact surface of genus g with b boundary components, the equation reads

$$\chi(\mathcal{S}) = 2 - 2g - b.$$

1.5 Real and complex germs of singularity

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. An *analytic variety* V over $\mathbb{K}$ is the set of points that satisfy a finite number of analytic equations defined in some open set $U \subset \mathbb{K}^n$. That is, one has analytic equations $f_1, \ldots, f_r : U \to \mathbb{K}$ and $V := \cap_{i=1}^{r} f_i^{-1}(0)$.

If $r = 1$ we say that V is a *hypersurface* in $U \subset \mathbb{K}^n$. A hypersurface V in $\mathbb{C}^n$ has codimension 1, that is, $dim(V) = n-1$. If $\mathbb{K} = \mathbb{R}$, then one has $dim(V) \leq n-1$.

Now consider an analytic application $f = (f_1, \ldots, f_r) : U \subset \mathbb{K}^n \to \mathbb{K}^r$, the variety $V = f^{-1}(0)$ and its Jacobian matrix $Df(x)$. Let $\rho(f)$ be the maximum rank of $Df(x)$ for $x \in \mathbb{K}^n$.

Definition 1.5.1 *A point $x_0 \in V$ is said to be regular if* $\operatorname{rank} Df(x_0) = \rho(f)$*. Otherwise, x_0 is said to be a singular point of V.*

Definition 1.5.2 *An analytic variety V is said to be singular if it has a singular point; otherwise V is said to be smooth.*

If V is a hypersurface, the definition of singular point coincides with the definition of critical point, that is, the singular points of V are the points in V where the gradient vector $\nabla f(x_0)$ of $f : U \subset \mathbb{K}^n \to \mathbb{K}$ is zero.

Definition 1.5.3 *An analytic variety V is said to be irreducible if for any analytic varieties V_1, V_2 such that $V = V_1 \cup V_2$, one has either $V = V_1$ or $V = V_2$. Otherwise, V is said to be reducible.*

Lemma 1.5.4 *An analytic variety V can always be written as an union*

$$V = V_1 \cup \cdots \cup V_k,$$

where each V_i is irreducible and is not a subset of V_j, for $i \neq j$. Moreover, $V_1, \cdots, V_k$ are uniquely determined by V.

The proof of this lemma can be found in [39].

Definition 1.5.5 *Consider the set of pairs (U_α, V_α), where U_α is a neighbourhood of the origin in $\mathbb{K}^n$ and V_α is a subset of U_α that contains 0. Two pairs (U_1, V_1) and (U_2, V_2) are said to be equivalent if there exist a neighbourhood $W \subset U_1 \cap U_2$ containing the origin such that $V_1 \cap W = V_2 \cap W$. An equivalence class of such pairs is said to be a germ at the origin of $\mathbb{K}^n$.*

Definition 1.5.6 *A germ of analytic variety at the origin, denoted by $(V, 0)$, is the equivalence class of an analytic set $V \subset \mathbb{K}^n$.*

Definition 1.5.7 *The germ $(V, 0)$ is said to be singular if the origin is a singular point of some (and therefore of every) analytic variety V containing $(V, 0)$. Otherwise, $(V, 0)$ is said to be regular.*

1.6 Whitney stratification

The idea of Whitney stratifications is to decompose a singular analytic subset of a smooth manifold M into smooth submanifolds that glue together in a "good" way, satisfying the so-called *Whitney conditions*.

Definition 1.6.1 *Let X be a subset of a smooth manifold M. A stratification of X is a partition of X in smooth submanifolds S_α of M, called strata, in such a way that for each point $x \in X$ there exists a neighbourhood U of x in M that intersects at most a finite number of strata.*

Definition 1.6.2 *The stratification $\{S_\alpha\}$ is said to have the boundary condition if given any two strata S_α and S_β with $S_\alpha \cap \overline{S}_\beta \neq \emptyset$, one has $S_\alpha \subseteq \overline{S_\beta}$.*

Definition 1.6.3 *We say that a stratification $\{S_\alpha\}$ of a subset X in $\mathbb{K}^n$ with the boundary condition is a Whitney stratification if it satisfies the Whitney conditions, that is, if for any pair (S_α, S_β) with $S_\beta \subset \overline{S_\alpha}$ and for any $y \in S_\beta$ one has:*

a) Given any sequence of points (x_i) in S_α converging to y such that the limit

$$\lim_{i\to\infty} T_{x_i}(S_\alpha) = T$$

exists in the corresponding grassmannian, then

$$T_y(S_\beta) \subseteq T.$$

b) Given any sequence of points (y_i) in S_β converging to y such that the limit of the secants $\overline{x_i y_i}$ exists, with

$$\lambda := \lim_{i\to\infty} \overline{x_i y_i},$$

then

$$\lambda \subset T.$$

The importance of Whitney stratifications lies in the following Whitney's 1965 theorem, proved in [44]:

Theorem 1.6.4 *Any complex or real analytic variety V admits a Whitney stratification.*

Later, Verdier proved in [43] the following:

Theorem 1.6.5 (Bertini-Sard) *If $\{S_\alpha\}$ is a Whitney stratification of $X \subset M$ and let x be a point in X. There exists a real number $\epsilon := \epsilon(x) > 0$ sufficiently small such that, for any $0 < \epsilon' \leq \epsilon$, the sphere $\boldsymbol{S}_{\epsilon'}$ is transversal to any stratum S_α.*

The following theorem is an easy consequence of the Bertini-Sard theorem (see [28] and [7], for example).

Theorem 1.6.6 (Local conical structure) *Let V be an analytic variety in $\mathbb{K}^n$ and let $x \in V$. There exists a real number $\epsilon := \epsilon(x) > 0$ sufficiently small such that the pair $(\boldsymbol{B}_\epsilon(x), \boldsymbol{B}_\epsilon(x) \cap V)$ is homeomorphic to the pair $\big(cone[\boldsymbol{S}_\epsilon(x)], cone[\boldsymbol{S}_\epsilon(x) \cap V]\big)$.*

Definition 1.6.7 *If x is taken to be the origin and if ϵ is as in the theorem above, then any ball $\boldsymbol{B}_{\epsilon'}$ of radius $\epsilon' \leq \epsilon$ centered in $0 \in \mathbb{K}^n$ is said to be a Milnor ball for V.*

Definition 1.6.8 *The intersection $V \cap \boldsymbol{S}_\epsilon$ is called the link of V.*

Remark 1.6.9 *The diffeomorphism type of the link is independent of the choice of the sphere $\boldsymbol{S}_\epsilon$, for ϵ sufficiently small.*

Now consider $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ a germ of analytic function with $0 \in \mathbb{K}$ an isolated critical value.

Definition 1.6.10 *We say that f has the Thom a_f-property in $0 \in \mathbb{K}$ if there exist a Whitney stratification $\{S_\alpha\}$ of $\mathbb{K}^n$ and a neighbourhood U of 0 in $\mathbb{K}^n$ such that $f^{-1}(0) \cap U$ is an union of strata satisfying the following condition:*

If $(p_n) \in U \backslash f^{-1}(0)$ is a sequence of points converging to $p \in f^{-1}(0)$ such that the tangent spaces $T_{p_n} f^{-1}(f(p_n))$ converge to some limit T, then one has $T \supseteq T_p(S_\alpha(p))$.

In [13] Hironaka proved the following theorem:

Theorem 1.6.11 *Any holomorphic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ has the Thom a_f-property.*

1.7 Milnor fibration theorems

In 1947, the french mathematician Charles Ehresmann proved in [10] the famous Ehresmann's fibration lemma, which turned to be very useful in differential topology:

Lemma 1.7.1 (Ehresmann fibration lemma) *Let M and N be differentiable manifolds of dimensions $n + k$ and k, respectively. Assume that M has no boundary and let*

$$f : M \to N$$

be a proper differentiable map which is a submersion everywhere, i.e., the Jacobian matrix $Df(x)$ has rank k for each $x \in M$. Then f is the projection map of a locally trivial fibre bundle.

There exists a version of the Ehresmann fibration lemma for a map defined on a differentiable manifold M with non-empty boundary ∂M, as above:

Lemma 1.7.2 *Let M and N be differentiable manifolds of dimensions $n+k$ and k, respectively. Assume that M has non-empty boundary ∂M, and let $f : M \to N$ be a proper differentiable map such that*

$$f : M \to N$$

and the restriction

$$f_{|} : \partial M \to N$$

are submersions. Then f is the projection map of a locally trivial fibre bundle.

The following lemma is going to be useful later, and it can be found in [29], for instance.

Lemma 1.7.3 *Let $f : M \to N$ be a differentiable map and let $x \in M$ be a regular point of f, and set $y =: f(x)$. Then $ker D(f)_x = T_x f^{-1}(y)$.*

Then one has the following corollary:

Corollary 1.7.4 *If V is a submanifold of a manifold M and if $f : M \to N$ is a differentiable map, then the critical points of $f_{|V}$ are the critical points of f which are in V and the regular points x of f such that $T_x V \subseteq T_x f^{-1}(y)$, where $y := f(x)$.*

1.7.1 Complex Milnor fibration theorems

In [20], Lê Dung Trang proved that the previous lemma together with lemma 1.6.11 above give the following classical theorem:

Theorem 1.7.5 (Milnor fibration theorem on the tube) *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function. Then there exist positive real numbers $0 < \eta << \epsilon$ sufficiently small such that the restrictions*

$$f_{|} : f^{-1}(\boldsymbol{D}^*_\eta) \cap \boldsymbol{B}_\epsilon \to \boldsymbol{D}^*_\eta$$

is the projection of a locally trivial fibre bundle.

Remark 1.7.6

(*i*) Note that f holomorphic implies that $0 \in \mathbb{C}$ is either a regular value or an isolated critical value;

(*ii*) ϵ is taken sufficiently small such that $\mathbf{B}_\epsilon$ is a Milnor ball for f and η is taken sufficiently small such that $f^{-1}(t)$ intersects $\mathbf{S}_\epsilon$ transversally, for any $t \in \mathbf{D}_\eta$ (theorem 1.6.11 guarantees that such number δ exists);

(*iii*) The topological manifold $f^{-1}(\mathbf{D}_\eta) \cap \mathbf{B}_\epsilon$ is usually called the *Milnor tube* of f and it is denoted by $N(\epsilon, \eta)$ or $N(\mathbf{B}_\epsilon, \mathbf{D}_\eta)$;

In his classical book [28], J. Milnor proved the following theorem:

Theorem 1.7.7 (Milnor fibration theorem on the sphere) *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function and let $\epsilon > 0$ be a sufficiently small real number such that $\boldsymbol{B}_\epsilon$ is a Milnor ball for f. Then the map $\phi : \boldsymbol{S}_\epsilon - \big(f^{-1}(0) \cap \boldsymbol{S}_\epsilon\big) \to \boldsymbol{S}^1$ given by*

$$\phi(z) = \frac{f(z)}{|f(z)|}$$

is the projection map of a locally trivial fibre bundle.

Remark 1.7.8 *The Milnor fibration in the tube (of theorem 1.7.5) is equivalent to the Milnor fibration on the sphere (of theorem 1.7.7), in the sense that their fibres are diffeomorphic.*

Definition 1.7.9

(*i*) *$F_t := f^{-1}(t) \cap \boldsymbol{B}_\epsilon$ is the Milnor fibre of f, for any $t \in \boldsymbol{D}^*_\eta$;*

(*ii*) *The set $F_0 := f^{-1}(0) \cap \boldsymbol{B}_\epsilon$ is the singular fibre of f;*

(*iii*) *The set $L_t := \partial F_t = f^{-1}(t) \cap \boldsymbol{S}_\epsilon$ is the boundary of the Milnor fibre of f;*

(*iv*) *The set $L_0 := f^{-1}(0) \cap \boldsymbol{S}_\epsilon$ is the link of f.*

We saw on theorem 1.6.6 that $V(f) = f^{-1}(0)$ is locally a cone over L_0.

1.7.2 Real Milnor fibration theorems

In [35] A. Pichon and J. Seade observed that Lê's arguments in [21] for holomorphic mappings extend to every real analytic map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n > p$, with an isolated critical value, provided it has the Thom a_f-property and $V := f^{-1}(0)$ has dimension more than 0. Hence one has in that setting a Milnor fibration on the tube, that is, the restriction

$$f_| : f^{-1}(\mathbf{B}^*_\eta) \cap \mathbf{B}_\epsilon \to \mathbf{B}^*_\eta$$

is the projection of a locally trivial fibre bundle, where $\mathbf{B}_\epsilon$ is a Milnor ball for f in $\mathbb{R}^n$ and $\mathbf{B}_\eta$ is a sufficiently small ball in $\mathbb{R}^p$.

If f is a real analytic germ as above, we shall use the same notations of definition 1.7.9.

1.8 The monodromy of a function

Consider M a differentiable manifold and let $p : M \to \mathbf{S}^1$ be the projection of a locally trivial fibre bundle. Given a C^∞ never-zero tangent vector field on $\mathbf{S}^1$, one can lift it to an integrable vector field on M which is transversal to the fibres of p. Using the flow lines of this vector field one can define a "first return map" on the fibres, which is well defined up to isotopy. This diffeomorphism is known as the *monodromy* of the corresponding fibre bundle.

Now consider a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. In the previous section we saw that f defines a locally trivial fibre bundle

$$f_| : f^{-1}(\mathbf{S}^1) \cap \mathbf{B}^{2n}_\epsilon \to \mathbf{S}^1.$$

The monodromy of the fibration $f_|$ is called the *geometric horizontal monodromy* of f. It is a diffeomorphism $m : F_t \to F_t$, where F_t is the Milnor fibre of f.

For each integer $q \geq 0$, it induces an isomorphism $m_q^* : H^q(F_t, \mathbb{C}) \to H^q(F_t, \mathbb{C})$ on the q-cohomology group of F_t with coefficients in $\mathbb{C}$. This isomorphism is called the *q-algebraic horizontal monodromy* of f.

Since $H^q(F_t, \mathbb{C})$ is a b_q-dimensional vector space, where b_q is the Betti number of $H^q(F_t, \mathbb{C})$, it follows that the isomorphism m_q^* defines a characteristic polynomial $\Delta_q(t)$:

$$\Delta_q(t) = det(tI - A_q),$$

where I is the $(b_q \times b_q)$ identity matrix and A_q is the matrix defined by m_q^*.

It is a monic polynomial (its leading coefficient is 1) of degree b_q, and it encodes several important properties of the matrix A_q, most notably its eigenvalues, its determinant and its trace. In fact, the roots of $\Delta_q(t)$ are precisely the eigenvalues of A_q; its constant coefficient is equal to $(-1)^{b_q} det(A_q)$, and the coefficient of $t^{b_q - 1}$ is equal to $-tr(A_q)$, the trace of $-A_q$. For example, the characteristic polynomial of a 2×2 matrix A is therefore given by $t^2 - tr(A)t + det(A)$.

It is well known that $H^q(F_t, \mathbb{C}) = 0$ for $q \geq n$ (see [28]). The *monodromy zeta function* $Z(t)$ of f is the alternating product of all characteristic polynomials $\Delta_q(t)$:

$$Z(t) = -\prod_{q=0}^{n-1} \Delta_q(t)^{(-1)^{q+1}}.$$

When $0 \in \mathbb{C}^n$ is an isolated singularity of f, it is well known that $H^q(F_t, \mathbb{C}) = 0$ for $q \neq 0$ and $q \neq n-1$; and that $\Delta_0(t) = t - 1$. Then in this case one has

$$Z(t) = -\frac{\Delta_{n-1}(t)^{(-1)^n}}{t-1}.$$

In other worlds, if f has an isolated singularity, we can express the characteristic polynomial of the monodromy in degree $n-1$ in terms of its monodromy zeta function by the formula

$$\Delta_{n-1}(t) = \big[(1-t)Z(t)\big]^{(-1)^n}.$$

1.9 Resolution of singularities

The subjects of this section are very classical and can be found in many books like [2], [6] and [17], for example. Here we follow the presentation of the book of J. Seade [38].

1.9.1 Resolution of plane curves

Consider $(\mathcal{C}, 0)$ a complex curve germ in some smooth complex surface germ $(X, 0)$. Take local coordinates so that we can identify $(X, 0)$ with $(\mathbb{C}^2, 0)$. Let us take a small neighbourhood U around 0 and consider the map $\gamma : U\backslash\{0\} \to \mathbb{C}P^1$ which associates to each $y \in U\backslash\{0\}$ the point in $\mathbb{C}P^1$ represented by the line determined by 0 and y.

The graph of γ is an analytic subset of $(U\backslash\{0\}) \times \mathbb{C}P^1$, whose closure

$$\tilde{X} = \overline{graph(\gamma)} \subset (U\backslash\{0\}) \times \mathbb{C}P^1$$

turns out to be a smooth complex surface. Notice that $\tilde{X}$ is obtained by removing 0 from X and replacing it by the limits of lines converging to 0. Thus, we have replaced 0 by a copy of $\mathbb{C}P^1$. There is a projection map $\pi : \tilde{X} \to X$ which is biholomorphic away from $E := \pi^{-1}(0) \simeq \mathbb{C}P^1$. This transformation is called the *blow-up* of X at 0, sometimes also called a *σ-process.*

If 0 is a singular point of $\mathcal{C}$, we call the closure $\overline{\pi^{-1}(\mathcal{C}\backslash\{0\})}$ in $\tilde{X}$ the *strict (or proper) transform* of $\mathcal{C}$ under the blow-up, denoted by $\tilde{\mathcal{C}}$. Notice that $\tilde{\mathcal{C}}$ is obtained by removing 0 from $\mathcal{C}$ and replacing it by the limits of lines which are tangent to $\mathcal{C}\backslash\{0\}$. This curve $\tilde{\mathcal{C}}$ is analytic in $\tilde{X}$ and projects to $\mathcal{C}$ under π; this curve may still be singular, but its singularities are simpler.

We may now repeat the process, choosing a singular point in $\tilde{\mathcal{C}}$, blowing up $\tilde{X}$ at this point to get $\pi_2 : \tilde{X}_2 \to \tilde{X}$ and then consider the proper transform of $\mathcal{C}$ in $\tilde{X}_2$, which is the closure of $(\pi_2 \circ \pi)^{-1}(\mathcal{C}\backslash\{0\})$, and so on.

Theorem 1.9.1 *Let X be a smooth complex surface (that is, a 2-dimensional complex manifold) and $\mathcal{C} \subset X$ an embedded reduced curve. Then there is a smooth complex surface Y and a proper map $\tau : Y \to X$, obtained by a finite sequence of blow-ups, such that the strict transform $\tilde{\mathcal{C}}$ of $\mathcal{C}$ in Y is smooth and the total transform $E = \tau^{-1}(\mathcal{C})$ has* *only ordinary normal double points as singularities, that is, the singular points are locally defined by the equation* $\{xy = 0\}$.

Definition 1.9.2 *The smooth complex surface* Y *is said to be a resolution of the curve* $\mathcal{C}$ *at* 0 *and* τ *is said to be a resolution map. The standard resolution of* $\mathcal{C}$ *is the smallest resolution with normal crossings (in this case of curves, there is a unique "smallest" resolution with normal crossings, namely, the one obtained when we blow-up as few times as possible).*

1.9.2 The topology of the Milnor fibre of a curve singularity

Let $(\mathcal{C}, 0) \subset (\mathbb{C}^2, 0)$ be a plane curve with equation $f(z) = 0$, and let $\mathbf{B}_\epsilon$ be a Milnor ball for f. Following [6], we next describe the topology of the Milnor fibre F_t of $\mathcal{C}$ with the help of its standard resolution $\pi : \tilde{X} \to \mathbb{C}^2$.

Let $E_1, \ldots, E_r$ denote the irreducible components of the exceptional divisor $E = \pi^{-1}(0) \subset \tilde{X}$, which are copies of the complex line $\mathbb{C}P^1$, and let $\tilde{\mathcal{C}}_1, \ldots, \tilde{\mathcal{C}}_k$ denote the irreducible components of the strict transform $\tilde{\mathcal{C}} \subset \tilde{X}$. If D_i is an irreducible component of the total pre-image $(f \circ \pi)^{-1}(0)$ (that is, D_i is either some E_i or some $\tilde{\mathcal{C}}_i$), let m_i be the multiplicity of D_i in $(f \circ \pi)^{-1}(0)$.

Under resolution, the ball $\mathbf{B}_\epsilon$ becomes a neighbourhood $\tilde{\mathbf{B}}$ of the exceptional divisor E in $\tilde{X}$ and the Milnor fibre $F_t = f^{-1}(t) \cap \mathbf{B}_\epsilon$ is homeomorphic to $\tilde{F}_t := (f \circ \pi)^{-1}(t) \cap \tilde{\mathbf{B}}$.

In a neighbourhood of a regular point x_0 of D_i and in suitable coordinates, the composition $(f \circ \pi)$ has the equation

$$(z_1, z_2) \mapsto z_1^{m_i},$$

and the fibre $\tilde{F}_t$ is locally homeomorphic to the set

$$\{(z_1, z_2) \mid z_1^{m_i} = t\}.$$

Hence, around x_0, $\tilde{F}_t$ is locally an m_i-fold covering of the component D_i through x_0 of the total pre-image of 0.

At a crossing point x_1 of two components D_i and D_j of the total pre-image of 0, $f \circ \pi$ looks, in suitable coordinates, like

$$(z_1, z_2) \mapsto z_1^{m_i} z_2^{m_j}.$$

Around x_1 we construct the polydisk

$$\Delta := \{(z_1, z_2) \mid |z_1| \leq 1, |z_2| \leq 1\}.$$

Then $\tilde{F}_t \cap \Delta$ consists of the disjoint union of $gcd(m_i, m_j)$-many cylinders.

Now we want to put together $\tilde{F}_t$ from the pieces just described. Around each of the double points of $(f \circ \pi)^{-1}(0)$, we choose a small polydisk as above. Let K be the union of all these polydisks. Thus if the exceptional curve E_i meets exactly r_i other curves of the total pre-image, then $E_i \backslash K$ is an r_i-tuply perforated 2-sphere.

$\tilde{F}_t$ is made up of three kind of pieces:

- pieces of $\tilde{F}_t \backslash K \cap \tilde{F}_t$ that lie over the components E_i, which are m_i-fold covering of the perforated spheres $E_i \backslash K$, and hence a Riemann surface with holes; we shall call these pieces M_i;
- pieces of $\tilde{F}_t \backslash K \cap \tilde{F}_t$ that lie over the components $\tilde{\mathcal{C}}_i$, which are simple coverings of the perforated disk $\tilde{\mathcal{C}}_i \backslash K$, and hence are cylinders;
- pieces of the intersection $\tilde{F}_t \cap K$, which are unions of cylinders.

The $gcd(m_i, m_j)$ cylinders corresponding to the intersection point of E_i with E_j are attached to the holes of M_i, at one boundary component of the cylinders, and to the holes of M_j at the other. One proceeds analogously for the intersection of E_i with a component $\tilde{\mathcal{C}}_j$.

As a consequence, the Euler characteristic of $\tilde{F}_t$ is given by

$$\chi(\tilde{F}_t) = \sum_{i=1}^{r} \chi(M_i).$$

Since each M_i is an m_i-tuple covering of the r_i-tuply perforated sphere $E_i \backslash K$, it follows that

$$\chi(M_i) = m_i(2 - r_i).$$

Then the Euler characteristic of the Milnor fibre F_t is given by

$$\chi(F_t) = \chi(\tilde{F}_t) = \sum_{i=1}^{r} m_i(2 - r_i).$$

1.9.3 Normal surfaces

We say that a complex analytic variety V is *normal* if every bounded holomorphic function on $V \backslash \Sigma$ extends to a holomorphic function on V, where Σ denotes the singular set of V. If V is normal, then the codimension of Σ is more than one (see [31] for instance).

A complex surface is a 2-dimensional complex analytic variety. When $(V, 0)$ is a complex surface embedded in $\mathbb{C}^3$, we have that $(V, 0)$ is a normal surface singularity if, and only if, it has an isolated singularity at 0.

It is well known that any complex analytic variety can be normalized, in the following sense: given a complex surface $V \subset \mathbb{C}^3$, there exist a normal surface $\tilde{V}$ and a finite (proper and with fibre a finite set of points), surjective, analytic morphism (holomorphism) $n : \tilde{V} \to V$ such that the restriction

$$n_| : \tilde{V} \backslash n^{-1}(\Sigma) \to V \backslash \Sigma$$

is an isomorphism (biholomorphism). The normal surface $\tilde{V}$ is called the *normalization* of V, and it is unique up to analytic isomorphism.

1.9.4 Resolution of surface singularities

Following [2] for instance, consider a germ $(V, 0)$ of a normal complex surface singularity.

Theorem 1.9.3 *Let $(V, 0)$ be a normal complex surface singularity in $\mathbb{C}^3$. Then there exist a non-singular complex surface $\tilde{V}$ and a proper analytic map $\pi : \tilde{V} \to V$ such that:*

(*i*) *$E := \pi^{-1}(0)$ is a (connected, reduced) divisor in $\tilde{V}$, that is, a union of 1-dimensional compact curves in $\tilde{V}$; and*

(*ii*) *the restriction of π to $\pi^{-1}(V \backslash \{0\})$ is a biholomorphic map between $\tilde{V} \backslash E$ and $V \backslash \{0\}$.*

The surface $\tilde{V}$ is called a *resolution* of the singularity of V, and $\pi : \tilde{V} \to V$ is the *resolution map*. The divisor E is called the *exceptional divisor*.

Notice that "the" resolution of $(V, 0)$ is not unique: given a resolution $\tilde{V}$ we can obtain new resolutions by performing blow-ups at points in E. By Theorem 1.9.1 above, given a resolution, we can make blow-ups on it, if necessary, so that the divisor E in Theorem 1.9.3 is good, i.e.:

(*iii*) each irreducible component E_i of E is non-singular; and

(*iv*) E has normal crossings, i.e., E_i intersects E_j, $i \neq j$, in at most one point, where they meet transversally, and no three of them intersect.

Definition 1.9.4 *A resolution $\pi : \tilde{V} \to V$ is good if its exceptional divisor is good, i.e., if it satisfies conditions (iii) and (iv) above.*

Definition 1.9.5 *Given a smooth complex surface $\tilde{V}$ and a complex curve $\mathcal{S}$ in it, the self-intersection of $\mathcal{S}$, usually denoted by $\mathcal{S} \cdot \mathcal{S}$, or simply by $\mathcal{S}^2$, is the Euler class of its normal bundle $\nu(\mathcal{S})$ in $\tilde{V}$ (which coincides with its Chern class) evaluated in the fundamental cycle $[\mathcal{S}]$. Equivalently, $\mathcal{S} \cdot \mathcal{S}$ is the number of zeroes, counted with signs, of a generic section of the normal bundle $\nu(\mathcal{S})$.*

Every time we make a blow-up on a smooth complex 2-manifold, we get a copy of $\mathbb{C}P^1$ with self-intersection -1.

Consider now a divisor $E = \cup_{i=1}^{r} E_i$ in a complex 2-manifold X, whose irreducible components E_i are non-singular, they all meet transversally and no three of them intersect (for example, the exceptional divisor of a good resolution as in theorem 1.9.3).

To such a divisor we can associate an $r \times r$ integral matrix $A = \big((E_{ij})\big)$, called the *intersection matrix* of E, as follows: on the diagonal Δ of A we put the self-intersection numbers E_i^2; and if a curve E_i meets E_j at E_{ij} points, we put this number as the corresponding coefficient of A.

So this is a symmetric matrix, whose coefficients away from the diagonal Δ are non-negative integers and in Δ we have the self-intersection numbers of the E_i, called the *weights* of these curves.

We have the following theorems of Mumford and Grauert (see [2]):

Theorem 1.9.6 *If E is the exceptional divisor of a resolution $\pi : \tilde{V} \to V$, where V is a normal surface, then the intersection matrix A is negative definite (all of its eigenvalues are negative) and the weights of the curves E_i are all negative numbers.*

Conversely:

Theorem 1.9.7 *If the divisor E in X is such that the intersection matrix A is negative definite, then we can blow down E analytically; we get a normal complex surface V, in general with a singularity at the image 0 of E, and the projection $\pi : X \to V$ is a resolution of $(V, 0)$ with exceptional divisor E.*

A divisor E in X as above is usually called an *exceptional divisor*, meaning by this that it can be blown down.

Now we associate a weighted graph $\mathcal{G} = \mathcal{G}(E)$ to a good exceptional divisor E in a complex 2-manifold X as follows: to each irreducible component E_i of E we associate a vertex (i), and if the curves E_i and E_j meet, then we join the vertices (i) and (j) by an edge. Each vertex has two integers attached to it:

- the genus $g_i \geq 0$ of the corresponding Riemann surface E_i;
- the weight $w_i = E_i^2 \in \mathbb{Z}$, which is the self-intersection number of E_i in X.

This weighted graph is called the *dual graph* of the exceptional divisor E, or the *dual graph of the resolution* when E is regarded as the exceptional set of a good resolution of a normal singularity.

1.9.5 Plumbed manifolds

Given the dual graph of a resolution, one can re-build the topology of the resolution, and hence the topology of the link of the singularity, using a tool known as *plumbing*, which we shall briefly describe bellow. Plumbing was first introduced by Milnor for constructing exotic spheres and then used systematically by Hirzebruch and others, to describe the topology of surface singularities.

Let E be a real 2-dimensional oriented vector bundle over a Riemann surface $\mathcal{S}$, and denote by $D(E)$ its unit disk bundle for some metric. The total space of $D(E)$, that we denote by the same symbol, is a 4-dimensional smooth manifold with boundary the unit sphere bundle $S(E)$. Notice that, restricted to a small disk $\mathbf{D}_\epsilon$ in $\mathcal{S}$, the manifold $D(E)$ is a product of the form $\mathbf{D}_2 \times \mathbf{D}_2$, where the first disk is $\mathbf{D}_\epsilon \subset \mathcal{S}$ and the second disk is in the fibres of E.

Now suppose we are given two such bundles E_i, E_j, over Riemann surfaces $\mathcal{S}_i, \mathcal{S}_j$. To perform plumbing on them we consider the total spaces of the corresponding unit disk bundles $D(E_i), D(E_j)$, we choose small disks $\mathbf{D}_{i,\epsilon}, \mathbf{D}_{j,\epsilon}$ in $\mathcal{S}_i, \mathcal{S}_j$, and take the restriction of $D(E_i), D(E_j)$ to these disks. Each of them is of the form $\mathbf{D}_2 \times \mathbf{D}_2$ as above. We now identify each point $(x, y) \in \mathbf{D}_{i,\epsilon} \times \mathbf{D}_2 \subset D(E_i)$ with the corresponding point $(y, x) \in \mathbf{D}_{j,\epsilon} \times \mathbf{D}_2 \subset D(E_j)$, i.e., interchanging base points in one of them with fibre points in the other.

The result is a 4-dimensional, oriented manifold with boundary and with corners, which can be smoothed off in a unique way up to isotopy. We denote this manifold by $P(E_i, E_j)$. One says that $P(E_i, E_j)$ is obtained by plumbing the bundles E_i and E_j over the Riemann surfaces $\mathcal{S}_i$ and $\mathcal{S}_j$.

The boundary $S(E_i, E_j) = \partial P(E_i, E_j)$ of this 4-manifold is obtained by plumbing the corresponding sphere bundles $S_i(E)$ and $S_j(E)$: we remove from $S_i(E)$ the interior of the solid tori $D_{i,\epsilon} \times \mathbf{S}^1$, and similarly we remove from $S_j(E)$ the interior of the solid tori $D_{j,\epsilon} \times \mathbf{S}^1$. Thus we get two 3-manifolds with boundary a torus $\mathbf{S}^1 \times \mathbf{S}^1$ in each; we then identify these boundaries by gluing the meridians in one torus to the parallels in the other. The result is a 3-manifold with corners, which can be smoothed off in a unique way up to isotopy.

