UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
PROGRAMA DE MAESTRÍA Y DOCTORADO EN CIENCIAS MATEMÁTICAS Y
DE LA ESPECIALIZACIÓN EN ESTADÍSTICA APLICADA

BIHOLOMORRPHISMS OF THE COMPLEX PROJECTIVE PLANE

QUE PARA OPTAR POR EL GRADO DE:
DOCTORA EN CIENCIAS

PRESENTA:
ADRIANA GONZÁLEZ URQUIZA

TUTORES PRINCIPALES:
DR. ANGEL CANO CORDERO, INSTITUTO DE MATEMÁTICAS, UNAM.
Dr. JOSÉ SEADE KURI, INSTITUTO DE MATEMÁTICAS, UNAM.

COMITÉ TUTOR:
DRA. LAURA ORTIZ BOBADILLA, INSTITUTO DE MATEMÁTICAS, UNAM
DR. GUILLERMO SIENRA LOERA, FACULTAD DE CIENCIAS, UNAM

CUERNAVACA, MORELOS, MARZO 2018.

Contents

Introducción

El estudio de los grupos Kleinianos comenzó alrededor de 1875 cuando Lazarus Fuch, un matemático alemán se propuso estudiar las soluciones de las ecuaciones diferenciales ordinarias lineales, quería saber bajo qué condicones éstas son soluciones algebraicas. La solución a este problema comenzó cuando Schwarz lo resolvió para la ecuación hipergeométrica:

$$x(1-x)\frac{d^2y}{dx^2}+(c-(a+b+1)x)\frac{dy}{dx}-aby=0.$$

Después, el mismo Lazarus Fuch resolvió el problema para la ecuación diferencial general de segundo orden, reduciendo el problema a una teoría de invariantes. Sin embargo su solución no fue completa y Felix Klein la simplificó y la corrigió usando técnicas de geometría y de teoría de grupos.

Mientras tanto, independientemente, Poincaré comenzó en 1880 a desarrollar una teoría más general de Superficies de Riemann, grupos discontinuos y ecuaciones diferenciales, a parti de la idea de un trabajo de Fuchs. Primero, él no conocía los trabajos de Schwarz y de Klein, y fue sólo después de correspondencia intercambiada entre Klein y Poincaré que ellos dos atacaron el problema desde un punto de vista de teoría de grupos. Este enfoque llevó a desarrollar posteriormente la teoría de Grupos Kleinianos, como Poincaré los llamó en su trabajo "Sur les groupes Kleinéens" [30] en 1881. Esta historia se puede encontrar en [15] explicada de una manera amplia y detallada.

El grupo $\mathrm{SL}(2,\mathbb{C})$ es el grupo de matrices de 2×2 con entradas complejas y determinante 1. El centro de este grupo está formado por la identidad I y $-I$. El grupo cociente $\mathrm{SL}(2,\mathbb{C})/\{I,-I\}$ se denota como $\mathrm{PSL}(2,\mathbb{C})$ y sus subgrupos discretos son llamados *grupos Kleinianos*.

El grupo $\mathrm{PSL}(2,\mathbb{C})$ tiene muchas representaciones, por ejemplo: sus elementos son las transformaciones conformes de la esfera de Riemann $\mathbb{S}^2$; o las isometrías que preservan orientación del espacio hiperbólico 3-dimensional $\mathbf{H}^3_{\mathbb{R}}$; y como los mapeos conformes que preservan orientación de la bola unitaria $\mathrm{B}^3\subset\mathbb{R}^3$ en sí misma.

Se ha investigado mucho sobre la extensión de estas representaciones en dimensiones superiores, es decir, se han estudiado las transformaciones conformes de la esfera $n-$dimensional, o las isometrías que preservan orientación del espacio real hiperbólico $\mathbf{H}^n_{\mathbb{R}}$, o los mapeos conformes que preservan orientación de la bola unitaria n-dimensional $\mathrm{B}^n\subset\mathbb{R}^n$ en sí misma.

No obstante, hay otra manera de hacer la teoría de grupo Kleinianos, más grande. Ya que la esfera de Riemann $\mathbb{S}^2$ es biholomorfa a la línea proyectiva compleja $\mathbf{P}^1_{\mathbb{C}}$, y los automorfismos holomorfos de $\mathbf{P}^1_{\mathbb{C}}$ son los elementos en $\mathrm{PSL}(2,\mathbb{C})$, uno podría preguntarse: ¿Cuáles son los automorfismos del espacio complejo proyectivo n-dimensional $\mathbf{P}^n_{\mathbb{C}}$?

Dado que en dimension mayor que uno, los mapeos conformes no son siempre holomorfos, ni los mapeos holomorfos son siempre conformes, uno puede estudiar los automorfismos holomorfos de $\mathbf{P}^n_{\mathbb{C}}$. Esta teoría fue estudiada en primer lugar por José Seade and Alberto Verjovsky al rededor de 1999, en [32], los autores definen a *los grupos Kleinianos complejos* como subgrupos discretos de $\mathrm{PSL}(n+1,\mathbb{C})$, cuya región de disconuidad en $\mathbf{P}^n_{\mathbb{C}}$ es no vacía.

Debemos resaltar que la región de discontinuidad a la que ellos hacen referencia está definida en términos del conjunto límite de la acción del grupo en $\mathbf{P}^n_{\mathbb{C}}$. El conjunto límite que los autores usan en su trabajo fue definida por R. Kulkarni [21] en el contexto de acciones de grupos discretos en cualquier espacio topológico. A lo largo de esta tesis llamaremos a este conjunto límite *el conjuto límite de Kulkarni* y sus propiedades de presentarán en la Subsección 1.4.1.

Seade y Verjovsky construyen grupos Kleinianos complejos usando teoría de Twistores, en particular, muestran que la dinámica conforme de los grupos Kleinianos en esferas puede encajarse en la dinámica holomorfa de los grupos Kleinianos complejos actuando en espacios proyectivos complejos. En [31], los mismo autores dam muchos ejemplos de tipos y construcciones de grupos Kleinianos complejos, incluyendo subgrupos de $\mathrm{PU}(n,1)$ que actúan en $\mathbf{P}^n_{\mathbb{C}}$, preservando una bola,

Los subgrupos de $\mathrm{PU}(n,1)$, las suspenciones y los subgrupos que preservan un hiperplano proyectivo fueron de los primeros ejemplos de grupos Kleinianos complejos que presentaron J.Seade and A.Verjovsky; A. Cano también estudió a los subgrupos con un grupo de control. Estos ejemplos existen actuando en $\mathbf{P}^n_{\mathbb{C}}$ para n mayor que 2, pero para $\mathbf{P}^2_{\mathbb{C}}$ ha sido complicado encontrar ejemplos con dinámica "interesante".

El objetivo de esta Tesis es presentar ejemplos de grupos que actúan intrínsecamente en el plano complejo proyectivo. Es conveniente entender la acción de los subgrupos en esta dimensión ya que se espera sea una parte fundamental para el entendimiento de las acciones de grupos en espacios de dimensiones mayores. Logramos el objetivo usando herramientas que serán explicadas a lo largo del trabajo.

∘

De los primeros trabajos en el área de grupos Kleinianos complejos fue realizado por J. P. Navarrete [28], quién dio una descripción de los tres tipos de elementros en $\mathrm{PSL}(3,\mathbb{C})$ de acuerdo a la forma en la que transforman una foliación de 3-esferas en $\mathbf{P}^2_{\mathbb{C}}$. Un elemento en $g \in \mathrm{PSL}(3,\mathbb{C})$ tiene un levantamiento a $\mathrm{SL}(3,\mathbb{C})$: $\mathbf{g}$; el autor encontró la forma canónica que un elemento puede tener dependiendo si es loxodrómica, parabólica o elíptica, y encontró el conjunto límite del grupo generado por ese elemento dependiendo de la clase de conjugación de dicho elemento. Más aún, el autor clasificó los elementos $g \in \mathrm{PSL}(3,\mathbb{C})$ dependiendo de la traza del levantamiento $\mathbf{g}$.

En un trabajo subsecuente [27], el mismo autor establece la relación entre el conjunto límite de Kulkarni y el conjunto límite de Chen-Greenberg de un subgrupo de $\mathrm{PU}(2,1)$ que actúa en $\mathbf{P}^2_{\mathbb{C}}$.

Cuando se comenzó a estudiar a los grupos Kleinianos complejos, no se sabía nada sobre los conjuntos límite, así que el descubrimiento de J. P. Navarrete fue fundamental, aunque el enfoque

que usó para encontrar el conjunto límite de los grupos se vuelve ineficiente cuando se requiere calcular el conjunto límite para subgrupos de automorfismos de espacios de dimensiones mayores; entonces sería conveniente encontrar otros enfoques para calcular dichos conjuntos límite.

En esta Tesis, trabajamos con otra herramienta para calcular el conjunto límite de subgrupos de $\mathrm{PSL}(3,\mathbb{C})$, éstas son las *transformaciones pseudo-proyectivas* (Subsección 1.2.3). Entonces, calcular el conjunto límite se vuelve una tarea más sencilla; en esta Tesis repetimos el trabajo de J. P. Navarrete de encontrar el conjunto límite de subgrupos cíclicos de $\mathrm{PSL}(3,\mathbb{C})$, esta vez, usando transformaciones pseudo-proyectivas. También, damos una pruba más corta del hecho de que el conjunto límite de Kulkarni de un subgrupo de $\mathrm{PU}(2,1)$ que actúa en $\mathbf{P}^2_{\mathbb{C}}$ es la unión de líneas tangentes a $\partial\mathbf{H}^2_{\mathbb{C}}$ en puntos del conjunto límite de Chen-Greenberg.

∘ ∘

Dentro de los grupos Kleinianos, están los grupos de Schottky. Estos grupos son una buena fuente de ejemplos que ilustran diferentes propiedades que tienen los grupos Kleinianos. Estos grupos son libres, discretos y tienen región de discontinuidad no vacía. Más aún, la variedad cociente obtenida de la región de discontinuidad Ω módulo la acción de un grupo de Schottky es un k-Toro, donde k es el número de generadores del grupo.

En [33], J. Seade y A. Verjovsky dan una definición de subgrupos de Schottky de $\mathrm{PSL}(2n,\mathbb{C})$ que actúan en espacios proyectivos complejos de dimensión impar. Sin embargo, para espacios proyectivos complejos de dimensión par, A. Cano mostró en [7] que no pueden existir subgrupos de Schottky de $\mathrm{PSL}(2n+1,\mathbb{C})$ como se definieron en [33].

Para acciones de grupos en espacios proyectivos complejos de dimension par están los *grupos tipo Schottky* que fueron definidos por J. P. Conze y Y. Guivarc'h en [11]; los autores dan una definición de conjunto límite para la accion de un grupo en un espacio lineal. Ellos consideran la cerradura de los puntos fijos atractores de elementos proximales en el grupo (un elemento es llamado *proximal* si tiene un eigenvalor de módulo estríctamente mayor que el módulo de los demás eigenvalores).

En este trabajo observamos que un elemento proximal en $\mathrm{PSL}(3,\mathbb{C})$ es un elemento loxodrómico, pero no todo elemento loxodrómico es proximal. Por otro lado, no todos los subgrupos de $\mathrm{PSL}(3,\mathbb{C})$ contienen elementos proximales; entonces, inspirados en la definición de conjunto límite de Conze y Guivarc'h, definimos el conjunto límite $\hat{L}(G)$ de un grupo G que actúa en $(\mathbf{P}^2_{\mathbb{C}})^*$. Esta definición funciona para muchos más subgrupos que la definición dada por Conze y Guivarc'h, ya que no siempre un grupo tiene elementos proximales. Probamos que el nuevo conjunto límite denotado por $\hat{L}(G)$, tiene propiedades análogas a las del conjunto límite para acciones de subgrupos de $\mathrm{PSL}(2,\mathbb{C})$. Con la ayuda de este nuevo conjunto límite, mostramos una relación entre el conjunto límite $\hat{L}(G)$ de G que actúa en $(\mathbf{P}^2_{\mathbb{C}})^*$ y el conjunto límite de Kulkarni para la acción de G en $\mathbf{P}^2_{\mathbb{C}}$.

Más aún, dado un subconjunto cerrado C de $(\mathbf{P}^2_{\mathbb{R}})^*$, construimos un grupo Kleiniano complejo tal que la distancia de Hausdorff, entre el conjunto límite $\hat{L}(G)$ y el subconjunto C, es tan pequeña como se quiera.

∘ ∘ ∘

Saber si un grupo es discreto es un problema importante en la teoría de grupos Kleinianos; esto no es una tarea fácil, incluso para los subgrupos de $\mathrm{PSL}(2,\mathbb{C})$ generados por dos elementos. Uno de los primeros criterios al respecto fue creado por Troels Jørgensen en [17], dio una condición necesaria para que un grupo generado por dos elementos f y g, sea discreto.

Siguiendo con esta ardua tarea, Maskit en [25], y Delin Tan en [34], estudiaron las condiciones para que un grupo sea discreto, en ambos casos, tratándose de un grupo generado por dos elementos. Con el objetivo de generalizar la desigualdad de Jørgensen, ellos toman elementos específicos de $\mathrm{PSL}(2,\mathbb{C})$, por ejemplo, Maskit estudia los grupos generados por una transformación elíptica a de órden seis y una transformación parabólica b, que mapea un punto fijo de a al otro, y encuentra que el grupo $\langle a, b\rangle$ es discreto.