The surfaces $\mathcal{S}_i, \mathcal{S}_j$ are naturally embedded in $P(E_i, E_j)$ as the zero-sections of the corresponding bundles, and they meet transversally in one point. Notice that the manifolds one gets in this way are entirely described, up to diffeomorphism, by the genera of the Riemann surfaces $\mathcal{S}_i, \mathcal{S}_j$, and by the Euler classes of the corresponding bundles, since these classes determine the isomorphism class of the bundles.

Definition 1.9.8 *A plumbing graph is a triple* (Σ, w, g) *consisting of a finite graph* Σ *with vertices* $(1), \ldots, (r)$, $r \geq 1$, *with no loops; a vector* w *of weights ,* $w = (w_1, \ldots, w_r)$, $w_i \in \mathbb{Z}$, *and a vector* $g = (g_1, \ldots, g_r)$ *of genera,* $g_i \in \mathbb{N}$.

So the dual graph of a good resolution of a normal singularity is a plumbing graph with negative definite intersection matrix . In this definition, by a loop we mean an arrow that begins and ends at the same vertex, and we do not allow this (geometrically this means a singular curve in the exceptional divisor that has a double crossing). There can be cycles, i.e., a chain of vertices and edges that returns to itself after a certain time.

Now, given a plumbing graph we may perform plumbing according to the graph: for each vertex (i) take a Riemann surface $\mathcal{S}_i$ of genus g_i and an oriented 2-plane bundle E_i over $\mathcal{S}_i$ with Euler class w_i. If there is an edge between the vertices (i) and (j), we plumb the corresponding bundles as above. If a vertex (i) is joined with other vertices, we choose pairwise disjoint small disks in each surface, as many as one has adjacent vertices, and perform plumbing by pairs as above. The result is a 4-dimensional manifold $P(E)$ with boundary $S(E)$. It follows from the construction that the manifold $P(E)$ contains the union $E = \cup \mathcal{S}_i$ as a deformation retract, and these surfaces are contained in $P(E)$ with self-intersection w_i. Hence the homology of $P(E)$ is that of E.

A manifold obtained in this way is known as a *plumbed manifold* (or *graph manifold*), and this term may refer either to the 4-manifold $P(E)$ with boundary, or to its boundary, which is a 3-manifold.

Remark 1.9.9 *Notice that if the plumbing graph* (Σ, w, g) *is the dual graph of a resolution* $\pi : \tilde{V} \to V$, *then the manifold* $P(E)$ *is diffeomorphic to a regular neighborhood of the exceptional set* E *in the resolution, which may be taken to be of the form* $\pi^{-1}(V \cap \boldsymbol{D}_\epsilon)$, *where* $\boldsymbol{D}_\epsilon$ *is a Milnor ball for* $(V, 0)$. *Since the resolution map is a biholomorphism away from* E, *it follows that the boundary* $S(E)$ *is diffeomorphic to the link of* V.

1.10 Waldhausen manifolds

Definition 1.10.1 *A standard fibered torus corresponding to a pair of coprime integers* (a, b), *with* $a > 0$, *is the surface bundle of the automorphism of a disk given by rotation by an angle of* $2\pi b/a$ *(with the natural fibering by circles). If* $a = 1$, *the middle fiber is called ordinary, while if* $a > 1$, *the middle fiber is called exceptional.*

Definition 1.10.2 *A Seifert manifold is a closed* 3*-manifold together with a decomposition into a disjoint union of circles (called fibers) such that each fiber has a tubular neighborhood that forms a standard fibered torus.*

Remark 1.10.3

(*i*) *A compact Seifert fiber space has only a finite number of exceptional fibers;*

(*ii*) *Any* $\boldsymbol{S}^1$*-bundle over a compact surface is a Seifert manifold;*

(*iii*) *A fibre bundle over* $\boldsymbol{S}^1$ *with fibre a cylinder is a Seifert manifold (since any automorphism of the cylinder is isotopic to a periodic one - see [16] for instance).*

Definition 1.10.4 *We say that a differentiable* 3*-manifold* M *is a Waldhausen manifold if there exist a finite decomposition* $M = \cup M_i$ *such that each* M_i *is a Seifert manifold and for* $i \neq j$*, the intersection* $M_i \cap M_j$ *is either empty or homeomorphic to a disjoint union of tori* $\boldsymbol{S}^1 \times \boldsymbol{S}^1$.

Remark 1.10.5 *Notice that every graph manifold is a Waldhausen manifold. The reciprocal was proved by Neumman in [33], where he assigns a canonical plumbing graph to each Waldhausen manifold.*

Chapter 2

The degeneration of the Milnor fibre of complex singularities

In this chapter, our object of study is a holomorphic germ of function

$$f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0).$$

We are interested in describing how the Milnor fibre of f degenerates to the singular one. Such degeneration is described in terms of the *Lê Polyhedra*, as we will define later.

Let ϵ and η be two small positive reals such that $0 < \eta << \epsilon << 1$ as in theorem 1.7.5, and set $F_t := f^{-1}(t) \cap \mathbf{B}_\epsilon$ and $\mathring{F}_t := f^{-1}(t) \cap \mathring{\mathbf{B}}_\epsilon$, for any $t \in \mathbf{D}_\eta$, where $\mathring{\mathbf{B}}_\epsilon$ denotes the interior of the ball $\mathbf{B}_\epsilon$.

Definition 2.0.6 *For any $t \in \boldsymbol{D}^*_\eta$, we say that a polyhedron P_t is a Lê Polyhedron for f and that a polyhedron P_0 is the respective special polyhedron if:*

(i) P_t is contained in F_t and F_t deformation retracts to P_t;

(ii) P_0 is contained in F_0 and F_0 deformation retracts to P_0;

(iii) there exists a continuous map $\Psi_t : \mathring{F}_t \to \mathring{F}_0$ which sends P_t to P_0 and such that Ψ_t restricts to a homeomorphism from $\mathring{F}_t \backslash P_t$ to $\mathring{F}_0 \backslash P_0$.

We say that f admits a Lê Polyhedron if there exist such polyhedra P_t and P_0 in F_t and F_0, respectively, and such continuous map Ψ_t, which we call a collapsing map for f, for any $t \in \boldsymbol{D}^*_\eta$.

Lê proved in [22] that any complex isolated singularity germ of function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ admits a Lê Polyhedron P_t, which has real dimension $n - 1$, with corresponding special polyhedron P_0 being the origin $0 \in \mathbb{C}$. We reproduce his proof in section 2.1.1, when $n = 2$, and then in section 2.1.2 we generalize his construction to a more "global" situation.

In section 2.2 we prove that any germ of line singularity $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ (that is, a holomorphic germ of function with critical locus a complex line) admits a Lê Polyhedron P_t, which has real dimension 3, with corresponding special polyhedron P_0 having real dimension either 2 or 3.

2.1 Isolated singularities

In this section we consider $f : \mathbb{C}^n \to \mathbb{C}$ a holomorphic function such that $0 \in \mathbb{C}^n$ is an isolated singularity (that is, there exists a small ball $\mathbf{B}_\epsilon$ in $\mathbb{C}^n$ such that the restriction of f to $\mathbf{B}_\epsilon^*$ is a submersion). To simplify notation, we also suppose that $f(0) = 0$.

In 2.1.1 we study the problem in a local sense, that is, we describe how the Milnor fibre $F_t := f^{-1}(t) \cap \mathbf{B}_\epsilon$, for $t \neq 0$ small, degenerates to the singular fibre $F_0 := f^{-1}(0) \cap \mathbf{B}_\epsilon$, where ϵ is a Milnor radius for f (see 1.7). This is equivalent to considering the germ of function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ (see 1.5).

In 2.1.2 we study the problem in a more global sense, that is, we describe how the smooth manifold $F_t := f^{-1}(t) \cap \mathbf{B}_\epsilon$, for $t \neq 0$ small, degenerates to the singular hypersurface given by $F_0 := f^{-1}(0) \cap \mathbf{B}_\epsilon$, when ϵ is not a Milnor radius for f and $n = 2$. This is equivalent to considering the restriction $f_| : \mathbf{B}_\epsilon \to \mathbb{C}$ when ϵ is big enough not to be a Milnor radius, but small enough for $\mathbf{B}_\epsilon$ to contain only one singular point of f.

2.1.1 Isolated singularity germ

In this section we shall prove the following theorem:

Theorem 2.1.1 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a complex isolated singularity germ of function. Then there exist sufficiently small real numbers ϵ and η with $0 < \eta << \epsilon << 1$ such that for any $t \in \boldsymbol{D}_\eta^*$, there exist:*

(i) a polyhedron P_t in F_t, of real dimension $n-1$, such that F_t deformation retracts to P_t;

(ii) a continuous map $\Psi_t : F_t \to F_0$ which sends P_t to $\{0\}$ and such that Ψ_t restricts to a diffeomorphism from $F_t \backslash P_t$ to $F_0 \backslash \{0\}$.

This is done using the concept of polar curves, which can be defined thanks to the following lemma of Lê Dung Trang (see [20]). Since its proof is quite technical and leaves the context of this thesis, we do not show it here.

For any linear form (transformation)

$$l : \mathbb{C}^n \to \mathbb{C}$$

taking $0 \in \mathbb{C}^n$ to $0 \in \mathbb{C}$, we define the analytic morphism

$$\phi_l : \mathbb{C}^n \to \mathbb{C}^2$$

defined by $\phi_l(z) = \big(l(z), f(z)\big)$, for any $z \in \mathbb{C}^n$

Lemma 2.1.2 *There exists a non-empty Zariski open set Ω in the space of non-zero linear forms of $\mathbb{C}^n$ to $\mathbb{C}$ that take $0 \in \mathbb{C}^n$ to $0 \in \mathbb{C}$ (that is, considering the dual space, the closed subsets are the algebraic subsets), such that for any $l \in \Omega$, the analytic morphism $\phi_l : \mathbb{C}^n \to \mathbb{C}^2$ satisfies:*

(i) if C is the critical locus of ϕ_l and $\Gamma_l \subset \mathbb{C}^n$ is the union of the irreducible components of C which are not contained in $f^{-1}(0)$, then Γ_l is either empty or a reduced complex curve;

(ii) the restriction of ϕ_l to the germ $(\Gamma_l, 0)$ defines an analytic morphism $(1-1)$ from $(\Gamma_l, 0)$ to its image $(\Delta_l, 0) := \big(\phi(\Gamma_l), 0\big)$.

If $l \in \Omega$, we say that l is a *good linear form* relative to f, and we say that the curve Γ_l is the *polar curve of f* relative to l and that the curve Δ_l is the *polar image of f* relative to l. From now on, we shall fix a good linear form l, and in order to simplify notation, we shall denote $\Gamma := \Gamma_l$ and $\Delta := \Delta_l$.

If ϵ, η_1 and η_2 are small positive reals with $0 < \eta_2 << \eta_1 << \epsilon$, where ϵ is a Milnor radius for f, consider the restriction

$$\phi_| : \phi^{-1}(\mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2}) \cap \mathbf{B}_\epsilon \to \mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2},$$

which induces a topological fibre bundle of $\phi^{-1}(\mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2} \setminus \Delta) \cap \mathbf{B}_\epsilon$ over $(\mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2}) \setminus \Delta$, whose composition with the projection on $\mathbf{D}_{\eta_2}$ induces a fibre bundle isomorphic to that of theorem 1.7.5. Then the Milnor fibre $f^{-1}(t) \cap \mathbf{B}_\epsilon$ is homeomorphic to $f^{-1}(t) \cap \mathbf{B}_\epsilon \cap l^{-1}(\mathbf{D}_{\eta_1})$, and we shall denote both of them by F_t.

Set

$$D_t := \mathbf{D}_{\eta_1} \times \{t\}.$$

Then $\phi_|$ induces a projection

$$\varphi_t : F_t \to D_t,$$

which is a fibre bundle over $D_t \backslash (\Delta \cap D_t)$.

We now prove theorem 2.1.1 when $n = 2$. The proof for $n > 2$ is done by induction on n (see [22]).

The construction of the Lê Polyhedron

Note that

$$\varphi_{t|} : F_t \backslash \varphi_t^{-1}(\Delta \cap D_t) \to D_t \backslash (\Delta \cap D_t)$$

is a topological covering, and let m be the degree of this covering.

Now fix $t \in \mathbf{D}^*_{\eta_2}$ and let $y_1(t), \dots, y_k(t)$ be the points of the intersection $\Delta \cap D_t$. Let λ_t be the barycenter of the set of points $\{y_1(t), \dots, y_k(t)\}$ in D_t. For each $j = 1, \dots, k$, let $\delta(y_j(t))$ be a simple path (differentiable and with no double points) starting at λ_t and ending at $y_j(t)$, such that two of them intersect only at λ_t.

Set

$$Q_t := \bigcup_{j=1}^{k} \delta(y_j(t))$$

and

$$P_t := \varphi_t^{-1}(Q_t),$$

which is clearly a polyhedron. See figure 2.1 bellow.

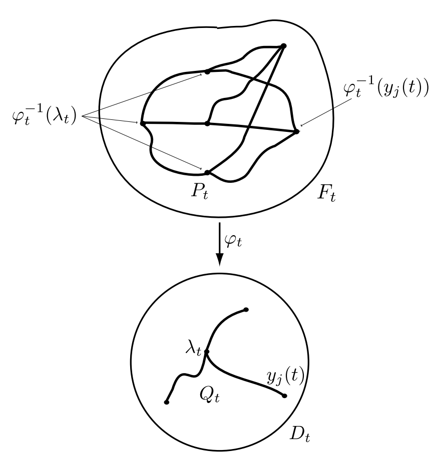


Figure 2.1:

Now, let v_t be a vector field in D_t such that v_t is:

- C^∞;
- null over Q_t;
- transversal to ∂D_t and pointing inwards.

Then the associated flow $q_t : [0,\infty[\ \times\ (D_t \backslash Q_t) \to D_t$ defines a map

$$\begin{array}{rcll} \xi_t & : & \partial D_t & \longrightarrow & Q_t \\ & & u & \longmapsto & \lim_{\tau\to\infty} q_t(\tau,u) \end{array}$$

such that ξ_t is continuous, surjective and differentiable.

Since φ_t is a differentiable covering over $D_t\backslash Q_t$, we can lift v_t to a vector field E_t in F_t such that E_t is:

- continuous over F_t;
- differentiable over $F_t\backslash P_t$;
- null over P_t;
- integrable;
- transversal to ∂F_t and points inwards.

Then the associated flow $\tilde{q}_t : [0,\infty[\ \times\ (F_t\backslash P_t) \to F_t$ defines a map

$$\begin{array}{rcll} \tilde{\xi}_t & : & \partial F_t & \longrightarrow & P_t \\ & & z & \longmapsto & \lim_{\tau\to\infty} \tilde{q}_t(\tau,z) \end{array}$$

such that $\tilde{\xi}_t$ is continuous, surjective and differentiable.

So now we have to show that F_t is homeomorphic to the mapping cylinder of $\tilde{\xi}_t$. In fact, the integration of the vector field E_t gives a surjective continuous map

$$\alpha : [0,\infty]\ \times\ \partial F_t \to F_t$$

that restricts to a diffeomorphism

$$\alpha_| : [0,\infty[\ \times \partial F_t \to F_t\backslash P_t.$$

Since the restriction $\alpha_\infty : \{\infty\} \times \partial F_t \to P_t$ is equal to $\tilde{\xi}_t$, which is differentiable and surjective, it follows that the induced map

$$[\alpha_\infty] : \big((\{\infty\} \times \partial F_t)/\sim\big) \to P_t$$

is a homeomorphism, where $\sim$ is the equivalence relation given by the identification $(\infty, z) \sim (\infty, z')$ if $\alpha_\infty(z) = \alpha_\infty(z')$. Hence the map

$$[\alpha] : \big(([0,\infty] \times \partial F_t)/\sim\big) \to F_t$$

induced by α defines a homeomorphism between F_t and the mapping cylinder of $\tilde{\xi}_t$.

This proves (i) of theorem 2.1.1, when $n = 2$.

The collapse along a path

We can do the construction of the vector field E_t simultaneously for all t in a simple path γ in $\mathbf{D}_{\eta_2}$ joining 0 and some $t_0 \in \partial\mathbf{D}_{\eta_2}$, such that γ is transverse to $\partial\mathbf{D}_{\eta_2}$. To simplify, we shall assume that γ is the closed segment of line in $\mathbf{D}_{\eta_2}$ joining 0 and t_0.

The natural projection $\pi : \mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2} \to \mathbf{D}_{\eta_2}$ restricted to Δ induces a ramified covering

$$\pi_| : \Delta \to \mathbf{D}_{\eta_2}$$

whose ramification locus is $\{0\}$.

Hence the inverse image of $\gamma \backslash \{0\}$ by this covering defines k disjoint simple paths in Δ, and each one of them is diffeomorphic to $\gamma \backslash \{0\}$. Moreover, the set $\Lambda = \bigcup_{t\in\gamma} \lambda_t$ defines a simple path in $\mathbf{D}_{\eta_1} \times \gamma$ such that $\Lambda \cap \Delta = \{0\}$.

We can choose the paths $\delta(y_j(t))$ in such a way that

$$T_j := \bigcup_{t\in\gamma} \delta(y_j(t))$$

forms a triangle differentially embedded in

$$\bigcup_{t\in\gamma} D_t = \mathbf{D}_{\eta_1} \times \gamma,$$

outside $\{0\}$. For any $j, j' \in \{1, \dots, k\}$ with $j \neq j'$, note that $T_j \cap T_{j'} = \Lambda$. Set

$$Q := \bigcup_{j=1}^{k} T_j.$$

See figure 2.2 above.

Now, let V be a vector field in $\mathbf{D}_{\eta_1} \times \gamma$ such that V is:

- continuous;
- null over Q;

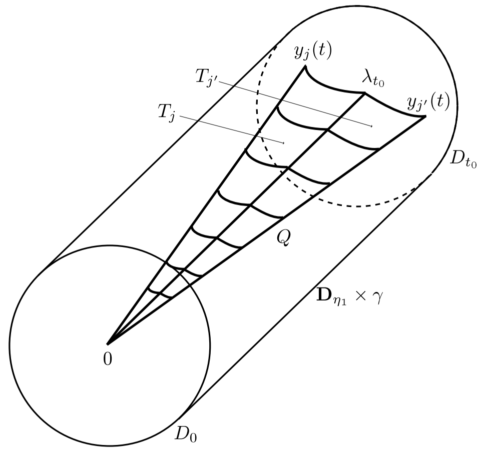


Figure 2.2:

- differentiable over $(\mathbf{D}_{\eta_1} \times \gamma)\backslash Q$;
- transversal to $\partial\mathbf{D}_{\eta_1} \times \gamma$; and such that
- the projection of V on γ is null.

Then the associated flow $w : [0, \infty[\ \times\ \big((\mathbf{D}_{\eta_1} \times \gamma)\backslash Q\big) \to \mathbf{D}_{\eta_1} \times \gamma$ defines a map

$$\begin{array}{rcccc} \xi & : & \partial\mathbf{D}_{\eta_1} \times \gamma & \longrightarrow & Q \\ & & z & \longmapsto & \lim_{\tau\to\infty} w(\tau, z) \end{array}$$

such that ξ is continuous, surjective and differentiable. It is easy to see that $\mathbf{D}_{\eta_1} \times \gamma$ is the mapping cylinder of ξ.

Set

$$F_\gamma := \phi^{-1}(\mathbf{D}_{\eta_1} \times \gamma) \cap \mathbf{B}_\epsilon.$$

and

$$P_\gamma := \phi^{-1}(Q),$$

which we call the *collapse polyhedron of f along γ*.

Lemma 2.1.3 *P_γ constructed above is a polyhedron.*

Proof: The flow w induces a surjective analytic map

$$\alpha : [0, \infty]\ \times\ (\partial\mathbf{D}_{\eta_1} \times \gamma) \to \mathbf{D}_{\eta_1} \times \gamma$$

that restricts to an analytic isomorphism

$$\alpha_{|} : [0, \infty[\times (\partial \mathbf{D}_{\eta_1} \times \gamma) \to (\mathbf{D}_{\eta_1} \times \gamma) \backslash Q.$$

Since the restriction $\alpha_\infty : \{\infty\} \times (\partial \mathbf{D}_{\eta_1} \times \gamma) \to Q$ is equal to ξ, which is analytic and surjective, it follows that the induced map

$$[\alpha_\infty] : \left[(\{\infty\} \times (\partial \mathbf{D}_{\eta_1} \times \gamma))/\sim\right] \to Q$$

is an analytic isomorphism, where $\sim$ is the equivalence relation given by the identification $(\infty, z) \sim (\infty, z')$ if $\alpha_\infty(z) = \alpha_\infty(z')$. Hence the map

$$[\alpha] : \left[([0, \infty] \times (\partial \mathbf{D}_{\eta_1} \times \gamma))/\sim\right] \to \mathbf{D}_{\eta_1} \times \gamma$$

induced by α defines an analytic isomorphism.
Now define $\pi : \left[([0, \infty] \times (\partial \mathbf{D}_{\eta_1} \times \gamma))/\sim\right] \to [0, \infty]$ the natural projection. Then we have the analytic map $(\pi \circ [\alpha]^{-1}) : \mathbf{D}_{\eta_1} \times \gamma \to [0, \infty]$ and $Q = (\pi \circ [\alpha]^{-1})^{-1}(\infty)$. Hence Q is an analytic set.
Moreover, since $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ is analytic and $P_\gamma = \phi^{-1}(Q)$, it follows that $P = (\pi \circ [\alpha]^{-1} \circ \phi)^{-1}(\infty)$ and that P is an analytic set. Since any analytic set is a polyhedron ([14]), it follows that both P and Q are polyhedra.

∎

Now, for any real $A > 0$, set

$$V_A(Q) := (\mathbf{D}_{\eta_1} \times \gamma) \backslash w\big([0, A[\times \partial \mathbf{D}_{\eta_1} \times \gamma\big),$$

a closed neighbourhood of Q in $\mathbf{D}_{\eta_1} \times \gamma$. Note that $\partial V_A(Q)$ is a differentiable manifold that fibres over γ with fibre a circle, by the restriction of the projection π.

Since

$$\phi_{|} : F_\gamma \backslash P_\gamma \to (\mathbf{D}_{\eta_1} \times \gamma) \backslash Q$$

is a fibre bundle, it follows that $\phi^{-1}(\partial V_A(Q))$ is a differentiable submanifold of F_γ which is a fibre bundle over γ. Set

$$Z := F_\gamma \backslash P_\gamma.$$

Now we will construct a vector field on Z as follows. Let θ be a vector field in γ that goes from t_0 to 0 in time $a > 0$. Since

$$Z = \phi^{-1}\big((\mathbf{D}_{\eta_1} \times \gamma) \backslash Q\big) \xrightarrow{\phi} (\mathbf{D}_{\eta_1} \times \gamma) \backslash Q \xrightarrow{\pi} \gamma$$

and

$$\phi^{-1}\big(\partial V_A(Q)\big) \xrightarrow{\phi} \partial V_A(Q) \xrightarrow{\pi} \gamma$$

are (differentiable) fibre bundles, we can lift θ to obtain a vector field E on Z such that:

- E is differentiable and
- E is tangent to $\phi^{-1}\big(\partial V_A(Q)\big)$, for any $A > 0$.

Then the associated flow $g : [0, a] \times Z \to Z$ defines a C^∞-diffeomorphism Ψ from $F_{t_0} \backslash P_{t_0}$ to $F_0 \backslash \{0\}$ that extends to a continuous map from F_{t_0} to F_0 and that sends P_{t_0} to $\{0\}$.

This proves (ii) of theorem 2.1.1 when $n = 2$. We can also prove the following lemma:

Lemma 2.1.4 *F_γ deformation retracts to P_γ.*

Proof: We can lift the vector field V to a vector field $\tilde{V}$ on $\phi^{-1}(\mathbf{D}_{\eta_1} \times \gamma)$, which is

- continuous;
- null over P_γ;
- differentiable over Z;
- transversal to $\phi^{-1}(\partial \mathbf{D}_{\eta_1} \times \gamma) = \partial F_\gamma$ and points inwards.

Then the associated flow $\tilde{q} : [0, \infty[\, \times Z \to Z$ defines a map $\tilde{\xi} : \partial F_\gamma \to P_\gamma$. Then just as before we have that F_γ is homeomorphic to the mapping cylinder of $\tilde{\xi}$.

2.1.2 Isolated singularity defined on a (not necessarily Milnor) ball

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a holomorphic function that takes $0 \in \mathbb{C}^2$ to $0 \in \mathbb{C}$ and consider the restriction $f_| : \mathbf{B}_\epsilon \to \mathbb{C}$, where $\epsilon > 0$ is not necessarily a Milnor radius for f and $\mathbf{B}_\epsilon$ contains exactly one critical point of f (at $0 \in \mathbf{B}_\epsilon$). We will describe how the smooth manifold $F_t := f^{-1}(t) \cap \mathbf{B}_\epsilon$, for $t \neq 0$ small, degenerates to the singular hypersurface given by $F_0 := f^{-1}(0) \cap \mathbf{B}_\epsilon$, in the same fashion of the last section. In other words, we will prove the following theorem:

Theorem 2.1.5 *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a holomorphic function that takes $0 \in \mathbb{C}^2$ to $0 \in \mathbb{C}$. Then for any positive real ϵ such that the restriction $f_| : \boldsymbol{B}_\epsilon \to \mathbb{C}$ has exactly one critical point (at $0 \in \boldsymbol{B}_\epsilon$), there exists a sufficiently small real number η with $0 < \eta << \epsilon << 1$ such that:*

(i) *for any $t \in \boldsymbol{D}^*_\eta$, there exist a polyhedron P_t in $F_t := f_|^{-1}(t)$ of real dimension 1 such that F_t deformation retracts to P_t;*

(ii) *if ϵ is a Milnor radius for f, there exists a continuous map $\Psi_t : F_t \to F_0$ which sends P_t to $\{0\}$ and such that Ψ_t restricts to a homeomorphism from $F_t \backslash P_t$ to $F_0 \backslash \{0\}$;*

(iii) *if ϵ is not a Milnor radius for f, there is a polyhedron P_0 in $F_0 := f_{|}^{-1}(0)$ of real dimension 1 such that F_0 deformation retracts to P_0 and a continuous map $\Psi_t : F_t \to F_0$ which sends P_t to P_0 and such that Ψ_t restricts to a homeomorphism from $F_t \backslash P_t$ to $F_0 \backslash P_0$.*

For any linear form

$$l : \mathbb{C}^2 \to \mathbb{C}$$

taking $0 \in \mathbb{C}^2$ to $0 \in \mathbb{C}$, the restriction of both f and l to $\mathbf{B}_\epsilon$ induces an analytic morphism

$$\phi_l : \mathbf{B}_\epsilon \to \mathbb{C}^2$$

defined by $\phi_l(z) = \big(l(z), f(z)\big)$, for any $z \in \mathbf{B}_\epsilon$. We have the following lemma, which is analogous to lemma 2.1.2 (see [20] for the proof):

Lemma 2.1.6 *There exists a non-empty Zariski open set Ω in the space of non-zero linear forms of $\mathbb{C}^2$ to $\mathbb{C}$ that take $0 \in \mathbb{C}^2$ to $0 \in \mathbb{C}$, such that for any $l \in \Omega$, the analytic morphism $\phi_l : \boldsymbol{B}_\epsilon \to \mathbb{C}^2$ satisfies:*

(i) *if C is the critical locus of ϕ_l and $\Gamma_l \subset \boldsymbol{B}_\epsilon$ is the union of the irreducible components of C which are not contained in $f^{-1}(0)$, then Γ_l is either empty or a reduced complex curve;*

(ii) *If $\Gamma_l \cap f^{-1}(0) = \{p_1, \ldots, p_r\}$, then for each p_i there exists a small neighbourhood V_i of p_i in $\boldsymbol{B}_\epsilon$ such that the restriction of ϕ_l to $\Gamma_l \cap V_i$ defines an analytic morphism $(1-1)$ of $\Gamma_l \cap V_i$ to its image $\Delta_{l,i} := \phi(\Gamma_l \cap V_i)$.*

As before, we say that $l \in \Omega$ is a *good linear form* relative to f. From now on, we shall fix a good linear form l, and in order to simplify notation, we shall denote $\Gamma := \Gamma_l$ and $\Delta := \Delta_l$.

Also as before, if ϵ, η_1 and η_2 are small positive reals with $0 < \eta_2 << \eta_1 << \epsilon$, consider the restriction

$$\phi_| : \phi^{-1}(\mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2}) \to \mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2},$$

which induces a topological fibre bundle of $\phi^{-1}(\mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2} \backslash \Delta) \cap \mathbf{B}_\epsilon$ over $\mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2} \backslash \Delta$, whose composition with the projection on $\mathbf{D}_{\eta_2}$ induces a fibre bundle isomorphic to that of theorem 1.7.5. Then the Milnor fibre $f^{-1}(t) \cap \mathbf{B}_\epsilon$ is homeomorphic to $f^{-1}(t) \cap \mathbf{B}_\epsilon \cap l^{-1}(\mathbf{D}_{\eta_1})$, and we shall denote both of them by F_t.

Now set

$$F_{\epsilon,\eta_2} := \mathbf{B}_\epsilon \cap f^{-1}(\mathbf{D}_{\eta_2}).$$

If we choose $\eta_2 > 0$ sufficiently small, we can suppose that $\Gamma' := \Gamma \cap F_{\epsilon,\eta_2}$ is contained in the union $\cup_{i=1}^r V_i$. For just a moment, we shall denote by f' the restriction of f to F_{ϵ,η_2}, and consider ϕ' defined on F_{ϵ,η_2} by setting $\phi'(z) := \big(l(z), f'(z)\big)$. Without lost of generality, we can also suppose that $l(p_i) \neq l(p_j)$, for any $i \neq j$, $i, j \in \{1, \ldots, r\}$, and then it follows from the previous lemma that the restriction of ϕ' to Γ' is finite and it defines an analytic morphism $(1-1)$ of Γ' to its image $\Delta' := \phi(\Gamma')$. In fact, if η_2 is sufficiently small, then Γ' has exactly r-connected components $\Gamma'(p_i)$, which are curves in p_i, that is, Γ' is the disjoint union

$$\Gamma' = \bigsqcup_{i=1}^{r} \Gamma'(p_i),$$

and Δ' is the disjoint union

$$\Delta' = \bigsqcup_{i=1}^{r} \phi(\Gamma'(p_i)) := \bigsqcup_{i=1}^{r} \Delta'(p_i).$$

In order to simplify notation, from now on we shall denote $f := f'$ (defined on $F_{\epsilon,\eta}$), $\Gamma := \Gamma'$ and so on. See figure 2.3.