Delin Tan en su trabajo analiza varias posibilidades de los sumandos de la desigualdad de Jørgensen, y dice, siempre asumiendo que el grupo generado por dos elementos es discreto, y jugando con las trazas del conmutador $fgf^{-1}g^{-1}$ o de f, qué pasa con las diferentes sumas; en resumen, provee de más casos para saber si un grupo no es discreto en caso de que no satisfaga las desigualdades.

Para subgrupos de $\mathrm{PSL}(3,\mathbb{C})$ no hay una desigualdad análoga a la desigualdad de Jørgensen. Sin embargo, para espacios distintos, diversos autores tienen resultados en ésta dirección, por ejemplo, para subgrupos del grupo de isometrías del espacio complejo hiperbólico (see [16]).

Finalmente, A.V. Masley, en [26] encontró condiciones suficientes para saber si un subgrupo de $\mathrm{PSL}(2,\mathbb{C})$, generado por dos elementos es discreto, ésto casi cuarenta años después del paper de Jørgensen. El tiempo que tomó llegar a este resultado puede ser un buen parámetro para asumir el problema como uno complicado.

En esta Tesis, intentamos estudiar también las condiciones para que un subgrupo de $\mathrm{PSL}(3,\mathbb{C})$, generado por dos elementos, sea discreto. Nuestro enfoque es puramente algebraico, y consideramos siempre que uno de los generadores es una transformación loxodrómica. Encontramos condicones sobre el conjunto límite de Kulkarni de ambos generadores para concluir que el grupo que generan no es discreto.

También presentamos la construcción de una familia de subgrupos de PSL$(3,\mathbb{R})$ que son generados por dos transformaciones loxodrómicas, éstas no satisfacen las condiciones del resultado mencionado en el párrafo anterior; esas transformaciones generan un grupo tipo Schottky, entonces el grupo es discreto, libre y con región de discontinuidad no vacía. Este trabajo, y el mencionado en el apartado ∘ ∘, fue realizado en coolaboración con W. Barrera and J. P. Navarrete.

∘∘∘∘

Ya que los elementos de $\mathrm{PSL}(2,\mathbb{C})$ están clasificados como elípticos, parabólicos o loxodrómicos, los subgrupos generados por algún tipo particular de elemento pueden ser estudiados. Por ejemplo, en [23], M. Lyubich y V. Suvorov estudian los subgrupos libres de $\mathrm{SL}(2,\mathbb{C})$ generados por dos elementos parabólicos. También L. Keen y C. Series en [19] estudian la familia de grupos Kleinianos generados por dos elementos parabólicos, y ese paper es revisado por C. Series y Y. Komori en [20].

Para subgrupos de $\mathrm{PU}(2,1)$, J. Parker y P. Will, en [29] estudian subgrupos generados por elementos unipotentes (que son elementos parabólicos), cuyo producto es también unipotente.

Inspirados en los resultados presentados en [24, I.D] estudiamos un resultado que involucra al conmutador de dos elementos de $\mathrm{PSL}(3,\mathbb{C})$, también pidiendo al conjunto límite de Kulkarni de cada uno de los generadores ciertas condiciones para encontrar que su conmutador es parabólico.

En los siguientes párrafos se especificará el contenido de cada capítulo de esta Tesis.

En el Capítulo 1 de esta Tesis, como un marco de referencia, presentamos algunas propiedades importantes de los subgrupos de $\mathrm{PSL}(2,\mathbb{C})$ y del conjunto límite de la acción de estos subgrupos en $\mathbf{P}^1_{\mathbb{C}}$. Como trabajaremos con grupos Kleinianos complejos actuando en el plano proyectivo complejo $\mathbf{P}^2_{\mathbb{C}}$, establecemos las definiciones y la notación que usaremos para para $\mathbf{P}^2_{\mathbb{C}}$ y $(\mathbf{P}^2_{\mathbb{C}})^*$, como también el grupo de transformaciones holomorfas de estos espacios.

También, presentamos a las *transformaciones pseudo-proyectivas*, que son transformaciones que son límite en el espacio de matrices 3×3 con coeficientes en $\mathbb{C}$, $\mathcal{M}_{3\times 3}(\mathbb{C})$ de una sucesión de transformaciones en $\mathrm{PSL}(3,\mathbb{C})$, las transformaciones pseudo-proyectivas serán importantes en el Segundo Capítulo de esta Tesis.

Introducimos brevemente al plano hiperbólico complejo, también a su grupo de transformaciones $\mathrm{PU}(2,1)$, y las propiedades a las que haremos referencia más adelante en el desarrollo del trabajo.

Presentaremos cuatro conceptos de conjunto límite: el conjunto límite de Kulkarni, el conjunto límite de Chen-Greenberg, el conjunto límite de Conze y Guivarc'h y el complemento de la región de equicontinuidad; enntre ellos no hay una relación de contención, sin embargo, con cada definición es posible probar diferentes propiedades de minimalidad del conjunto límite y pueden ser comparados de alguna manera.

En la última sección de este Capítulo, establecemos más notación que usaremos a lo largo de la Tesis.

En el Capítulo 2, estudiamos el conjunto límite de los subgrupos cíclicos de $\mathrm{PSL}(3,\mathbb{C})$ que actúan en $\mathbf{P}^2_{\mathbb{C}}$. Como se mencionó anteriormente, J. P. Navarrete encontró el conjunto límite de los subgrupos cíclicos en uno de los primeros artículos de la teoría de grupos Kleinianos complejos. Aquí, usamos transformaciones pseudo-proyectivas como otra forma de encontrar el conjunto límite de subgrupos generados por un elemento de $\mathrm{PSL}(3,\mathbb{C})$. Nos fijamos en la forma canónica que $\mathbf{g} \in \mathrm{PSL}(3,\mathbb{C})$ pueda tener y verificamos en cada caso, cual es el conjunto límite.

Adicionalmente, las transformaciones pseudo-proyectivas con útiles para establecer una relación entre el conjunto límite de Kulkarni y el de Chen-Greenberg de un subgrupo de $\mathrm{PU}(2,1)$ que actúa en $\mathbf{P}^2_{\mathbb{C}}$. Esta herramienta provee un método más simple de calcular el conjunto límite de subgrupos de automorfismos de espacios de dimensiones mayores.

Introducimos en el Capítulo 3 los grupos tipo Schottky y sus propiedades. Recordamos algunas Proposiciones que Conze y Guivarc'h usan en su artículo, así como el conjunto límite que ellos usan para la acción de subgrupos que actúan en espacios lineales. Probamos algunas proposiciones que nos permiten relacionar su trabajo con la acción de subgrupos de $\mathrm{PSL}(3,\mathbb{C})$.

En artículos previos escritos por W. Barrera, A. Cano y J.P. Navarrete, se definieron dos conjuntos de líneas distintos. El conjunto $\mathcal{E}(G)$ está formado por el kernel de transformaciones pseudo-proyectivas del grupo. Mientras $E(G)$ es el conjunto de líneas complejas ℓ para las cuales existe una transformación $g \in G$ tal que $\ell \subset \Lambda_{\mathrm{K}}(g)$. Enunciamos teoremas que los autores probaron en [6] y en [3] para relacionar ambos conjuntos en la Proposición 3.7.

Introducimos una nueva defineción de conjunt límite $\hat{L}(G)$ (Definición 3.8), esta vez para la acción de subgrupos que actúan en el espacio de líneas del plano proyectivo complejo $(\mathbf{P}^2_{\mathbb{C}})^*$, presentamos un ejemplo de cómo se puede encontrar este conjunto límite para un grupo cíclico generado por una transformación fuertemente loxodrómica. Después de esto, probamos quien es el conjunto límite $\hat{L}(g)$ para cada forma canónica de Jordan que g pueda tener. Luego probamos, con la ayuda de algunos resultados establecidos en [6] y [3], que $\mathcal{E}(G) = \hat{L}(G)$.

El conjunto $\hat{L}(G)$ tiene propiedades análogas a las del conjunto límite para la acción de subgrupos de $\mathrm{PSL}(2,\mathbb{C})$, (Teorema 1.2), y los presentamos en los Corolarios 3.15, 3.13 y 3.14. Con la ayuda de este nuevo conjunto límite, mostramos una relación entre $\hat{L}(G)$ de G que actúa en $(\mathbf{P}^2_{\mathbb{C}})^*$ y el conjunto límite de Kulkarni $\Lambda_{\mathrm{K}}(G)$ para la acción de G en $\mathbf{P}^2_{\mathbb{C}}$.

Teorema 3.16. Sea $G \leqslant \mathrm{PSL}(3,\mathbb{C})$ un subgrupo infinito discreto que actúa en $\mathbf{P}^2_{\mathbb{C}}$ sin puntos fijos ni líneas invariantes. Sea $\hat{L}(G)$ el conjunto límite de G que actúa en $(\mathbf{P}^2_{\mathbb{C}})^*$, entonces

$$\Lambda_{\mathrm{K}}(G) = \bigcup_{\ell \in \hat{L}(G)} \ell.$$

En el Capítulo 4, después de conocer la dualidad entre el conjunto límite de Kulkarni y el conjunto límite en el espacio de líneas de $\mathbf{P}^2_{\mathbb{C}}$, explicamos a través de los Lemas 4.6, 4.8, 4.9 y 4.10 el siguiente Teorema:

Teorema 4.11. Dado $\epsilon > 0$ y un subconjunto cerrado $C \subset (\mathbf{P}^2_{\mathbb{R}})^*$ tal que C tiene al menos tres puntos en posición general y $\bigcup_{\ell \in C} \ell \neq \mathbf{P}^2_{\mathbb{C}}$, entonces existe un grupo Kleiniano complejo G_ϵ, tal que la distancia de Hausdorff entre $\hat{L}(G_\epsilon)$ y C es menor que ϵ. smaller than ϵ.

Como segundo ejemplo de grupos Kleinianos, construimos un subgrupo de $\mathrm{PSL}(3,\mathbb{C})$ con dos generadores proximales g y f, con la propiedad de que el conjunto límite de Kulkarni de ambas transformaciones no comparta ninguna bandera. Este grupo es un grupo tipo Schottky, por lo cual es libre, discreto y tiene región de discontinuidad no vacía.

Teorema 4.17. Si g es una transformación como en la Proposición 4.13 y si f es una transformación como en la Proposición 4.16, entonces $G = \langle f, g \rangle$ es un grupo tipo Schottky que actúa en $\mathbf{P}^2_{\mathbb{R}}$ con región de discontinuidad no vacía.

En el Capítulo 5 comenzamos con el análisis de la discretez de subgrupos de $\mathrm{PSL}(3,\mathbb{C})$ generados por dos elementos, con la condición de que una de las transformaciones sea loxodrómica y que la otra transformación no preserve el conjunto límite de Kulkanir de la transformación loxodrómica y que tengan una línea invariante común con el mismo punto fijo por ambas transformaciones.

Probamos la siguientes Proposiciones:

Proposición 5.9. Sea $f \in \mathrm{PSL}(3,\mathbb{C})$ una transformación loxodrómica con levantamiento $\mathbf{f} \in \mathrm{PSL}(3,\mathbb{C})$, $\mathbf{f} = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3)$, y $|\lambda_1| < |\lambda_2| < |\lambda_3|$. Si $g \in \mathrm{PSL}(3,\mathbb{C})$ es tal que f y g tienen una bandera en común (Definición 5.2) y el conjunto límite de Kulkarni $\Lambda_{\mathrm{K}}(f)$, no es invariante bajo g, y g es triangular, entonces $\langle \mathbf{f}, \mathbf{g} \rangle$ no es discreto.

Proposición 5.11. Sea $\mathbf{f} \in \mathrm{PSL}(3,\mathbb{C})$ una homotecia compleja con levantamiento $\mathbf{f} \in \mathrm{SL}(3,\mathbb{C})$, $\mathbf{f} = \mathrm{Diag}(\lambda, \lambda, \lambda^{-2})$ y $|\lambda| > 1$. Si $g \in \mathrm{PSL}(3,\mathbb{C})$ es tal que f y g tienen una bandera en común (Definición 5.2), y el conjunto límite de Kulkarni $\Lambda_{\mathrm{K}}(f)$, no es invariante bajo g, entonces $\langle \mathbf{f}, \mathbf{g} \rangle$ no es discreto.

Proposición 5.12. Sean $\mathbf{f}$ y $\mathbf{g}$ transformaciones en $\mathrm{PSL}(3,\mathbb{C})$ tales que $\mathbf{f}$ es un tornillo con levantamiento $\mathbf{f} \in \mathrm{SL}(3,\mathbb{C})$, $\mathbf{f} = \mathrm{Diag}(\lambda, \mu, (\lambda\mu)^{-1})$ y $|\lambda| = |\mu| > 1$. Si $g \in \mathrm{PSL}(3,\mathbb{C})$ es tal que f y g tienen una bandera en común (Definición 5.2), y el conjunto límite de Kulkarni $\Lambda_{\mathrm{K}}(f)$, no es invariantes bajo g, entonces $\langle \mathbf{f}, \mathbf{g} \rangle$ no es discreto.