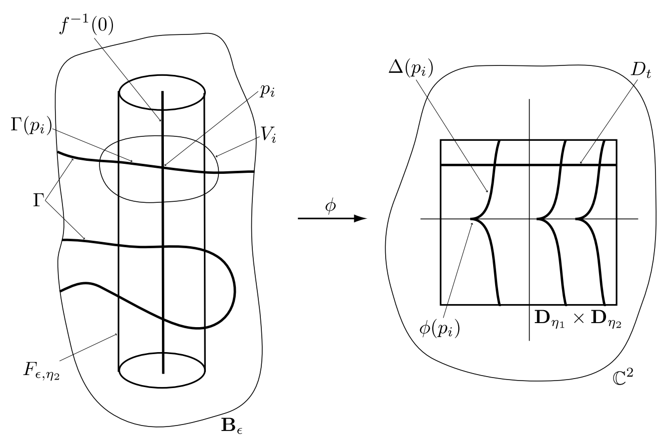


Figure 2.3:

Now fix $l \in \Omega$ and for any $t \in \mathbf{D}_{\eta_2}$, set

$$D_t := \mathbf{D}_{\eta_1} \times \{t\}.$$

Then $\phi : F_{\epsilon,\eta_2} \to \mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2}$ induces a projection

$$\varphi_t : F_t \to D_t,$$

which is a fibre bundle over $D_t \backslash (\Delta \cap D_t)$.

The polyhedron

We have seen that, for each $t \in \mathbf{D}_{\eta_2}$,

$$\varphi_{t|} : F_t \backslash \varphi_t^{-1}(\Delta \cap D_t) \to D_t \backslash (\Delta \cap D_t)$$

is a topological fibre bundle, and therefore it is a topological covering of degree m_t.

Let $y_1(t), \ldots, y_k(t)$ be the points of the intersection $\Delta \cap D_t$. Note that each $y_j(t)$, for $j = 1, \ldots, k$, is contained in some $\Delta(p_i)$, for some $i = 1, \ldots, r$.

Let λ_t be the barycenter of the set of points $\{y_1(t), \ldots, y_k(t)\}$ in D_t and for each $j = 1, \ldots, k$, let $\delta(y_j(t))$ be a simple path (differentiable and with no double points) starting at λ_t and ending at $y_j(t)$, such that two of them intersect only at λ_t.

Set

$$Q_t := \bigcup_{j=1}^{k} \delta(y_j(t))$$

and

$$P_t := \varphi_t^{-1}(Q_t).$$

See figure 2.1.

Now, let v_t be a vector field in D_t such that v_t is:

- C^∞;
- null over Q_t;
- transversal to ∂D_t and points inwards.

Then the associated flow $q_t : [0, \infty[\times (D_t \backslash Q_t) \to D_t$ defines a map

$$\begin{array}{rcccl} \xi_t & : & \partial D_t & \longrightarrow & Q_t \\ & & u & \longmapsto & \lim\limits_{\tau \to \infty} q_t(\tau, u) \end{array},$$

such that ξ_t is continuous, surjective and differentiable.

Since φ_t is a covering over $D_t \backslash Q_t$, which is differentiable in this case of dimension $n = 2$, we can lift v_t to a vector field E_t in F_t such that E_t is:

- continuous over F_t;
- differentiable over $F_t \backslash P_t$;
- null over P_t;
- integrable;

- transversal to ∂F_t and points inwards.

Then the associated flow $\tilde{q}_t : [0, \infty[\times (F_t \backslash P_t) \to F_t$ defines a map

$$\begin{array}{cccc} \tilde{\xi}_t & : & \partial F_t & \longrightarrow & P_t \\ & & z & \longmapsto & \lim\limits_{\tau \to \infty} \tilde{q}_t(\tau, z) \end{array},$$

such that $\tilde{\xi}_t$ is continuous, surjective and differentiable.

Then proceeding as in the previous section we show that F_t is homeomorphic to the mapping cylinder of $\tilde{\xi}_t$.

The collapse along a path

Now we do the construction of the vector field E_t simultaneously for all t in a simple path γ in $\mathbf{D}_\eta$ joining 0 and some $t_0 \in \partial \mathbf{D}_\eta$, such that γ is transverse to $\partial \mathbf{D}_\eta$. To simplify, we shall assume that γ is the closed segment of line in $\mathbf{D}_\eta$ joining 0 and t_0.

The natural projection $\pi : \mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2} \to \mathbf{D}_{\eta_2}$ restricted to Δ induces a ramified covering

$$\pi_| : \Delta \to \mathbf{D}_{\eta_2}$$

whose ramification locus is $D_0 \cap \Delta = \{\phi(p_1), \ldots, \phi(p_r)\}$.

Hence the inverse image of $\gamma \backslash \{0\}$ by this covering defines k disjoint simple paths in Δ, and each one of them is diffeomorphic to $\gamma \backslash \{0\}$. Each of these paths have $\phi(p_i)$ in its closure, for some $i = 1, \ldots, r$, and it contains the points $y_j(t)$, for some $j = 1, \ldots, k$ and any $t \in \gamma \backslash \{0\}$. We shall denote by $\varsigma_{i,j}$ the respective path that has $\phi(p_i)$ in its closure and contains $y_j(t)$. In particular, we have that $r \leq k$. See figure 2.4.

Moreover, the set $\Lambda = \bigcup_{t \in \gamma} \lambda_t$ defines a simple path in $\mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2}$ such that either $\Lambda \cap \Delta = \phi(p_1)$, if $r = 1$, or $\Lambda \cap \Delta = \emptyset$, if $r > 1$.

We can choose the paths $\delta(y_j(t))$ in such a way that

$$T_j := \bigcup_{t \in \gamma} \delta(y_j(t))$$

forms either a triangle, if $r = 1$, or a square, if $r > 1$, differentiably immersed in

$$\bigcup_{t \in \gamma} D_t = \mathbf{D}_{\eta_1} \times \gamma$$

outside $\delta(y_j(0))$. For any $j, j' \in \{1, \ldots, k\}$ with $j \neq j'$, note that either $T_j \cap T_{j'} = \Lambda$, if both $\varsigma_{i,j}$ and $\varsigma_{i',j'}$ are defined for some $i, i' \in \{1, \ldots, r\}$ with $i \neq i'$; or $T_j \cap T_{j'} = \Lambda \cup \gamma(y_j(0)) =$

$\Lambda \cup \gamma(y_{j'}(0))$, if both $\varsigma_{i,j}$ and $\varsigma_{i,j'}$ are defined for some $i \in \{1, \ldots, r\}$. See figure 2.5. Set

$$Q := \bigcup_{j=1}^{k} T_j.$$

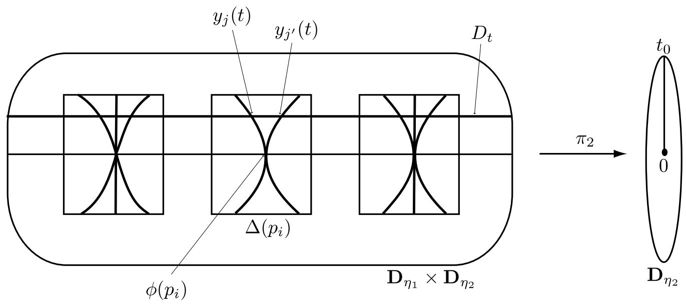


Figure 2.4:

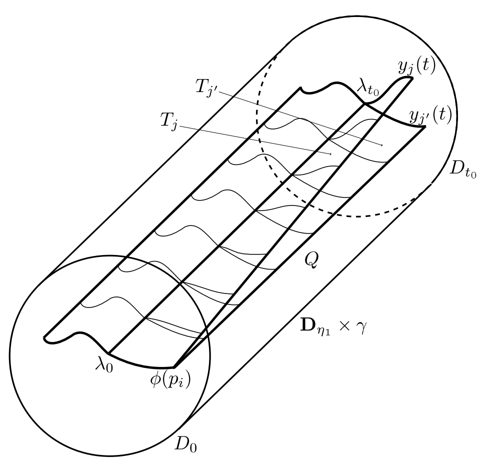


Figure 2.5:

Now, let V be a vector field in $\mathbf{D}_{\eta_1} \times \gamma$ such that V is:

- continuous;

- null over Q;
- differentiable over $(\mathbf{D}_{\eta_1} \times \gamma)\backslash Q$;
- transversal to $\partial\mathbf{D}_{\eta_1} \times \gamma$; and such that
- the projection of V on γ is null.

Then the associated flow $w : [0, \infty[\times \big((\mathbf{D}_{\eta_1} \times \gamma)\backslash Q\big) \to \mathbf{D}_{\eta_1} \times \gamma$ defines a map

$$\begin{array}{cccc} \xi & : & \partial\mathbf{D}_{\eta_1} \times \gamma & \longrightarrow & Q \\ & & z & \longmapsto & \lim\limits_{\tau\to\infty} w(\tau, z) \end{array},$$

such that ξ is continuous, surjective and differentiable.

For any real $A > 0$, set

$$V_A(Q) := (\mathbf{D}_{\eta_1} \times \gamma)\backslash w\big([0, A[\times\partial\mathbf{D}_{\eta_1} \times \gamma\big),$$

a closed neighbourhood of Q in $\mathbf{D}_{\eta_1} \times \gamma$. Note that $\partial V_A(Q)$ is a differentiable manifold that fibres over γ with fibre a circle, by the restriction of the projection π. Moreover, $\mathbf{D}_{\eta_1} \times \gamma$ is clearly the mapping cylinder of ξ.

Set

$$F_\gamma := \phi^{-1}(\mathbf{D}_{\eta_1} \times \gamma) \cap \mathbf{B}_\epsilon.$$

Since

$$\phi_| : F_\gamma\backslash\phi^{-1}(Q) \to (\mathbf{D}_{\eta_1} \times \gamma)\backslash Q$$

is a fibre bundle, it follows that $\phi^{-1}(\partial V_A(Q))$ is a differentiable submanifold of F_γ which is a fibre bundle over γ.

Now, set

$$P_\gamma := \phi^{-1}(Q),$$

which we call the *collapse polyhedron of f along γ*. It is a polyhedron in F_γ of real dimension 2. Let θ be a vector field in γ that goes from t_0 to 0 in time $a > 0$.

Set

$$Z := F_\gamma\backslash P_\gamma.$$

Since

$$Z = \phi^{-1}\big((\mathbf{D}_{\eta_1} \times \gamma)\backslash Q\big) \xrightarrow{\phi} (\mathbf{D}_{\eta_1} \times \gamma)\backslash Q \xrightarrow{\pi} \gamma$$

and

$$\phi^{-1}\big(\partial V_A(Q)\big) \xrightarrow{\phi} \partial V_A(Q) \xrightarrow{\pi} \gamma$$

are (differentiable) fibre bundles, we can lift θ to obtain a vector field E such that:

- E is differentiable;
- E is tangent to $\phi^{-1}\big(\partial V_A(Q)\big)$, for any $A > 0$.

Then the associated flow $g : [0, a] \times Z \to Z$ defines a C^∞-diffeomorphism Ψ from $M_{t_0} \backslash P_{t_0}$ to $M_0 \backslash P_0$ that extends to a continuous map from M_{t_0} to M_0 and that sends P_{t_0} to P_0.

2.2 Line singularities

When one wishes to generalize a property of isolated singularities for non-isolated singularities, the most natural class to be studied is that of line singularities, which were first defined by Siersma in [40] as the class of holomorphic germs of function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with critical locus a smooth germ of curve $(\Sigma, 0)$. In this section we show that any line singularity $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ admits a Lê Polyhedron, in the following sense:

Definition 2.2.1 *Let f be a line singularity as above and let H_s be a family of hyperplane sections of $\mathbb{C}^3$ transversal to Σ at s. We say that a real number $\epsilon > 0$ is a good Milnor radius for f and that $\boldsymbol{B}_\epsilon$ is a good Milnor ball for f if, for any $s \in \Sigma_\epsilon := \Sigma \cap \boldsymbol{B}_\epsilon$, the intersection $\boldsymbol{B}_\epsilon \cap H_s$ is a Milnor ball for the restriction of f to H_s. We say that f admits a good Milnor radius if there is $\epsilon > 0$ which is a good Milnor radius for f.*

Theorem 2.2.2 *If $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is a line singularity, then there exist ϵ and η sufficiently small, with $0 < \eta << \epsilon << 1$, such that for any $t \in \boldsymbol{D}^*_\eta$:*

(*i*) *There exists a polyhedron P_t, of real dimension 3, in the Milnor fibre F_t such that F_t deformation retracts to P_t;*

(*ii*) *If f does not admit a good Milnor radius, there exists a (contractible) polyhedron P_0 of real dimension 3 in the singular fibre F_0 such that F_0 deformation retracts to P_0; and there is a continuous map $\Psi_t : \mathring{F}_t \to \mathring{F}_0$ which sends P_t to P_0 and such that Ψ_t restricts to a homeomorphism from $\mathring{F}_t \backslash P_t$ to $\mathring{F}_0 \backslash P_0$;*

(*iii*) *If f admits a good Milnor radius, there exists a collapsing map $\Psi_t : \mathring{F}_t \to \mathring{F}_0$ sending P_t to $\Sigma \cap \boldsymbol{B}_\epsilon$ and such that Ψ_t restricts to a homeomorphism from $\mathring{F}_t \backslash P_t$ to $\mathring{F}_0 \backslash (\Sigma \cap \boldsymbol{B}_\epsilon)$.*

The proof of this theorem is presented in subsection 2.2.1, and it is based on lemmas 2.2.3, 2.2.4, 2.2.5 and 2.2.6 of that subsection. Lemmas 2.2.3 and 2.2.4 are particular cases of propositions 2.2.8 and 2.2.9, respectively, presented in subsection 2.2.2, and which concern the construction of a Lê Polyhedron for complex analytic singularities $f : M \to \mathbb{C}$ defined on a compact n-dimensional complex manifold with boundary ∂M, such that the critical locus of f is the whole special fibre $M_0 := f^{-1}(0)$. Here, by a compact complex manifold with boundary we mean that M is compact, smooth and its interior is a complex manifold.

We prove propositions 2.2.8 and 2.2.9 in subsection 2.2.3. Finally, in subsection 2.2.4 we prove lemmas 2.2.3, 2.2.4, 2.2.5 and 2.2.6.

2.2.1 Lê Polyhedron for line singularities

Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a line singularity, that is, a holomorphic germ of function whose critical locus Σ is a complex line. Without lost of generality, we can suppose that Σ is the z_3-axis, that is,

$$\Sigma = \{z_1 = z_2 = 0\},$$

Consider a generic family of parallel hyperplane sections $H_s \subset \mathbb{C}^3$ transversal to Σ at each $s \in \Sigma$ such that the each H_s is transversal to the smooth part of the hypersurface defined by f.

Then considering the restrictions

$$f_s := f_{|H_s} : H_s \to \mathbb{C}$$

we obtain a family, in the parameter $s \in \Sigma \overset{\text{diff.}}{=} \mathbb{C}$, of holomorphic functions with isolated singularity.

It is well known that one can define the Milnor fibration of f using polydisks (see section 5.1 for instance), that is, if ϵ_1 and ϵ_2 are sufficiently small positive reals with $0 < \epsilon_1 < \epsilon_2 << 1$, one has that the restriction

$$f_| : f^{-1}(\mathbf{D}^*_\eta) \cap (\mathbf{B}_{\epsilon_1} \times \mathbf{B}_{\epsilon_2}) \to \mathbf{D}^*_\eta$$

is a fibre bundle and $f^{-1}(t) \cap \mathbf{B}_\epsilon$ is homeomorphic to $f^{-1}(t) \cap (\mathbf{B}_{\epsilon_1} \times \mathbf{B}_{\epsilon_2})$, where

- $\mathbf{D}_\eta$ denotes the disk in $\mathbb{C}$ centered at zero and with radius $\eta > 0$ sufficiently small;
- $\mathbf{D}^*_\eta$ is the punctured disk $\mathbf{D}_\eta \backslash \{0\}$;
- $\mathbf{B}_{\epsilon_1}$ denotes the ball in $\Sigma \overset{\text{diff.}}{=} \mathbb{C}$ centered at zero and with radius ϵ_1;
- $\mathbf{B}_{\epsilon_2}$ denotes the ball in $H_0 \overset{\text{diff.}}{=} \mathbb{C}^2$ centered at zero and with radius ϵ_2, and $\{s\} \times \mathbf{B}_{\epsilon_2}$ is the ball in H_s parallel to $\mathbf{B}_{\epsilon_2}$, for any $s \in \Sigma$;

– $\mathbf{B}_\epsilon \subset \mathbb{C}^3$ is an usual Milnor ball for f.

Now, since either Σ or $\Sigma\backslash\{0\}$ is a stratum of a Whitney stratification for f, we can take ϵ_1 and ϵ_2 sufficiently small such that:

- $\big(f^{-1}(0)\backslash f_0^{-1}(0)\big)\cap(\mathbf{B}_{\epsilon_1}\times\mathbf{B}_{\epsilon_2})$ is a fibre bundle over $(\Sigma\cap\mathbf{B}_{\epsilon_1})\backslash\{0\}$ with fibre $f_s^{-1}(0)\cap(\{s\}\times\mathbf{B}_{\epsilon_2})$, for some $s\in\Sigma\backslash\{0\}$;
- $(\{0\}\times\mathbf{B}_{\epsilon_2})$ is a Milnor ball for f_0.

Then for any $t\in\mathbf{D}^*_\eta$, let F_t denote the Milnor fibre of f, that is,

$$F_t := f^{-1}(t)\cap(\mathbf{B}_{\epsilon_1}\times\mathbf{B}_{\epsilon_2}),$$

which is a compact smooth manifold of dimension 2 with boundary

$$\partial F_t = f^{-1}(t)\cap\partial(\mathbf{B}_{\epsilon_1}\times\mathbf{B}_{\epsilon_2}),$$

and let F_0 denote the singular fibre of f, that is,

$$F_0 := f^{-1}(0)\cap(\mathbf{B}_{\epsilon_1}\times\mathbf{B}_{\epsilon_2}).$$

To simplify notation, from now on we shall denote by f_s the restriction of f to $\{s\}\times\mathbf{B}_{\epsilon_2}$. Also, set $\Sigma_\epsilon := \Sigma\cap\mathbf{B}_{\epsilon_1}$.

Now, for any $t\in\mathbf{D}^*_\eta$, let

$$\pi_t : \mathring{F}_t \to \Sigma_\epsilon$$

be the analytic function given by the projection of $\mathbb{C}^3$ on Σ induced by the hyperplane sections H_s, restricted to $\mathring{F}_t$. There are two possibilities, considering ϵ_1 and ϵ_2 sufficiently small:

(*a*) π_t is a (trivial) topological fibre bundle over Σ_ϵ (for example, if Σ_ϵ is a stratum of a Whitney stratification of f);

(*b*) π_t is not a topological fibre bundle over Σ_ϵ. Then $\Sigma_\epsilon\backslash\{0\}$ is a stratum of a Whitney stratification of f, and hence π_t is a topological fibre bundle over $\Sigma_\epsilon\backslash\{0\}$ and $0\in\mathbb{C}$ is a critical value for π_t; there are two cases to consider:

(b_1) $Sing(\pi_t)\subsetneq\pi_t^{-1}(0)$ and therefore $\pi_t^{-1}(0)$ is singular;

(b_2) $Sing(\pi_t)=\pi_t^{-1}(0)$ and therefore $\pi_t^{-1}(0)$ is smooth.

Note that, for any $t \in \mathbf{D}^*_\eta$ and $s \in \Sigma_\epsilon$, we have

$$\pi_t^{-1}(s) = int[f_s^{-1}(t)],$$

the interior of $f_s^{-1}(t)$. But then $\pi_t^{-1}(0) = int[f_0^{-1}(t)]$, the interior of the Milnor fibre of the isolated singularity given by f_0, and therefore it is smooth. So case (b_1) cannot occur.

One can also check that case (a) happens only if f admits a good Milnor radius.

Case (a)

In this case, if $\epsilon_2 > 0$ is sufficiently small, clearly $\{s\} \times \mathbf{B}_{\epsilon_2}$ is a Milnor ball for any any f_s with $s \in \Sigma_\epsilon$, and then we can construct a Lê Polyhedron $P_t \subset \mathring{F}_t$ as follows:

For each $s \in \Sigma_\epsilon$, let $P_{t,s} \subset f_s^{-1}(t)$ be a Lê Polyhedron for the isolated singularity function f_s as in [20]. Then the fibre bundle

$$\begin{array}{ccc} int[f_s^{-1}(t)] & \hookrightarrow & \mathring{F}_t \\ & & \downarrow{\scriptstyle \pi_t} \\ & & \Sigma_\epsilon \end{array}$$

induces a fibre bundle

$$\begin{array}{ccc} P_{t,s} & \hookrightarrow & P_t \\ & & \downarrow \\ & & \Sigma_\epsilon \end{array}$$

where the total space P_t is a *Lê Polyhedron for* f.

In fact, since $f_s^{-1}(t)$ deformation retracts to $P_{t,s}$, it follows that F_t deformation retracts to P_t. Moreover, we know that for each $s \in \Sigma_\epsilon$, there exists a collapsing map

$$\Psi_{t,s} : f_s^{-1}(t) \to f_s^{-1}(0)$$

which restricts to a homeomorphism $f_s^{-1}(t) \backslash P_{t,s} \to f_s^{-1}(0) \backslash \{s\}$ and such that $\Psi_{t,s}(P_{t,s}) = \{s\}$. Then we can define a continuous collapsing map

$$\Psi_t : \mathring{F}_t \to \mathring{F}_0$$

for f setting

$$\Psi_t(z) := \Psi_{t,\pi_t(z)}(z).$$

Hence Ψ_t restricts to a homeomorphism $\mathring{F}_t \backslash P_t \to \mathring{F}_0 \backslash \Sigma_\epsilon$ and that $\Psi_t(P_t) = \Sigma_\epsilon$.

Case (b_2)

In this case we have to understand how

$$int[f_s^{-1}(t)] = \pi_t^{-1}(s)$$

degenerates to

$$int[f_0^{-1}(t)] = \pi_t^{-1}(0)$$

as $s \in \Sigma_\epsilon \backslash \{0\}$ goes to $0 \in \Sigma_\epsilon$, for any $t \in \mathbf{D}^*_\eta$ fixed. To do that, we use lemmas 2.2.3 and 2.2.4 bellow, which give a Lê Polyhedron for the projection π_t, and lemmas 2.2.5 and 2.2.6. We shall prove these lemmas in the next sections.

Lemma 2.2.3 *For any $t \in \boldsymbol{D}^*_\eta$ and $s \in \Sigma_\epsilon$, there exists a polyhedron $P_{t,s}$ in $\pi_t^{-1}(s)$, of real dimension* 1*, such that $\pi_t^{-1}(s)$ deformation retracts to $P_{t,s}$.*

Lemma 2.2.4 *For any $t \in \boldsymbol{D}^*_\eta$ and $s \in \Sigma_\epsilon \backslash \{0\}$, there exists a continuous map $\tilde{\Psi}_{t,s}$: $\pi_t^{-1}(s) \to \pi_t^{-1}(0)$ such that $\tilde{\Psi}_{t,s}$ restricts to a homeomorphism $\pi_t^{-1}(s)\backslash P_{t,s} \to \pi_t^{-1}(0)\backslash P_{t,0}$ and takes $P_{t,s}$ to $P_{t,0}$.*

Lemma 2.2.5 *The construction of the polyhedra $P_{t,s}$ and of the collapses $\tilde{\Psi}_{t,s} : \pi_t^{-1}(s) \to \pi_t^{-1}(0)$ can be done simultaneously for any $s \in \Sigma_\epsilon$. Then we obtain a polyhedron P_t in F_t, of real dimension* 3*, such that F_t deformation retracts to P_t and such that, for any $s \in \Sigma_\epsilon$, the fibre $\pi_t^{-1}(s)$ deformation retracts to the intersection $P_t \cap \pi_t^{-1}(s)$.*

Lemma 2.2.6 *For any $t \in \boldsymbol{D}^*_\eta$ and $s \in \Sigma_\epsilon$, the polyhedron $P_{t,s}$ is also a Lê Polyhedron for f_s, in the global sense of section 2.1.2.*

Now we will show that the polyhedron P_t is a Lê Polyhedron for f. Define the collapsing map

$$\Psi_t : \mathring{F}_t \to \mathring{F}_0$$

for f setting

$$\Psi_t(z) := \Psi_{t,\pi_t(z)}(z),$$

where $\Psi_{t,s} : f_s^{-1}(t) \to f_s^{-1}(0)$ is a collapsing map for f_s (in the global sense), which restricts to a homeomorphism from $f_s^{-1}(t)\backslash P_{t,s}$ to $f_s^{-1}(0)\backslash P_{0,s}$ and sends $P_{t,s}$ to $P_{0,s}$ (where $P_{0,s}$ is a special polyhedron for f_s in the global sense).

Suppose that f does not admit a good Milnor radius and set

$$P_0 := \bigcup_{s \in \Sigma_\epsilon} P_{0,s},$$

which is a polyhedron in F_0 containing Σ_ϵ and such that F_0 deformation retracts to P_0. Then Ψ_t clearly restricts to a homeomorphism $\mathring{F}_t \backslash P_t \to \mathring{F}_0 \backslash P_0$ and $\Psi_t(P_t) = P_0$, and then we have proved (ii) of Theorem 2.2.2.

Now suppose that f admits a good Milnor radius. Then f_s is an isolated singularity and we can consider $\Psi_{t,s} : int[f_s^{-1}(t)] \to int[f_s^{-1}(0)]$ to be a collapsing map for f_s (in the local sense of [20]), which restricts to a homeomorphism from $int[f_s^{-1}(t)]\backslash P_{t,s}$ to $int[f_s^{-1}(0)]\backslash\{s\}$ and sends $P_{t,s}$ to $\{s\}$. Hence Ψ_t restricts to a homeomorphism $\mathring{F}_t \backslash P_t \to \mathring{F}_0 \backslash \Sigma_\epsilon$ and $\Psi_t(P_t) = \Sigma_\epsilon$, so we have proved (iii) of Theorem 2.2.2.

2.2.2 A special class of singularities

Definition 2.2.7 *If M is a compact, smooth manifold with non-empty boundary whose interior is a complex manifold, we say that M is a compact complex manifold with boundary.*

Let M be a compact oriented smooth 2-dimensional complex manifold in $\mathbb{C}^3$, with boundary ∂M, and let $f : M \to \mathbb{C}$ be a holomorphic function such that:

(i) the critical locus of f is given by $\Sigma = f^{-1}(0) \neq \emptyset$;

(ii) $f^{-1}(t)$ is a connected hypersurface in M which intersects ∂M transversally, for any $t \in \mathbb{C}$ sufficiently small.

Later we will take M to be the Milnor fibre of a complex hypersurface. For any positive real $\eta > 0$, define

$$M_\eta := f^{-1}(\mathbf{D}_\eta).$$

By Ehresmann's fibration lemma , the restriction

$$f_| : M_\eta \backslash (f^{-1}(0) \cap M_\eta) \to \mathbf{D}^*_\eta$$

is a topological fibre bundle. We want to describe how $M_t := f^{-1}(t)$ degenerates to $M_0 := f^{-1}(0)$, as $t \in \mathbb{C}$ goes to zero. We have the following propositions, which generalize lemmas 2.2.3 and 2.2.4 of the previous section:

Proposition 2.2.8 *For any $t \in \boldsymbol{D}_\eta$, one has that:*

(i) *there exist a polyhedron P_t in M_t, of real dimension 1, and an integrable vector field over M_t, null over P_t, which defines a continuous map $\tilde{\xi}_t : \partial M_t \to P_t$;*

(ii) *the fibre M_t is the mapping cylindre of $\tilde{\xi}_t$, and therefore M_t deformation retracts to P_t.*

Proposition 2.2.9 *For any $t \in \boldsymbol{D}_\eta^*$, there exist a continuous map $\Psi_t : M_t \to M_0$ such that Ψ_t restricts to a homeomorphism $M_t \backslash P_t \to M_0 \backslash P_0$ and Ψ_t takes P_t to P_0.*

We call Ψ_t a collapsing map for f. To prove these propositions, we need to construct a good projection of the Milnor fibre F_t of f to a disk, denoted by $\varphi_t : F_t \to D_t$. This is what we do in the rest of this section.

For any linear form

$$l : \mathbb{C}^3 \to \mathbb{C}$$

taking $0 \in \mathbb{C}^3$ to $0 \in \mathbb{C}$, the restriction of both f and l to M induces an analytic morphism

$$\phi_l : M \to \mathbb{C}^2$$

defined by $\phi_l(z) = \big(l(z), f(z)\big)$, for any $z \in M$. As before, we have the following lemma:

Lemma 2.2.10 *There exists a non-empty Zariski open set Ω in the space of non-zero linear forms of $\mathbb{C}^3$ to $\mathbb{C}$ that take $0 \in \mathbb{C}^3$ to $0 \in \mathbb{C}$, such that for any $l \in \Omega$, the analytic morphism $\phi_l : M \to \mathbb{C}^2$ satisfies:*

(*i*) *if C is the critical locus of ϕ_l and $\Gamma_l \subset M$ is the union of the irreducible components of C which are not contained in $f^{-1}(0)$, then Γ_l is either empty or a reduced complex curve;*

(*ii*) *If $\Gamma_l \cap f^{-1}(0) = \{p_1, \ldots, p_r\}$, then for each p_i there exists a small neighbourhood V_i of p_i in M such that the restriction of ϕ_l to $\Gamma_l \cap V_i$ defines an analytic morphism $(1-1)$ of $\Gamma_l \cap V_i$ to its image $\Delta_{l,i} := \phi(\Gamma_l \cap V_i)$.*

If $l \in \Omega$, we say that l is a *general linear form* relative to f, and we say that the curve Γ_l is the *polar curve of f* related to l and that the curve Δ_l is the *polar image of f* related to l.

From now on, we shall fix a good linear form l, and in order to simplify notation, we shall denote $\Gamma := \Gamma_l$ and $\Delta := \Delta_l$.

If we choose $\eta > 0$ sufficiently small, we can suppose that $\Gamma' := \Gamma \cap M_\eta$ is contained in the union $\cup_{i=1}^r V_i$. For a moment, let us denote by f' the restriction of f to M_η, and consider ϕ' defined on M_η by setting $\phi'(z) := \big(l(z), f'(z)\big)$. Without lost of generality, we can also suppose that $l(p_i) \neq l(p_j)$, for any $i \neq j$, $i, j \in \{1, \ldots, r\}$, so the previous lemma implies that the restriction of ϕ' to Γ' is finite and it defines an analytic morphism $(1-1)$ of Γ' to its image $\Delta' := \phi(\Gamma')$. In fact, if η is sufficiently small, then Γ' has exactly r-connected components $\Gamma'(p_i)$, which are curves in p_i, that is, Γ' is the disjoint union

$$\Gamma' = \bigsqcup_{i=1}^{r} \Gamma'(p_i),$$

and Δ' is the disjoint union

$$\Delta' = \bigsqcup_{i=1}^{r} \phi(\Gamma'(p_i)) := \bigsqcup_{i=1}^{r} \Delta'(p_i).$$

In order to simplify notation, from now on we shall denote $f := f'$ (defined on M_η), $\Gamma := \Gamma'$ and so on. See figure 2.6.

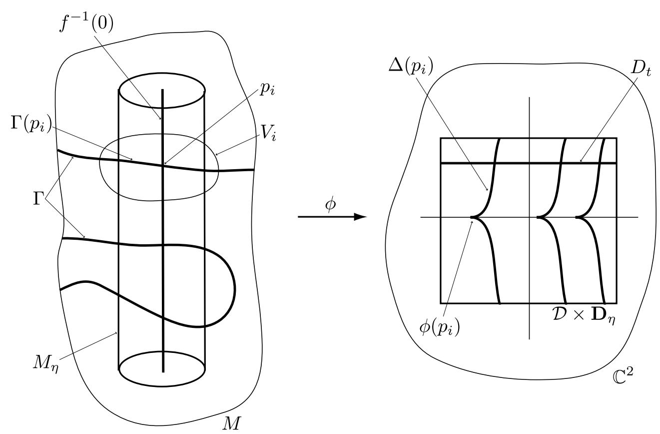


Figure 2.6:

Now fix $l \in \Omega$ such that the set $\mathcal{D} := l(M_\eta)$ is diffeomorphic to a disk in $\mathbb{C}$, and for any $t \in \mathbf{D}_\eta$, set

$$D_t := \mathcal{D} \times \{t\}.$$

Then $\phi : M_\eta \to \mathcal{D} \times \mathbf{D}_\eta$ induces a projection

$$\varphi_t : M_t \to D_t,$$

which is a fibre bundle over $D_t \backslash (\Delta \cap D_t)$.