Proposition 5.13. Sean $\mathbf{f}$ y $\mathbf{g}$ transformaciones en $\mathrm{PSL}(3,\mathbb{C})$ tales que $\mathbf{f}$ es una transformación loxoparabólica con levantamiento $\mathbf{f} \in \mathrm{SL}(3,\mathbb{C})$,

$$\mathbf{f} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}$$

donde $|\lambda| > 1$. Si f y g tienen una bandera en común (Defínción 5.2) es $e_1 \in \overleftrightarrow{e_1 e_2}$, y el conjunto límite de Kulkarni $\Lambda_{\mathrm{K}}(f)$, no es invariante bajo g, entonces $\langle \mathbf{f}, \mathbf{g} \rangle$ no es discreto.

En la segunda Sección del Capítulo 5, estudiamos las condiciones suficientes para que el conmutador de dos elementos sea parabólico. Probamos la siguiente Proposición:

Proposición 5.16. Sean $\mathbf{f}$ y $\mathbf{g}$ dos transformaciones en $\mathrm{PSL}(3,\mathbb{C})$ tales que $\mathbf{f}$ es un elemento loxodrómico, y $\mathbf{g}(\Lambda_{\mathrm{K}}(\mathbf{f})) \neq \Lambda_{\mathrm{K}}(\mathbf{f})$. Suponiendo que $\mathbf{f}$ y $\mathbf{g}$ tienen una bandera en común, entonces $[\mathbf{f}, \mathbf{g}]$ es parabólico.

En la última sección del Capítulo 5 damos un panorama de la utilidad de los resultados presentados en la primera y segunda Sección de este Capítulo. En particular, los resultados serán útiles para probar diferentes teoremas que permitiran tener una clasificación completa de los subgrupos elementales de $\mathrm{PSL}(3,\mathbb{C})$.

Introduction

The study of Kleinian groups began around 1875, when Lazarus Fuchs, a German mathematician wanted to understand when the solutions of a linear ordinary differential equation are algebraic. The solution of the problem started when H. A. Schwarz solved it for the hypergeometric equation:

$$x(1-x)\frac{d^2y}{dx^2} + (c-(a+b+1)x)\frac{dy}{dx} - aby = 0.$$

Then L. Fuchs himself, solved it for the general second-order equation by reducing it to a problem of invariant theory. But L. Fuchs solution was imperfect and F. Klein simplified and corrected it by a mixture of geometric and group-theoretic techniques.

Meanwhile, independently, H. Poincaré began in 1880 to develop a more general theory of Riemann surfaces, discontinuous groups and differential equations, picking up the idea from the paper of L. Fuchs. At first, he did not know about what H. A. Schwarz and F. Klein had done, and only after F. Klein and H. Poincaré interchanged correspondence they tried to solve the problem from the group-theoretic approach. This group-theoretic approach led to Kleinian groups, as H. Poincaré called them in his work "Sur les groupes Kleinéens" [30] in 1881. A wide and detailed story on the subject can be read in [15].

The group $\mathrm{SL}(2,\mathbb{C})$ is the group of $2\times 2-$matrices with complex entries and determinant 1. The center of this group is formed by the identity I and $-I$. The quotient group $\mathrm{SL}(2,\mathbb{C})/\{I,-I\}$ is denoted as $\mathrm{PSL}(2,\mathbb{C})$ and its discrete subgroups are called *Kleinian groups*.

The group $\mathrm{PSL}(2,\mathbb{C})$ has many descriptions, for example: the elements are the conformal transformations of the Riemann sphere $\mathbb{S}^2$; or the orientation-preserving isometries of the 3-dimensional, real hyperbolic space $\mathbf{H}^3_{\mathbb{R}}$; and as orientation-preserving conformal maps of the open unit ball $\mathrm{B}^3 \subset \mathbb{R}^3$ to itself.

A lot of work has been done extending any of these representations to higher dimensions, that is, the conformal transformations of the n-dimensional sphere, or the orientation-preserving isometries of the n-dimensional real hyperbolic space $\mathbf{H}^n_{\mathbb{R}}$, or the orientation preserving conformal maps of the open unit n-dimensional ball $\mathrm{B}^n \subset \mathbb{R}^n$ to itself.

However there is still another way to enlarge the theory of Kleinian groups. Since the Riemann sphere $\mathbb{S}^2$ is biholomorphic to the complex projective line $\mathbf{P}^1_{\mathbb{C}}$, and the holomorphic automorphisms of $\mathbf{P}^1_{\mathbb{C}}$ are the elements in $\mathrm{PSL}(2,\mathbb{C})$, one can ask: What about the automorphisms of the n-dimensional complex projective space $\mathbf{P}^n_{\mathbb{C}}$?

Given that in dimension greater than one, the conformal maps are not always holomorphic, nor holomorphic maps are necessarily conformal, one can study the holomorphic automorphisms of $\mathbf{P}^n_{\mathbb{C}}$. This theory was first studied by J. Seade and A. Verjovsky around 1999, in [32], the authors define *complex Kleinian groups* as discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$, whose region of discontinuity on $\mathbf{P}^n_{\mathbb{C}}$ is non-empty.

We must point out, that the region of discontinuity that they make reference to, is defined in terms of the limit set of the action, and the limit set they use in that paper was defined by R. Kulkarni [21] in the context of actions of discrete groups in any topological space, along this thesis we will call this set the *Kulkarni limit set*, and it is properly defined in Subsection 1.4.1.

J. Seade and A. Verjovsky use twistor theory to construct complex Kleinian groups, in particular, they show that the conformal dynamics of Kleinian groups on spheres can be embedded in the holomorphic dynamics of complex Kleinian groups acting on complex projective spaces. In [31] the same authors give several examples of types and constructions of complex Kleinian groups, including subgroups of $\mathrm{PU}(n,1)$ which act on $\mathbf{P}^n_{\mathbb{C}}$ preserving a ball.

Subgroups of $\mathrm{PU}(n,1)$, suspensions and subgroups preserving a projective hyperplane were the examples that J. Seade and A. Verjovsky first presented, A. Cano also studied subgroups with a control group. These examples exist acting in $\mathbf{P}^n_{\mathbb{C}}$ for n greater than 2, but for $\mathbf{P}^2_{\mathbb{C}}$ has been hard to find examples of groups with "interesting" dynamics.

The aim of this thesis is to present examples of groups which act intrinsecally on the complex projective plane. The understanding of the action of subgroups in this dimension is convinient for understanding actions in spaces of higher dimension. We get to the goal, using some new tools that will be explained along the text.

∘

One of the foremost works in complex Kleinian groups was done by J. P. Navarrete [28], who gave a dynamical description of the different types of elements in $\mathrm{PSL}(3,\mathbb{C})$ according on how any of them would transform a foliation given by 3-spheres in $\mathbf{P}^2_{\mathbb{C}}$. An element in $g \in \mathrm{PSL}(3,\mathbb{C})$ has a lift to $\mathrm{SL}(3,\mathbb{C}) : \mathbf{g}$; the author found the Jordan canonical form that an element might have depending if it is loxodromic, parabolic or elliptic, and he found the limit set of an element depending on its class of conjugation. Moreover, he classified the elements $g \in \mathrm{PSL}(3,\mathbb{C})$ depending on the trace of the lift $\mathbf{g}$.

In a subsequent paper [27], the same author establishes the relationship between the Kulkarni limit set and the Chen-Greenberg limit set of a subgroup of $\mathrm{PU}(2,1)$ acting on $\mathbf{P}^2_{\mathbb{C}}$.

When complex Kleinian groups began to be studied, nothing was known about the limit sets, so J. P. Navarrete's discovery was fundamental, although the approach to find the limit sets of groups becomes inefficient when calculating the limit set of subgroups of automorphisms of higher dimensional spaces, so remains the doubt whether it exist any other approach to calculate the limit set.

In this thesis we work with another tool to calculate the limit set of subgroups of $\mathrm{PSL}(3,\mathbb{C})$, these are the *pseudo-projective tranformations* (Subsection 1.2.3). Then, calculating the limit set becomes an easier task; we repeat the work done by J. P. Navarrete of finding the limit set of cyclic subgroups of $\mathrm{PSL}(3,\mathbb{C})$, this time using pseudo-projective transformations. Also, a shorter proof is given, of the fact that the Kulkarni limit set of a subgroup of PU$(2,1)$ acting on $\mathbf{P}^2_{\mathbb{C}}$ is the union of lines tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ in points of the Chen-Greenberg limit set.

∘ ∘

Within the Kleinian groups, there exist the Schottky groups. These groups provide a good source of examples that ilustrate different properties that Kleinian groups can have. They are free, discrete groups with nonempty region of discontinuity. Moreover, the quotient manifold obtained by the region of discontinuity Ω modulo the action of a Schottky group is a k-torus, where k is the number of generators of the group.

In [33], J. Seade and A. Verjovsky gave a definition of Schottky subgroups of $\mathrm{PSL}(2n,\mathbb{C})$ acting on odd dimensional complex projective spaces. However for even dimensional complex projective spaces, A. Cano showed in [7] that there cannot exists Schottky subgroups of $\mathrm{PSL}(2n+1,\mathbb{C})$ as defined in [33].

For actions of groups in even dimensional complex projective spaces, there are the *Schottky type groups*, which were defined by J. P. Conze and Y. Guivarc'h in [11]; the authors give a definition of limit set for the action of a group on a linear space. They consider the closure of attracting fixed points of proximal elements in the group (an element is called *proximal* whenever it has an eigenvalue with modulus strictly greater than the modulus of all other eigenvalues).

In this work, we observe that a proximal element in $\mathrm{PSL}(3,\mathbb{C})$ is a loxodromic element, but a loxodromic element is not necessarily a proximal element. On the other hand, not every subgroup of $\mathrm{PSL}(3,\mathbb{C})$ contains proximal elements; so inspired in the definition of limit set by J. P. Conze and Y. Guivarc'h we define the limit set $\hat{L}(G)$ of a group G acting on the space of lines in $\mathbf{P}^2_{\mathbb{C}}$: $(\mathbf{P}^2_{\mathbb{C}})^*$. This definition works for many more subgroups than the one given by J. P. Conze and Y. Guivarc'h, since not always a group has proximal elements. We prove that the new limit set, denoted as $\hat{L}(G)$, has analogous properties as the limit set in actions of subgroups of $\mathrm{PSL}(2,\mathbb{C})$. With the help of this new limit set, we show a relation between the limit set $\hat{L}(G)$ of G acting on $(\mathbf{P}^2_{\mathbb{C}})^*$, and the Kulkarni limit set $\Lambda_{\mathrm{K}}(G)$ for the action of G on $\mathbf{P}^2_{\mathbb{C}}$.

Evenmore, for some closed subset C of $(\mathbf{P}^2_{\mathbb{R}})^*$, we could build a complex Kleinian group such that the distance between the limit set $\hat{L}(G)$ and the subset C is arbitrarily small, considering the Hausdorff distance.

∘ ∘ ∘

The discreteness of a group has always been an issue in the theory of Kleinian groups; decide whether a group is discrete has not been easy, even for the subgroups of $\mathrm{PSL}(2,\mathbb{C})$ generated by two elements. One of the firsts criterion was created by T. Jørgensen in [17], he gave a necessary condition for a group generated by two elements f and g, to be discrete.

Following with this ardous task, B. Maskit in [25], and D. Tan in [34], studied the condition for a group to be discrete, in both cases, studying a group generated by two elements. With the aim of generalize Jørgensen's inequality, they take specific elements, for example B. Maskit studies the group generated by an elliptic transformation a of order six and a parabolic element b, that maps one fixed point of a onto the other, and he finds that the group $\langle a,b\rangle$ is discrete.

D. Tan in his work analyzes various possibilities of the summands of Jørgensen's inequality, and says, always with the assumption that the group generated by two elements is discrete, and playing with the traces of either the commutator $fgf^{-1}g^{-1}$ or f what happen with different sums, providing more cases to decide whether a group is discrete or not, in case the inequalities are not satisfied.

For subgroups of $\mathrm{PSL}(3, \mathbb{C})$ there is not an analogous of Jørgensen's inequality. However, for different dimensions, many authors have results in this direction, for example, concerning subgroups of the isometry group of complex hyperbolic spaces (see [16]).

It was A. V. Masley, who finally, in [26] found sufficient discreteness conditions for 2-generator subgroups in $\mathrm{PSL}(2, \mathbb{C})$, almost forty years after the publication of Jørgensen's paper. The time that took to get to this knowledge, can be a good parameter of how hard the problem is.

In this thesis, we attempt to study also discreteness conditions for 2-generator subgroups of $\mathrm{PSL}(3, \mathbb{C})$. Our approach is merely algebraic, and considering one of the generators a loxodromic transformation. What we find are conditions on the Kulkarni limit set of both generators, to conclude that the group generated by them is not discrete.

We also present the construcion of a family of subgroups of PSL$(3, \mathbb{R})$ which are generated by two loxodromic transformations, they do not satisfy the conditions of the result mentioned in the previous paragraph; they generate a Schottky type group, therefore the group is discrete, free and with nonempty region of discontinuity. This work, and the work mentioned in ∘ ∘, was done together with W. Barrera and J. P. Navarrete.

Once the elements of $\mathrm{PSL}(2, \mathbb{C})$ are classified as elliptic, parabolic or loxodromic, the subgroups generated by some specific type of element can be studied. For example, in [23], M. Lyubich and V. Suvorov study the free subgroups of $\mathrm{SL}(2, \mathbb{C})$ with two parabolic generators. Also L. Keen and C. Series in [19] study the family of Kleinian groups generated by two parabolic elements, and that paper is revisited by C. Series and Y. Komori in [20].

For subgroups of $\mathrm{PU}(2, 1)$, J. Parker and P. Will, in [29] study subgroups generated by unipotent elements, which are parabolic elements, whose product is also unipotent.

Inspired in the results presented in [24, I.D] we study a result concerning the commutator of two elements in $\mathrm{PSL}(3, \mathbb{C})$, also asking for conditions in the individual Kulkarni limit set of the commutator to be parabolic.

In the following paragraphs the content of each chapter of this thesis will be specified.

In Chapter 1 in this thesis, as a frame of reference, we present some important properties of the subgroups of $\mathrm{PSL}(2, \mathbb{C})$ and of the limit set of the action of these subgroups in $\mathbf{P}^1_{\mathbb{C}}$. As we will work with complex Kleinian groups acting in the complex projective plane $\mathbf{P}^2_{\mathbb{C}}$, we establish the notation and definitions that we will use of $\mathbf{P}^2_{\mathbb{C}}$ and $(\mathbf{P}^2_{\mathbb{C}})^*$, as well as the group of holomorphic transformations of these spaces.

Also, we present the *pseudo-projective transformations*, which are transformation that are limit in the space of matrices 3×3, with coefficients in $\mathbb{C}$, $\mathcal{M}_{3\times3}(\mathbb{C})$ of a sequence of transformations in $\mathrm{PSL}(3,\mathbb{C})$, the pseudo-projective transformations will be of important use in the second chapter of this work.

We briefly introduce the complex hyperbolic plane, as well as its group of transformations $\mathrm{PU}(2,1)$, and the properties to the ones we will make reference.

We will present four concepts of limit sets: the Kulkarni limit set, the Chen-Greenberg limit set, the Conze-Guivarc'h limit set and the complement of the region of equicontinuity; among them, there is not a relation of contention, however with each definition one can prove different properties of minimality of the limit set and can be compared in some way.

The last section of this Chapter is committed to stablish notation that we will use through this work.

In Chapter 2, we study the limit set of cyclic subgroups of $\mathrm{PSL}(3,\mathbb{C})$ acting on $\mathbf{P}^2_{\mathbb{C}}$. As we have said, J. P. Navarrete found the limit set of cyclic subgroups in a pioneer paper. Here we us pseudo-projective transformations as another way of finding the limit set of subgroups generated by one element of $\mathrm{PSL}(3,\mathbb{C})$. We look at the canonical form that $g \in \mathrm{PSL}(3,\mathbb{C})$ can have and check in each case, which is the limit set.

Additionally, these transformations are useful for establishing a relationship between the Kulkarni limit set and the Chen-Greenberg limit set of a subgroup of $\mathrm{PU}(2,1)$ acting on $\mathbf{P}^2_{\mathbb{C}}$. This tool provides a simpler method for calculating the limit set of subgroups of automorphisms of higher dimensional spaces.

In Chapter 3, we beging by introducing the Schottky type groups and their properties. We recall some Propositions that J. P. Conze and Y. Guivarc'h use in their paper, and the definition of Schottky type groups we introduce the limit set they use for the action of subgroups. We prove some propositions that allow us to relate their work with the actions of subgroups of $\mathrm{PSL}(3,\mathbb{C})$.

On previous articles by W. Barrera, A. Cano and J.P. Navarrete, two different subsets of lines have been defined. The set $\mathcal{E}(G)$ is formed by the kernel of the pseudo-projective transformations of the group. And $E(G)$ is the set of complex lines ℓ for which there exist a transformation $g \in G$ such that $\ell \subset \Lambda_{\mathrm{K}}(g)$. We state Theorems that were proved in [6] and in [3] to finally relate both subsets in Proposition 3.7.

A new definition of limit set $\hat{L}(G)$ is introduced (Definition 3.8), this time for action of subgroups in the space of lines of the complex projective space $(\mathbf{P}^2_{\mathbb{C}})^*$, we present an example of how this limit set works for a cyclic group generated by a strongly loxodromic transformation. After this, we prove which is the limit set $\hat{L}(g)$ for every Jordan canonical form that g can have. Then we prove, with the help from some results also in [6] and [3] that $\mathcal{E}(G) = \hat{L}(G)$.

The set $\hat{L}(G)$ has analogous properties of the limit set for actions of subgroups of $\mathrm{PSL}(2,\mathbb{C})$, (Theorem 1.2), and we present them in Corollaries 3.15, 3.13 and 3.14. With the help of this new limit set, we show a relation between $\hat{L}(G)$ of G acting on $(\mathbf{P}^2_{\mathbb{C}})^*$ and the Kulkarni limit set $\Lambda_{\mathrm{K}}(G)$ for the action of G on $\mathbf{P}^2_{\mathbb{C}}$.

Theorem 3.16. Let $G \leqslant \mathrm{PSL}(3,\mathbb{C})$ be an infinite discrete subgroup acting on $\mathbf{P}^2_{\mathbb{C}}$ without fixed points nor invariant lines. Let $\hat{L}(G)$ be the limit set of G acting on $(\mathbf{P}^2_{\mathbb{C}})^*$, then

$$\Lambda_{\mathrm{K}}(G) = \bigcup_{\ell \in \hat{L}(G)} \ell.$$

In Chapter 4, after knowing the duality between the Kulkarni limit set and the limit set in the space of lines of $\mathbf{P}^2_{\mathbb{C}}$, we explain through Lemmas 4.6, 4.8, 4.9 and 4.10 the following theorem:

Theorem 4.11. Given $\epsilon > 0$ and a closed subset $C \subset (\mathbf{P}^2_{\mathbb{R}})^*$ such that C has at least three points in general position and $\bigcup_{\ell \in C} \ell \neq \mathbf{P}^2_{\mathbb{C}}$, then there is a complex Kleinian group G_ϵ, such that the Hausdorff distance between $\hat{L}(G_\epsilon)$ and C is smaller than ϵ.

As a second example of Kleinian groups, we build a subgroup of $\mathrm{PSL}(3, \mathbb{C})$ with two proximal generators g and f. This group has the property that the Kulkarni limit set of both transformations, do not share any flag. This group is a Schottky type group, which means it is free, discrete and has nonempty region of discontinuity.

Theorem 4.17. If g is a transformation as in Proposition 4.13 and if f is a transformation as in Proposition 4.16, then $G = \langle f, g \rangle$ is a Schottky type group acting on $\mathbf{P}^2_{\mathbb{R}}$ with nonempty region of discontinuity.

In Chapter 5 we begin the analysis of the discretness for subgroups of $\mathrm{PSL}(3, \mathbb{C})$ generated by two elements, in the case that one of the elements is loxodromic, that the other element does not preserve the Kulkarni limit set of the loxodromic transformation, and they share an invariant line and a fixed point in their limit sets.

Definition 5.2. Let f and g be two different transformations in $\mathrm{PSL}(3, \mathbb{C})$. We say that *$f$ and g have a flag $p \in \ell$ in common* if whenever $\ell \subset \Lambda_{\mathrm{K}}(f)$ and p is a fixed point for $\langle f \rangle$, then the line ℓ is invariant under the action of g and p is a global fixed point.

Having the previous definition, we state the following propositions, which we will prove:

Proposition 5.9. Let $f \in \mathrm{PSL}(3, \mathbb{C})$ be a strongly loxodromic transformation with a lift $\mathbf{f} \in \mathrm{PSL}(3, \mathbb{C})$, $\mathbf{f} = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3)$ with $|\lambda_1| < |\lambda_2| < |\lambda_3|$. For $g \in \mathrm{PSL}(3, \mathbb{C})$ such that f and g have a flag in common (Definition 5.2) and the Kulkarni limit set $\Lambda_{\mathrm{K}}(f)$, is not invariant under g, and g is triangular then $\langle \mathbf{f}, \mathbf{g} \rangle$ is not discrete.

Proposition 5.11. Let $\mathbf{f} \in \mathrm{PSL}(3, \mathbb{C})$ be a complex homothety with a lift $\mathbf{f} \in \mathrm{SL}(3, \mathbb{C})$, $\mathbf{f} = \mathrm{Diag}(\lambda, \lambda, \lambda^{-2})$ and $|\lambda| > 1$. For $g \in \mathrm{PSL}(3, \mathbb{C})$ such that f and g have a flag in common (Definition 5.2), and the Kulkarni limit set $\Lambda_{\mathrm{K}}(f)$, is not invariant under g, then $\langle \mathbf{f}, \mathbf{g} \rangle$ is not discrete.

Proposition 5.12. Let $\mathbf{f}$ and $\mathbf{g}$ be transformations in $\mathrm{PSL}(3, \mathbb{C})$ such that $\mathbf{f}$ is a screw transformation with a lift $\mathbf{f} \in \mathrm{SL}(3, \mathbb{C})$, $\mathbf{f} = \mathrm{Diag}(\lambda, \mu, (\lambda\mu)^{-1})$ and $|\lambda| = |\mu| > 1$. If $g \in \mathrm{PSL}(3, \mathbb{C})$ is such that f and g have a flag in common (Definition 5.2), and the Kulkarni limit set $\Lambda_{\mathrm{K}}(f)$, is not invariant under g, then $\langle \mathbf{f}, \mathbf{g} \rangle$ is not discrete.

Proposition 5.13. Let $\mathbf{f}$ and $\mathbf{g}$ be transformations in $\mathrm{PSL}(3,\mathbb{C})$ such that $\mathbf{f}$ is a loxoparabolic transformation with a lift $\mathbf{f} \in \mathrm{SL}(3,\mathbb{C})$,

$$\mathbf{f} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}$$

where $|\lambda| > 1$. If $g \in \mathrm{PSL}(3,\mathbb{C})$ is such that f and g have the flag in common (Defintion 5.2) is $e_1 \in \overleftrightarrow{e_1 e_2}$, and the Kulkarni limit set $\Lambda_{\mathrm{K}}(f)$, is not invariant under g, then $\langle \mathbf{f}, \mathbf{g} \rangle$ is not discrete.

In the second Section of Chapter 5, we study sufficient conditions for the commutator of two elements $[f, g] = fgf^{-1}g^{-1}$ to be parabolic. We prove the following Proposition:

Proposition 5.16. Let $\mathbf{f}$ and $\mathbf{g}$ be two transformations in $\mathrm{PSL}(3,\mathbb{C})$ such that $\mathbf{f}$ is a loxodromic element, and $\mathbf{g}(\Lambda_{\mathrm{K}}(\mathbf{f})) \neq \Lambda_{\mathrm{K}}(\mathbf{f})$. Suppose that $\mathbf{f}$ and $\mathbf{g}$ share exactly one flag in their Kulkarni limit set, then $[\mathbf{f}, \mathbf{g}]$ is parabolic.

In the last section of Chapter 5 we give an overview of the utility of the results presented in the first and second Sections of this Chapter. In particular, the results will be useful to prove different theorems that will help to have a complete classification of the elementary groups in $\mathrm{PSL}(3,\mathbb{C})$.

Chapter 1

Kleinian Groups

In this Chapter we will present the definitions and lemmas with the ones we will use through out the rest of this work.

1.1 Classical Kleinian groups and some properties of $\mathrm{PSL}(2,\mathbb{C})$

The work done in the theory of complex Kleinian groups began as an extension of the theory of classical Kleinian groups, that is, subgroups of the group of automorphisms of the complex projective line. These automorphisms are fractional linear transformations: and if $z \in \mathbb{C}$,

$$g(z) = \frac{az+b}{cz+d}, \tag{1.1}$$

where a, b, c, d are complex numbers and $ad - bc \neq 0$. These transformations can be related to 2×2-matrices

$$\mathbf{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with determinant different from zero. Observe that if we multiply the matrix by a complex number $t \in \mathbb{C} - \{0\}$ the corresponding transformation remains the same. Using this property we conclude that the fractional linear transformations are in a bijective correspondance with the elements in $\mathrm{PSL}(2,\mathbb{C})$.

As we will be interested in conjugacy classes of subgroups of $\mathrm{PSL}(2,\mathbb{C})$ we point out that the trace function is invariant under conjugation. As the trace in $\mathrm{PSL}(2,\mathbb{C})$ it is not well defined because if g is an element in $\mathrm{PSL}(2,\mathbb{C})$ it has two representatives:

$$\mathbf{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad -\mathbf{g} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix},$$

then the square of the trace is considered: $\mathrm{tr}^2(\mathbf{g}) = (a+d)^2$.

The transformations are classified according to the fixed points they have. A fractional lineal transformation with exactly one fixed point is called *parabolic*. If the fractional lineal transformation has two fixed points can be conjugated to a rotation or to a dilation. If it is conjugated to a rotation then the transformation is called *elliptic* and if it is conjugated to a dilation is called *hyperbolic*. A non-elliptic transformation with exactly two fixed points is called *loxodromic* (these include hyperbolic transformations).

We present one of the results concerning the classification of the elements in $\mathrm{PSL}(2,\mathbb{C})$ and the trace of the related matrix, this results can be found in [24, p.6].