2.2.3 Proof of propositions 2.2.8 and 2.2.9

The construction of the Lê Polyhedron

We have seen that, for each $t \in \mathbf{D}_\eta$,

$$\varphi_{t|} : M_t \backslash \varphi_t^{-1}(\Delta \cap D_t) \to D_t \backslash (\Delta \cap D_t)$$

is a topological covering of degree m_t.

Let $y_1(t), \ldots, y_k(t)$ be the points of the intersection $\Delta \cap D_t$. Note that each $y_j(t)$, for $j = 1, \ldots, k$, is contained in some $\Delta(p_i)$, for some $i = 1, \ldots, r$.

Let λ_t be the barycenter of the set of points $\{y_1(t), \ldots, y_k(t)\}$ in D_t and for each $j = 1, \ldots, k$, let $\delta(y_j(t))$ be a simple path (differentiable and with no double points) starting at λ_t and ending at $y_j(t)$, such that two of them intersect only at λ_t.

Set

$$Q_t := \bigcup_{j=1}^{k} \delta(y_j(t))$$

and

$$P_t := \varphi_t^{-1}(Q_t).$$

See figure 2.1.

Now, let v_t be a vector field in D_t such that v_t is:

- C^∞;
- null over Q_t;
- transversal to ∂D_t and points inwards.

Then the associated flow $q_t : [0, \infty[\times (D_t \backslash Q_t) \to D_t$ defines a map

$$\begin{array}{cccc} \xi_t : & \partial D_t & \longrightarrow & Q_t \\ & u & \longmapsto & \lim_{\tau \to \infty} q_t(\tau, u) \end{array},$$

such that ξ_t is continuous, surjective and differentiable.

Since φ_t is a covering over $D_t \backslash Q_t$, which is differentiable in this case of dimension $n = 2$, we can lift v_t to a vector field E_t in M_t such that E_t is:

- continuous over M_t;
- differentiable over $M_t \backslash P_t$;
- null over P_t;
- integrable;
- transversal to ∂M_t and points inwards.

Then the associated flow $\tilde{q}_t : [0,\infty[\ \times\ (M_t \backslash P_t) \to M_t$ defines a map

$$\begin{array}{rcccc} \tilde{\xi}_t & : & \partial M_t & \longrightarrow & P_t \\ & & z & \longmapsto & \lim\limits_{\tau\to\infty} \tilde{q}_t(\tau, z) \end{array},$$

such that $\tilde{\xi}_t$ is continuous, surjective and differentiable.

So now we have to show that M_t is homeomorphic to the mapping cylinder of $\tilde{\xi}_t$. In fact, the integration of the vector field $\tilde{v}_t$ gives a surjective continuous map

$$\alpha : [0,\infty]\ \times\ \partial M_t \to M_t$$

that restricts to a diffeomorphism

$$\alpha_| : [0,\infty[\ \times\ \partial M_t \to M_t \backslash P_t.$$

Since the restriction $\alpha_\infty : \{\infty\} \times \partial M_t \to P_t$ is equal to $\tilde{\xi}_t$, which is differentiable and surjective, it follows that the induced map

$$[\alpha_\infty] : \big((\{\infty\} \times \partial M_t)/\sim\big) \to P_t$$

is a homeomorphism, where $\sim$ is the equivalence relation given by the identification $(\infty, z) \sim (\infty, z')$ if $\alpha_\infty(z) = \alpha_\infty(z')$. Hence the map

$$[\alpha] : \big(([0,\infty] \times \partial M_t)/\sim\big) \to M_t$$

induced by α defines a homeomorphism between M_t and the mapping cylinder of $\tilde{\xi}_t$.

The collapse along a path

We can do the construction of the vector field E_t simultaneously for all t in a simple path γ in $\mathbf{D}_\eta$ joining 0 and some $t_0 \in \partial\mathbf{D}_\eta$, such that γ is transverse to $\partial\mathbf{D}_\eta$. To simplify, we shall assume that γ is the closed segment of line in $\mathbf{D}_\eta$ joining 0 and t_0.

The natural projection $\pi : \mathcal{D} \times \mathbf{D}_\eta \to \mathbf{D}_\eta$ restricted to Δ induces a ramified covering

$$\pi_| : \Delta \to \mathbf{D}_\eta$$

whose ramification locus is $D_0 \cap \Delta = \{\phi(p_1), \ldots, \phi(p_r)\}$.

Hence the inverse image of $\gamma\backslash\{0\}$ by this covering defines k disjoint simple paths in Δ, and each one of them is diffeomorphic to $\gamma\backslash\{0\}$. Each of these paths have $\phi(p_i)$ in its closure, for some $i = 1, \ldots, r$, and it contains the points $y_j(t)$, for some $j = 1, \ldots, k$ and any $t \in \gamma \backslash \{0\}$. We shall denote by $\varsigma_{i,j}$ the respective path that has $\phi(p_i)$ in its closure and contains $y_j(t)$. In particular, we have that $r \leq k$. See figure 2.7 bellow.

Moreover, the set $\Lambda = \bigcup_{t\in\gamma} \lambda_t$ defines a simple path in $\mathcal{D} \times \mathbf{D}_\eta$ such that either $\Lambda \cap \Delta = \phi(p_1)$, if $r = 1$, or $\Lambda \cap \Delta = \emptyset$, if $r > 1$.

We can choose the paths $\delta(y_j(t))$ in such a way that

$$T_j := \bigcup_{t\in\gamma} \delta(y_j(t))$$

forms either a triangle, if $r = 1$, or a square, if $r > 1$, differentially immersed in

$$\bigcup_{t\in\gamma} D_t = \mathcal{D} \times \gamma$$

outside $\delta(y_j(0))$. For any $j, j' \in \{1, \ldots, k\}$ with $j \neq j'$, note that either $T_j \cap T_{j'} = \Lambda$, if both $\varsigma_{i,j}$ and $\varsigma_{i',j'}$ are defined for some $i, i' \in \{1, \ldots, r\}$ with $i \neq i'$; or $T_j \cap T_{j'} = \Lambda \cup \gamma(y_j(0)) = \Lambda \cup \gamma(y_{j'}(0))$, if both $\varsigma_{i,j}$ and $\varsigma_{i,j'}$ are defined for some $i \in \{1, \ldots, r\}$. See figure 2.8. Set

$$Q := \bigcup_{j=1}^{k} T_j.$$

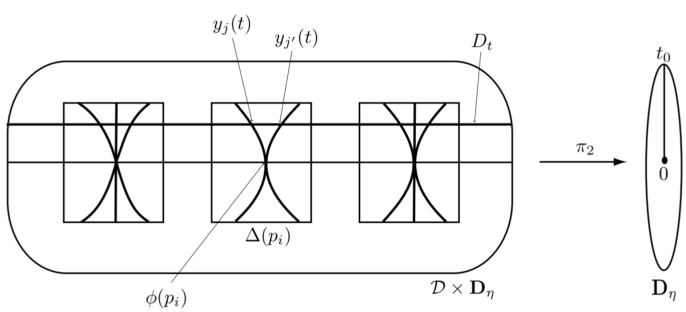


Figure 2.7:

Now, let V be a vector field in $\mathcal{D} \times \gamma$ such that V is:

- continuous;

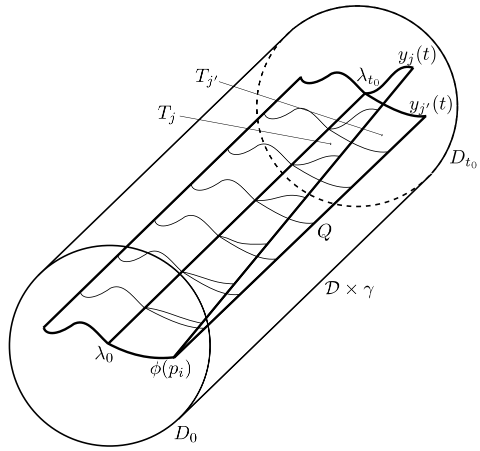


Figure 2.8:

- null over Q;
- differentiable over $(\mathcal{D} \times \gamma) \backslash Q$;
- transversal to $\partial \mathcal{D} \times \gamma$; and such that
- the projection of V on γ is null.

Then the associated flow $w : [0, \infty[\, \times \, \big((\mathcal{D} \times \gamma) \backslash Q\big) \to \mathcal{D} \times \gamma$ defines a map

$$\begin{array}{rcccc} \xi & : & \partial \mathcal{D} \times \gamma & \longrightarrow & Q \\ & & z & \longmapsto & \lim_{\tau \to \infty} w(\tau, z) \end{array},$$

such that ξ is continuous, surjective and differentiable.

For any real $A > 0$, set

$$V_A(Q) := (\mathcal{D} \times \gamma) \backslash w\big([0, A[\times \partial \mathcal{D} \times \gamma\big),$$

a closed neighbourhood of Q in $\mathcal{D} \times \gamma$. Note that $\partial V_A(Q)$ is a differentiable manifold that fibres over γ with fibre a circle, by the restriction of the projection π. Moreover, $\mathcal{D} \times \gamma$ is clearly the mapping cylinder of ξ.

Set

$$M_\gamma := \phi^{-1}(\mathcal{D} \times \gamma).$$

Since

$$\phi_| : M_\gamma \backslash \phi^{-1}(Q) \to (\mathcal{D} \times \gamma) \backslash Q$$

is a fibre bundle, it follows that $\phi^{-1}(\partial V_A(Q))$ is a differentiable submanifold of M_γ which is a fibre bundle over γ.

Now, set

$$P_\gamma := \phi^{-1}(Q),$$

which we call the *collapse polyhedron of f along γ*. It is a polyhedron in M_γ of real dimension 2. Let θ be a vector field in γ that goes from t_0 to 0 in time $a > 0$.

Set

$$Z := M_\gamma \backslash P_\gamma.$$

Since

$$Z = \phi^{-1}\big((\mathcal{D} \times \gamma)\backslash Q\big) \xrightarrow{\phi} (\mathcal{D} \times \gamma)\backslash Q \xrightarrow{\pi} \gamma$$

and

$$\phi^{-1}\big(\partial V_A(Q)\big) \xrightarrow{\phi} \partial V_A(Q) \xrightarrow{\pi} \gamma$$

are (differential) fibre bundles, we can lift θ to obtain a vector field E such that:

- E is differentiable;
- E is tangent to $\phi^{-1}\big(\partial V_A(Q)\big)$, for any $A > 0$.

Then the associated flow $g : [0, a] \times Z \to Z$ defines a C^∞-diffeomorphism Ψ from $M_{t_0} \backslash P_{t_0}$ to $M_0 \backslash P_0$ that extends to a continuous map from M_{t_0} to M_0 and that sends P_{t_0} to P_0. This proves proposition 2.2.9.

The collapse polyhedron

Consider the intersection of Δ with $\mathcal{D} \times \mathbf{D}^*_\eta$. Then we obtain k punctured disks Υ_j. The barycenter points of these punctured disks also give a punctured disk Λ such that the intersection of the closure of all this punctured disks is either $\phi(p_1)$, if $r = 1$, or empty, if $r > 1$. Fix $j \in \{1, \ldots, k\}$. The union of paths $\delta(y_j(t))$ for all $t \in \mathbf{D}_\eta$ gives a 3-dimensional polyhedron T_j in $\mathcal{D} \times \mathbf{D}_\eta$. Note that either $T_j \cap T_{j'} = \Lambda$ or $T_j \cap T_{j'} = \Lambda \cup \gamma(y_j(0)) = \Lambda \cup \gamma(y_{j'}(0))$.

Then we define

$$Q := \bigcup_{j=1}^{k} T_j$$

and, as before, we construct a vector field V in $\mathcal{D} \times \mathbf{D}_\eta$ that retracts $\mathcal{D} \times \mathbf{D}_\eta$ onto Q. Now set

$$P := \phi^{-1}(Q),$$

which we call the *collapse polyhedron of f along a disk*. It is a polyhedron of real dimension $n+1$, contained in M_η.

Also set

$$Z := M_\eta \backslash P.$$

Since ϕ is a submersion over $(\mathcal{D} \times \mathbf{D}_\eta)\backslash Q$, it follows that V lifts to a vector field E in Z with the desired properties, exactly as we did before. Then we have:

Corollary 2.2.11 *M_η deformation retracts to P.*

2.2.4 Proof of lemmas 2.2.3, 2.2.4, 2.2.5 and 2.2.6

Lemma 2.2.3 and Lemma 2.2.4 follow immediately, applying propositions 2.2.8 and 2.2.9, respectively, to the projection π_t. Lemma 2.2.5 follows setting P_t to be the collapse polyhedron P of π_t along Σ_ϵ, since we have showed that M_η deformation retracts to P (see corollary 2.2.11).

Now, since the construction of the Lê Polyhedron of both π_t and f_s depends only on the projection φ_t, which is the restriction of the linear form l to the Milnor fibre of the function, Lemma 2.2.6 is immediate: when constructing a Lê Polyhedron for f_s, one just have to consider the linear form l to be the same of that used in the construction of $P_{t,s}$. Note that since the set of general linear forms l, denoted by Ω, is a non-empty Zariski set in the space of all the linear forms of $\mathbb{C}^n$ to $\mathbb{C}$, we can assume that l is a general one (otherwise, just take another linear form l in the construction of $P_{t,s}$).

Chapter 3

Real analytic map-germs of the type $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$

In this chapter we study the topology of real analytic map-germs of the type $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, that is, the product of a holomorphic function and the complex conjugated of another holomorphic function.

Real analytic map-germs of this type have been studied by J. Seade and A. Pichon ([34], [35], [36], [37]), and their importance lies mainly in the fact that they are real analytic maps with many properties of complex functions. Hence, studying their topology is a good way of trying to understand the topology of real analytic singularities.

Precisely, consider $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ two holomorphic functions such that the real analytic map-germ given by

$$f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$$

has an isolated critical value at $0 \in \mathbb{C}$.

In [36], Pichon and Seade proved that such map-germs as above has the Thom a_f-property, and hence it admits a Milnor fibration in the Milnor tube, that is, there exist sufficient small positive reals $0 < \eta << \epsilon$ such that the restriction

$$f\bar{g}_| : (f\bar{g})^{-1}(\mathbf{D}^*_\eta) \cap \mathbf{B}_\epsilon \to \mathbf{D}^*_\eta$$

is a locally trivial fibre bundle.

In this chapter, we consider the Milnor fibre F_t and the singular fibre F_0 of the map-germ $f\bar{g}$ as defined in section 1.7. In section 3.1, we study the topology of F_t in terms of the embedded resolution of $f\bar{g}$. Then in section 3.2 we compare it to the Milnor fibre of the holomorphic germ of function $fg : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$.

In section 3.3, we generalize the A'Campo formula ([1]) to compute the zeta-function of the monodromy of $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ in terms of its embedded resolution (see sections 1.8 and 1.9).

Finally, in section 3.4 we construct a Lê Polyhedron for the isolated real analytic singularity $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with the extra hypothesis that g depends only on one variable, in a fashion very similar to section 2.1. This is theorem 3.4.1

3.1 The topology of the Milnor fibre of $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$

Let $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a real analytic map-germ as above, with a Milnor fibration in the tube. Let $\pi : \tilde{M} \to \mathbf{B}_\epsilon$ be a good embedded resolution of F_0, as in section 1.9, where ϵ is a Milnor radius for f. Then consider:

- $\pi^{-1}(0) := E = \bigcup_{i=1}^{s} E_i$, the exceptional divisor with its decomposition in irreducible components;
- $(f\bar{g} \circ \pi)^{-1}(0) = (fg \circ \pi)^{-1}(0) = \sum_{i=1}^{s} k_i E_i + \tilde{C}$ the total transform, where $\tilde{C}$ is the strict transform, which has a decomposition into connected components $\tilde{C} = \bigcup_{p=1}^{w} \tilde{C}_p$;
- $F_\eta := (f\bar{g} \circ \pi)^{-1}(\eta)$, the Milnor fibre of $f\bar{g}$.

For each i, $1 \leq i \leq s$, let U_i be a (closed) tubular neighbourhood of E_i, and for each p, $1 \leq p \leq w$ let $\tilde{U}_p$ be a tubular neighbourhood of $\tilde{C}_p$. Then we define the following closed sets (see figure 3.1):

- $V_{ij} := U_i \cap U_j$;
- $\tilde{V}_{ip} := U_i \cap \tilde{U}_p$;
- $V_i := U_i \backslash \big(\bigcup_{\substack{i,j=1 \\ i \neq j}}^{s} V_{ij} \cup \bigcup_{p=1}^{w} \tilde{V}_{ip} \big)$;
- $\tilde{V}_p := \tilde{U}_p \backslash \bigcup_{i=1}^{s} \tilde{V}_{ip}$.

We decompose the Milnor fibre F_η as follows:

$$F_\eta = \left(\bigcup_{i=1}^{s} (V_i \cap F_\eta) \right) \cup \left(\bigcup_{\substack{i,j=1 \\ i \neq j}}^{s} (V_{ij} \cap F_\eta) \right) \cup \left(\bigcup_{p=1}^{w} (\tilde{V}_p \cap F_\eta) \right) \cup \left(\bigcup_{\substack{1 \leq i \leq s \\ 1 \leq p \leq w}} (\tilde{V}_{ip} \cap F_\eta) \right)$$

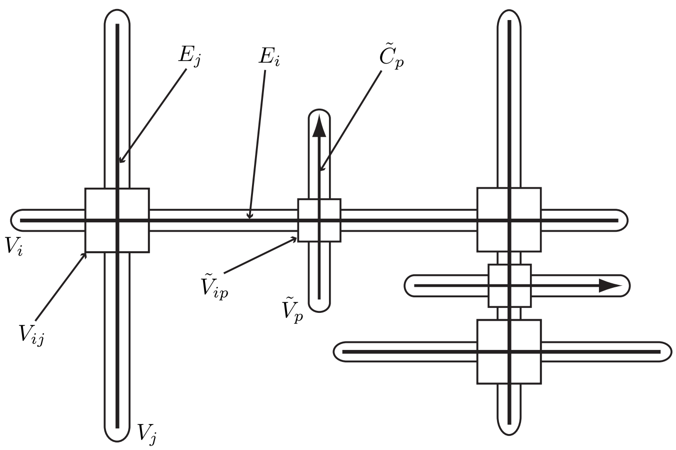


Figure 3.1:

Note that doing some convenient change of coordinates, each part $V_{ij} \cap F_\eta$ or $\tilde{V}_{ip} \cap F_\eta$ of the Milnor fibre F_η has equation of the form

$$x^{a_i}\bar{x}^{b_i}y^{a_j}\bar{y}^{b_j}\varphi_1\overline{\varphi_2} = \eta$$

and each part $V_i \cap F_\eta$ or $\tilde{V}_p \cap F_\eta$ of the Milnor fibre has equation of the form

$$x^{a_i}\bar{x}^{b_i}\varphi_1\overline{\varphi_2} = \eta,$$

where $\varphi_1(x, y)$ and $\varphi_2(x, y)$ are units in $\mathbb{C}\{x, y\}$, a_i is the multiplicity of f corresponding to E_i and b_i is the multiplicity of g corresponding to E_i (obviously either $a_p = 1$ and $b_p = 0$ or $a_p = 0$ and $b_p = 1$, for each $p = 1, \ldots w$).

Lemma 3.1.1 *The intersection of the Milnor fibre F_η with each neighbourhood V_i, V_{ij}, $\tilde{V}_p$ or $\tilde{V}_{ip}$ is homeomorphic to either a finite disjoint union of cylinders (cases (i), (iii) and (iv) in the proof) or to a finite covering over a disk minus some disks (case (ii) in the proof).*

Proof: There are four cases to consider:

(i) $F_\eta \cap V_{ij}$ with $a_i \neq b_i$ and $a_j \neq b_j$; and $F_\eta \cap \tilde{V}_{ip}$ with $a_i \neq b_i$:
By some change of coordinates we can locally consider

$$(f\bar{g} \circ \pi) = x^{a_i}\bar{x}^{b_i}y^{a_j}\bar{y}^{b_j}\varphi_1\overline{\varphi_2},$$

where

$$\begin{cases} \varphi_1(x,y) = \alpha + \psi_1(x,y) \\ \varphi_2(x,y) = \beta + \psi_2(x,y) \end{cases},$$

with $\alpha, \beta \in \mathbb{C}^*$ and $\psi_1(0) = \psi_2(0) = 0$. If we set

$$\begin{cases} \varphi_{1,t}(x,y) = \alpha + t\psi_1(x,y) \\ \varphi_{2,t}(x,y) = \beta + t\psi_2(x,y) \end{cases},$$

we can define the 1-parameter family

$$h_t = f_t\overline{g_t} = x^{a_i}\bar{x}^{b_i}y^{a_j}\bar{y}^{b_j}\varphi_{1,t}\overline{\varphi_{2,t}},$$

defined on V_{ij}, which gives a homotopy between $h_1 = (f\bar{g}\circ\pi)$ and $h_0 = x^{a_i}\bar{x}^{b_i}y^{a_j}\bar{y}^{b_j}\alpha\bar{\beta}$. We want to show that $h_1^{-1}(\eta)$ is homeomorphic to $h_0^{-1}(\eta)$.

Consider the real analytic mapping

$$\begin{array}{rccc} H: & V_{ij}\times[0,1] & \to & \mathbb{C}\times[0,1] \\ & (z,t) & \mapsto & \big(h_t(z),t\big) \end{array}$$

and define

$$M := \big(V_{ij}\times[0,1]\big) \cap H^{-1}\big(\mathbf{D}_\eta^*\times[0,1]\big).$$

Then consider the restriction

$$H_| : M \to \mathbf{D}_\eta^* \times [0,1].$$

We will show that $H_|$ has three properties:

- Clearly, it is proper.
- It is a submersion on M:

  The Jacobian matrix of H is given by

$$\begin{pmatrix} \frac{\partial h_t}{\partial x} & \frac{\partial h_t}{\partial \bar{x}} & \frac{\partial h_t}{\partial y} & \frac{\partial h_t}{\partial \bar{y}} & \frac{\partial h_t}{\partial t} \\ \frac{\partial \overline{h_t}}{\partial x} & \frac{\partial \overline{h_t}}{\partial \bar{x}} & \frac{\partial \overline{h_t}}{\partial y} & \frac{\partial \overline{h_t}}{\partial \bar{y}} & \frac{\partial \overline{h_t}}{\partial t} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial \bar{x}} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial \bar{y}} & \frac{\partial t}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_t}{\partial x}\overline{g_t} & f_t\overline{\frac{\partial g_t}{\partial x}} & \frac{\partial f_t}{\partial y}\overline{g_t} & f_t\overline{\frac{\partial g_t}{\partial y}} & x^{a_i}y^{a_j}\psi_1\bar{g}_t \\ \bar{f}_t\frac{\partial g_t}{\partial x} & \overline{\frac{\partial f_t}{\partial x}}g_t & \bar{f}_t\frac{\partial g_t}{\partial y} & \overline{\frac{\partial f_t}{\partial y}}g_t & x^{b_i}y^{b_j}\psi_2\bar{f}_t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

  Then $H_|$ is a submersion in a point p if, and only if, it is not a solution of at least one of the four following equations:

$$\begin{cases} (1) \quad |\frac{\partial f_t}{\partial x}|^2|g_t|^2 - |\frac{\partial g_t}{\partial x}|^2|f_t|^2 = 0 \\ (2) \quad |\frac{\partial f_t}{\partial y}|^2|g_t|^2 - |\frac{\partial g_t}{\partial y}|^2|f_t|^2 = 0 \\ (3) \quad |f_t|^2\frac{\partial g_t}{\partial x}\overline{\frac{\partial g_t}{\partial y}} - |g_t|^2\frac{\partial f_t}{\partial x}\overline{\frac{\partial f_t}{\partial y}} = 0 \\ (4) \quad fg\big(\frac{\partial f_t}{\partial x}\frac{\partial g_t}{\partial y} - \frac{\partial f_t}{\partial y}\frac{\partial g_t}{\partial x}\big) = 0 \end{cases}$$

Note that $f_t = x^{a_i} y^{a_j} \varphi_{1,t}$ and $g_t = x^{b_i} y^{b_j} \varphi_{2,t}$. Then setting

$$\begin{cases} \zeta_1 = a_i \varphi_{1,t} + x \frac{\partial \varphi_{1,t}}{\partial x} \\ \zeta_2 = a_j \varphi_{1,t} + y \frac{\partial \varphi_{1,t}}{\partial y} \\ \zeta_3 = b_i \varphi_{2,t} + x \frac{\partial \varphi_{2,t}}{\partial x} \\ \zeta_4 = b_j \varphi_{2,t} + y \frac{\partial \varphi_{2,t}}{\partial y} \end{cases}$$

we have that

$$\begin{cases} \frac{\partial f_t}{\partial x} = x^{a_i - 1} y^{a_j} \zeta_1 \\ \frac{\partial f_t}{\partial y} = x^{a_i} y^{a_j - 1} \zeta_2 \\ \frac{\partial g_t}{\partial x} = x^{b_i - 1} y^{b_j} \zeta_3 \\ \frac{\partial g_t}{\partial y} = x^{b_i} y^{b_j - 1} \zeta_4 \end{cases}$$

Substituting on equations (1) to (4), we have the equations:

$$\begin{cases} (1) \quad |x|^{2(a_i+b_i-1)} |y|^{2(a_j+b_j)} \begin{vmatrix} |\zeta_1|^2 & |\zeta_3|^2 \\ |\varphi_{1,t}|^2 & |\varphi_{2,t}|^2 \end{vmatrix} = 0 \\ (2) \quad |x|^{2(a_i+b_i)} |y|^{2(a_j+b_j-1)} \begin{vmatrix} |\zeta_2|^2 & |\zeta_4|^2 \\ |\varphi_{1,t}|^2 & |\varphi_{2,t}|^2 \end{vmatrix} = 0 \\ (3) \quad \bar{x} y |x|^{2(a_i+b_i-1)} |y|^{2(a_j+b_j-1)} \begin{vmatrix} \zeta_1 \overline{\zeta_2} & \zeta_3 \overline{\zeta_4} \\ |\varphi_{1,t}|^2 & |\varphi_{2,t}|^2 \end{vmatrix} = 0 \\ (4) \quad x^{2a_i+2b_i-1} y^{2a_j+2b_j-1} \varphi_{1,t} \varphi_{2,t} \begin{vmatrix} \zeta_1 & \zeta_2 \\ \zeta_3 & \zeta_4 \end{vmatrix} = 0 \end{cases}$$

Since

$$\begin{vmatrix} |\zeta_1(0)|^2 & |\zeta_3(0)|^2 \\ |\varphi_{1,t}(0)|^2 & |\varphi_{2,t}(0)|^2 \end{vmatrix} = |a_i^2 - b_i^2|.|\varphi_{1,t}(0)\varphi_{2,t}(0)|^2$$

and

$$\begin{vmatrix} |\zeta_2(0)|^2 & |\zeta_4(0)|^2 \\ |\varphi_{1,t}(0)|^2 & |\varphi_{2,t}(0)|^2 \end{vmatrix} = |a_j^2 - b_j^2|.|\varphi_{1,t}(0)\varphi_{2,t}(0)|^2,$$

the result follows.

- It is a submersion on the boundary ∂M:
  By corollary 1.7.4, we have to show that $T_x \partial M$ intersects $T_x H^{-1}(H(x))$ transversally, for any $x \in \partial M$. Then one can check that it is enough to show that $T_x W_t$

intersects $T_x h_t^{-1}(h_t(x))$ transversally, for any $x \in W_t$ and $t \in [0,1]$, where $W_t := \partial V_{ij} \cap h_t^{-1}(\mathbf{D}_\eta^*)$.
So we just have to show that the fibres $h_t^{-1}(x)$ intersect ∂V_{ij} transversally, for any $x \in \mathbf{D}_\eta^*$ and $t \in [0,1]$. But this follows immediately from the fact that $h_t = f_t \bar{g}_t$ has the Thom a_f-property.

Then if follows from Ehresmann's Fibration Lemma that the Milnor fibre of $h_1 = (f\bar{g}) \circ \pi$ is diffeomorphic to the Milnor fibre of $h_0 = x^{a_i}\bar{x}^{b_i}y^{a_j}\bar{y}^{b_j}\alpha\bar{\beta}$, which is diffeomorphic to

$$\left\{ x^{a_i}\bar{x}^{b_i} = \frac{\eta}{y^{a_j}\bar{y}^{b_j}} \right\} \cap (D_{\epsilon_1} \times D_{\epsilon_2}),$$

for some small $\epsilon_1, \epsilon_2 << \epsilon$, which is a covering over an annulus, and therefore it is a disjoint union of cylinders.

(ii) $F_\eta \cap V_i$ with $a_i \neq b_i$ and $F_\eta \cap \tilde{V}_p$:
We can apply exactly the same proof of case (i), considering $(f\bar{g} \circ \pi) = x^{a_i}\bar{x}^{b_i}\varphi_1\overline{\varphi_2}$. Then we get that the Milnor fibre of $f\bar{g}$ inside V_i is diffeomorphic to the set

$$\{x^{a_i}\bar{x}^{b_i} = \eta\},$$

and therefore it is an $|a_i - b_i|$-covering of $V_i \cap E_i$, which is a disk minus some disks.

(iii) $F_\eta \cap V_{ij}$ with $a_i = b_i$ and $a_j \neq b_j$ or with $a_i \neq b_i$ and $a_j = b_j$; and $F_\eta \cap \tilde{V}_{ip}$ with $a_i = b_i$:
Consider the mapping germ

$$\begin{array}{rccc} (f \circ \pi, g \circ \pi) : & (\mathbb{C}^2, 0) & \to & (\mathbb{C}^2, 0) \\ & (x, y) & \mapsto & (x^{a_i}y^{a_j}\varphi_1, x^{b_i}y^{b_j}\varphi_2) \end{array}$$

We want to find a change of coordinates $\Theta = (\Theta_1, \Theta_2) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that

$$(f \circ \pi, g \circ \pi) = (x^{a_i}y^{a_j}, x^{b_i}y^{b_j}) \circ \Theta,$$

which happens if and only if $(f \circ \pi, g \circ \pi) = (\Theta_1^{a_i}\Theta_2^{a_j}, \Theta_1^{b_i}\Theta_2^{b_j})$. If we set $\Theta_1 = x\theta_1$ and $\Theta_2 = y\theta_2$, with $\theta_1(0) \neq 0$ and $\theta_2(0) \neq 0$, then our problem is to find $\theta_1, \theta_2 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ such that

$$(f \circ \pi, g \circ \pi) = (x^{a_i}y^{a_j}\theta_1^{a_i}\theta_2^{a_j}, x^{b_i}y^{b_j}\theta_1^{b_i}\theta_2^{b_j}).$$

This happens if and only if the system

$$\begin{cases} x^{a_i}y^{a_j}\theta_1^{a_i}\theta_2^{a_j} = x^{a_i}y^{a_j}\varphi_1 \\ x^{b_i}y^{b_j}\theta_1^{b_i}\theta_2^{b_j} = x^{b_i}y^{b_j}\varphi_2 \end{cases}$$

has solution (θ_1,θ_2). This is equivalent to

$$\begin{cases}\theta_1^{a_i}\theta_2^{a_j}=\varphi_1\\ \theta_1^{b_i}\theta_2^{b_j}=\varphi_2,\end{cases}$$

which has solution, if and only if the linear system with variables $log\theta_1, log\theta_2$

$$\begin{cases}a_i\log\theta_1+a_j\log\theta_2=\log\varphi_1\\ b_i\log\theta_1+b_j\log\theta_2=\log\varphi_2\end{cases}$$

has solution. But we have the non-vanishing of the determinant

$$\begin{vmatrix}a_i & a_j\\ b_i & b_j\end{vmatrix}\neq 0,$$

and so the system has solutions. Then $(f\bar{g}\circ\pi)=(x^{a_i}\bar{x}^{b_i}|y|^{2b_i})\circ\Theta$ and therefore its Milnor fibre is given by the equation

$$\{|y|^{2b_j}=\frac{\eta}{x^{a_i}\bar{x}^{b_i}}\}\cap V_{ij},$$

which is a disjoint union of cylinders (which intersect $\partial\overline{V}_{ij}$ transversally).