Proposition 1.1. *Let g be a transformation in $\mathrm{PSL}(2,\mathbb{C})$, and $\mathbf{g} \in \mathrm{SL}(2,\mathbb{C})$ a lift of g.*

- *(i)* $\mathrm{tr}^2(\mathbf{g})$ *is real and* $0 \leqslant \mathrm{tr}^2(\mathbf{g}) < 4$ *if and only if g is elliptic.*
- *(ii)* $\mathrm{tr}^2(\mathbf{g}) = 4$ *if and only if g is either parabolic or the identity.*
- *(iii)* $\mathrm{tr}^2(\mathbf{g})$ *is real and greater than 4 if and only if g is hyperbolic.*
- *(iv)* $\mathrm{tr}^2(\mathbf{g})$ *is not in the interval* $[0,\infty)$ *if and only if g is loxodromic, but not hyperbolic.*

1.1.1 Limit set of subgroups of $\mathrm{PSL}(2,\mathbb{C})$

As several definitions of limit set are used along this thesis I would like to begin presenting the definition of limit set used for groups acting on the complex projective line, that is, subgroups of $\mathrm{PSL}(2,\mathbb{C})$ acting on $\mathbf{P}^1_{\mathbb{C}}$. This in order to have a reference and have a point of comparison between the different definitions we will use for limit sets of subgroups of $\mathrm{PSL}(3,\mathbb{C})$ acting on $\mathbf{P}^2_{\mathbb{C}}$.

According to Lehner in [22, p.86] If G is a subgroup of $\mathrm{PSL}(2,\mathbb{C})$, a point α is called a *limit point* of G provided there is a point $z \in \mathbb{C}$ and an infinite sequence $\{g_n\}$ of different elements of G such that $g_n(z) \to \alpha$. The set of all limit points of G is the *limit set* and it is denoted by $\mathcal{L}$ or $\mathcal{L}(\mathrm{G})$ (if necessary).

A point that is not a limit point is called and *ordinary point* of G, and denote the set of ordinary points as $\mathcal{O}$, the *ordinary set*. Observe that by definition $\mathcal{O} = \mathcal{L}^c$, where $\mathcal{L}^c$ stands for the complement of $\mathcal{L}$.

Theorem 1.2. *Let G be a subgroup of* $\mathrm{PSL}(2,\mathbb{C})$ *acting on* $\mathbf{P}^1_{\mathbb{C}}$*. Then*

1. *[22, 1C] For all $g \in$ G, $g(\mathcal{L}) = \mathcal{L}$ and $g(\mathcal{O}) = \mathcal{O}$.*
2. *[22, 1G]$\mathcal{O}$ is an open set and $\mathcal{L}$ is a closed set.*
3. *[22, 4A]For either*
   (a) z an ordinary point, or
   (b) $z \in \mathcal{L}$ with the possible exception of $A = \{z\}$ and for some other point,

   if $A \subset \mathcal{L}$ and $z \in \mathbf{P}^2_{\mathbb{C}}$, the orbit of z under the group G, Gz is dense at A.
4. *[22, 4G] If $\mathcal{L}$ contains more than two points it is a perfect set.*
5. *[22, 4H] If $\mathcal{L}$ is not a single point, it is the closure of the set of fixed points of the hyperbolic or loxodromic transformations of G.*

6. *[22, 4I] If G has more than one limit point and has parabolic elements, $\mathcal{L}$ is the closure of the set of parabolic fixed points.*

7. *[22, 4J] If S is a closed set containing at least two points such that $g(S) \subset S$ for all $g \in G$, then $S \supset \mathcal{L}$.*

8. *[22, 5F] If G is a group of linear transformations of the plane, either $\mathcal{L}(G)$ is the complex sphere $\mathbf{P}^1_{\mathbb{C}}$ or $\mathcal{L}(G)$ is nowhere dense in $\mathbf{P}^1_{\mathbb{C}}$.*

1.2 The complex projective plane $\mathbf{P}^2_{\mathbb{C}}$

Let us move to the next dimension, that is, $\mathbb{C}^3$. The material of this section is based in the book Introduction to Algebraic Geometry written by Kenji Ueno [36, Chapter 2].

Take the triples (x, y, z) and (u, v, w) in $\mathbb{C}^3$ and set $W = \mathbb{C}^3 - \{(0, 0, 0)\}$. Consider the following equivalence relation $\sim$ in W

$$(x, y, z) \sim (u, v, w) \iff (x, y, z) = \alpha(u, v, w) \quad \text{for a complex number} \quad \alpha \neq 0.$$

The *complex projective plane*, denoted $\mathbf{P}^2_{\mathbb{C}}$, is the set of all equivalence classes. A point in $\mathbf{P}^2_{\mathbb{C}}$ will be denoted $[x : y : z]$ and notice that $[x : y : z] = [\alpha x : \alpha y : \alpha z]$. The plane $\mathbf{P}^2_{\mathbb{C}}$ has the structure of a smooth, compact 2-complex-dimensional manifold.

Three complex charts can be defined: U_1, U_2, and U_3:

$$\begin{aligned} U_1 &= \{[x : y : z] \in \mathbf{P}^2_{\mathbb{C}} : x \neq 0\} \\ U_2 &= \{[x : y : z] \in \mathbf{P}^2_{\mathbb{C}} : y \neq 0\} \\ U_3 &= \{[x : y : z] \in \mathbf{P}^2_{\mathbb{C}} : z \neq 0\}. \end{aligned} \tag{1.2}$$

For any point $[x : y : z] \in U_1$,

$$[x : y : z] = \left[1 : \frac{y}{x} : \frac{z}{x}\right],$$

and the mapping

$$\phi_1 \colon [x : y : z] \in U_1 \longrightarrow \left(\frac{y}{x}, \frac{z}{x}\right) \in \mathbb{C}^2,$$

is a bijection from U_1 to $\mathbb{C}^2$, whose inverse is given by:

$$\phi_1^{-1} \colon (v, w) \in \mathbb{C}^2 \longrightarrow [1 : v : w] \in U_1.$$

Similarly, the mappings $\phi_2 \colon U_2 \to \mathbb{C}^2$ and $\phi_3 \colon U_3 \to \mathbb{C}^2$ are defined.

Now, let us look at the set $\mathbf{P}^2_{\mathbb{C}} - U_1$. The points in this space has the form $[0 : y : z]$ with $(y, z) \neq (0, 0)$. Then, the point (y, z) determines a point in $\mathbf{P}^1_{\mathbb{C}}$. Also, for any point $[v : w] \in \mathbf{P}^1_{\mathbb{C}}$ we get the point $[0 : v : w] \in \mathbf{P}^2_{\mathbb{C}} - U_1$. The mapping

$$[0 : y : z] \in \mathbf{P}^2_{\mathbb{C}} \longrightarrow [y : z] \in \mathbf{P}^1_{\mathbb{C}}$$

is a bijection. So we identify $\mathbf{P}^2_{\mathbb{C}} - U_1$ and $\mathbf{P}^1_{\mathbb{C}}$. If we also identify U_1 and $\mathbb{C}^2$ using ϕ_1, the complex projective plane is the union:

$$\mathbf{P}^2_{\mathbb{C}} = \mathbb{C}^2 \cup \mathbf{P}^1_{\mathbb{C}}.$$

The set $\mathbf{P}^2_{\mathbb{C}} - U_1$ is called the *line at infinity* and $\mathbb{C}^2$ is the *complex affine plane* whose coordinates are (x, y).

A point (x, y) in the complex affine plane can be considered in $\mathbf{P}^2_{\mathbb{C}}$ as $[1 : x : y]$. Through this construction the line at infinity is not represented, but can be expressed by:

$$x = 0.$$

A line in the complex affine plane $\mathbb{C}^2$ can be written in the form

$$\alpha + \beta x + \gamma y = 0, \quad \text{where} \quad (\beta, \gamma) \neq (0,0), \quad \alpha, \beta, \gamma \in \mathbb{C}. \tag{1.3}$$

Thinking in the variables x and y as the quotients u_2/u_1 and u_3/u_1 and substituting the values in (1.3), we get the equation:

$$\alpha u_1 + \beta u_2 + \gamma u_3 = 0. \tag{1.4}$$

The last equation makes sense in $\mathbf{P}^2_{\mathbb{C}}$, because if $\lambda \neq 0$, the multiples $[\lambda u_1 : \lambda u_2 : \lambda u_3]$ of $[u_1 : u_2 : u_3]$, also satisfy the equation.

So the linear homogeneous equation (1.4) determines a *projective line* in the complex projective plane $\mathbf{P}^2_{\mathbb{C}}$.

1.2.1 The dual $(\mathrm{P}^2_{\mathbb{C}})^*$ of the complex projective plane

We can say more about equation (1.4), the set it represents can be expressed as:

$$\ell_{\alpha,\beta,\gamma} = \{[\alpha_1 : \alpha_2 : \alpha_3] \in \mathbf{P}^2_{\mathbb{C}} : \alpha\alpha_1 + \beta\alpha_2 + \gamma\alpha_3 = 0\}.$$

Observe that for any non-zero $\lambda \in \mathbb{C}$ the equation

$$\lambda\alpha u_1 + \lambda\beta u_2 + \lambda\gamma u_3 = 0,$$

defines the same line as in equation (1.4). So the following result can be stated:

Lemma 1.3.

(i) The lines $\ell_{\alpha,\beta,\gamma}$ and $\ell_{\alpha',\beta',\gamma'}$ are the same if and only if $[\alpha : \beta : \gamma] = [\alpha' : \beta' : \gamma']$.

(ii) Two distinct lines intersect in one point.

As a corollary of the lemma we get:

Corolary 1.4. *The set of all lines in $\mathbf{P}^2_{\mathbb{C}}$ are in a one to one correspondence with points $\mathbf{x} \in \mathbf{P}^2_{\mathbb{C}}$ by*

$$\ell_{\alpha,\beta,\gamma} = [\alpha : \beta : \gamma].$$

We are identifying the set of lines in $\mathbf{P}^2_{\mathbb{C}}$ with the complex projective plane itself. We denote the space of lines of $\mathbf{P}^2_{\mathbb{C}}$ as $(\mathbf{P}^2_{\mathbb{C}})^*$, and call it the *dual complex plane*.

1.2.2 Projective transformations

Definition 1.5. Given a 3×3 matrix with complex coefficients

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \det(A) \neq 0,$$

the map $f_A \colon \mathbf{P}^2_{\mathbb{C}} \longrightarrow \mathbf{P}^2_{\mathbb{C}}$ given by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{bmatrix}, \tag{1.5}$$

is called *projective transformation* determined by A.

The determinant of A is different from zero and the system of equations

$$a_{11}x + a_{12}y + a_{13}z = 0$$
$$a_{21}x + a_{22}y + a_{23}z = 0$$
$$a_{31}x + a_{32}y + a_{33}z = 0.$$

does not have a solution different from $(0, 0, 0)$. The transformation presented in (1.5) is well defined.

Lemma 1.6.

(i) For a non-zero complex number λ, we have $f_{\lambda A} = f_A$.

(ii) For 3×3 matrices A, B with $\det(A) \neq 0$, $\det(B) \neq 0$, we have $f_A \circ f_B = f_{AB}$. In particular $f_{A^{-1}} = {f_A}^{-1}$.

The set of projective transformations of $\mathbf{P}^2_{\mathbb{C}}$ forms a group and it is denoted as $\mathrm{PGL}(3, \mathbb{C})$. If we take $\mathbb{C}^+ = \mathbb{C} - \{0\}$ and I_3 as the cubic roots of the unity, then $\mathrm{PGL}(3, \mathbb{C})$ corresponds to the quotient:

$$\mathrm{PGL}(3, \mathbb{C}) = \mathrm{GL}(3, \mathbb{C})/\mathbb{C}^+\mathrm{I}_3.$$

So, we will talk about transformations $g \in \mathrm{PGL}(3, \mathbb{C})$, and we will work with matrices $\mathbf{g} \in \mathrm{GL}(3, \mathbb{C})$ representing the transformation g. We will talk about transformations g, and make calculations with its lift $\mathbf{g}$.

The action of $\mathrm{PGL}(3,\mathbb{C})$ in $(\mathbf{P}^2_{\mathbb{C}})^*$

The projective transformations maps lines in $\mathbf{P}^2_{\mathbb{C}}$ into lines: consider $\mathbf{g}$ a lift of g, and the line $\ell_{\alpha,\beta,\gamma}$ given by

$$\alpha x_1 + \beta x_2 + \gamma x_3 = 0.$$

Set

$$(\alpha', \beta', \gamma') = (\alpha, \beta, \gamma)\mathbf{g}^{-1}. \tag{1.6}$$

For $a = [a_1 : a_2 : a_3] \in \ell_{\alpha,\beta,\gamma}$, let $b = g(a) = [b_1 : b_2 : b_3]$. The following equation is satisfied:

$$\alpha' b_1 + \beta' b_2 + \gamma' b_3 = 0,$$

meaning that $[b_1 : b_2 : b_3]$ is a point in the line $\ell_{\alpha',\beta',\gamma'}$. Conversely, for $b \in \ell_{\alpha',\beta',\gamma'}$, set $a = b\mathbf{g}^{-1} = [a_1 : a_2 : a_3]$. Then we see that $a \in \ell_{\alpha,\beta,\gamma}$. So $\mathbf{g}$ takes the line $\ell_{\alpha,\beta,\gamma}$ into $\ell_{\alpha',\beta',\gamma'}$. The relationship between the coeficients of the lines is given by equation (1.6).

If we think as the group $\mathrm{PGL}(3,\mathbb{C})$ acting on $(\mathbf{P}^2_{\mathbb{C}})^*$, the action is by the right, with the inverse of their elements. That is, if $\ell = [x_1 : x_2 : x_3]$ then

$$g \cdot \ell := (x_1\, x_2\, x_3)g^{-1}. \tag{1.7}$$

1.2.3 Pseudo-Projective Transformations

The *pseudo-projective transformations* will be very important to describe the limit set of subgroups of automorphisms of $\mathbf{P}^2_{\mathbb{C}}$. We introduce them.

Let $M\colon \mathbb{C}^3 \to \mathbb{C}^3$ be a nonzero linear transformation. Consider the kernel of the transformation $\ker(M) \subset \mathbb{C}^3$. Denote by $[\ ]\colon \mathbb{C}^3 - \{0\} \to \mathbf{P}^2_{\mathbb{C}}$ the canonical projection. Let $[\ker(M)] \subset \mathbf{P}^2_{\mathbb{C}}$ be the projection of $\ker(M)$. Precisely, $[\ker(M)] = \ker(M)/\mathbb{C}^+$.

M induces a well-defined transformation $\widetilde{M}\colon \mathbf{P}^2_{\mathbb{C}} - [\ker(M)] \to \mathbf{P}^2_{\mathbb{C}}$, given by

$$\widetilde{M}([v]) = [M(v)].$$

Indeed, it is well defined because $M(v) \neq 0$, and it is a projective transformation on its domain: for every $\lambda \in \mathbb{C}^+$, $\widetilde{M}[\lambda v] = [\lambda M(v)]$, and coincide with $[M(v)]$ in $\mathbf{P}^2_{\mathbb{C}}$.

We call $\widetilde{M}$ a *pseudo-projective transformation*, and we denote the set of these transformations of $\mathbf{P}^2_{\mathbb{C}}$ as $PS(3,\mathbb{C})$; this space is the closure of $\mathrm{PSL}(3,\mathbb{C})$ in the space of $3\times 3-$matrices with complex entries. It is known that the pointwise convergence is equivalent to the convergence as a space of transformations. The following proposition is proved in [9, proposition 7.4.1].

Proposition 1.7. *Let $(g_m)_{m\in\mathbb{N}} \subset \mathrm{PSL}(3,\mathbb{C})$ be a sequence of distinct elements; then there exists a subsequence, still denoted $(g_m)_{m\in\mathbb{N}}$ and a transformation $g \in PS(3,\mathbb{C})$ such that $g_m \to g$ when $m \to \infty$ in compact subsets of $\mathbf{P}^2_{\mathbb{C}} - [\ker(g)]$.*

We will use this pseudo-projective transformations after noting that there is at least a lift to $\mathrm{GL}(3,\mathbb{C})$ for each transformation in $\mathrm{PSL}(3,\mathbb{C})$; we can, in fact, multiply g^n by a scalar $\alpha \in \mathbb{C}^+$, and, after projecting, we get the same transformation. If multiplied by an adequate scalar: usually the norm of the entry with greater norm of g, the sequence $[\alpha_n g^n]$ converges to a pseudo-projective transformation.

1.3 The complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$

For this section we will follow the book written by W. Goldman [12, Chapter 3]. Let $\mathbb{C}^{2,1}$ be the $3-$dimensional complex vector space consisting of triples $\mathbf{z} = (z_1, z_2, z_3)$ in $\mathbb{C}^3$ with the Hermitian inner product:

$$\langle \mathbf{z}, \mathbf{w} \rangle := z_1\overline{w_1} + z_2\overline{w_2} - z_3\overline{w_3}. \tag{1.8}$$

With this Hermitian product, the space $\mathbb{C}^{2,1}$ is divided into three sets. A vector $\mathbf{z} \in \mathbb{C}^{2,1}$ is called *positive* (resp. *negative, null*) if and only if the Hermitian product $\langle \mathbf{z}, \mathbf{z} \rangle$ is positive, (resp. negative or zero). The *complex hyperbolic plane*, $\mathbf{H}^2_{\mathbb{C}}$ is defined to be the subset of $\mathbf{P}(\mathbb{C}^{2,1})$ consisting of negative lines in $\mathbb{C}^{2,1}$.

The set of null lines in $\mathbb{C}^{2,1}$ is called the *boundary* of $\mathbf{H}^2_{\mathbb{C}}$, denoted $\partial\mathbf{H}^2_{\mathbb{C}}$.

1.3.1 Transformations of $\mathbf{H}^2_{\mathbb{C}}$: $\mathrm{PU}(2,1)$

We denote by $\mathrm{Diag}(a, b, c)$ the 3×3 diagonal matrix, whose values in the diagonal are, a, b and $c \in \mathbb{C}$.

Let $H = \mathrm{Diag}(1, 1, -1)$ be the matrix from which the Hermitian product (1.8) is defined. That is:

$$\langle \mathbf{z}, \mathbf{w} \rangle = (\overline{w_1}\ \overline{w_2}\ \overline{w_3}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

H has signature $(2, 1)$ being 2 the number of positive eigenvalues and 1 the number of the negative eigenvalues that the matrix H has. Consider all the matrices A in $\mathrm{GL}(3, \mathbb{C})$ such that

$$A^\dagger H A = H, \tag{1.9}$$

where $A^\dagger = \bar{A}^t$. The group of matrices satisfying (1.9) is called an *indefinite unitary group* (see [14, p.23]). In this particular case, it is denoted $\mathrm{U}(2, 1)$.

We identify $\mathbf{H}^2_{\mathbb{C}}$ with the unit ball

$$\mathbf{B}^2 = \{\mathbf{z} \in \mathbb{C}^2 | \langle \mathbf{z}, \mathbf{z} \rangle < 1\},$$

through the map

$$\begin{aligned} \mathbf{A} \colon \mathbb{C}^2 &\longrightarrow \mathbf{P}(\mathbb{C}^{2,1}) \\ z' &\mapsto \begin{bmatrix} z' \\ 1 \end{bmatrix}. \end{aligned} \tag{1.10}$$

The map $\mathbf{A}$ is a biholomorphic embedding of $\mathbb{C}^2$ onto the affine chart of $\mathbf{P}(\mathbb{C}^{2,1})$ defined by $z_3 \neq 0$.

As any vector in $\mathbb{C}^{2,1}$ with homogeneous coordinate $z_3 = 0$ is positive then $\mathbf{H}^2_{\mathbb{C}}$ is a subset of the image of $\mathbb{C}^2$ under $\mathbf{A}$. And $\mathbf{A}$ identifies $\mathbf{B}^2$ con $\mathbf{H}^2_{\mathbb{C}}$ and $\partial\mathbf{H}^2_{\mathbb{C}} = \mathbb{S}^3 \subset \mathbb{C}^2$ with $\partial\mathbf{H}^2_{\mathbb{C}}$.

The subgroup $\mathrm{PU}(2,1)$ acts transitively on the set $\mathbf{H}^2_{\mathbb{C}}$ of negative lines in $\mathbb{C}^{2,1}$ and it also acts transitively on the set $\partial\mathbf{H}^2_{\mathbb{C}}$, of null lines in $\mathbb{C}^{2,1}$, [12, Lemmas 3.1.3 and 3.1.6].

Remark 1.8. 1. $\mathbf{B}^2$ is contained in one affine chart.

2. $\mathbf{B}^2$ is the unitary ball in that chart.

$3-$spheres in $\mathbf{P}^2_{\mathbb{C}}$

Definition 1.9. The $3-$*spheres* in $\mathbf{P}^2_{\mathbb{C}}$ are defined as the subsets

$$T(r) = \{[x_1 : x_2 : x_3] \in \mathbf{P}^2_{\mathbb{C}} : |x_1|^2 + |x_2|^2 - r^2|x_3|^2 = 0\}$$

and their images under the action of $\mathrm{PSL}(3,\mathbb{C})$.

Remark 1.10. There is a foliation of $\mathbf{P}^2_{\mathbb{C}} - (\overleftrightarrow{\mathbf{e_1 e_2}} \cup \mathbf{e_3})$ given by the family of $3-$spheres

$$T(r) = \{[x_1 : x_2 : x_3] \in \mathbf{P}^2_{\mathbb{C}} : |x_1|^2 + |x_2|^2 = r^2|x_3|^2\},$$

where $\overleftrightarrow{\mathbf{e_1 e_2}}$ denotes the complex line $\{[x_1 : x_2 : x_3] \in \mathbf{P}^2_{\mathbb{C}} : x_3 = 0\}$ and $\mathbf{e_3} = [0 : 0 : 1]$.

Observe that the sphere T is given by the Hermitian form with signature $(2,1)$, namely $H = \mathrm{Diag}(1,1,-1)$, as

$$T = \{\mathbf{p} = [x : y : z] \in \mathbf{P}^2_{\mathbb{C}} : \mathbf{p}^\dagger H \mathbf{p} = 0\}.$$

If $[z_1 : z_2 : z_3] \in T \subset \mathbf{P}^2_{\mathbb{C}}$ for some sphere T given by the Hermitian form H and $g \in \mathrm{PSL}(3,\mathbb{C})$, the image of $[z_1 : z_2 : z_3]$ under g can be found after noting the following:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = g \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}. \tag{1.11}$$

Equation (1.11) is equivalent to the following:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = g^{-1} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \left[\mathbf{g}^{-1} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right].$$

Therefore $\left[\mathbf{g}^{-1} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right]$ satisfies:

$$\Big(g^{-1} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \Big)^* H \Big(g^{-1} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \Big) = (\overline{w_1}\ \overline{w_2}\ \overline{w_3})(\mathbf{g}^{-1})^* H (\mathbf{g}^{-1}) \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

And the form $(\mathbf{g}^{-1})^* H(\mathbf{g}^{-1})$, is the Hermitian form that defines the image under g of the sphere T.

Remark 1.11. When $g \in \mathrm{PSL}(3,\mathbb{C})$ acts on $(\mathbf{P}^2_{\mathbb{C}})^*$, the corresponding Hermitian form to stablish which is the image sphere is: $\mathbf{g}^* H \mathbf{g}$.

1.4 Limit sets

1.4.1 Kulkarni Limit Set

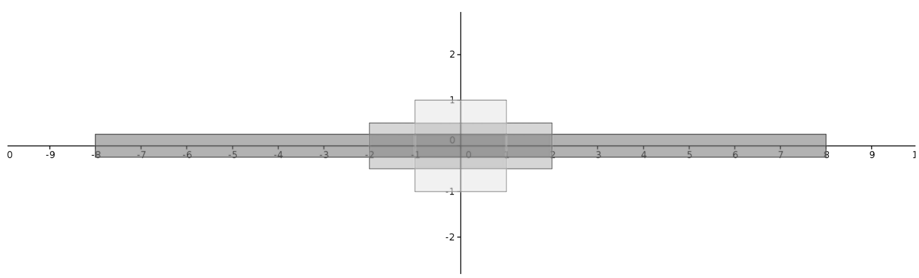
In this section we will follow the article of Ravi Kulkarni [21]. In 1978 he defined a limit set for a group acting on very general topological spaces. We will write the definition only for subgroups of $\mathrm{PSL}(3,\mathbb{C})$ acting on $\mathbf{P}^2_{\mathbb{C}}$.

Definition 1.12. Let G be a subgroup of $\mathrm{PSL}(3,\mathbb{C})$ acting on $\mathbf{P}^2_{\mathbb{C}}$. The action of G is said to be *properly discontinuous* on a $G-$invariant subset Ω of $\mathbf{P}^2_{\mathbb{C}}$ if for any two compact subsets C and D of Ω, $g(C) \cap D \neq \varnothing$ for a finite number of $g \in G$.

The definition of limit set for classical Kleinian groups of fractional linear transformations is as we said in Subsection 1.1.1, the closure of cluster points of $G-$orbits of points. However, some special features of conformal geometry are involved in the definition, features that no longer exists in higher dimension.

From the following example arose the need to consider also cluster points of $G-$orbits of compact sets:

Example 1.13. In $\mathbb{R}^2$, take the action $(x,y) \mapsto \left(2x, \frac{1}{2}y\right)$. The only cluster point of orbits is $(0,0)$, but the action is not properly discontinuous on $\mathbb{R}^2 - \{(0,0)\}$: take the square of side length one around the origin, then the iteration of the transformation of this compact set gets thinner in the y axis and wider on the x axis.



So the squares accumulate on the x axis. The behavior is analogous when considering the inverse of the transformation.

We introduce three subsets of $\mathbf{P}^2_{\mathbb{C}}$.

$L_0(G)$ the closure of points in $\mathbf{P}^2_{\mathbb{C}}$ with infinite isotropy group,

$L_1(G)$ the closure of accumulation points of the orbits of points in $\mathbf{P}^2_{\mathbb{C}} - L_0(G)$,

$L_2(G)$ the closure of accumulation points of $G-$orbits of compact subsets contained in $\mathbf{P}^2_{\mathbb{C}} - (L_0(G) \cup L_1(G))$.

Definition 1.14. The union $L_0(G) \cup L_1(G) \cup L_2(G)$ is called the *Kulkarni limit set* and it is denoted $\Lambda_{\mathrm{K}}(G)$. The complement of this union $\Omega_K(G) := \mathbf{P}^2_{\mathbb{C}} - \Lambda_{\mathrm{K}}(G)$ is the *discontinuity region* of G.