(iv) $F_\eta\cap V_i$ with $a_i=b_i$ and $F_\eta\cap V_{ij}$ with $a_i=b_i$ and $a_j=b_j$:
A. Pichon and J. Seade [35] proved that $a_i=b_i$ implies that E_i does not represent a rupture vertex of the dual graph $\mathcal{G}$ of the total transform of $(fg)^{-1}(0)$ by π, that is, E_i represents a vertex with only one or two incident edges.
Let S be the union of all the exceptional divisors E_i such that $a_i=b_i$ and consider its decomposition in connected components $S=S_1\cup\cdots\cup S_k$. For each S_l, let Ω_l be the union of all the V_i, V_{ij} and $\tilde{V_{ip}}$ that intersect S_l, excluding the V_{ij}'s that intersect some E_i with $a_i\neq b_i$ (there are at least one and at most two of them). There are only two cases to consider:

- (a) There is just one V_{ij} in S_l that intersects some E_i with $a_i\neq b_i$.
- (b) There are exactly two V_{ij}'s in S_l that intersect some E_i's with $a_i\neq b_i$.

We claim that case (a) does never occur. To show that, we shall consider the subgraph $\mathcal{G}'\subset\mathcal{G}$ containing S_l and the corresponding $E_j\notin S_l$ such that $E_{jk}=1$, for some $E_k\in S_l$. Note that then $a_j\neq b_j$.
Now, we know that if M denotes the $(s\times s)$-intersection matrix of $S_l\cup E_j$, that is,

$$m_{ij}=\begin{cases}E_i^2,\ if\ i=j;\\ 1,\ if\ i\neq j\ and\ E_i\ intersects\ E_j\\ 0,\ otherwise\end{cases},$$

and if u_i denotes the number of intersection points between E_i and the strict transform of f minus the number of intersection points between E_i and the strict transform of g, then we have (see [11], Theorem 18.2):

$$M \cdot \begin{pmatrix} a_j - b_j \\ \vdots \\ a_s - b_s \end{pmatrix} + \begin{pmatrix} u_j \\ \vdots \\ u_s \end{pmatrix} = 0$$

Hence we have

$$\begin{pmatrix} E_j^2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & E_k^2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & E_{s-1}^2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & E_s^2 \end{pmatrix} \cdot \begin{pmatrix} a_j - b_j \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u_j \\ 0 \\ \vdots \\ 0 \\ u_s \end{pmatrix} = 0,$$

which implies that $a_j = b_j$, a contradiction. In fact, it follows that there cannot be two consecutive vertices with $a_i = b_i$, that is, for each l one has $S_l = E_i$ and $\Omega_l = V_i$.

Now, it is not difficult to see that the piece

$$(f\bar{g} \circ \pi)^{-1}(\mathbf{S}_\eta) \cap \Omega_l$$

of the Milnor tube $(f\bar{g} \circ \pi)^{-1}(\mathbf{S}_\eta)$ has the homotopy type of a torus $\mathbb{T}^2$ (in fact, it is a $\mathbf{S}^1$-bundle over a punctured disk). Since the restriction

$$(f\bar{g} \circ \pi)_| : (f\bar{g} \circ \pi)^{-1}(\mathbf{S}_\eta) \cap \Omega_l \to \mathbf{S}_\eta$$

is the projection of a locally trivial fibre bundle, and supposing that $F_\eta \cap \Omega_l$ is connected, we get the following induced sequence of homotopy groups:

$$\pi_2(\mathbf{S}^1) \to \pi_1(F_\eta \cap \Omega_l) \to \pi_1(\mathbb{T}^2) \to \pi_1(\mathbf{S}^1) \to \pi_0(F_\eta \cap \Omega_l),$$

which is isomorphic to

$$0 \to \pi_1(F_\eta \cap \Omega_l) \to \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \to 0.$$

Hence $\pi_1(F_\eta \cap \Omega_l)$ is isomorphic to $\mathbb{Z}$, and since $F_\eta \cap \Omega_l$ has two boundary components (two circles), it follows that $F_\eta \cap \Omega_l$ is a cylinder (see section 1.4).

If the piece of the Milnor fibre $F_\eta \cap \Omega_l$ has more than one connected component, since the part of the Milnor tube $(f\bar{g} \circ \pi)^{-1}(\mathbf{S}_\eta) \cap \Omega_l$ is connected, the monodromy m must define a one-cycle permutation $(F_1, \ldots, F_r)$ on the connected components of $F_\eta \cap \Omega_l$. In particular, they are all diffeomorphic. For $i \neq r$, let m_i denote the diffeomorphism from F_i to F_{i+1}, and let m_r be the diffeomorphism from F_r to F_1. Then we can construct a fibre bundle over the circle with total space

$$(f\bar{g} \circ \pi)^{-1}(\mathbf{S}_\eta) \cap \Omega_l$$

and with fibre F_1 in the following way: consider the circle given by the identification of the boundary of the interval $[0, r]$. The space $(f\bar{g} \circ \pi)^{-1}(\mathbf{S}_\eta) \cap \Omega_l$ is diffeomorphic to the quotient of $\coprod_{i=0}^{r-1} [i, i+1] \times F_{i-1}$ by the identification of (i, x) with $(i+1, m_i(x))$ for any $i < r$ and (r, x) with $(0, m_r(x))$. The projection to the circle is now obvious.

∎

We shall need the following Remark on families of functions of the form $f_t\bar{g}_t$, which is an adaptation of the corresponding result for families of holomorphic functions.

Remark 3.1.2 *Let $f_t : (\mathbb{C}^2, 0) \to \mathbb{C}$, $g_t : (\mathbb{C}^2, 0) \to \mathbb{C}$ be families of holomorphic germs depending holomorphically on a parameter t which varies in a disk D. Suppose that $f_t g_t$ has an isolated singularity at the origin, whose Milnor number is independent of t. For any $t_0 \in D$, there exists a neighbourhood $t_0 \in U \subset D$, and a positive number ϵ which is a Milnor radius for $f_t\bar{g}_t$, for any $t \in U$.*

Proof: A Milnor radius for a real analytic germ of the form $f\bar{g}$ is a positive radius ϵ such that for any other radius $\epsilon' \leq \epsilon$, the sphere of radius ϵ' centered at the origin meets $fg^{-1}(0)$ transversely. Thus ϵ is a Milnor radius for the real analytic germ $f\bar{g}$ if and only if it is a Milnor radius for the holomorphic germ fg. The assertion of the Remark follows from the corresponding one for μ-constant families of holomorphic germs of functions in two variables: its well known that a μ-constant family of plane curves is Whitney equisingular and admits a uniform Milnor radius.

∎

3.2 Comparing the Milnor fibres of fg and $f\bar{g}$

Now we would like to compare the Milnor fibre of a real analytic map-germ with an isolated critical value of the type $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with the Milnor fibre of the holomorphic germ $fg : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, supposing that g is not constant.

In order to simplify notation, we shall denote the Milnor fibre of fg by F^{fg} and the Milnor fibre of $f\bar{g}$ by $F^{f\bar{g}}$. When we just write F, one can read F^{fg} or $F^{f\bar{g}}$.

Let F_i denote the part of the Milnor fibre inside V_i, F_{ij} denote the part of the Milnor fibre inside V_{ij} and $\tilde{F_{is}}$ denote the part of the Milnor fibre inside $\tilde{V_{is}}$, as before. We know that the Euler characteristic of F is

$$\chi(F) = \sum_{i=1}^{n} \chi(F_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \chi(F_{ij}) + \sum_{s} \chi(\tilde{F_{is}}).$$

In the holomorphic case, we know that each F_i^{fg} is a $(a_i + b_i)$-covering of the sphere minus r_i points, where r_i is the number of double points of E on E_i plus the number of points of the intersection $E_i \cap \tilde{C}$, and that each F_{ij} and each $\tilde{F_{is}}$ is a disjoint union of cylinders (see [6]). Then $\chi(F_{ij}^{fg}) = \chi(\tilde{F_{is}}) = 0$ and

$$\chi(F^{fg}) = \sum_{i=1}^{n} (a_i + b_i)(2 - r_i).$$

In the real analytic case $f\bar{g}$, it follows from Lemma 3.1.1 that

$$\chi(F^{f\bar{g}}) = \sum_{i=1}^{n} |a_i - b_i|(2 - r_i).$$

Since F^{fg} and $F^{f\bar{g}}$ are surfaces with the same boundary, they are homeomorphic if, and only if, $\chi(F^{fg}) = \chi(F^{f\bar{g}})$, that is, if and only if

$$\sum_{r_i > 2} \big((a_i + b_i) - |a_i - b_i|\big)(2 - r_i) + \sum_{r_i = 1} \big((a_i + b_i) - |a_i - b_i|\big) = 0.$$

This global approach does not seem to help in the matter of deciding if the Milnor fibres of fg and of $f\bar{g}$ are homeomorphic or not. So we should split the dual graph of the resolution π into smaller parts:

For each rupture vertex (j) of the dual graph of the resolution (that is, $r_j \geq 3$), denote by A_j the subgraph consisting of (j) and all the adjacent bamboos having an extremity (k) with $r_k = 1$. Then

$$\chi(F^{fg}) = \sum_{j} \chi(A_j^{fg})$$

and

$$\chi(F^{f\bar{g}}) = \sum_{j} \chi(A_j^{f\bar{g}}),$$

where $\chi(A_j^{fg})$ (resp. $\chi(A_j^{f\bar{g}})$) denotes the Euler characteristic of the part of the Milnor fibre of fg (resp. $f\bar{g}$) that lies inside the boxes that intersect the part of the resolution graph corresponding to A_j.

Lemma 3.2.1 *For each rupture vertex* (j) *of the dual graph of the resolution, one has*

$$\chi(A_j^{fg}) < \chi(A_j^{f\bar{g}}).$$

Proof: Let I be the set of rupture vertex (j) such that A_j is given by only one point (which means that (j) does not have any adjacent bamboo with extremity (k) with $r_k = 1$). It is easy to see that if $(j) \in I$, then

$$\chi(A_j^{fg}) = (a_j + b_j)(2 - r_j) < |a_j - b_j|(2 - r_j) = \chi(A_j^{f\bar{g}}).$$

Now, if $(j) \notin I$, we define

$$a_i' := \sum_{\substack{(i)\in A_j \\ r_i=1}} a_i \quad \text{and} \quad b_i' := \sum_{\substack{(i)\in A_j \\ r_i=1}} b_i,$$

and observe that $a_i' < a_j$ and $b_i' < b_j$. Then

$$\begin{aligned}
\Delta(\chi) &:= \chi(A_j^{fg}) - \chi(A_j^{f\bar{g}}) \\
&= -(r_j - 2)(a_j + b_j) + \textstyle\sum_{\substack{(i)\in A_j \\ r_i=1}} (a_i + b_i) + (r_j - 2)|a_j - b_j| - \textstyle\sum_{\substack{(i)\in A_j \\ r_i=1}} |a_i - b_i| \\
&= (a_i' + b_i') - (r_j - 2)(a_j + b_j) + (r_j - 2)|a_j - b_j| - \textstyle\sum_{\substack{(i)\in A_j \\ r_i=1}} |a_i - b_i|.
\end{aligned}$$

There are only two possibilities:

(1) Suppose there is just one bamboo in A_j, that is, there is just one $(k) \in A_j$ with $r_k = 1$. Then:

- (i) If $a_k > b_k$ and $a_j > b_j$, then $\Delta(\chi) = 2\big(b_k - (r_j - 2)b_j\big) \leq 2(b_k - b_j) < 0$, since $b_k < b_j$;
- (ii) If $a_k < b_k$ and $a_j < b_j$, then $\Delta(\chi) = 2\big(a_k - (r_j - 2)a_j\big) \leq 2(a_k - a_j) < 0$;
- (iii) If $a_k < b_k$ and $a_j > b_j$, then $\Delta(\chi) = 2\big(a_k - (r_j - 2)b_j\big) < 2\big(b_k - (r_j - 2)b_j\big) \leq 2(b_k - b_j) < 0$;
- (iv) If $a_k > b_k$ and $a_j < b_j$, then $\Delta(\chi) = 2\big(b_k - (r_j - 2)a_j\big) < 2\big(a_k - (r_j - 2)a_j\big) \leq 2(a_k - a_j) < 0$, since $a_k < a_j$.

(2) Suppose there are two bamboos in A_j, that is, there are $E_{k_1}, E_{k_2} \in A_j$ with $r_{k_1} = r_{k_2} = 1$. Since

$$|a_{k_1} - b_{k_1}| + |a_{k_2} - b_{k_2}| \geq |a_i' - b_i'|,$$

it follows that

$$-(|a_{k_1} - b_{k_1}| + |a_{k_2} - b_{k_2}|) \leq -|a_i' - b_i'|,$$

and then

$$\Delta(\chi) \leq (a_i' + b_i') - (r_j - 2)(a_j + b_j) + (r_j - 2)|a_j - b_j| - |a_i' - b_i'|,$$

which is < 0 by case (1).

■

Hence, since

$$\chi(F^{fg}) = \sum_{j \in I} \chi(A_j^{fg}) + \sum_{j \notin I} \chi(A_j^{fg})$$

and

$$\chi(F^{f\bar{g}}) = \sum_{j \in I} \chi(A_j^{f\bar{g}}) + \sum_{j \notin I} \chi(A_j^{f\bar{g}}),$$

we have proved the following theorem:

Theorem 3.2.2 *The Milnor fibre of a real analytic germ with an isolated critical value of the type $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is homeomorphic to the Milnor fibre of the holomorphic germ $fg : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ if, and only if, g is a constant function.*

3.3 The horizontal monodromy of $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$

Now we want to give a formula to calculate the zeta function (see section 1.8) of the monodromy h of $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with an isolated critical value, in terms of the combinatorics of the embedded resolution of fg, in the same spirit of the work of A'Campo [1], where he calculates the zeta function of the monodromy for holomorphic functions.

If we set $F_\theta := (f\bar{g})^{-1}(\eta e^{i\theta})$, for η sufficiently small, then for each $q \geq 0$, the monodromy $h : F_\theta \to F_\theta$ defines an isomorphism between vector spaces (the homology groups) given by $h^* : H^q(F_\theta; \mathbb{C}) \to H^q(F_\theta; \mathbb{C})$, the so called *algebraic monodromy.*

There is a classical way of calculating the zeta function of h in terms of the Lefschetz numbers of h, as follows: For each $k \geq 1$, the Lefschetz number of the k-iteration of h is defined by

$$\Lambda(h^k) = \sum_{q \geq 0} (-1)^q trace[(h^*)^k : H^q(F_\theta, \mathbb{C}) \to H^q(F_\theta, \mathbb{C})].$$

If we define the integers $s_1, s_2, \dots$ by the relations

$$\Lambda(h^k) = \sum_{i|k} s_i,$$

then the zeta function of h is given by

$$Z(t) = \prod_{i \geq 1} (1 - t^i)^{-s_i/i}.$$

So all we have to do is to calculate the Lefschetz numbers of h. First we recall the following lemma:

Lemma 3.3.1 *Consider the following commutative chain map on an exact sequence:*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & G_0 & \xrightarrow{\alpha_0} & G_1 & \xrightarrow{\alpha_1} & \dots & \xrightarrow{\alpha_{n-1}} & G_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & & & \downarrow{\scriptstyle \varphi_n} & & \\ 0 & \longrightarrow & G_0 & \xrightarrow{\alpha_0} & G_1 & \xrightarrow{\alpha_1} & \dots & \xrightarrow{\alpha_{n-1}} & G_n & \longrightarrow & 0 \end{array}$$

Then

$$\sum_{i=0}^{n} (-1)^i \mathrm{trace}[\varphi_i] = 0$$

Proof: We do it by induction on n. So first we suppose $n = 1$, and then we have the following exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & G_0 & \xrightarrow{\alpha_0} & G_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & \\ 0 & \longrightarrow & G_0 & \xrightarrow{\alpha_0} & G_1 & \longrightarrow & 0 \end{array}$$

Since $\varphi_1 \circ \alpha_0 = \alpha_0 \circ \varphi_0$ and since α_0 is an isomorphism, it follows that $\varphi_1 = \alpha_0 \circ \varphi_0 \circ \alpha_0^{-1}$, and therefore $trace(\varphi_0) = trace(\varphi_1)$.

Now suppose that the result is true for $n-1$ and write $G_{n-1} = Im(\alpha_{n-2}) \oplus G_{n-1}/Im(\alpha_{n-2})$. Let φ^a_{n-1} be the restriction of φ_{n-1} to $Im(\alpha_{n-2})$ and let φ^b_{n-1} be the restriction of φ_{n-1} to the quocient $\big(G_{n-1}/Im(\alpha_{n-2})\big)$.

Since $Ker(\alpha_{n-1}) = Im(\alpha_{n-2})$, it follows that $\varphi^a_{n-1}\big(Im(\alpha_{n-2})\big) = \big(Im(\alpha_{n-2})\big)$ and that $\varphi^b_{n-1}\big(G_{n-1}/Im(\alpha_{n-2})\big) = \big(G_{n-1}/Im(\alpha_{n-2})\big)$. Then we can split the commutative diagram in two:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & G_0 & \xrightarrow{\alpha_0} & G_1 & \xrightarrow{\alpha_1} & \dots & \xrightarrow{\alpha_{n-2}} & Im(\alpha_{n-2}) & \xrightarrow{\alpha_{n-1}} & 0 \\ & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & & & \downarrow{\scriptstyle \varphi^a_{n-1}} & & \\ 0 & \longrightarrow & G_0 & \xrightarrow{\alpha_0} & G_1 & \xrightarrow{\alpha_1} & \dots & \xrightarrow{\alpha_{n-2}} & Im(\alpha_{n-2}) & \xrightarrow{\alpha_{n-1}} & 0 \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & G_{n-1}/Im(\alpha_{n-2}) & \xrightarrow{\alpha_{n-1}} & G_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi^b_{n-1}} & & \downarrow{\scriptstyle \varphi_n} & & \\ 0 & \longrightarrow & G_{n-1}/Im(\alpha_{n-2}) & \xrightarrow{\alpha_{n-1}} & G_n & \longrightarrow & 0 \end{array}$$

By the induction hypothesis, we know that

$$\sum_{i=0}^{n-2}(-1)^i trace[\varphi_i]+(-1)^{n-1}trace[\varphi_{n-1}^a]=0.$$

Moreover, one can see exactly as in the case $n=1$ that $trace[\varphi_{n-1}^b]=trace[\varphi_n]$, and since $trace[\varphi_{n-1}]=trace[\varphi_{n-1}^a]+trace[\varphi_{n-1}^b]$, the result follows.

■

Lemma 3.3.2 *If X is the union of the interiors of topological spaces A and B and if h is an automorphism of X such that $h(A)=A$ and $h(B)=B$, then*

$$\Lambda(h_X)=\Lambda(h_A)+\Lambda(h_B)-\Lambda(h_{A\cap B}).$$

Proof: Consider the following chain map on the Mayer-Vietoris sequence:

$$\begin{array}{ccccccccc}
\longrightarrow & H^q(X) & \longrightarrow & H^q(A)\oplus H^q(B) & \longrightarrow & H^q(A\cap B) & \longrightarrow & H^{q+1}(X) & \longrightarrow\\
 & \downarrow{\scriptstyle h_X^*} & & \downarrow{\scriptstyle (h_A^*,h_B^*)} & & \downarrow{\scriptstyle h_{A\cap B}^*} & & \downarrow{\scriptstyle h_X^*} & \\
\longrightarrow & H^q(X) & \longrightarrow & H^q(A)\oplus H^q(B) & \longrightarrow & H^q(A\cap B) & \longrightarrow & H^{q+1}(X) & \longrightarrow
\end{array}$$

Then it follows from the previous lemma that

$$\sum_{q\geq 0}(-1)^q trace[h_X^*:H^q(X;\mathbb{C})\to H^q(X;\mathbb{C})]=$$

$$\Big(\sum_{q\geq 0}(-1)^q trace[(h_A^*,f_B^*):H^q(A;\mathbb{C})\oplus H^q(B;\mathbb{C})\to H^q(A;\mathbb{C})\oplus H^q(B;\mathbb{C})]\Big)-$$

$$\Big(\sum_{q\geq 0}(-1)^q trace[h_{A\cap B}^*:H^q(A\cap B;\mathbb{C})\to H^q(A\cap B;\mathbb{C})]\Big),$$

that is,

$$\Lambda(h_X)=\Lambda\big((h_A,h_B)\big)-\Lambda(h_{A\cap B}).$$

But since

$$trace[(h_A^*,h_B^*):H^q(A;\mathbb{C})\oplus H^q(B;\mathbb{C})]=$$

$$trace[h_A^*:H^q(A;\mathbb{C})\to H^q(A;\mathbb{C})]+trace[h_B^*:H^q(B;\mathbb{C})\to H^q(B;\mathbb{C})],$$

it follows that

$$\Lambda(h_X)=\Lambda(h_A)+\Lambda(h_B)-\Lambda(h_{A\cap B}).$$

■

Now let $\pi:\tilde{M}\to\mathbb{C}^2$ be an embedded resolution of the curve fg at the origin and let $E=\cup_{i=1}^{s}E_i$ be the decomposition of the exceptional divisor of π in irreducible components. Let a_i and b_i denote the multiplicity of E_i in the total transform of f and g, respectively.

Applying the previous lemma to the decomposition of the Milnor fibre of $f\bar{g}$ in the boxes V_i, V_{ij} and Ω_l as in Lemma 3.1.1 (here, in order to simplify notation, we do not make distinction between V_i and $\tilde{V_p}$ nor between V_{ij} and $\tilde{V_{ip}}$), we get:

$$\Lambda(h^k)=\sum_{\substack{i=1\\ a_i\neq b_i}}^{s}\Lambda(h^k_{V_i\cap F_\theta})+\sum_{l}\Lambda(h^k_{\Omega_l\cap F_\theta})+\sum_{\substack{i,j=1\\ i\neq j}}^{s}\Lambda(h^k_{V_{ij}\cap F_\theta})-\sum_{i=1}^{s}\Lambda(h^k_{\partial V_i\cap F_\theta}).$$

Recall that we have grouped the components V_i with $a_i=b_j$ and V_{ij} with $a_i=b_i$ and $a_j=b_j$ in larger domains Ω_l, and we have proved in Lemma 3.1.1 that the part of the Milnor number contained in these components is a finite union of cylinders. Moreover, we have seen that each of these domains Ω_l contains only one V_i, that is, $\Omega_l=V_i$, with $a_i=b_i$.

Lemma 3.3.3 *We have:*

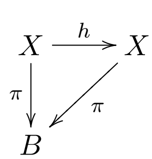
(i) $\Lambda(h^k_{V_{ij}\cap F_\theta})=0$*, for any* $i\neq j$*;*

(ii) $\Lambda(h^k_{\Omega_l\cap F_\theta})=0$*, for any* l*;*

(iii) $\Lambda(h^k_{\partial V_i\cap F_\theta})=0$*, for any* i*.*

Proof: By Lemma 3.1.1, each piece of the Milnor fibre considered in this lemma is a finite union of either cylinders (cases (i) and (ii)) or circles (case (iii)). The cylinders of cases (i) and (ii) have always a circle of case (iii) as a boundary component. Since a cylinder can be retracted to its boundary, it is enough to prove the result for case (iii). The finite union of circles $\partial V_i\cap F_\theta$ is a finite covering over a circle of E_i and the monodromy is compatible with the covering projection. Since the Euler characteristic of the circle vanishes, the result then follows from the following lemma: ∎

Lemma 3.3.4 *Let* $\pi:X\to B$ *be a* m*-covering of a compact manifold with boundary and let* h *be an automorphism of* X *such that the diagram*



commutes. For $b \in B$, denote by $h_b : \pi^{-1}(b) \to \pi^{-1}(b)$ the permutation induced by h, and suppose this permutation is cyclic and transitive. Then

$$\Lambda(h^k) = \chi(B).\Lambda(h_b^k) = \begin{cases} \chi(B).m, & \text{if } m \mid k; \\ 0, & \text{if } m \nmid k \end{cases}$$

Proof: Suppose that B is contractible. Then $X \overset{\text{homeo}}{\simeq} B \times \pi^{-1}(b)$ and then $H^q(X) \overset{\text{isom}}{\simeq} H^q(\pi^{-1}(b))$ and therefore $\Lambda(h^k) = \Lambda(h_b^k)$. If B is not contractible, we can write it as a (finite) union of contractible sets B_i, $i \in \{1, \ldots, c\}$, such that $B_i \cap B_j$ is contractible, for any $i, j \in \{1, \ldots, c\}$. Then we proceed by induction on c.
If the result is true for $c-1$, we define $\hat{B} := \cup_{i=1}^{c-1} B_i$, $X_{cup} := \pi^{-1}(\hat{B})$ and $X_c := \pi^{-1}(B_c)$. Then write $B = \hat{B} \cup B_c$ and applying the Mayer-Vietoris sequence associated to this decomposition one gets

$$\Lambda(h) = \Lambda(h_{|X_{cup}}) + \Lambda(h_{|X_c}) - \Lambda(h_{|X_{cup} \cap X_c}) =$$
$$\chi(\hat{B})\Lambda(h_b^k) + \chi(B_c)\Lambda(h_b^k) - \chi(\hat{B} \cap B_c)\Lambda(h_b^k) =$$
$$\chi(B)\Lambda(h_b^k).$$

Now observe that $\Lambda(h_b^k) = \sum_{q \geq 0} (-1)^q trace[(h_b^k)^* : H^q(\pi^{-1}(b)) \to H^q(\pi^{-1}(b))]$, which is the trace of the induced isomorphism $(h_b^k)^* : \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{m-times} \to \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{m-times}$, which is equal to m if $m \mid k$, or zero otherwise. ∎

Now, if $a_i \neq b_i$, then $V_i \cap F_\theta$ is a covering of degree $d_i := |a_i - b_i|$ over E_i minus r_i-disks, where r_i is the number of double points of the total transform of fg on E_i. Moreover, the monodromy h is compatible with the covering projection. Hence using the two previous lemmas we get the following formula:

$$\Lambda(h^k) = \sum_{\substack{i=1 \\ a_i \neq b_i}}^{s} \Lambda(h_{X_i}^k) = \sum_{\substack{i=1 \\ a_i \neq b_i \\ d_i \mid k}}^{s} d_i(2 - r_i).$$

Moreover, since h^0 denotes the identity, we also have

$$\Lambda(h^0) = \sum_{i=1}^{s} \Lambda(h^0_{V_i \cap F_\theta}) = \sum_{i=1}^{s} \chi(V_i \cap F_\theta) = \sum_{i=1}^{s} d_i(2 - r_i).$$

Now, since for each $k \geq 1$ we have

$$\sum_{d_i \mid k} d_i(2 - r_i) = \Lambda(h^k) = \sum_{d_i \mid k} s_{d_i},$$

it follows that

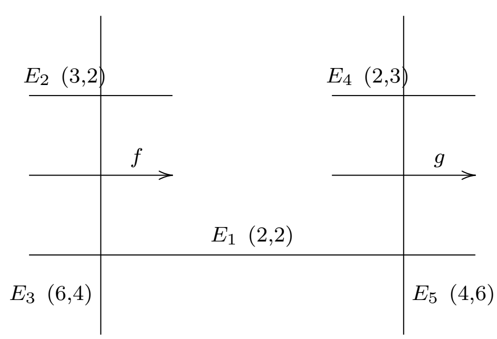
$$s_{d_i} = d_i(2 - r_i)$$

and then we have the following theorem:

Theorem 3.3.5 *Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two holomorphic functions such that the real analytic map-germ $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ has an isolated critical value. Let $\pi : \tilde{M} \to \mathbb{C}^2$ be an embedded resolution of the curve fg at the origin and let $E = \cup_{i=1}^{s} E_i$ be a decomposition of the exceptional divisor of π in irreducible components. Let a_i and b_i denote the multiplicity of E_i in the total transform of f and g, respectively. Set $d_i := |a_i - b_i|$ and let r_i be the number of double points of the total transform of fg in E_i. Then the zeta function of the monodromy of the Milnor fibration of $f\bar{g}$ is given by*

$$Z(t) = \prod_{i=1}^{s} (1 - t^{d_i})^{r_i - 2}.$$

Example 3.3.6 Consider the holomorphic functions $f(x, y) = x^2 + y^3$ and $g(x, y) = x^3 + y^2$. Then the real analytic germ $f\bar{g} : (\mathbb{R}^4, 0) \to (\mathbb{R}^2, 0)$ has an isolated singularity at $0 \in \mathbb{C}^2$. The graph of the resolution of the complex curve $\{fg = 0\} = \{f\bar{g} = 0\}$ is given below:



In the holomorphic case fg, the part of the Milnor fibre inside each box V_{ij} is a disjoint union of $gcd(a_i + b_i, a_j + b_j)$ cylinders, and inside each box V_i it is an $(a_i + b_i)$-covering of a sphere minus r_i disks, with Euler characteristic $(a_i + b_i)(2 - r_i)$. Then the part of the Milnor fibre inside: V_{13} and V_{15} are two cylinders; V_{23} and V_{45} are five cylinders; V_1 is two cylinders; V_2 and V_4 are five disks; V_3 and V_5 are compact surfaces of genus 2 with boundaries eight circles. So the Milnor fibre of fg is a twice-perforated surface of genus 5.

In the real analytic case $f\bar{g}$, according to Lemma 3.1.1, the part of the Milnor fibre inside: V_{13} and V_{15} are two cylinders; V_{23} and V_{45} are one cylinder; V_1 are two cylinders; V_2 and V_4 are one disk; V_3 and V_5 are compact surfaces of genus 0, that is, spheres, with boundaries four circles each one. Hence the Milnor fibre of $f\bar{g}$ is a twice-perforated surface of genus 1, that is, a torus with boundary two disjoint circles.

Moreover, by Theorem 3.3.5 the zeta function of h^{fg} is given by

$$Z(t) = (1-t^5)^{-2}(1-t^{10})^2$$

and the zeta function of $h^{f\bar{g}}$ is given by

$$Z(t) = (1-t)^{-2}(1-t^2)^2.$$

3.4 The degeneration of the Milnor fibre of $f\bar{g}:(\mathbb{C}^2,0)\to(\mathbb{C},0)$

Consider a real analytic map-germ of the type $f\bar{g}:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ with an isolated critical point at $0\in\mathbb{C}^2$. We are going to describe the degeneration of the Milnor fibre $F_t := (f\bar{g})^{-1}(t)\cap\mathbf{B}_\epsilon$ to $F_0 := (f\bar{g})^{-1}(0)\cap\mathbf{B}_\epsilon$, where ϵ and t are sufficiently small and g is not constant.