José Seade and Alberto Verjovsky in [31] and [32] introduced the next definition:

Definition 1.15. Let $G < \mathrm{PSL}(3, \mathbb{C})$ be a discrete subgroup, G is a *complex Kleinian group* if $\Omega_K(G) \neq \varnothing$.

Proposition 1.16. *[21, Prop. 1.3] Let $G < \mathrm{PSL}(3, \mathbb{C})$ acting on $\mathbf{P}^2_{\mathbb{C}}$. G equipped with the compact-open topology. Then $L_0(G)$, $L_1(G)$, $L_2(G)$, $\Lambda_K(G)$, $\Omega_K(G)$ are $G-$invariant subsets of $\mathbf{P}^2_{\mathbb{C}}$ and the action of G is properly discontinuous on $\Omega_K(G)$.*

Quasi-minimality Lemma

This lemma is relevant when we calculate the limit set of cyclic groups. It is introduced in [28, Lemma 5.3].

Lemma 1.17. *Let G be a subgroup of $\mathrm{PSL}(3, \mathbb{C})$. If $C \subset \mathbf{P}^2_{\mathbb{C}}$ is a closed subset such that, for every compact subset $K \subset \mathbf{P}^2_{\mathbb{C}} - C$, the accumulation points of the family $\{g(K)\}_{g \in G}$ are contained in $L_0(G) \cup L_1(G)$, then $L_2(G) \subseteq C$.*

Proof. On the contrary suppose that there is a point $\mathbf{x} \in L_2(G) - C$. By definition of $L_2(G)$, there is a compact subset $K_1 \subset \mathbf{P}^2_{\mathbb{C}} - (L_0(G) \cup L_1(G))$, such that $\mathbf{x}$ is its accumulation point. Let K_1 be the sequence $(\mathbf{k}_n) \cup \{\mathbf{k}\}$, where $\mathbf{k} = \lim_{n \to \infty} \mathbf{k}_n$. Then, there is a sequence $(g_n) \subset G$ such that $g_n(\mathbf{k}_n)$ converges to $\mathbf{x}$. As $\mathbf{x} \notin C$, for some $N \in \mathbb{N}$ and $n > N$, $g_n(\mathbf{k}_n)$ are not elements in C. So, for $n > N$, the set $K_2 = (g_n(\mathbf{k}_n)) \cup \{\mathbf{x}\}$ is a compact subset in $\mathbf{P}^2_{\mathbb{C}} - C$, by hypothesis its accumulation points are in $L_0(G) \cup L_1(G)$. However, if we evaluate the sequence (g_n^{-1}) in K_2, we have that $g_n^{-1}(g_n(\mathbf{k}_n)) = \mathbf{k}_n$ converges to $\mathbf{k} \notin L_0(G) \cup L_1(G)$, a contradiction. □

1.4.2 Chen-Greenberg Limit Set

The information in this Subsection can be found in [10].

Lemma 1.18. *[10, Lemma 4.3.1] Let $p \in \mathbf{H}^2_{\mathbb{C}}$ and let $\{g_n\}$ be a sequence in $U(2, 1)$ such that $\lim_{n\to\infty} g_n(p) = q \in \partial\mathbf{H}^2_{\mathbb{C}}$. The limit $\lim_{n\to\infty} g_n(p')$ is also q for all $p' \in \mathbf{H}^2_{\mathbb{C}}$.*

Definition 1.19. Let G be a subgroup of $\mathrm{U}(2, 1)$ and let $p \in \mathbf{H}^2_{\mathbb{C}}$. We will call the set $\mathbf{L}(G) = \overline{G(p)} \cap \partial\mathbf{H}^2_{\mathbb{C}}$, the *Chen-Greenberg limit set*.

Observe that by Lemma 1.18, this limit set in independent from the point p.

Now, we present some properties of the Chen-Greenberg limit set.

Lemma 1.20. *[10, Lemma 4.3.2] Let G be a subgroup of $U(2, 1)$:*

(a) $\mathbf{L}(G)$ *is invariant under G.*

(b) *If G' is a subgroup of G, then $\mathbf{L}(G') \subset \mathbf{L}(G)$.*

(c) *If G' is a subgroup of finite index of G, then $\mathbf{L}(G') = \mathbf{L}(G)$.*

(d) *If $\overline{G}$ is the closure of G in $U(2,1)$, then $\mathbf{L}(\overline{G}) = \mathbf{L}(G)$.*

1.4.3 Conze-Guivarch Limit Set

In [11] J.P. Conze and Y. Guivarc'h work with matrices in $\mathrm{GL}(n,\mathbb{R})$ and present many results we can use in the context of matrices with complex entries.

Following their paper, we will call an element $A \in \mathrm{SL}(3,\mathbb{C})$ *proximal* if it has an eigenvalue λ_0 such that $|\lambda_0| > |\lambda|$ for all other eigenvalues λ of A. For that matrix A, the eigenvector v_0 corresponding to the eigenvalue λ_0 is called a *dominant vector.*

In the same paper, they ask the subgroup G of $\mathrm{GL}(3,\mathbb{C})$ to satisfy two conditions so the results are valid, namely:

(H1) The action of G is strongly irreducible, that is, there does not exist any proper nonzero subspace of $\mathbb{C}^3$ invariant under the action of the subgroup of finite index in G.

(H2) G contains an element which is proximal.

Remark 1.21. Conditions (H1) and (H2) hold for the action of G on the dual space $(\mathbf{P}^2_{\mathbb{C}})^*$ whenever they hold for the action of G in $\mathbf{P}^2_{\mathbb{C}}$.

Definition 1.22. Let G be a subgroup of $\mathrm{SL}(3,\mathbb{C})$ and consider its action on $\mathbf{P}^2_{\mathbb{C}}$. We denote by $L(G)$ the closure of the subset of $\mathbf{P}^2_{\mathbb{C}}$ consisting of the projectivization of all the eigenvectors corresponding to the eigenvalues with norm greater than the others of every $g \in G$, that is

$$L(G) = \{[v] : v\,\text{dominant vector of}\, g \in G\}.$$

We will call the set $L(G)$ the *Conze-Guivarc'h limit set.*

Recall that a closed invariant subset F of a space X with an action of a group G is said to be *minimal* if it contains no non-empty proper subset which is closed and $G-$invariant.

Proposition 1.23. *$L(G)$ is a $G-$ invariant subset of $\mathbf{P}^2_{\mathbb{C}}$. And if the $G-$action satisfies (H1) and (H2), then $L(G)$ is minimal and it is the only minimal set for the action.*

1.4.4 Set of Equicontinuity

There is another set it is used when working with complex Kleinian groups and this is the set of Equicontinuity.

Recall that a family $\mathcal{F}$ of continuous functions f defined on some complete metric space X with values in another complete metric space Y is called normal if every sequence of functions in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of X to a continuous function from X to Y.

Definition 1.24. The *equicontinuity set for a family* $\mathcal{F}$ of endomorphisms of $\mathbf{P}^2_{\mathbb{C}}$, denoted $Eq(\mathcal{F})$, is defined to be the set of points $\mathbf{x} \in \mathbf{P}^2_{\mathbb{C}}$ for which there is an open neighborhood U of $\mathbf{x}$ such that $\{f|_U : f \in \mathcal{F}\}$ is a normal family.

The equicontinuity set is open and in the particular case when the family $\mathcal{F}$ consists of the elements of a group $G \subset \mathrm{PSL}(3, \mathbb{C})$, the equicontinuity set is $G-$invariant.

1.4.5 Notation

- Whenever we write $\mathbb{F}$, we will make reference to either $\mathbb{R}$ or $\mathbb{C}$.
- The vectors $\mathbf{e_1}, \mathbf{e_2}$ and $\mathbf{e_3}$ are the projection to $\mathbf{P}^2_{\mathbb{C}}$ of the canonical basis of $\mathbb{C}^3$.
- $\overleftrightarrow{\mathbf{pq}}$ will denote the line passing trough the points $\mathbf{p}$ and $\mathbf{q}$ in $\mathbf{P}^2_{\mathbb{C}}$.
- We will say that a line $\ell \subset \mathbf{P}^2_{\mathbb{C}}$ is an *attracting (or repelling) line* if there exists a sequence $(g_n) \subset G$ and $\eta \in (\mathbf{P}^2_{\mathbb{C}})^*$ such that $g_n \cdot \eta \to \ell$, (or $g_n^{-1} \cdot \eta \to \ell$) when n tends to infinity.
- The set of accumulation points of any set A is denoted A'.
- When talking about cyclic subgroups $\langle g \rangle$ generated by some element $g \in \mathrm{PSL}(3, \mathbb{C})$, we will write $L_0(g), L_1(g), \ldots$ for referring to the sets related to the limit sets.
- For either an element $\mathbf{x} \in \mathbf{P}^2_{\mathbb{C}}$ or a matrix $\mathbf{g} \in \mathrm{SL}(3, \mathbb{C})$, we will denote its transpose with the exponent "T" $\mathbf{x}^T$, $\mathbf{g}^T$.

Chapter 2

Kulkarni limit sets of cyclic groups in $\mathrm{PSL}(3, \mathbb{C})$

In this chapter we study the transformations in $\mathrm{PSL}(3, \mathbb{C})$. In 2008, J.P. Navarrete [28] made a clasification of the elements in the group according to their trace. In that paper he defined which elements are elliptic, parabolic or loxodromic according to a dynamical definition, the limit sets of cyclic groups were analized and described in each case.

As we said in Subsection 1.3.1 the group that preserves the complex hyperbolic plane is PU$(2, 1)$, however this subgroup of $\mathrm{PSL}(3, \mathbb{C})$ can be thought as acting on the complex projective plane. In another paper [27], the same author studied the Kulkarni limit set for subgroups of PU$(2, 1)$ acting on $\mathbf{P}^2_{\mathbb{C}}$.

In this second chapter we review the work done by J.P. Navarrete about the two previous results, this time studied from a more topological point of view. We use pseudo-projective transformations to find out in the first case which is the Kulkarni limit set for cyclic subgroups of $\mathrm{PSL}(3, \mathbb{C})$ and in the second case, that the Kulkarni limit set for subgroups G of PU$(2, 1)$ acting on $\mathbf{P}^2_{\mathbb{C}}$ is equal to the tangent lines to $\partial \mathbf{H}^2_{\mathbb{C}}$ in points of the Chen-Greenberg limit set for the group G. Using pseudo-projective transformations seems a better tool to study the limit set of groups in higher dimensions.

We begin analysing the limit sets of the cyclic groups.

2.1 Limit sets of Loxodromic Transformations

In [28], the author gives the definition of loxodormic transformation in $\mathrm{PSL}(3, \mathbb{C})$, namely:

Definition 2.1. An element $g \in \mathrm{PSL}(3, \mathbb{C})$ is called *loxodromic transformation* if there is a $3-$sphere S in $\mathbf{P}^2_{\mathbb{C}}$ such that $g(S \cup \mathbb{X}_i) \subset \mathbb{X}_i$ for some $i = 1, 2$. Where $\mathbb{X}_1$, and $\mathbb{X}_2$ are the connected components of $\mathbf{P}^2_{\mathbb{C}} - S$.

An example of an element in $\mathrm{PSL}(3, \mathbb{C})$ satisfying Definition 2.1 can be induced by a matrix Diag$(\lambda_1, \lambda_2, \lambda_3)$, with $|\lambda_1| < |\lambda_2| < |\lambda_3|$. If $S = \{\mathbf{x} = [x : y : z] \in \mathbf{P}^2_{\mathbb{C}} : |x|^2 + |y|^2 - |z|^2 = 0\}$ then

$\mathbb{X}_1 = \{\mathbf{x} \in \mathbf{P}^2_{\mathbb{C}} : |x|^2 + |y|^2 - |z|^2 < 0\}$ is one of the components of $\mathbf{P}^2_{\mathbb{C}} - S$.

Applying the transformation g to a point in $\mathbb{X}_1$, we get $[\lambda_1 x : \lambda_2 y : \lambda_3 z]$, then

$$|\lambda_1 x|^2 + |\lambda_2 y|^2 < |\lambda_2|^2(|x|^2 + |y|^2) < |\lambda_2|^2|z|^2 < |\lambda_3 z|^2.$$

We conclude that $g(\mathbf{x}) \in \mathbb{X}_1$, and g is a loxodromic transformation.

According to the Jordan canonical form, a loxodoromic element has at least two eigenvalues of different module, [28, Proposition 6.7]. So if an element is loxodromic its lift is conjugate to one of the following possible matrices.

$$\underset{|\lambda| \neq 1}{\mathbf{g_1} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}}, \quad \underset{\substack{\lambda \neq \mu \\ |\lambda| = |\mu| \neq 1}}{\mathbf{g_2} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & (\lambda\mu)^{-1} \end{pmatrix}}, \quad \underset{|\lambda| \neq 1}{\mathbf{g_3} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}}, \quad \underset{\substack{|\lambda_1| < |\lambda_2| < |\lambda_3| \\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}.}}{\mathbf{g_4} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}}. \tag{2.1}$$

The transformation $\mathbf{g}_1$ is called a *complex homothety,* $\mathbf{g}_2$ is called a *screw,* the name of the elements conjugate to $\mathbf{g}_3$ is *loxoparabolic* and $\mathbf{g}_4$ are *strongly loxodromic* elements.