3.4.1 The polar curve of $f\bar{g}$

With the same notation of the previous chapter, we define the map $\phi:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ by setting $\phi(x,y) := \big(y, f\bar{g}(x,y)\big)$. The Jacobian matrix of ϕ is given by

$$J(\phi) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ \frac{\partial f}{\partial x}\bar{g} & f\overline{\frac{\partial g}{\partial x}} & \frac{\partial f}{\partial y}\bar{g} & f\overline{\frac{\partial g}{\partial y}} \\ \bar{f}\frac{\partial g}{\partial x} & \overline{\frac{\partial f}{\partial x}}g & \bar{f}\frac{\partial g}{\partial y} & \overline{\frac{\partial f}{\partial y}}g \end{pmatrix}$$

and the critical locus of ϕ is given by

$$C := \left\{ \left|f\frac{\partial g}{\partial x}\right| - \left|g\frac{\partial f}{\partial x}\right| = 0 \right\}.$$

Then C is a real analytic variety in $\mathbb{R}^4$ of dimension ≤ 3. This shows that, in the general case, it is not possible to make an analogous construction of a Lê Polyhedron for $f\bar{g}:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ as in [20], since the dimension of the set C could be too big for that.

Hence, in order to be able to obtain some result, we shall ask another hypothesis: Suppose that

$$g(x,y) = g(y),$$

that is, that g depends only on the variable y. Then $\frac{\partial g}{\partial x} = 0$ and hence

$$C = \{g = 0\} \cup \left\{ \frac{\partial f}{\partial x} = 0 \right\},$$

that is, the set of points $(x_0, y_0) \in \mathbb{C}^2$ such that $g(x_0, y_0) = 0$ and $\frac{\partial f}{\partial x}(x_0, y_0) = 0$, which is a complex curve in $\mathbb{C}^2$. Since $\{g = 0\} \subset (f\bar{g})^{-1}(0)$, it follows that the union of the irreducible components of C that are not contained in $(f\bar{g})^{-1}(0)$, denoted by Γ, is given by the complex curve

$$\Gamma = \left\{ \frac{\partial f}{\partial x} = 0 \right\}.$$

Now set $\Delta := \phi(\Gamma)$, the polar image of $f\bar{g}$. Since ϕ is not holomorphic, Δ in general is not a complex curve in $\mathbb{C}^2$, but it can be given a Puiseux parametrization, as we show bellow.

3.4.2 Parametrizing Δ

By some change of coordinate, we can consider $g = y^c$, for some real number $c \neq 0$ (the case $c = 0$ is the holomorphic case, which we are not studying in this section). Also, since f is holomorphic, we can write it as a convergent power series

$$f = \sum_{a,b=0}^{\infty} \kappa_{a,b} x^a y^b,$$

with $\kappa_{a,b} \in \mathbb{C}$. Therefore the real analytic germ $f\bar{g}$ can be write as the convergent power series

$$f\bar{g} = \sum_{a,b=0}^{\infty} \kappa_{a,b} x^a y^b \bar{y}^c$$

Now consider the decomposition of the complex curve Γ into irreducible components

$$\Gamma = \gamma_1 \cup \cdots \cup \gamma_q.$$

Then for each $i = 1, \ldots, q$ we can give γ_i a Puiseux parametrization

$$x = \alpha_i y^{m_i/n_i} + \sum_{j \geq 0} \beta_{i,j} y^{m_{i,j}/n_i},$$

where $\alpha_i, \beta_{i,j} \in \mathbb{C}$, with $\alpha_i \neq 0$, and $n_i, m_i, m_{i,j} \in \mathbb{N}^*$. For each $i = 1, \ldots, q$, set

$$\tau_i := \phi(\gamma_i) \subset \mathbb{C}^2.$$

Then each τ_i admitis a Puiseux parametrization in the complex coordinates (u, v) of $\mathbb{C}^2$:

$$v(u) = \sum_{a,b=0}^{\infty} [\kappa_{a,b} u^b \bar{u}^c (\alpha_i u^{m_i/n_i} + \sum_{j \geq 0} \beta_{i,j} u^{m_{i,j}/n_i})^a].$$

In particular, if we consider the real analytic function

$$\begin{array}{cccc} \vartheta : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & u & \longmapsto & v(u) \end{array},$$

we get that $\tau_i \cap \{v = t\} = \vartheta^{-1}(t)$, for any complex number t, which gives a finite union of points.

3.4.3 Constructing a Lê Polyhedron

Consider small enough positive real numbers ϵ, η_1, η_2 with $0 < \eta_2 << \eta_1 << \epsilon << 1$ such that the real analytic map $\phi = (y, f\bar{g})$ induces a real analytic map

$$\phi_| : \mathbf{B}_\epsilon \cap \phi^{-1}(\mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2}) \to \mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2}$$

which restricts to a fibre bundle over $(\mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2}) \backslash \Delta$.

As before, we can study the topology of the real surface $F_t := \mathbf{B}_\epsilon \cap \phi^{-1}(\mathbf{D}_{\eta_1} \times \{t\})$ instead of studying the topology of the Milnor fibre $F_t = \mathbf{B}_\epsilon \cap (f\bar{g})^{-1}(t)$.

For any $t \in \mathbf{D}^*_\eta$, set

$$D_t := \mathbf{D}_{\eta_1} \times \{t\}.$$

Then $\phi_|$ induces a projection

$$\varphi_t : F_t \to D_t,$$

which is a finite covering over $D_t \backslash (\Delta \cap D_t)$. Note that $\Delta \cap D_t$ is a finite set of points, as we have seen above, and then we denote

$$\Delta \cap D_t = \{y_1, \ldots, y_k\}.$$

Let λ_t be the barycenter of the set of points $\{y_1(t), \ldots, y_k(t)\}$ in D_t and for each $j = 1, \ldots, k$, let $\delta(y_j(t))$ be a simple path (differentiable and with no double points) starting at λ_t and ending at $y_j(t)$, such that two of them intersect only at λ_t.

Set

$$Q_t := \bigcup_{j=1}^{k} \delta(y_j(t))$$

We define the Lê Polyhedron P_t to be

$$P_t := \varphi_t^{-1}(Q_t).$$

See figure 3.2 bellow.

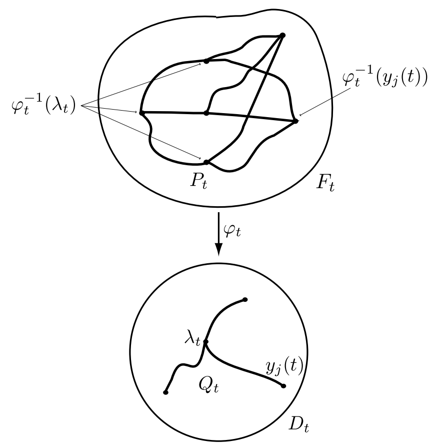


Figure 3.2:

Now, let v_t be a vector field in D_t such that v_t is:

- C^∞;
- null over Q_t;
- transversal to ∂D_t and points inwards.

Then the associated flow $q_t : [0, \infty[\times (D_t \backslash Q_t) \to D_t$ defines a map

$$\begin{array}{rcccc} \xi_t & : & \partial D_t & \longrightarrow & Q_t \\ & & u & \longmapsto & \lim\limits_{\tau \to \infty} q_t(\tau, u) \end{array},$$

such that ξ_t is continuous, surjective and differentiable.

Since φ_t is a covering over $D_t \backslash Q_t$, which is differentiable (in the real sense), we can lift v_t to a vector field E_t in F_t such that E_t is:

- continuous over F_t;
- differentiable over $F_t \backslash P_t$;

- null over P_t;
- integrable;
- transversal to ∂F_t and points inwards.

Then the associated flow $\tilde{q}_t : [0, \infty[\, \times\, (F_t \backslash P_t) \to F_t$ defines a map

$$\begin{array}{rcccc} \tilde{\xi}_t & : & \partial F_t & \longrightarrow & P_t \\ & & z & \longmapsto & \lim\limits_{\tau \to \infty} \tilde{q}_t(\tau, z) \end{array},$$

such that $\tilde{\xi}_t$ is continuous, surjective and differentiable.

So now we show that F_t is homeomorphic to the mapping cylinder of $\tilde{\xi}_t$. In fact, the integration of the vector field $\tilde{v}_t$ gives a surjective continuous map

$$\alpha : [0, \infty] \ \times \ \partial F_t \to F_t$$

that restricts to a diffeomorphism

$$\alpha_| : [0, \infty[\, \times\, \partial F_t \to F_t \backslash P_t.$$

Since the restriction $\alpha_\infty : \{\infty\} \times \partial F_t \to P_t$ is equal to $\tilde{\xi}_t$, which is differentiable and surjective, it follows that the induced map

$$[\alpha_\infty] : \big((\{\infty\} \times \partial F_t)/ \sim \big) \to P_t$$

is a homeomorphism, where $\sim$ is the equivalent relation given by identifying $(\infty, z) \sim (\infty, z')$ if $\alpha_\infty(z) = \alpha_\infty(z')$. Hence the map

$$[\alpha] : \big(([0, \infty] \times \partial F_t)/ \sim \big) \to F_t$$

induced by α defines a homeomorphism between F_t and the mapping cylinder of $\tilde{\xi}_t$.

3.4.4 The collapsing map

We can do the construction of the vector field E_t simultaneously for all t in a simple path γ in $\mathbf{D}_{\eta_2}$ joining 0 and some $t_0 \in \partial \mathbf{D}_{\eta_2}$, such that γ is transverse to $\partial \mathbf{D}_{\eta_2}$. The natural projection $\pi : \mathbf{D}_{\eta_1} \times \mathbf{D}_{\eta_2} \to \mathbf{D}_{\eta_2}$ restricted to Δ induces a ramified covering

$$\pi_| : \Delta \to \mathbf{D}_{\eta_2}$$

whose ramification locus is $\{0\} \subset \Delta$.

Hence the inverse image of $\gamma\backslash\{0\}$ by this covering defines k disjoint simple paths in Δ, and each one of them is diffeomorphic to $\gamma\backslash\{0\}$. Moreover, the set

$$\Lambda = \bigcup_{t\in\gamma} \lambda_t$$

defines a simple path in $\mathbf{D}_{\eta_1} \times \gamma$ such that $\Lambda \cap \Delta = \{0\}$. We can choose the paths $\delta(y_j(t))$ in such a way that

$$T_j := \bigcup_{t\in\gamma} \delta(y_j(t))$$

forms a triangle differentially immersed in

$$\bigcup_{t\in\gamma} D_t = \mathbf{D}_{\eta_1} \times \gamma$$

outside $\{0\}$. The triangles T_j, for $j = 1, \ldots, k$, have exactly the path Λ in common. Set

$$Q := \bigcup_{j=1}^{k} T_j.$$

Now, let V be a vector field in $\mathbf{D}_{\eta_1} \times \gamma$ such that V is:

- continuous;
- null over Q;
- differentiable over $(\mathbf{D}_{\eta_1} \times \gamma)\backslash Q$;
- transversal to $\partial\mathbf{D}_{\eta_1} \times \gamma$; and such that
- the projection of V on γ is null.

Then the associated flow $w : [0, \infty[\, \times \big((\mathbf{D}_{\eta_1} \times \gamma)\backslash Q\big) \to \mathbf{D}_{\eta_1} \times \gamma$ defines a map

$$\begin{array}{rccc} \xi & : & \partial\mathbf{D}_{\eta_1} \times \gamma & \longrightarrow & Q \\ & & z & \longmapsto & \lim\limits_{\tau\to\infty} w(\tau, z) \end{array},$$

such that ξ is continuous, surjective and differentiable.

For any real $A > 0$, set

$$V_A(Q) := (\mathbf{D}_{\eta_1} \times \gamma)\backslash w\big([0, A[\times \partial\mathbf{D}_{\eta_1} \times \gamma\big),$$

a closed neighbourhood of Q in $\mathbf{D}_{\eta_1} \times \gamma$. Note that $\partial V_A(Q)$ is a differentiable manifold that fibres over γ with fibre a circle, by the restriction of the projection π. Moreover, $\mathbf{D}_{\eta_1} \times \gamma$ is clearly the mapping cylinder of ξ.

Set

$$F_\gamma := \phi^{-1}(\mathbf{D}_{\eta_1} \times \gamma) \cap \mathbf{B}_\epsilon.$$

Since

$$\phi_| : F_\gamma \backslash \phi^{-1}(Q) \to (\mathbf{D}_{\eta_1} \times \gamma) \backslash Q$$

is a fibre bundle, it follows that $\phi^{-1}(\partial V_A(Q))$ is a differentiable submanifold of F_γ which is a fibre bundle over γ.

Now, set

$$P_\gamma := \phi^{-1}(Q),$$

which we call the *collapse polyhedron of $f\bar{g}$ along γ*. It is a polyhedron in F_γ of real dimension 2. Let θ be a vector field in γ that goes from t_0 to 0 in time $a > 0$.

Set

$$Z := F_\gamma \backslash P_\gamma.$$

Since

$$Z = \phi^{-1}\big((\mathbf{D}_{\eta_1} \times \gamma) \backslash Q\big) \xrightarrow{\phi} (\mathbf{D}_{\eta_1} \times \gamma) \backslash Q \xrightarrow{\pi} \gamma$$

and

$$\phi^{-1}\big(\partial V_A(Q)\big) \xrightarrow{\phi} \partial V_A(Q) \xrightarrow{\pi} \gamma$$

are differential fibre bundles, we can lift θ to obtain a vector field E such that:

- E is differentiable;
- E is tangent to $\phi^{-1}\big(\partial V_A(Q)\big)$, for any $A > 0$.

Then the associated flow $g : [0, a] \times Z \to Z$ defines a C^∞-diffeomorphism Ψ from $X_{t_0} \backslash P_{t_0}$ to $F_0 \backslash \{0\}$ that extends to a continuous map from X_{t_0} to F_0 and that sends P_{t_0} to $\{0\}$. Hence we have proved:

Theorem 3.4.1 *Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two holomorphic germs of function with no common irreducible components and such that the real analytic germ given by $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ has an isolated singularity $0 \in \mathbb{C}^2$. Suppose that $g(x, y) = g(y)$. Then there exist ϵ and η sufficiently small, with $0 < \eta << \epsilon << 1$, such that for any $t \in \boldsymbol{D}_\eta^*$ there exists a polyhedron P_t, of real dimension 1, in the Milnor fibre F_t of f such that F_t deformation retracts to P_t. Moreover, there exists a continuous map $\Psi_t : F_t \to F_0$ which sends P_t to $\{0\}$ and such that Ψ_t restricts to a homeomorphism from $F_t \backslash P_t$ to $F_0 \backslash \{0\}$.*

Chapter 4

The vanishing zone

If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a holomorphic map-germ with $0 \in \mathbb{C}^n$ an isolated critical point, Milnor proved in [28] that the boundary of the Milnor fibre L_t of f is diffeomorphic to the link L_0 of f. But if f has a non-isolated singularity, that is, if the critical locus Σ of f has non-zero dimension, it follows that L_0 is no longer a differentiable manifold, as it contains a non-empty singular locus

$$L(\Sigma) := \Sigma \cap \mathbf{S}_\epsilon$$

which we call the *link of the critical locus of* f. So we would like to understand how the smooth manifold L_t degenerates to the singular variety L_0.

It turns out that, when Σ is a complex curve, such degeneration happens only inside a small neighbourhood of the link of the critical locus in the Milnor sphere $\mathbf{S}_\epsilon$. Such neighbourhood is called the *vanishing zone of* f and is denoted by W. In other words, there exists a small neighbourhood W of $L(\Sigma)$ in $\mathbf{S}_\epsilon$ such that $L_t \backslash \mathring{W}$ (which is a smooth manifold with boundary, called the *trunk* of f) is diffeomorphic to $L_0 \backslash \mathring{W}$, where $\mathring{W}$ denotes the interior of W. So the degeneration of L_t to L_0 is given by the degeneration of $W_t := W \cap L_t$ to $W_0 := W \cap L_0$.

The idea of the vanishing zone was developed by Siersma in [40] and it is very nicely described by Michel and Pichon in [23] for $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ with critical locus Σ a complex curve. We present this description in section 4.1.

One can also think about the problem of understanding the degeneration of L_t to L_0 for a real analytic map-germ

$$f : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$$

with $n \geq m$, such that $0 \in \mathbb{R}^m$ is an isolated critical value with the Thom a_f-property (which admits a Milnor fibration in the tube - see section 1.7.2).

In section 4.2, we define the vanishing zone for such real analytic map-germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$, supposing that the critical locus Σ of f is either a smooth manifold or an isolated singularity in $\mathbb{R}^n$. As in the holomorphic case, the vanishing zone is a neighbourhood W of the link of the critical locus $L(\Sigma)$ in the Milnor sphere $\mathbf{S}_\epsilon$ such that $L_t \backslash \mathring{W}$ is diffeomorphic to $L_0 \backslash \mathring{W}$.

When f is holomorphic and Σ is a complex curve, Siersma also showed in [40] that each connected component $W_t[i]$ of $W_t : W \cap L_t$ is a fibre bundle over a circle $\mathbf{S}^1$ with fibre the so called *transversal type Milnor fibre*, which is the Milnor fibre of the restriction of f to a hyperplane section H_s of $\mathbb{C}^n$ passing through a point $s \in \Sigma \backslash \{0\}$ and transversal to Σ. The monodromy h_i of such fibre bundle is called the *vertical monodromy of f relative to $W_t[i]$*.

In section 4.3, we investigate if this result can be generalized for real analytic map-germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ as above, allowing Σ to be either a smooth manifold or an isolated singularity in $\mathbb{R}^n$ of any dimension. Then we obtain the following result, which gives a necessary and sufficient condition for $W_t := W \cap L_t$ and $W_0 := W \cap L_0$ to be fibre bundles over a sphere with fibre the transversal type Milnor fibre of f:

Theorem 4.0.2 *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$, with $n \geq m$, be a real analytic map-germ such that $0 \in \mathbb{R}^m$ is an isolated critical value with the Thom a_f-property and such that the critical locus Σ of f is either a smooth manifold or an isolated singularity in $\mathbb{R}^n$. Then W_t and W_0 are fibre bundles over $L(\Sigma)$ with fibre the transversal type Milnor fibre of f if, and only if, one has that either Σ or $\Sigma \backslash \{0\}$ is a stratum of a Whitney stratification of f.*

4.1 The vanishing zone of $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$

Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a reduced holomorphic germ whose critical locus Σ is a complex curve. We know that the link L_0, the Milnor fibre F_t and its boundary L_t can also be defined up to homeomorphism by using the ball with corners

$$\mathbf{B}^2_\alpha \times \mathbf{B}^4_\epsilon = \{(x, y, z) \in \mathbb{C}^3 \ / \ x \in \mathbf{B}^2_\alpha, (y, z) \in \mathbf{B}^4_\epsilon\}.$$

Then for $0 \leq |t| \leq \eta << \alpha << \epsilon$, we have that

$$F_t \overset{\text{diffeo}}{\simeq} f^{-1}(t) \cap (\mathbf{B}^2_\alpha \times \mathbf{B}^4_\epsilon)$$

and

$$L_t = \partial F_t \overset{\text{diffeo}}{\simeq} f^{-1}(t) \cap \partial(\mathbf{B}^2_\alpha \times \mathbf{B}^4_\epsilon).$$

For a generic choice of the z-axis, one has that

$$\Gamma := \{\frac{\partial f}{\partial z} = 0\} \cap \{f = 0\}$$

is a germ of curve such that $\Sigma \subset \Gamma$ and $\{\frac{\partial f}{\partial z} = 0\}$ is a reduced germ of hypersurface in $(\mathbb{C}^3, 0)$.

Conventions 4.1.1 Let H_a be the hyperplane $\{x = a\}$ in $\mathbb{C}^3$. We choose the x-axis in such a way that for any $a \in \mathbf{B}^2_\alpha$, the planes H_a are transversal to the curve Γ in the ball $\mathbf{B}^2_\alpha \times \mathbf{B}^4_\epsilon$. Then we can take α and ϵ, $0 < \alpha < \epsilon$, small enough such that:

1. $\{0\} \times B^4_\epsilon$ is a Milnor ball for the germ of curve defined by the equation $f(0, y, z) = 0$ in H_0 (with k irreducible components);
2. The link $L(\Gamma) = \Gamma \cap \partial(\mathbf{B}^2_\alpha \times \mathbf{B}^4_\epsilon)$ is contained in $\mathbf{S}^1_\alpha \times \mathbf{B}^4_\epsilon$;
3. For each a, $0 \leq |a| \leq \alpha$, the hyperplane section $\{f(a, y, z) = 0\}$ has only isolated singular points in $\{a\} \times \mathbf{B}^4_\epsilon$.

Proposition 4.1.2 *For η small enough and $t \in \boldsymbol{B}^2_\eta$, the intersections $L_t \cap (\boldsymbol{B}^2_\alpha \times \boldsymbol{S}^3_\epsilon)$ are disjoint union of k solid tori.*

Proof: Since $L(\Sigma) \subset L(\Gamma) \subset \mathbf{S}^1_\alpha \times \mathbf{B}^4_\epsilon$, the function f restricted to $\{a\} \times \mathbf{S}^3_\epsilon$, $a \in \mathbf{B}^2_\alpha$, is a proper submersion. Then by Ehresmann's fibration lemma,

$$f_a : (\{a\} \times \mathbf{S}^3_\epsilon) \cap f^{-1}(\mathbf{B}^2_\eta) \to \mathbf{B}^2_\eta$$

is a fibre bundle. Besides, since $f_a^{-1}(t)$ is an 1-dimensional compact real manifold, it is a disjoint union of (k) circles, $\forall a \in \mathbf{B}^2_\alpha$, $\forall t \in \mathbf{B}^2_\eta$. Then if we set $\tilde{f}(a, x) := f_a(x)$, it follows that

$$\tilde{f} : (\mathbf{B}^2_\alpha \times \mathbf{S}^3_\epsilon) \cap \tilde{f}^{-1}(\mathbf{B}^2_\eta) \to \mathbf{B}^2_\eta$$

is a fibre bundle and

$$L_t \cap (\mathbf{B}^2_\alpha \times \mathbf{S}^3_\epsilon) = \tilde{f}^{-1}(t) = \{(a, f_a^{-1}(t))/a \in \mathbf{B}^2_\alpha\}$$

is a disjoint union of (k) solid tori.

■

Set

$$g = \frac{\partial f}{\partial z} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$$

and

$$\psi : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0),$$

the holomorphic germ defined by $\psi = (g, f)$.

Definition 4.1.3 *Denoted by W the union of the connected components of*

$$\psi^{-1}(\boldsymbol{B}^2_\theta \times \boldsymbol{B}^2_\eta) \cap (\boldsymbol{S}^1_\alpha \times \boldsymbol{B}^4_\epsilon)$$

which intersect the link $L(\Sigma)$. W is called the vanishing zone of f. Also define

$$\partial W = W \cap g^{-1}(S^1_\theta)$$

and for $0 \leq |t| \leq \eta$, define

$$W_t = W \cap L_t.$$

Now consider the germ $(x, \psi) : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ and let h be the reduction of the germ

$$\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}.$$

We define $(V, 0)$ the germ in $(\mathbb{C}^3, 0)$ with equation $h = 0$, which is the reduced critical locus of (x, ψ), since

$$Jac(x, \psi) = Jac(x, g, f) = \begin{bmatrix} 1 & 0 & 0 \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}.$$

For a general choice of the coordinates, $(V, 0)$ is a germ of normal surface. If α and ϵ are small enough, then $\mathbf{B}^2_\alpha \times \mathbf{B}^4_\epsilon$ is also a Milnor ball for h.

Lemma 4.1.4 *For sufficiently small $\epsilon, \alpha, \theta, \eta$ with $0 < \eta << \theta << \alpha < \epsilon$, one has $V \cap \partial W = \emptyset$.*

Proof: Let $\sigma_1, \cdots, \sigma_l$ be the irreducible components of $\Sigma \subset V$ and consider

$$\pi : (\tilde{V}, 0) \to (V, 0)$$

a good resolution of $(V, 0)$ and of σ_i, for all $i \in \{1, \cdots, l\}$. Then for each i, $\pi^{-1}(\sigma_i)$ is a normal crossing curve, which intersects $\pi^{-1}(0)$ at p_i.
We know that $(g_{|V})^{-1}(0)$ and $(f_{|V})^{-1}(0)$ are complex curves in V that contain $\Sigma(f)$. For each σ_i, let V_i be a small neighborhood of $\sigma_i \cap (\mathbf{S}^1_\alpha \times \mathbf{B}^4_\epsilon)$ in $\mathbf{S}^1_\alpha \times \mathbf{B}^4_\epsilon$.
Taking local coordinates (x, y) around the point p_i in the resolution graph , we get

$$(f \circ \pi)(x, y) = y^{m_f} x^\nu u(x, y)$$

and

$$(g \circ \pi)(x, y) = y^{m_g} x^\mu v(x, y),$$

where $u(x, y)$ and $v(x, y)$ are unities in $\mathbb{C}\{x, y\}$.
By some suitable change of coordinates, we can consider $v(x, y) = 1$. Then for θ small enough we have that

$$(g_{|V})^{-1}(\mathbf{B}^2_\theta) \cap V_i \approx (g \circ \pi)^{-1}(\mathbf{B}^2_\theta) \cap \pi^{-1}(V_i) = \{(x, y) \in V_i / |x^\mu| \leq \frac{\theta}{\alpha^{m_g}}, |y| = \alpha\},$$

a solid torus.
In the same way we see that for η small enough, $(f_{|V})^{-1}(\mathbf{B}^2_\eta) \cap V_i$ is a solid torus.
Since $u(0, 0) \neq 0$, we can take α and ϵ small enough such that for all $(x, y) \in \pi^{-1}(\mathbf{B}^2_\alpha \times \mathbf{B}^4_\epsilon)$ one has $0 < C \leq |u(x, y)|$, for some constant $C \in \mathbb{R}$. Then if we take

$$\eta < \frac{\alpha^{m_f} \beta\theta}{\alpha^{m_g}}$$

we get

$$[f^{-1}(\mathbf{B}_\eta) \cap V_i] \subset [int(g^{-1}(\mathbf{B}_\theta)) \cap V_i] = [g^{-1}(\mathring{\mathbf{B}}_\theta) \cap V_i].$$

Therefore $V \cap \partial W = \emptyset$.

∎

Theorem 4.1.5 *There exist sufficiently small positive real numbers $\epsilon, \alpha, \theta, \eta$ with $0 < \eta << \theta << \alpha < \epsilon$ such that f induces an isotopy $L_t \backslash W_t \to L_0 \backslash W_0$.*

Proof: Set

$$M := f^{-1}(\mathbf{B}^2_\eta) \cap \left[(\mathbf{S}^1_\alpha \times \mathbf{B}^4_\epsilon) \backslash \mathring{W}\right],$$

a manifold with boundary

$$\partial M = \left[f^{-1}(\mathbf{S}^2_\eta) \cap (\mathbf{S}^1_\alpha \times \mathbf{B}^4_\epsilon)\right] \cup \left[f^{-1}(\mathbf{B}^2_\eta) \cap \partial W\right].$$

By Ehresmann's fibration lemma for manifolds with boundary, the restriction

$$f_| : M \to \mathbf{B}^2_\eta$$

is a fibre bundle if one has the following conditions:

(1) $\forall p = (x, y, z) \in M$, $D(f_{|M})_p : T_p(\mathbf{S}^1_\alpha \times \mathbf{B}^4_\epsilon) \to T_{f(p)}\mathbb{C}$ is a surjection;

(2) $\forall p \in \partial M$, $D(f_{|\partial M})_p : T_p(\partial M) \to T_{f(p)}\mathbb{C}$ is a surjection.

By corollary 1.7.4, this conditions are equivalent to the following:

(1) the fibres of f are transversal to $\mathbf{S}^1_\alpha \times \mathbf{B}^4_\epsilon$;

(2) the fibres of f are transversal to ∂W.

Milnor theory (see [28]) asserts that (1) holds. So we just have to examine how the fibres of f intersect the boundary of W. Note that

$$\partial W = g^{-1}(\mathbf{S}^1_\theta) \pitchfork (\mathbf{S}^1_\alpha \times \mathbf{B}^4_\epsilon).$$

Then its normal space is the 3-vector subspace of $\mathbb{R}^6$ generated by n_1 and n_2, where n_1 is a normal vector to $g^{-1}(\mathbf{S}^1_\theta)$ in $\mathbb{C}^3$ and $n_2 = (x_1, x_2, 0, 0, 0, 0)$ is a normal vector to $\mathbf{S}^1_\alpha \times \mathbf{B}^4_\epsilon$ in $\mathbb{R}^6$.
Since $g^{-1}(\mathbf{S}^1_\theta)$ is the inverse image of θ by the composition

$$\mathbb{C}^3 \xrightarrow{g} \mathbb{C} \xrightarrow{\|\cdot\|} \mathbb{R}.$$

we can set $n_1 = grad(\|g\|)$. But

$$D(\|g\|)_{(x,y,z)} = g(x,y,z) D(g)_{(x,y,z)}$$

and since $g(x,y,z) \neq 0$ for $(x,y,z) \in \partial W$, we can set $n_1 = (\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z})$.
Therefore, (2) holds if for any $p \in \partial W$ we have

$$(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \notin \langle (\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}), (x, 0, 0) \rangle,$$

which is true by lemma 2.2.5.

■

Unfortunately, when one tries to extend the definition of a vanishing zone for the real analytic case $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ with isolated critical value and Thom a_f-property, it becomes too complicate mainly because of the lack of control on the dimension and structure of the set V. Then a new (but quite equivalent) definition of vanishing zone, as we do in the next section, becomes very useful.

4.2 The vanishing zone of real analytic map-germs

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$, with $n \geq m$, be a real analytic map-germ such that $0 \in \mathbb{R}^m$ is an isolated critical value with the Thom a_f-property. Suppose that Σ has at most an isolated singularity (that is, Σ is either a smooth manifold or it has an isolated singularity).

In this section, we construct a vanishing zone for such real analytic map-germ, using the idea of the *cellular tube* of a submanifold S of a manifold M, defined by Brasselet in [4], which generalizes the concept of tubular neighborhoods.

Lemma 4.2.1 *There exists a regular neighbourhood W of $L(\Sigma)$ in $\boldsymbol{S}_\epsilon$ such that its boundary ∂W intersects L_0 transversally. Moreover, W is a fibre bundle over $L(\Sigma)$ with fibre a $(n-k)$-dimensional disk in $\boldsymbol{S}_\epsilon$, where k is the dimension of Σ.*

Proof: Let us consider a Whitney stratification of $\mathbf{S}_\epsilon$ such that L_0 is a union of strata. Now let us consider a triangulation (K) of $\mathbf{S}_\epsilon$ such that each strata of the Whitney stratification is an union of simplices. Let (K') be the barycentric decomposition of (K).
Using (K') one constructs the associated cellular dual decomposition (D) of $\mathbf{S}_\epsilon$: given a simplex σ in (K) of dimension s, its dual $d(\sigma)$ is the union of all simplices τ in (K') whose closure meets σ exactly at its barycenter $\hat{\sigma}$, that is,

$$\bar{\tau} \cap \sigma = \hat{\sigma}.$$

It is a cell of dimension $(n-s-1)$. Taking the union of all these dual cells we get the dual decomposition (D) of (K). By construction, each cell σ intersects its dual $d(\sigma)$ transversally.
We let W be the union of cells in (D) which are dual of simplices in $L(\Sigma)$; it provides a cellular tube around $L(\Sigma)$ in $\mathbf{S}_\epsilon$, which means it satisfies the following properties:

(i) W is a compact neighbourhood of $L(\Sigma)$ containing $L(\Sigma)$ in its interior and ∂W is a retract of $W \backslash L(\Sigma)$;

(ii) W retracts to $L(\Sigma)$;

(iii) For any neighbourhood U of $L(\Sigma)$ in $\mathbf{S}_\epsilon$, if the triangulation of $\mathbf{S}_\epsilon$ is sufficiently "fine" then we can assume $W \subset U$.

Now consider

- $A := \{\sigma \in (K) \ / \ \sigma$ is a (1)-simplex of L_0 whose closure intersects $\partial W\}$ and
- $B := \{\sigma \in (K') \ / \ \sigma$ is a $(n-2)$ simplex in ∂W whose closure intersects $L_0\}$.

Then one can see that

$$B = \bigcup_{\sigma \in A} d(\sigma).$$

Therefore ∂W intersects L_0 transversally. Moreover, since $L(\Sigma)$ is a smooth manifold without boundary, it follows that W is a bundle on $L(\Sigma)$ whose fibres are disks (see [5]). ∎

Note that in such construction, W results to be a non-smooth manifold, since its boundary ∂W has corners. But in fact we can consider a smoothing of ∂W as in the following lemma (see [15] for the proof):

Lemma 4.2.2 *For any arbitrarily small neighbourhood U of ∂W in $\boldsymbol{S}_\epsilon^{n-1}$, there exist a smooth manifold C differentially imbedded in U and a homeomorphism $h : \partial W \to C$. Moreover, C intersects L_0 transversally.*

So from now on, when we mention the boundary ∂W we actually refer to the smoothing C.

Definition 4.2.3 *We say that W is a vanishing zone of f.*

Corollary 4.2.4 *For t sufficiently small, one has that $L_t \pitchfork \partial W$, that is, L_t intersects ∂W transversally. As a consequence, $\partial W_t := \partial W \cap L_t$ is a differentiable manifold.*

Proof: Suppose it is not true, that is, suppose that there exists a sequence of points (p_t) in $f^{-1}(\mathbf{B}_\eta^*) \cap \partial W$, with $p_t \in L_t \cap \partial W$, which converges to $p_0 \in L_0 \cap \partial W$, such that $T_{p_t}L_t$ intersects $T_{p_t}\partial W$ not transversally, for each p_t. Set

$$T := \lim_{t \to 0} T_{p_t}L_t$$

and let $Reg(L_0)$ denote the regular part of L_0. Since f has the Thom a_f-property, it follows that $T_{p_0}(RegL_0) \subset T$, and since $T_{p_0}(RegL_0) \pitchfork T_{p_0}\partial W$ by the previous lemma, it follows that $T \pitchfork T_{p_0}\partial W$. Consider d a metric in the correspondent grassmannian. Since transversality is an open property, it happens that if $d(T, T_{p_t}L_t)$ and $d(T_{p_0}\partial W, T_{p_t}\partial W)$ are sufficiently small, then $T_{p_t}L_t \pitchfork T_{p_t}\partial W$, a contradiction.
Therefore, if η is sufficiently small, one has that $T_{p_t}L_t \pitchfork T_{p_t}\partial W$, for any $t \in \mathbf{D}_\eta$ and $p_t \in (L_t \cap \partial W)$.

■

Lemma 4.2.5 *For any $t \in \boldsymbol{B}_\eta$, the smooth manifold $L_t \backslash \mathring{W}$ is diffeomorphic to $L_0 \backslash \mathring{W}$.*

Proof: Set $M = \mathbf{S}_\epsilon \cap f^{-1}(\mathbf{B}_\eta) \backslash \mathring{W}$. By Ehresmann's fibration lemma for manifolds with boundary, $f_{|_M} : \mathbf{S}_\epsilon \cap f^{-1}(\mathbf{B}_\eta) \backslash \mathring{W} \to \mathbf{B}_\eta$ is a fibre bundle if one has the following conditions:

(1) $\forall p \in M$, $D(f_{|_M})_p : T_p(\mathbf{S}_\epsilon) \to T_{f(p)}\mathbb{C}$ is a surjection;

(2) $\forall p \in \partial M$, $D(f_{|_{\partial M}})_p : T_p(\partial M) \to T_{f(p)}\mathbb{C}$ is a surjection.

By corollary 1.7.4, this conditions are equivalent to the following:

(1') the fibres of f are transversal to $\mathbf{S}_\epsilon$;

(2') the fibres of f are transversal to ∂M.

But $(1')$ follows from Milnor theory (see [28] or chapter 1) and $(2')$ follows from lemma 2.2.4. ■

4.3 The topology of f inside the vanishing zone

We want to describe the topology of L_0 inside $W_0 = W \cap L_0$, and the topology of L_t inside $W_t = W \cap L_t$. By construction, we know that W is a fibre bundle over $L(\Sigma)$, with projection p and fibre the $(n-k)$-dimensional ball $\mathbf{B}(s)$, centered at $s \in L(\Sigma)$.

$$\begin{array}{ccc} \mathbf{B}(s) & \hookrightarrow & W \\ & & \downarrow{\scriptstyle p} \\ & & L(\Sigma) \end{array}$$

For each $s \in L(\Sigma)$, consider the restriction

$$f_s : \mathbf{B}(s) \to \mathbb{R}^m.$$

Clearly, $f_s^{-1}(0)$ is a compact real variety in $\mathbf{B}(s)$ with dimension $\leq n-k-1$, for any $s \in L(\Sigma)$. Moreover, by lemma 4.2.1, it has an isolated singularity at $0 \in \mathbf{B}(s)$. Also, for any $t \in \mathbf{B}_\eta^*$, one has that $f_s^{-1}(t)$ is a differentiable manifold of real dimension $n-k-1$, since t is a regular value of f_s.

Lemma 4.3.1 *There exists δ sufficiently small such that if W is taken to be a fibre bundle with fibre the ball $\boldsymbol{B}_\delta(s)$ centered at $s \in L(\Sigma)$, then $\mathring{W}_t := \mathring{W} \cap L_t$ intersects $\boldsymbol{B}_\delta(s)$ transversally, for any $s \in L(\Sigma)$ and $t \in \boldsymbol{B}_\eta$.*

Proof:

(a) Consider $t = 0$.

We know that $L_0 \supseteq L(\Sigma)$ and that $L(\Sigma)$ intersects $\mathbf{B}_\delta(s)$ transversally for any $s \in L(\Sigma)$, that is,

$$T_s(L(\Sigma)) + T_s(\mathbf{B}_\delta(s)) = T_s(\mathbf{S}_\epsilon).$$

We want to show that $T_y(\mathring{W}_0) + T_y(\mathbf{B}_\delta(s)) = T_y(\mathbf{S}_\epsilon)$, for any $y \in [\mathring{W}_0 \cap \mathbf{B}_\delta(s)]$. Let (S_α) be a Whitney stratification of W_0 induced by a Whitney stratification of f. There are two possibilities:

Case I: Suppose that s belongs to a stratum of dimension $k-1$. Then by the (a)-condition of the Whitney stratification, we have that for y sufficiently close to s the tangent space $T_y(RegL_0)$ is very close to a plane P that contains $T_s(L(\Sigma))$ and so it is transversal to $\mathbf{B}_\delta(s)$. Since transversality is an open property, it follows that $\mathring{W}_0$ intersects $\mathbf{B}_\delta(s)$ in y transversally. So if δ is sufficiently small, we have that $\mathring{W}_0$ intersects $\mathbf{B}_\delta(s)$ transversally.

Case II: Suppose that s belongs to a stratum of dimension $< k-1$. We can consider $s' \in L(\Sigma)$ as close to s as we want such that s' belongs to a stratum of dimension $k-1$. Then for $y \in \mathbf{B}_\delta(s)$ sufficiently close to s, the tangent $T_y(RegL_0)$ is very close to a plane P that contains $T_{s'}(L(\Sigma))$ and so it is transversal to $\mathbf{B}_\delta(s')$. But since $\mathbf{B}_\delta(s)$ is very close to $\mathbf{B}_\delta(s')$, it follows that P is transversal to $\mathbf{B}_\delta(s)$. Therefore $\mathring{W}_0$ intersects $\mathbf{B}_\delta(s)$ in y transversally. So if δ is sufficiently small, we have that $\mathring{W}_0$ intersects $\mathbf{B}_\delta(s)$ transversally.

If we take δ sufficiently small such that $\mathring{W}_0$ intersects $\mathbf{B}_\delta(s)$ transversally, then clearly $\mathring{W}_0$ intersects $\mathbf{B}_\delta(s')$ transversally, for any $s' \in L(\Sigma)$ sufficiently close to s. Since $L(\Sigma)$ is compact, we are done.

(b) Consider $t \neq 0$ and let δ be as above. We know that

$$T_{y'}(\mathring{W}_0) + T_{y'}(\mathbf{B}_\delta(s)) = T_{y'}(\mathbf{S}_\epsilon),$$

for any $y' \in [\mathring{W}_0 \cap \mathbf{B}_\delta(s)]$. We want to show that

$$T_y(\mathring{W}_t) + T_y(\mathbf{B}_\delta(s)) = T_y(\mathbf{S}_\epsilon),$$

for any $y \in [\mathring{W}_t \cap \mathbf{B}_\delta(s)]$. Since f has the Thom a_f-property, we know that for $t > 0$ sufficiently small, $T_y(\mathring{W}_t)$ is very close to a plane P that contains $T_{y'}(Reg(L_0))$, for some $y' \in Reg(L_0) \cap \mathbf{B}_\delta(s)$ very close to y. But then P intersects $T_{y'}(\mathbf{B}_\delta(s))$ transversally and therefore $T_y(\mathring{W}_t)$ intersects $T_y(\mathbf{B}_\delta(s))$ transversally.

■

Proposition 4.3.2 *The following conditions are equivalent:*

(i) *W_0 is a fibre bundle over $L(\Sigma)$ with projection $p_0 = p_| : W_0 \to L(\Sigma)$ and fibre $f_s^{-1}(0)$;*

(ii) *$\partial W_0 = \partial W \cap L_0$ is a fibre bundle over $L(\Sigma)$ with projection $p_| : \partial W_0 \to L(\Sigma)$ and fibre $\partial\big(f_s^{-1}(0)\big)$, the link of f_s;*

(iii) *$\partial W_t = \partial W \cap L_t$ is a fibre bundle over $L(\Sigma)$ with projection $p_| : \partial W_t \to L(\Sigma)$ and fibre $\partial\big(f_s^{-1}(t)\big)$, the boundary of the Milnor fibre of f_s;*

(iv) *W_t is a fibre bundle over $L(\Sigma)$ with projection $p_t = p_| : W_t \to L(\Sigma)$ and fibre $f_s^{-1}(t)$.*

Proof: The implications $(i) \Rightarrow (ii)$ and $(iv) \Rightarrow (iii)$ are immediate. So we shall prove:

$(ii) \Rightarrow (i)$ First we show that $f_s^{-1}(0)$ is homeomorphic to $f_{s'}^{-1}(0)$, for any $s, s' \in L(\Sigma)$:
Fix $s \in L(\Sigma)$ and let the ball $\mathbf{B}_\theta(s) \subset \mathbf{B}_\delta(s)$ centered at s be a Milnor ball for f_s.

Claim: There exists a neighborhood V_s of s in $L(\Sigma)$ such that for any $s' \in V_s$, one has that $\mathbf{B}_\theta(s') \subset \mathbf{B}_\delta(s')$ is a Milnor ball for $f_{s'}$. To prove this claim, suppose it is not true, that is, that there exist s' as close to s as one whishes, and θ' with $0 < \theta' \leq \theta$ such that $f_{s'}^{-1}(0)$ intersects $\mathbf{S}_{\theta'}$ not transversally. Set $\delta = \theta'$. Since s' is close to s and since f is continuous, if $f_s^{-1}(0)$ intersects (transversally) $\mathbf{S}_{\theta'}$ in q connected components, then $f_{s'}^{-1}(0)$ intersects $\mathbf{S}_{\theta'}$ transversally in at least q connected components, and not transversally in at least one. Then ∂W_0 is not a fibre bundle over $L(\Sigma)$, a contradiction.

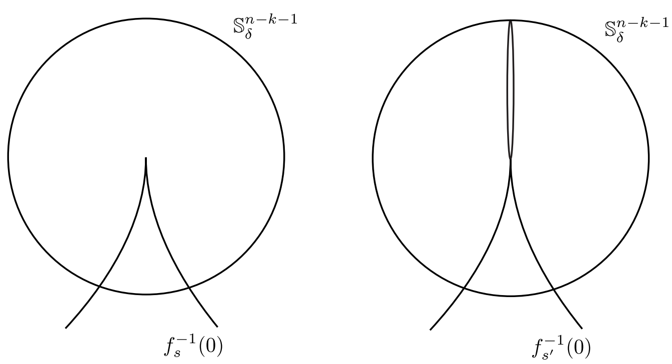


Figure 4.1:

Then we have that $f_s^{-1}(0) \cap \mathbf{B}_\theta \simeq Cone(f_s^{-1}(0) \cap \mathbf{S}_\theta)$ and $f_{s'}^{-1}(0) \cap \mathbf{B}_\theta \simeq Cone(f_{s'}^{-1}(0) \cap \mathbf{S}_\theta)$, for any $s' \in V_s$, and since we are supposing ∂W_0 is a fibre bundle, we have that $f_s^{-1}(0) \cap \mathbf{S}_\theta \simeq f_{s'}^{-1}(0) \cap \mathbf{S}_\theta$. Setting $\delta = \theta$, it follows that $f_s^{-1}(0) \simeq f_{s'}^{-1}(0)$, for any $s' \in V_s$. Since $L(\Sigma)$ is compact, we can choose $\delta > 0$ sufficiently small so that W_0 is a fibre bundle.

Now we just have to show the locally triviality of $p_|$. Given $s \in L(\Sigma)$, let V_s be a neighborhood of s in $L(\Sigma)$ such that p is trivial in V_s, that is, there exists a homeomorphism $\Psi = (\Psi_1, \Psi_2) : p^{-1}(V_s) \to V_s \times \mathbf{B}_\delta(s)$. We have to show that $p_|^{-1}(V_s)$ is homeomorphic to $V_s \times p_|^{-1}(s)$, and then its enough to show that $p^{-1}(V_s) \cap L_0 \simeq V_s \times \big(\mathbf{B}_\delta(s) \cap L_0\big)$. This homeomorphism is given by the map $\varphi : p^{-1}(V_s) \cap L_0 \to V_s \times \big(\mathbf{B}_\delta(s) \cap L_0\big)$ given by $\varphi(w) := (\Psi_1(w), h_{\Psi_1(w)}^{-1}(w))$, with $\varphi^{-1}(r_1, r_2) := h_{r_1}(r_2)$

$(ii) \Leftrightarrow (iii)$ It follows from lemma 4.2.5.

$(iii) \Rightarrow (iv)$ By lemma 4.3.1, we know that $\mathring{W}_t$ intersects $\mathbf{B}_\delta(s)$ transversally, for any $s \in L(\Sigma)$. Since $p_| : \partial W_t \to L(\Sigma)$ is the projection of a fibre bundle, it follows that it is a surjective submersion. Therefore, ∂W_t intersects $\mathbf{B}_\delta(s)$ transversally, for any $s \in L(\Sigma)$. Then it follows from the Ehresmann's fibration lemma that $p_| : W_t \to L(\Sigma)$ is the projection of a fibre bundle.

■

Proposition 4.3.3 *The following conditions are equivalent:*

(i) *W_0 is a fibre bundle over $L(\Sigma)$ with projection $p_0 : W_0 \to L(\Sigma)$ and fibre $f_s^{-1}(0)$;*

(ii) *$L(\Sigma)$ is a stratum of a stratification of W_0 induced by a Whitney stratification of f;*

(iii) *Either Σ or $\Sigma\backslash\{0\}$ is a stratum of a Whitney stratification of f;*

Proof: It is easy to see that (ii) and (iii) are equivalent. So we shall prove:

$(i) \Rightarrow (ii)$ Suppose that $f_s^{-1}(0) \simeq f_{s'}^{-1}(0)$, for any $s, s' \in L(\Sigma)$.
For any point $p \in W_0$, let $\vec{p}$ denote the vector in $\mathbb{C}^n$ defined by p. For any $p_1, p_2 \in W_0$, set $\overrightarrow{p_1p_2} = \vec{p}_1 + \vec{p}_2$ and denote $\overline{p_1p_2}$ the line in $\mathbb{C}^n$ given by $\overrightarrow{p_1p_2}$.
Let (x_i) be a sequence of points in $Reg(W_0)$ and (y_i) a sequence of points in $L(\Sigma)$ such that

$$\lim_{i\to\infty} x_i = s \in L(\Sigma), \quad \lim_{i\to\infty} y_i = s, \quad \lim_{i\to\infty} T_{x_i}(RegW_0) = T \ \text{ and } \ \lim_{i\to\infty} \overline{x_iy_i} = \lambda.$$

We have to show that $T \supset \lambda$. If we set $s_i := p(x_i)$, we claim that:

- $\lim_{i\to\infty} \overline{x_iy_i} \subseteq \lim_{i\to\infty} T_{x_i}(RegW_0)$ if, and only if, $\lim_{i\to\infty} \overline{x_is_i} \subseteq \lim_{i\to\infty} T_{x_i}(f_{s_i}^{-1}(0))$:
  In fact, for any $s \in L(\Sigma)$ set V_s a neighborhood of s in $L(\Sigma)$ such that $p_|^{-1}(V_s) \simeq f_s^{-1}(0) \times V_s$. Then

  $$T_{x_i}(RegW_0) = T_{x_i}(f_{s_i}^{-1}(0)) \times T_{s_i}(V_{s_i})$$

  and therefore

  $$\lim_{i\to\infty} T_{x_i}(RegW_0) = \lim_{i\to\infty} T_{x_i}(f_{s_i}^{-1}(0)) \times \lim_{i\to\infty} T_{s_i}(V_{s_i}).$$

  Note that $\overrightarrow{x_iy_i} = \overrightarrow{x_is_i} + \overrightarrow{s_iy_i}$ and therefore

  $$\lim_{i\to\infty} \overrightarrow{x_iy_i} = \lim_{i\to\infty} \overrightarrow{x_is_i} + \lim_{i\to\infty} \overrightarrow{s_iy_i}.$$

Then

$$\lim_{i\to\infty} \overline{x_i y_i} \subset \lim_{i\to\infty} T_{x_i}(RegW_0)$$

if, and only if,

$$\lim_{i\to\infty} \overrightarrow{x_i s_i} + \lim_{i\to\infty} \overrightarrow{s_i y_i} \subset \lim_{i\to\infty} T_{x_i}(f_{s_i}^{-1}(0)) \times \lim_{i\to\infty} T_{s_i}(V_{s_i}),$$

which happens if, and only if,

$$\lim_{i\to\infty} \overline{x_i s_i} \subset \lim_{i\to\infty} T_{x_i}(f_{s_i}^{-1}(0)).$$

Now consider $h_i : \mathbf{B}_\delta(s_i) \to \mathbf{B}_\delta(s)$ the homeomorphism such that $h_i(f_{s_i}^{-1}(0)) = f_s^{-1}(0)$. We claim that:

- $\lim_{i\to\infty} \overline{x_i s_i} = \lim_{i\to\infty} \overline{(h_i(x_i))s}$:
  Set $R := \lim_{i\to\infty} \overline{x_i s_i}$. Then for each $p \in R$, there exists a sequence of points (p_i) in $\mathbb{C}^n$ such that $p_i \in \overline{x_i s_i}$ and $p_i \to p$.
  Define $\tilde{p_i} := pr(h_i(p_i))$, where pr is the orthogonal projection of $\mathbf{B}_\delta(s)$ into $\overline{(h_i(x_i))s}$.
  Since $p_i \to p$, it happens that for any arbitrarily small ball $\mathbf{B}(p)$ in $\mathbb{C}^n$ centered at p there exists a some p_i in $\mathbf{B}(p)$, which implies that $h_i(p_i) \in \mathbf{B}(p) \cap \mathbf{B}_\delta(s)$, and then $\tilde{p_i} \in \mathbf{B}(p) \cap \mathbf{B}_\delta(s)$. Therefore $\tilde{p_i} \to p$.
  Then $\forall p \in R$ there exists a sequence of points $(\tilde{p_i})$ in $\mathbb{C}^n$ such that $\tilde{p_i} \in \overline{(h_i(x_i))s}$ and $\tilde{p_i} \to p$, and hence $\lim_{i\to\infty} \overline{(h_i(x_i))s} = R$.
- $\lim_{i\to\infty} T_{x_i}(f_{s_i}^{-1}(0)) = \lim_{i\to\infty} T_{h_i(x_i)}(f_s^{-1}(0))$:
  Set $S := \lim_{i\to\infty} T_{x_i}(f_{s_i}^{-1}(0))$. Then for each $p \in S$, there exists a sequence of points (p_i) in $\mathbb{C}^n$ such that $p_i \in T_{x_i}(f_{s_i}^{-1}(0))$ and $p_i \to p$.
  Consider $H_i : T_{x_i}(f_{s_i}^{-1}(0)) \to T_{h_i(x_i)}(f_s^{-1}(0))$ a homeomorphism between these two hyperplanes and define $\tilde{p_i} := H_i(p_i)$.
  Since $p_i \to p$, it happens that for any arbitrarily small ball $\mathbf{B}(p)$ in $\mathbb{C}^n$ centered at p there exists some p_i in $\mathbf{B}(p)$, which implies that $H_i(p_i) \in \mathbf{B}(p) \cap \mathbf{B}_\delta(s)$, and therefore $\tilde{p_i} \to p$.
  Then $\forall p \in S$ there exists a sequence of points $(\tilde{p_i})$ in $\mathbb{C}^n$ such that $\tilde{p_i} \in T_{h_i(x_i)}(f_s^{-1}(0))$ and $\tilde{p_i} \to p$, and hence $\lim_{i\to\infty} T_{h_i(x_i)}(f_s^{-1}(0)) = S$.

Then we have proved that

$$\lambda \subseteq T \Leftrightarrow \lim_{i\to\infty} \overline{(h_i(x_i))s} \subseteq \lim_{i\to\infty} T_{h_i(x_i)}(f_s^{-1}(0)).$$

But since $f_s^{-1}(0)$ has an isolated singularity, it follows that $Reg(f_s^{-1}(0))$ and $\{s\}$ are the two strata of a Whitney stratification of $f_s^{-1}(0)$. Hence $\lim_{i\to\infty} \overline{(h_i(x_i))s} \subseteq \lim_{i\to\infty} T_{h_i(x_i)}(f_s^{-1}(0))$, by the (b)-Whitney condition, and then we are done.

$(ii) \Rightarrow (i)$ By the properties of Whitney stratifications, we know that for each $s \in L(\Sigma)$ there exists a neighborhood U_s of s in W_0 such that $U_s \simeq V_s \times [f_s^{-1}(0) \cap U_s]$, where $V_s := U_s \cap L(\Sigma)$. Since $L(\Sigma)$ is compact, we can choose $\delta > 0$ sufficiently small such that $f_s^{-1}(0) \cap U_s = f_s^{-1}(0)$, for any $s \in L(\Sigma)$. Then we have that $p_|^{-1}(V_s) \simeq V_s \times f_s^{-1}(0)$. In particular, $f_s^{-1}(0) \simeq f_{s'}^{-1}(0)$, for any $s' \in V_s$.

∎

Corollary 4.3.4 *In particular, W_t is a fibre bundle over $L(\Sigma)$ for the following map-germs:*

(a) $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *holomorphic with critical locus Σ a complex curve;*

(b) $f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ *real analytic germ such that $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ are holomorphic and have no common irreducible components and such that $f\bar{g}$ has an isolated critical value (because then one clearly has that Σ is a complex curve in $(f\bar{g})^{-1}(0) = (fg)^{-1}(0)$, as we will see later in chapter 6);*

Combining all the results of this chapter, we obtain the following theorem:

Theorem 4.3.5 *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$, with $n \geq m$, be a real analytic map-germ such that $0 \in \mathbb{R}^m$ is an isolated critical value with the Thom a_f-property. Suppose that Σ has at most an isolated singularity. Then:*

(i) *There exists a neighbourhood W of $L(\Sigma)$ in $\boldsymbol{S}_\epsilon$, which is a fibre bundle over $L(\Sigma)$ with fibre a disk, such that $L_t \backslash \mathring{W}$ is homeomorphic to $L_0 \backslash \mathring{W}$;*

(ii) *The intersection $W_t := L_t \cap W_t$ is a fibre bundle over $L(\Sigma)$ if, and only if, the intersection $W_0 := L_0 \cap W_t$ is a fibre bundle over $L(\Sigma)$, which happens if, and only if, either Σ or $\Sigma \backslash \{0\}$ is a stratum of a Whitney stratification of f.*

Chapter 5

The boundary of the Milnor fibre of $f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$

With the tools developed in the previous chapters, we can finally describe the topology of the boundary of the Milnor fibre of a real analytic map-germ $f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ as a Waldhausen manifold.

Let $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be two holomorphic functions such that the real analytic map-germ $f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ has an isolated critical value at $0 \in \mathbb{C}$. Seade and Pichon proved in [36] that such a real analytic function $f\bar{g}$ as above has the Thom a_f-property. Hence there exist $\epsilon, \eta \in \mathbb{R}$ sufficiently small, $0 < \eta << \epsilon$, such that $f\bar{g}$ can be given a Milnor-Lê fibration in the tube as follows:

$$f\bar{g}_| : (f\bar{g})^{-1}(\mathbf{D}^*_\eta) \cap \mathbf{B}_\epsilon \to \mathbf{D}^*_\eta.$$

We will prove that the boundary of the Milnor fibre

$$L_t := (f\bar{g})^{-1}(t) \cap \mathbf{S}^5_\epsilon,$$

for $t \in \mathbf{D}^*_\eta$, is a Waldhausen manifold.

5.1 Systems of neighbourhoods

In the holomorphic case, it is well known that it is possible to define the Milnor fibration using different systems of neighbourhoods at the origin. Balls and polydisks are the most widely used systems of neighbourhoods. In the proof of Theorem 5.2.3 we need to work with a Milnor fibration defined using a polydisk instead of a ball. We shall prove now that the boundary of the Milnor fibre defined for polydisks is homeomorphic to the boundary of the Milnor fibre defined for balls.

Let Σ be the critical locus of $\{fg = 0\}$. It is easy to see (as we will show in the next chapter) that Σ is the complex curve given by

$$\Sigma = \Sigma(f) \cup \Sigma(g) \cup \big(f^{-1}(0) \cap g^{-1}(0)\big) = \Sigma(f\bar{g}).$$

Choose a coordinate system (x, y, z) of $\mathbb{C}^3$ such that there exists ϵ such that for any $\epsilon' \leq \epsilon$, the boundary of the polydisk

$$\Delta_\epsilon = \{(x, y, z) \in \mathbb{C}^3 : \max\{|x|, |y|, |z|\} \leq \epsilon\}$$

meets Σ transversely at the open face

$$\{(x, y, z) \in \mathbb{C}^3 : \max\{|x|, |y|\} < \epsilon, |z| = \epsilon\}.$$

Consider the family of norms in $\mathbb{C}^3$

$$||(x, y, z)||_s := (|x|^{1/s} + |y|^{1/s} + |z|^{1/s})^s,$$

$$||(x, y, z)||_0 := \max\{|x|, |y|, |z|\}$$

for $s \in [0, 1/2]$, which depends continuously on the parameter s.

A positive number ϵ is a Milnor radius for the function $\{fg = 0\}$ with respect to the norm $||\centerdot||_s$ if for any positive $\epsilon' \leq \epsilon$ the hypersurface $fg = 0$ is transverse in the stratified sense to the sphere $||(x, y, z)||_s = \epsilon'$. It is well known that for any $s \in [0, 1/2]$ there is a Milnor radius for $\{fg = 0\}$ with respect to the norm $||\centerdot||_s$. Moreover, by the continuity of the norms in the parameter s, if ϵ is a Milnor radius for $\{fg = 0\}$ with respect to the norm $||\centerdot||_s$, then there exists a neighbourhood U of $s \in [0, 1/2]$ such that ϵ is a Milnor radius for $\{fg = 0\}$ with respect to the norm $||\centerdot||_{s'}$ for any $s' \in U$. Using the compactness of $[0, 1/2]$ we find a radius ϵ which is a Milnor radius for $\{fg = 0\}$ with respect to the norm $||\centerdot||_s$ for any $s \in [0, 1/2]$.

Given any $U \subset [0, 1/2]$ we define the set

$$\mathcal{B}_\epsilon^U := \{(x, y, z, s) \in \mathbb{C}^3 \times U : ||(x, y, z)||_s \leq \epsilon\}.$$

It should be viewed as a family of Milnor balls for varying norms.

Since the function $f\bar{g}$ satisfies the Thom a_f-property, for any $s \in [0, 1/2]$ there exists a neighbourhood U of $s \in [0, 1/2]$ and a positive η such that the mapping

$$F^U : \mathcal{B}_\epsilon^U \cap (f\bar{g})^{-1}(D_\eta^*) \to D_\eta^* \times U$$

defined by $F^U(x,y,z,s) := (f(x,y,z)\bar{g}(x,y,z), s)$ is a topological fibre bundle.

By compactness of $[0,1/2]$ there exists a positive η such that the mapping

$$F : \mathcal{B}_\epsilon^{[0,1/2]} \cap (f\bar{g})^{-1}(D_\eta^*) \to D_\eta^* \times [0,1/2]$$

is a fibre bundle.

Consequently, the boundaries of fibres $F^{-1}(t,0)$ and $F^{-1}(t,1/2)$ are homeomorphic for any $t \in D_\eta^*$, and hence the boundary of the Milnor fibre defined for polydisks is homeomorphic to the boundary of the Milnor fibre defined for balls.

5.2 The Waldhausen structure

Our goal is to prove that the boundary of the Milnor fibre

$$L_t = (f\bar{g})^{-1}(t) \cap \mathbf{S}_\epsilon^5,$$

for $t \in \mathbf{D}_\eta^*$, is a Waldhausen manifold. Since we have proved that the boundary of the Milnor fibre defined using balls is homeomorphic to the boundary of the Milnor fibre using polydisk, from now on we will assume that $\mathbf{S}_\epsilon^5$ denotes the boundary of the ball of Milnor radius ϵ, with the norm $|| \centerdot ||_0$, as in the previous section.

The singular locus of

$$L_0 = (f\bar{g})^{-1}(0) \cap \mathbf{S}_\epsilon^5 = (fg)^{-1}(0) \cap \mathbf{S}_\epsilon^5$$

is the intersection of the sphere $\mathbf{S}_\epsilon^5$ with the complex curve Σ, that is, $L(\Sigma) := \Sigma \cap \mathbf{S}_\epsilon^5$. We know that $L(\Sigma)$ is a finite disjoint union of circles $\mathbf{S}^1$ contained in the open face

$$\{(x,y,z) \in \mathbb{C}^3 : \max\{|x|,|y|\} < \epsilon, |z| = \epsilon\}.$$

Let

$$n : \tilde{X} \to (fg)^{-1}(0)$$

be the normalization of $(fg)^{-1}(0)$. Set $\tilde{\Sigma} := n^{-1}(\Sigma)$ and $\tilde{L}_0 := n^{-1}(L_0)$.

Let W denote the vanishing zone of $f\bar{g}$, which is nothing but a tubular neighbourhood of $L(\Sigma)$ in $\mathbf{S}_\epsilon^5$ such that $L_t \backslash \mathring{W}$ is homeomorphic to $L_0 \backslash \mathring{W}$ (see chapter 4). Set $W_0 := W \cap L_0$ and $W_t := W \cap L_t$.

Lemma 5.2.1 *$L_t \backslash W_t$ is a Waldhausen manifold.*

Proof: Consider $n:(\overline{F_0},0)\to(F_0,0)$ a normalization of $(F_0,0)$. Then $\overline{\Sigma}:=n^{-1}(\Sigma)$ is a complex curve in $(\overline{F_0},0)$. So we can consider $\pi:(\tilde{F_0},0)\to(\overline{F_0},0)$ a good resolution of both $\overline{F_0}$ and $\overline{\Sigma}$. Also consider $\overline{L_0}:=n^{-1}(L_0)$, the link of $(\overline{F_0},0)$, and $\overline{L(\Sigma)}:=n^{-1}(L(\Sigma))$, which is diffeomorphic to $L(\overline{\Sigma})$, the link of $\overline{\Sigma}$ in $\overline{L_0}$.
Then the pair $(\pi^{-1}(\overline{L_0}),\pi^{-1}(\overline{L(\Sigma)})$ is diffeomorphic to the pair $(\overline{L_0},\overline{L(\Sigma)})$.
So we can consider a Waldhausen decomposition of $\overline{F_0}$ such that the link $\overline{L(\Sigma)}$ is a union of Seifert fibres and such the set $\overline{W_0}:=n^{-1}(W_0)$, which is a tubular neighborhood of the link $\overline{L(\Sigma)}$ in $\overline{L_0}$, is saturated with Seifert fibres (see remark 1.9.9). Then we can restrict this Waldhausen decomposition to a Waldhausen decomposition of $\overline{L_0}\backslash\overline{W_0}$.
Since the restriction $n_|:\overline{L_0}\backslash\overline{W_0}\to L_0\backslash W_0$ is an homeomorphism, it happens that $L_0\backslash W_0$ is also a Waldhausen manifold, and then it follows from theorem 4.2.5 that $L_t\backslash W_t$ is a Waldhausen manifold.

■

Now, since ∂W_t is a finite disjoint union of tori, all we have to prove is that W_t is a Waldhausen manifold. If $\Sigma=\Sigma_1\cup\cdots\cup\Sigma_k$ is the decomposition of Σ into irreducible components, we get the decomposition of W into disjoint connected components $W=W[1]\sqcup\cdots\sqcup W[k]$, where $W[l]$ is a small tubular neighbourhood of the circle $L(\Sigma_l):=\Sigma_l\cap\mathbf{S}^5_\epsilon$ in $\mathbf{S}^5_\epsilon$, for $l=1,\ldots,k$.

We have already seen in chapter 4 that W_t is a fibre bundle over $L(\Sigma)$. In this particular case of dimension $n=3$, there is an easier way of showing this fact, as follows:

Fix a component Σ_l. Given $p\in\Sigma_l\backslash\{0\}$ let H_p be the 2-dimensional affine hyperplane of $\mathbb{C}^3$ passing through p and parallel to $\{z=0\}$. Choosing ϵ small enough we may assume that the Milnor number of the germ $(fg|_{H_p},p)$ is independent of p (since either Σ or $\Sigma\backslash\{0\}$ is a stratum of a Whitney stratification of fg). Therefore, by Remark 3.1.2 and the compactness of $L(\Sigma_l)$ we deduce the existence of a positive δ such that for any $p\in L(\Sigma_l)$ the ball in H_p centered at p and of radius δ is a Milnor ball for $(f\bar{g})|_{H_p}$ at p. We may choose $W[l]$ to be the union of those balls when p varies in $L(\Sigma_l)$. With this definition, there is a natural fibration

$$\sigma_l:W[l]\to L(\Sigma_l)$$

with fibre a complex 2-ball.

Since $f\bar{g}$ satisfies the Thom a_f-property, there exists a positive η such that the mapping

$$\Psi_l:W[l]\cap(f\bar{g})^{-1}(D_\eta)\to D_\eta\times L(\Sigma_l)$$

defined by $\Psi_l:=(f\bar{g},\sigma_l)$ has only the circle $\{0\}\times L(\Sigma_l)$ as critical values. Therefore, for $t\in D^*_\eta$, the restriction

$$\sigma_{l|}:W_t[l]\to L(\Sigma_l)$$

is a fibre bundle, where $W_t[l] := L_t \cap W[l]$. Its fibre is called the Transversal Milnor fibre of $f\bar{g}$ at Σ_l and its monodromy h the vertical monodromy along Σ_l (see [40] or chapter 1).

To prove that W_t is a Waldhausen manifold, we have to show that each connected component $W_t[l]$, for $l = 1, \ldots, k$, is a Waldhausen manifold. To do that, it is sufficient to give a decomposition of each transversal Milnor fibre which is invariant under the corresponding vertical monodromy $h := h_l$ and such that the corresponding pieces of $W_t[l]$ are Seifert manifolds. We will prove that they are Seifert manifolds either by proving that they are fibre bundles over a circle and with fibre a cylinder, or by showing directly that the restriction of h to the corresponding piece of the transversal Milnor fibre is periodic.

Let us fix on an irreducible component Σ_l of Σ, which by an easy argument can be assumed, without loosing generality, to be the z-axis (see [25], Lemma 4.4). Let D be the disk of radius ϵ around the origin of Σ_l, with boundary $\mathbf{S}^1$. This coincides with the intersection of Σ_l with the polydisk of size ϵ. The region $W[l]$ coincides now with the product $\mathbf{S}^1 \times B_\delta$, where B_δ is the ball of radius δ in the (x, y)-complex 2-plane, and the mapping σ_l coincides with the projection on the first factor.

We can look at the restriction $(f\bar{g})_| : \mathbf{S}^1 \times \mathbb{C}^2 \to \mathbb{C}$ as a family in the parameter $\mathbf{S}^1$. For each $s \in \mathbf{S}^1$, we denote by $f_s\bar{g}_s$ the restriction $f\bar{g}_{|\{s\}\times\mathbb{C}^2}$. The corresponding holomorphic family $fg_| : \mathbf{S}^1 \times \mathbb{C}^2 \to \mathbb{C}$ is μ-constant over $\mathbf{S}^1$. Then it is well known that we can consider a minimal embedded resolution in family

$$\pi : \tilde{M} \to \mathbf{S}^1 \times \mathbb{C}^2,$$

which is an analytic morphism π where:

- $E := \pi^{-1}(\mathbf{S}^1 \times \{0\})$ is the exceptional divisor, with a decomposition in irreducible components $E = \cup_{i=1}^r E_i$, where an irreducible component is defined as the closure of a connected component of $E \backslash Sing(E)$;

- for each $s \in \mathbf{S}^1$, define $X_s := \pi^{-1}(\{s\} \times B_\delta)$. Then

$$\pi_s : X_s \to \mathbb{C}^2$$

  is the minimal embedded resolution of the plane curve singularity defined by the restriction of fg to H_s. We denote by E^s the exceptional divisor of π_s, and by E_i^s the set of irreducible components of E^s contained in E_i.

Note that if $p_1 : \mathbf{S}^1 \times \mathbb{C}^2 \to \mathbf{S}^1$ is the projection on the first factor, then $p_1 \circ \pi : \tilde{M} \to \mathbf{S}^1$ is a fibre bundle with monodromy $\tilde{H} : X_s \to X_s$.

Now, for each $s \in \mathbf{S}^1$, define the boxes V_i^s and V_{ij}^s in X_s as in section 3.1 (in order to simplify notation, we shall make no distinction between boxes of the type V_i and $\tilde{V}_p$ nor between boxes of the type V_{ij} and $\tilde{V}_{ip}$).

From now on, fix $s \in \mathbf{S}^1$ and let V_i (resp. V_{ij}) be the union of orbits of $\tilde{H}$ that intersect V_i^s (resp. V_{ij}^s). Note that one can have $V_i = V_j$ for some $i \neq j$ (resp. $V_{ij} = V_{i'j'}$ for some $i \neq i'$ and $j \neq j'$). Also note that each V_i (resp. V_{ij}) is a fibre bundle over $\mathbf{S}^1$ with fibre a finite disjoint union of boxes of the type V_i^s (resp. V_{ij}^s), since $\tilde{H}$ takes each box V_i^s (resp. V_{ij}^s) to some box V_j^s (resp. $V_{i'j'}^s$).

Clearly, the transversal Milnor fibre $(f\bar{g})^{-1}(t) \cap \sigma_l^{-1}(s)$ is diffeomorphic to

$$F_t(s) := (f_s\bar{g}_s \circ \pi_s)^{-1}(t),$$

and the vertical monodromy of $f\bar{g}$ can be taken to be the automorphism $\tilde{h} := \tilde{H}_| : F_t(s) \to F_t(s)$. Moreover, $F_t(s)$ can be decomposed as follows:

$$F_t(s) = \left(\bigcup_i \left(F_t(s) \cap V_i^s\right)\right) \cup \left(\bigcup_{i,j} \left(F_t(s) \cap V_{ij}^s\right)\right).$$

Clearly, this decomposition is preserved by the vertical monodromy.

Moreover, if we set $\tilde{W}_t[l] := \pi^{-1}\big((f\bar{g})^{-1}(t) \cap \mathbf{B}_\epsilon\big) = \pi^{-1}(W_t[l])$, which is diffeomorphic to $W_t[l]$, we have the decomposition

$$\tilde{W}_t[l] = \left(\bigcup_i \left(\tilde{W}_t[l] \cap V_i\right)\right) \cup \left(\bigcup_{i,j} \left(\tilde{W}_t[l] \cap V_{ij}\right)\right),$$

and clearly each $\tilde{W}_t[l] \cap V_i$ (resp. $\tilde{W}_t[l] \cap V_{ij}$) is a fibre bundle over $\mathbf{S}^1$ with fibre a finite number of disjoint copies of $F_t(s) \cap V_i^s$ (resp. $F_t(s) \cap V_{ij}^s$).

In Lemma 3.1.1, we showed that each part of the Milnor fibre of type $F_t(s) \cap V_{ij}^s$ is a finite disjoint union of cylinders. Hence $\tilde{W}_t[l] \cap V_{ij}$ is a fibre bundle over $\mathbf{S}^1$ with fibre a finite disjoint union of cylinders. The classification of such kind of fibrations yields that $\tilde{W}_t[l] \cap V_{ij}$ is a Seifert manifold (see Remark 1.10.3).

Now, if $V_i^s \subset X_s$ is such that $a_i = b_i$, by (iv) of Lemma 3.1.1 we have that $F_t(s) \cap V_i^s$ is a disjoint union of cylinders. As before, we conclude that the corresponding piece $\tilde{W}_t[l] \cap V_i$ is a Seifert manifold.

If $V_i^s \subset X_s$ is such that $a_i \neq b_i$, by (ii) of Lemma 3.1.1 we have that $F_t(s) \cap V_i^s$ is a finite covering of $E_i^s \cap V_i^s = \mathbb{P}^1 \backslash \mathrm{d}isks$, and hence the corresponding piece $\tilde{W}_t[l] \cap V_i$ is a finite covering of $E_i \cap V_i$.

If E_i^s does not represent a rupture vertex of the dual graph of the total transform of the transversal singularity by π_s, then $F_t(s) \cap V_i$ is a finite disjoint union of either disks or cylinders, and therefore $\tilde{W}_t[l] \cap V_i$ is a fibre bundle over $\mathbf{S}^1$ with fibre a finite disjoint union of either disks or cylinders, and therefore it is a Seifert manifold.

So now we shall see what happens in V_i if E_i^s represents a rupture vertex. Consider the restriction

$$\tilde{h}_i := \tilde{H}_| : E_i^s \to E_j^s.$$

We know that there exists an integer $m_i > 0$ such that the iterated $(\tilde{h}_i)^{m_i}$ is an automorphism of E_i^s. Consider $E_{j_1}^s, E_{j_2}^s, E_{j_3}^s \subset X_s$ such that $E_{j_1}^s \cap E_i^s = \{x_1\}$, $E_{j_2}^s \cap E_i^s = \{x_2\}$ and $E_{j_3}^s \cap E_i^s = \{x_3\}$. Then we must have that $(\tilde{h}_i)^{6m_i}$ is an automorphism of E_i^s that fixes x_1, x_2 and x_3. Since any automorphism of $\mathbb{P}^1$ that fixes three points is the identity, we conclude that $\tilde{h}_i$ is periodic, and hence $E_i \cap V_i$ is a Seifert manifold. Then the following proposition finishes our proof.

Proposition 5.2.2 *(i) A finite covering of a Seifert manifold is a Seifert manifold;*

(ii) A finite covering of a Waldhausen manifold is a Waldhausen manifold.

Proof: Let $\pi : M' \to M$ be a finite covering of a Waldhausen manifold. Write $M = \cup M_i$, where each Seifert piece M_i is a fibre bundle over a compact surface with boundary F_i and fibre $\mathbf{S}^1$ (with finitely many multiple special fibres), and projection $p_i : M_i \to F_i$. It is enough to prove that each piece $M_i' := \pi^{-1}(M_i)$ is a Seifert manifold. The composition

$$p_i \circ \pi : M_i' \to F_i$$

is a bundle (with finitely many multiple special fibres) with fibre a finite covering of $\mathbf{S}^1$. If such fibre is connected, then we have expressed M_i' as a bundle over a surface with fibre $\mathbf{S}^1$, and therefore it is a Seifert manifold.
If the fibre is not connected, we define the following equivalence relation in M_i': two points are identified if they belong to the same connected component of the same fibre of $p_i \circ \pi$. The quotient of M_i' by this equivalence relation is a surface F_i' that covers F_i, and the quotient application $q : M_i' \to F_i'$ expresses M_i' as a bundle over F_i' with fibre $\mathbf{S}^1$. ∎

Then we have proved:

Theorem 5.2.3 *Let $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be two holomorphic functions such that the real analytic germ given by $f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ has an isolated critical value at $0 \in \mathbb{C}$. Then the boundary of the Milnor fibre of $f\bar{g}$ is a Waldhausen manifold.*

Chapter 6

The degeneration of the boundary of the Milnor fibre

In this chapter, we finally describe the degeneration of the boundary of the Milnor fibre to the link of a real analytic map-germ $f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, with $n \geq 3$, given by two holomorphic germs $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ satisfying the following hypothesis:

(A) $f\bar{g}$ has an isolated critical value at $0 \in \mathbb{C}$;

(B) Either Σ or $\Sigma \backslash \{0\}$ is a stratum of a Whitney stratification of fg, with dimension $k \leq n-2$;

(C) $\frac{\partial g}{\partial z_1} = 0$;

(D) Either g is constant (and hence $f\bar{g}$ is holomorphic) or the map $(f, g) : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$ is a complete intersection, that is, the intersection $f^{-1}(0) \cap g^{-1}(0)$ has complex dimension $n-2$.

For example, consider

$$f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$$

holomorphic with critical locus Σ a complex curve. Then we shall prove the following theorem:

Theorem 6.0.4 *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two holomorphic germs of function such that the real analytic map-germ $f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ satisfies the hypothesis (A), (B), (C) and (D) as above. Then, for any $t \neq 0$ sufficiently small, there exist:*

(*i*) *a small neighbourhood* W *of the link of the critical locus* $L(\Sigma)$ *in the boundary of the Milnor ball* $\boldsymbol{S}_\epsilon$ *such that the part of* L_t *(the boundary of the Milnor fibre of* $f\bar{g}$*) that is not inside* $\mathring{W}$ *is diffeomorphic to the part of* L_0 *(the link of* $f\bar{g}$*) that is not inside* $\mathring{W}$*;*

(*ii*) *a polyhedron* P_t *in* $W_t = L_t \cap W$*, of real dimension* $n+k-2$*, such that* W_t *deformation retracts to* P_t*;*

(*iii*) *a continuous map* $\Psi_t : W_t \to W_0 = L_0 \cap W$ *which restricts to a homeomorphism from* $W_t \backslash P_t$ *to* $W_0 \backslash L(\Sigma)$ *and sends* P_t *to* $L(\Sigma)$*.*

The hypothesis (A) implies that:

- There exist sufficient small positive reals $0 < \eta << \epsilon$ such that the restriction

$$f\bar{g}_{|} : (f\bar{g})^{-1}(\mathbf{D}^*_\eta) \cap \mathbf{B}_\epsilon \to \mathbf{D}^*_\eta$$

  is a fibre bundle (as we have already seen before);

- $\Sigma = \Sigma(fg) = \Sigma(f) \cup \Sigma(g) \cup \big(f^{-1}(0) \cap g^{-1}(0)\big)$; in particular, Σ is a complex variety.

  In fact, if we consider the complex coordinates $z_1, \bar{z_1}, \ldots, z_n, \bar{z_n}$, then

$$f\bar{g} = (\Re f\bar{g}, \Im f\bar{g}) = \frac{1}{2}\left(f\bar{g} + \bar{f}g, \frac{1}{i}(f\bar{g} - \bar{f}g)\right)$$

  and the Jacobian matrix of the function $f\bar{g}$ is isomorphic by

$$\begin{pmatrix} \frac{\partial f}{\partial z_1}\bar{g} + \bar{f}\frac{\partial g}{\partial z_1} & f\frac{\bar{\partial} g}{\partial \bar{z_1}} + g\frac{\bar{\partial} f}{\partial \bar{z_1}} & \frac{\partial f}{\partial z_2}\bar{g} + \bar{f}\frac{\partial g}{\partial z_2} & f\frac{\bar{\partial} g}{\partial \bar{z_2}} + g\frac{\bar{\partial} f}{\partial \bar{z_2}} & \ldots & \frac{\partial f}{\partial z_n}\bar{g} + \bar{f}\frac{\partial g}{\partial z_n} & f\frac{\bar{\partial} g}{\partial \bar{z_n}} + g\frac{\bar{\partial} f}{\partial \bar{z_n}} \\ \frac{\partial f}{\partial z_1}\bar{g} - \bar{f}\frac{\partial g}{\partial z_1} & f\frac{\partial g}{\partial z_1} - g\frac{\partial f}{\partial z_1} & \frac{\partial f}{\partial z_2}\bar{g} - \bar{f}\frac{\partial g}{\partial z_2} & f\frac{\partial g}{\partial z_2} - g\frac{\partial f}{\partial z_2} & \ldots & \frac{\partial f}{\partial z_n}\bar{g} - \bar{f}\frac{\partial g}{\partial z_n} & f\frac{\partial g}{\partial z_n} - g\frac{\partial f}{\partial z_n} \end{pmatrix}$$

  So the points of Σ are the points that satisfy all the following equations:

  - $fg(\frac{\partial f}{\partial z_i}\frac{\partial g}{\partial z_j} - \frac{\partial f}{\partial z_j}\frac{\partial g}{\partial z_i}) = 0$, for any $i \neq j$;
  - $|f\frac{\partial g}{\partial z_i}| = |g\frac{\partial f}{\partial z_i}|$, for any $i = 0, \ldots, n$;
  - $|f|^2\frac{\partial g}{\partial z_i}\frac{\bar{\partial} g}{\partial \bar{z_j}} = |g|^2\frac{\partial f}{\partial z_i}\frac{\bar{\partial} f}{\partial \bar{z_j}}$, for any $i \neq j$.

  Since $f\bar{g}$ has an isolated critical value, then $\Sigma(f\bar{g}) \subset (fg)^{-1}(0)$. Define $\gamma = \big(f^{-1}(0) \cap g^{-1}(0)\big)$, a complex variety in $\mathbb{C}^n$ of codimension 2.

  From the equations above, we know that $\gamma \subseteq \Sigma$. If $x \in \Sigma(f)$, then $f(x) = 0$ and $\frac{\partial f}{\partial z_i} = 0$, for any $i = 1, \ldots, n$, and hence $x \in \Sigma$, by those equations. Therefore

$\Sigma(f) \subseteq \Sigma$. In the same way, $\Sigma(g) \subseteq \Sigma$. So $\Sigma \supseteq \gamma \cup \Sigma(f) \cup \Sigma(g)$.

Now, if $x \in \Sigma$ and $x \notin \gamma$, let's say $f(x) = 0$ and $g(x) \neq 0$, then by the third equation above we see that $\frac{\partial f}{\partial z_i} = 0$, for any $i = 1, \dots, n$, and hence $x \in \Sigma(f)$ (or if $f(x) \neq 0$ and $g(x) = 0$, then $x \in \Sigma(g)$). Therefore, $\Sigma = \gamma \cup \Sigma(f) \cup \Sigma(g)$.

Note that $\Sigma(fg) = \{(\frac{\partial f}{\partial z_1} g + f \frac{\partial g}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} g + f \frac{\partial g}{\partial z_n}) = \underline{0}\}$. Then $\Sigma(f) \subset \Sigma(fg)$, $\Sigma(g) \subset \Sigma(fg)$ and $\gamma \subset \Sigma(fg)$. So it follows from the arguments above that $\Sigma \subseteq \Sigma(fg)$.

On the other hand, if $x \in \Sigma(fg)$ and $x \notin \gamma$, then either $f(x) = 0$ or $g(x) = 0$. Suppose $f(x) = 0$. Then $\frac{\partial f}{\partial z_i}(x) = 0$, for any $i = 1, \dots, n$, and hence $x \in \Sigma(f)$. In the same way, if $g(x) = 0$, then $x \in \Sigma(g)$. So $\Sigma(fg) \subseteq \Sigma$. So $\Sigma = \Sigma(fg)$.

Hypothesis (B) implies that the critical locus Σ of $f\bar{g}$ is either a smooth or an isolated singularity complex analytic variety of dimension $k \leq n-2$, and therefore the vanishing zone W, defined as in section 4.2, is a fibre bundle over $L(\Sigma)$ with fibre a $(2n-2k)$-dimensional disk in $\mathbf{S}_\epsilon$. Hence, for any $t \in \mathbf{D}^*_\eta$, we can define the sets F_t, F_0, L_t, L_0, W_t and W_0 as before. Moreover, it implies that W_t and W_0 are fibre bundles over $L(\Sigma)$ (theorem 4.3.4).

Now let

$$\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$$

be the decomposition of Σ into irreducible components. Then $L(\Sigma)$ has a decomposition into disjoin components components given by

$$L(\Sigma) = L(\Sigma_1) \sqcup \cdots \sqcup L(\Sigma_r).$$

Hence W has a decomposition into disjoint connected components given by

$$W = W[1] \sqcup \cdots \sqcup W[r]$$

and W_t and W_0 have decompositions into disjoint connected components given by

$$W_t = W_t[1] \sqcup \cdots \sqcup W_t[r]$$

and

$$W_0 = W_0[1] \sqcup \cdots \sqcup W_0[r].$$

Then each connected component $W_0[i]$ is a fibre bundle over the $(2k-1)$-manifold $L(\Sigma_i)$ and each connected component $W_t[i]$ is a fibre bundle over $L(\Sigma_i)$:

$$\begin{array}{ccc} (f\bar{g})_s^{-1}(t) \hookrightarrow W_t[i] & \qquad & (f\bar{g})_s^{-1}(0) \hookrightarrow W_0[i] \\ \downarrow p & & \downarrow p \\ L(\Sigma_i) & & L(\Sigma_i) \end{array}$$

where the ball $\mathbf{B}_\theta(s)^{2n-2k} \subset \mathbf{S}_\epsilon^{2n-1}$ can be taken to be a Milnor ball for $(f\bar{g})_s$ (see the claim of the proof of theorem 4.3.2), and $(f\bar{g})_s : \mathbf{B}_\theta(s) \to \mathbb{C}$ has an isolated singularity at $0 \in \mathbf{B}_\theta(s)$.

From now on, we should consider the Milnor fibration of $f\bar{g}$ defined on a polydisk instead of a ball, that is, considering the polydisk

$$\Delta_\epsilon := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : max\{|z_1|, \ldots, |z_n|\} \leq \epsilon\}$$

instead of the ball $\mathbf{B}_\epsilon$ (see section 5.1). Then we can choose a coordinate system $(z_1, \ldots, z_n)$ of $\mathbb{C}^n$ such that there exists $\epsilon > 0$ such that for any $\epsilon' \leq \epsilon$ the boundary of the polydisk Δ_ϵ intersects Σ transversally at the open face

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^n : max\{|z_1|, \ldots, |z_{n-1}|\} < \epsilon, |z_n| = \epsilon\},$$

which contains $L(\Sigma)$.

Then for each $s \in L(\Sigma)$, we can consider $(f\bar{g})_s$ to be the real analytic isolated singularity function germ

$$(f\bar{g})_s : (H_s, s) \to (\mathbb{C}, 0)$$

given by the restriction of $f\bar{g}$ to H_s, where H_s is the $(n-k)$-dimensional affine hyperplane of $\mathbb{C}^n$ passing through s and parallel to $\{z_{n-k+1} = z_{n-k+2} \cdots = z_n = 0\}$.

Note that (i) of theorem 6.0.4 was proved in chapter 4. Now we shall prove (ii) and (iii) of that theorem. First we need to construct a Lê Polyhedron for each isolated singularity $(f\bar{g})_s$:

- If g is constant and hence $f\bar{g}$ is holomorphic, the construction of a Lê polyhedron for $(f\bar{g})_s$ is given by Theorem 2.1.1;
- If g is not constant, by hypothesis (D) we have that $\big(f^{-1}(0) \cap g^{-1}(0)\big)$ has complex dimension $(n-2)$. Since $\big(f^{-1}(0) \cap g^{-1}(0)\big)$ is contained in Σ, we must have $k \geq n-2$. But clearly one has $k \leq n-2$, and therefore $k = n-2$. Then H_s is a 2-dimensional affine hyperplane. Since $\frac{\partial g}{\partial z_1} = 0$, by hypothesis (C), we can assume, taking appropriate coordinates, that for each $s \in L(\Sigma)$ one has $\frac{\partial g_s}{\partial z_1} = 0$. Then the construction of a Lê polyhedron for $(f\bar{g})_s$ is given by Theorem 3.4.1.

Then we obtain:

- a polyhedron $P_{t,s}$ of real dimension $n-k-1$ in the compact real surface $(f\bar{g})_s^{-1}(t)$ such that $(f\bar{g})_s^{-1}(t)$ deformation retracts to $P_{t,s}$.

- a continuous map $\Psi_{t,s} : (f\bar{g})_s^{-1}(t) \to (f\bar{g})_s^{-1}(0)$ which sends $P_{t,s}$ to $\{0\}$ and such that $\Psi_{t,s}$ restricts to a homeomorphism from $(f\bar{g})_s^{-1}(t)\backslash P_{t,s}$ to $(f\bar{g})_s^{-1}(0)\backslash\{0\}$.

Now fix $s_i \in L(\Sigma_i)$ and $t \in \mathbf{D}_\eta^*$. Consider P_{t,s_i} a Lê Polyhedron for $(f\bar{g})_{s_i}$ and Ψ_{t,s_i} a collapsing map as in the previous lemma. Since $W_t[i]$ is a fibre bundle over $L(\Sigma_i)$, it follows that for any $s \in L(\Sigma_i)\backslash\{s_i\}$ there exists a homeomorphism

$$h_{t,s} : \big(\mathbf{B}_\theta(s), (f\bar{g})_s^{-1}(t)\big) \to \big(\mathbf{B}_\theta(s_i), (f\bar{g})_{s_i}^{-1}(t)\big)$$

and a homeomorphism

$$h_{0,s} : \big(\mathbf{B}_\theta(s), (f\bar{g})_s^{-1}(0)\big) \to \big(\mathbf{B}_\theta(s_i), (f\bar{g})_{s_i}^{-1}(0)\big),$$

where $\mathbf{B}_\theta(s_i) \subset H_{s_i}$ is a Milnor ball for $(f\bar{g})_{s_i}$ (and therefore $\mathbf{B}_\theta(s) \subset H_s$ is a Milnor ball for f_s). Then for each $s \in L(\Sigma_i)\backslash\{s_i\}$ we define

$$P_{t,s} := h^{-1}(P_{t,s_i}).$$

Lemma 6.0.5 *For each $s \in L(\Sigma_i)\backslash\{s_i\}$, the polyhedron $P_{t,s}$ defined as above is a Lê Polyhedron for $(f\bar{g})_s$.*

Proof: We have to show that:

(*i*) $(f\bar{g})_s^{-1}(t)$ deformation retracts to $P_{t,s}$:
Since $(f\bar{g})_{s_i}^{-1}(t)$ deformation retracts to P_{t,s_i}, there exists a family of maps $\alpha_\kappa : (f\bar{g})_{s_i}^{-1}(t) \to (f\bar{g})_{s_i}^{-1}(t)$ in the real parameter $\kappa \in I$, where I denotes the unit interval, such that α_0 is the identity, $\alpha_1((f\bar{g})_{s_i}^{-1}(t)) = P_{t,s_i}$ and the restriction of α_κ to P_{t,s_i} is the identity, for any $\kappa \in I$.
Then for each $\kappa \in I$, define the map $\beta_\kappa : (f\bar{g})_s^{-1}(t) \to (f\bar{g})_s^{-1}(t)$ by setting $\beta_\kappa = h^{-1} \circ \alpha_\kappa \circ h$. This family of maps clearly defines a deformation retraction of $(f\bar{g})_s^{-1}(t)$ onto $P_{t,s}$.

(*ii*) there exists a continuous map $\Psi_{t,s} : (f\bar{g})_s^{-1}(t) \to (f\bar{g})_s^{-1}(0)$ such that $\Psi_{t,s}$ restricts to a homeomorphism from $(f\bar{g})_s^{-1}(t)\backslash P_{t,s}$ to $(f\bar{g})_s^{-1}(0)\backslash\{s\}$ and such that $\Psi_{t,s}$ sends $P_{t,s}$ to $\{s\}$:
Setting $\Psi_{t,s} := h_{0,s}^{-1} \circ \Psi_{t,s_i} \circ h_{t,s}$ we clearly obtain the desired map.

$$\begin{array}{ccc} (f\bar{g})_s^{-1}(t) & \xrightarrow{h_{t,s}} & (f\bar{g})_{s_i}^{-1}(t) \\ \downarrow{\scriptstyle \Psi_{t,s}} & & \downarrow{\scriptstyle \Psi_{t,s_i}} \\ (f\bar{g})_s^{-1}(0) & \xrightarrow{h_{0,s}} & (f\bar{g})_{s_i}^{-1}(0) \end{array}$$

■

If we set $P_t[i]$ to be the union of P_{t,s_i} with the union of all the $P_{t,s}$ for $s \in L(\Sigma_i)\backslash\{s_i\}$ defined as above, that is,

$$P_t[i] := \bigcup_{s\in L(\Sigma_i)} P_{t,s},$$

then clearly $P_t[i]$ is a fibre bundle over $L(\Sigma_i)$ with fibre P_{t,s_i} and $W_t[i]$ deformation retracts to $P_t[i]$.

$$\begin{array}{ccc} P_{t,s_i} & \hookrightarrow & P_t[i] \\ & & \downarrow{\scriptstyle p} \\ & & L(\Sigma_i) \end{array}$$

Now define the continuous map $\Psi_t[i] : W_t[i] \to W_0[i]$ by setting

$$\Psi_t[i](x) := \Psi_{t,p(x)}[i](x).$$

Clearly, $\Psi_t[i]$ restricts to a homeomorphism from $W_t[i]\backslash P_t[i]$ to $W_0[i]\backslash L(\Sigma_i)$ and sends $P_t[i]$ to $L(\Sigma_i)$.

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