Kulkarni limit set of a complex homothety

Lemma 2.2. *If $g \in \mathrm{PSL}(3, \mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3, \mathbb{C})$ as $\mathbf{g}_1$ in equation* (2.1)*, then $L_0(g)$ consist of a line ℓ of fixed points and a global fixed point not in ℓ.*

Proof. We will find $L_0(\mathbf{g}_1)$. The elements in $G = \langle \mathbf{g}_1 \rangle$, are of the form:

$$\begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & 0 \\ 0 & 0 & \lambda^{-2n} \end{pmatrix},$$

with $n \in \mathbb{Z}$. Then, the points $\mathbf{x} = [x : y : z]$ with infinite isotropy group satisfy:

$$\mathbf{g}_1^n(\mathbf{x}) = \begin{bmatrix} \lambda^n x \\ \lambda^n y \\ \lambda^{-2n} z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{2.2}$$

Equality in Equation (2.2) is true whenever $\mathbf{x} = \mathbf{e}_3$ or $\mathbf{x} = [x : y : 0]$. That is, points in the line $\overleftrightarrow{\mathbf{e_1 e_2}}$ are fixed points for the group. □

Lemma 2.3. *If $g \in \mathrm{PSL}(3, \mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3, \mathbb{C})$ as $\mathbf{g}_1$ in equation* (2.1)*, then $L_1(g)$ consist of a line ℓ of fixed points and a global fixed point not in ℓ.*

Proof. Now we have to find the accumulation points of orbits of points in $\mathbf{P}^2_{\mathbb{C}} - L_0(\mathbf{g}_1)$.

Observe that if $\mathbf{g}_1$ is multiplied by some non-zero scalar $\alpha \in \mathbb{C}$, $\alpha \mathbf{g}_1^n$ induces projectively the same transformation as $\mathbf{g}_1$. Then we multiply $\mathbf{g}_1$ by λ^{-n}. Projectively, the following equalities are valid:

$$\mathbf{g}_1^n = \lambda^{-n}\mathbf{g}_1^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-3n} \end{pmatrix}. \tag{2.3}$$

And when n tends to infinity, $\lambda^{-n}\mathbf{g}_1^n$ tends to the constant transformation $M_1 = \mathrm{Diag}(1,1,0)$ whenever $|x| + |y| \neq 0$. Precisely, $\ker(M_1) = \{\mathbf{x} \in \mathbf{P}^2_{\mathbb{C}} : |x| + |y| = 0\} = \{\mathbf{e_3}\}$. Therefore, for every $[x : y : z] \in \mathbf{P}^2_{\mathbb{C}} - \ker M_1$, the sequence $\lambda^{-n}\mathbf{g}_1(\mathbf{x}^T)$ converges to $[x : y : 0]^T$ as $n \to \infty$.

Now, multiply $\mathbf{g}_1^{-n}$ by λ^{2n}. The sequence $\lambda^{2n}\mathbf{g}_1^{-n}$ converges to the constant transformation $M_2 = \mathrm{Diag}(0,0,1)$, whenever $z \neq 0$. Observe that $\ker(M_2) = \overleftrightarrow{\mathbf{e_1}\mathbf{e_2}}$.

Then, points in either $\{\mathbf{e_3}\}$ or in $\overleftrightarrow{\mathbf{e_1}\mathbf{e_2}}$ are accumulation points of orbits of points in $\mathbf{P}^2_{\mathbb{C}} - L_0(\mathbf{g}_1)$. □

Lemma 2.4. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{g}_1$ in equation* (2.1)*, then $L_2(g)$ consist of a line ℓ of fixed points and a global fixed point not in ℓ.*

Proof. It is easy to see that $\{\mathbf{e_3}\} \cup \overleftrightarrow{\mathbf{e_1}\mathbf{e_2}} \subset L_2(g)$: as we have seen in the Proof of Lemma 2.3, the points that are not contained in $\ker(M_1)$, nor in $\ker(M_2)$ converge to a point in the line $\overleftrightarrow{\mathbf{e_1}\mathbf{e_2}}$ for positive iterates of $\mathbf{g}_1$, and converge to $\mathbf{e_3}$ for negative iterates of $\mathbf{g}_1$. To prove the contention $L_2(g) \subset \{\mathbf{e_3}\} \cup \overleftrightarrow{\mathbf{e_1}\mathbf{e_2}}$, suppose that $C = \{\mathbf{e_3}\} \cup \overleftrightarrow{\mathbf{e_1}\mathbf{e_2}}$. By Lemmas 2.2 and 2.3, the orbits of points $\mathbf{x} \in \mathbf{P}^2_{\mathbb{C}} - C$, accumulate in $L_0(g) \cup L_1(g)$. By Lemma 1.17, we have the desired contention. □

After Lemmas 2.2, 2.3 and 2.4, we have the next proposition.

Proposition 2.5. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{g}_1$ in equation* (2.1)*, then $\Lambda_K(g)$ consist of a global fixed point and a line of fixed points.*

Kulkarni limit set of a screw transformation

Lemma 2.6. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{g}_2$ in equation* (2.1)*, then $L_0(g)$ is either a global fixed point and a line of fixed points or three fixed points.*

Proof. We will find $L_0(\mathbf{g}_2)$. The elements in $\langle \mathbf{g}_2 \rangle$, are of the form:

$$\begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \mu^n & 0 \\ 0 & 0 & (\lambda\mu)^{-n} \end{pmatrix},$$

with $n \in \mathbb{Z}$. As $|\lambda| = |\mu|$, then $\lambda/\mu = e^{2\pi i\alpha}$, for some $\alpha \in \mathbb{R}$. If $\alpha \in \mathbb{Q}$, for infinitely many values of n, $\mu^{-n}\mathbf{g}_2^n$ is equal to $\mathrm{Diag}(1,1,(\lambda\mu^2)^{-n})$, because $\lambda^n/\mu^n = 1$. Then for every point $\mathbf{x}$ in $\overleftrightarrow{\mathbf{e_1}\mathbf{e_2}}$, the isotropy group of $\mathbf{x}$ is infinite.

On the other hand, if $\alpha \in \mathbb{R} - \mathbb{Q}$, then the only points with infinite isotropy group are $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$. □

Lemma 2.7. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{g}_2$ in equation* (2.1)*, then $L_1(g)$ consist of a global fixed point and a line of fixed points.*

Proof. As observed in the proof of Lemma 2.6, for infinitely many values of n, the sequence $\mu^{-n}\mathbf{g}_2^n$ is equal to $\mathrm{Diag}(1,1,(\lambda\mu^2)^{-n})$, and converges to the pseudo-projective transformation $M_1 = \mathrm{Diag}(1,1,0)$, as $n \to \infty$, so $\overleftrightarrow{\mathbf{e_1 e_2}}$ is an attracting line whenever $\mathbf{x}$ does not have the first and second coordinate equal to zero.

Also, multiplying $\mathbf{g}_2^{-n}$ by $(\lambda\mu)^{-n}$, we have $(\lambda\mu)^{-n}\mathbf{g}_2^{-n} = \mathrm{Diag}((\lambda^2\mu)^{-n},(\lambda\mu^2)^{-n},1)$, and the sequence of these transformations tends to $M_2 = \mathrm{Diag}(0,0,1)$ as $n \to \infty$. Then, $\mathbf{e_3}$ is a repelling point. □

Lemma 2.8. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{g}_2$ in equation* (2.1)*, then $L_2(g)$ consist of a global fixed point and a line of fixed points.*

The proof of Lemma 2.8 is exactly the same as the proof of Lemma 2.4 and we omit it. After Lemmas 2.6, 2.7 and 2.8, we have the next proposition.

Proposition 2.9. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{g}_2$ in equation* (2.1)*, then $\Lambda_K(g)$ consist of a global fixed point and a line of fixed points.*

Kulkarni limit set of a loxoparabolic element

Lemma 2.10. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{g}_3$ in equation* (2.1)*, then $L_0(g)$ is the set of the two fixed points.*

Proof. We will find $L_0(\mathbf{g}_3)$. The elements in $\langle \mathbf{g}_3 \rangle$, are of the form:

$$\mathbf{g}_3^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & 0 \\ 0 & \lambda^n & 0 \\ 0 & 0 & \lambda^{-2n} \end{pmatrix}, \quad n \in \mathbb{Z}.$$

A point $[\mathbf{x}] \in \mathbf{P}^2_{\mathbb{C}}$ has infinite isotropy group if the equation $\mathbf{g}_3^n([\mathbf{x}]) = [\mathbf{x}]$ is satisfied for infinitely many values of $n \in \mathbb{Z}$. Then,

$$\begin{bmatrix} \lambda^n x + n\lambda^{n-1} y \\ \lambda^n y \\ \lambda^{-2n} z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2.4}$$

only when $\mathbf{x} = \mathbf{e_1}$ or $\mathbf{x} = \mathbf{e_3}$ □

Lemma 2.11. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{g}_3$ in equation* (2.1)*, then $L_1(g)$ is the set of the two fixed points.*

Proof. Observe that if we divide the element $\mathbf{g}_3^n$ by $n\lambda^{n-1}$ and $\mathbf{g}_3^{-n}$ by λ^{2n}, we get the following two sequences of transformations:

$$\frac{1}{n\lambda^{n-1}}\mathbf{g}_3^n = \begin{pmatrix} \frac{\lambda}{n} & 1 & 0 \\ 0 & \frac{\lambda}{n} & 0 \\ 0 & 0 & \frac{\lambda^{-3n+1}}{n} \end{pmatrix} \quad \text{or} \quad \lambda^{-2n}\mathbf{g}_3^{-n} = \begin{pmatrix} \lambda^{-3n} & -n\lambda^{-3n-1} & 0 \\ 0 & \lambda^{-3n} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The first sequence converge to the matrix

$$M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.5}$$

defining the constant transformation with value $\mathbf{e_3}$, whose kernel is $\ker(M_3) = \{[\mathbf{x}] \in \mathbf{P}^2_{\mathbb{C}} : y = 0\}$. The second sequence converges to the matrix M_2 for $[\mathbf{x}] \in \mathbf{P}^2_{\mathbb{C}}$, such that $z \neq 0$, as in proof of Lemma 2.3. So $L_1(g_3) = \{\mathbf{e_1}, \mathbf{e_3}\}$. □

Lemma 2.12. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{g}_3$ in equation (2.1), then $L_2(g)$ consist of an attracting and a repelling line.*

Proof. Take a line in $\mathbf{P}^2_{\mathbb{C}}$ passing trough $\mathbf{p} = [p_1 : p_2 : 0]$ and $q = [q_1 : q_2 : q_3]$. The equation of this line is $p_2q_3x - p_1q_3y - p_2q_1z = 0$ and corresponds to the point $[p_2q_3 : -p_1q_3 : -p_2q_1] \in (\mathbf{P}^2_{\mathbb{C}})^*$. The action of the group $\langle g \rangle$ in the space of lines is as shown in (1.7) and the sequence $\lambda^{-2n}g^n \cdot \overleftrightarrow{pq}$ converges to the point $[0 : 0 : -p_2q_1] = [0 : 0 : 1] \in (\mathbf{P}^2_{\mathbb{C}})^*$, which corresponds to the line $\overleftrightarrow{e_1e_2}$.

Also, the sequence $n\lambda^{n-1}g^{-n} \cdot \overleftrightarrow{pq}$ converges to the point $[0 : p_2q_3 : 0] = [0 : 1 : 0] \in (\mathbf{P}^2_{\mathbb{C}})^*$ and this point corresponds to the line $\overleftrightarrow{e_1e_3} \subset \mathbf{P}^2_{\mathbb{C}}$. Then $\overleftrightarrow{e_1e_2} \cup \overleftrightarrow{e_1e_3} \subset L_2(g)$.

Let $C = \overleftrightarrow{e_1e_2} \cup \overleftrightarrow{e_1e_3}$, for all compact subset $K \subset \mathbf{P}^2_{\mathbb{C}} - C$, the orbit $g^n(K)$ converges to $\mathbf{e_1} \in L_0(g) \cup L_1(g)$. By Lemma 1.17, $L_2(g) \subset \overleftrightarrow{e_1e_2} \cup \overleftrightarrow{e_1e_3}$, and the proof is finished. □

After Lemmas 2.10, 2.11 and 2.12, we have the next proposition.

Proposition 2.13. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{g}_3$ in equation (2.1), then $\Lambda_K(g)$ consist of an attracting and a repelling line.*

Kulkarni limit set of a strongly loxodromic element

Lemma 2.14. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{g}_4$ in equation (2.1), then $L_0(g)$ is the set of three fixed points.*

Proof. We will find $L_0(\mathbf{g}_4)$. The elements in $\langle \mathbf{g}_4 \rangle$, are of the form:

$$\mathbf{g}_4^n = \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix}$$

A point $[\mathbf{x}] \in \mathbf{P}^2_{\mathbb{C}}$ has infinite isotropy group if the equation $\mathbf{g}_4^n([\mathbf{x}]) = [\mathbf{x}]$ is satisfied for infinitely many values of $n \in \mathbb{Z}$. Then,

$$\begin{bmatrix} \lambda_1^n x \\ \lambda_2^n y \\ \lambda_3^n z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2.6}$$

only when $\mathbf{x} = \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$. □

Lemma 2.15. *If* $g \in \mathrm{PSL}(3,\mathbb{C})$ *is conjugate to a transformation with a lift in* $\mathrm{SL}(3,\mathbb{C})$ *as* $\mathbf{g}_4$ *in equation* (2.1)*, then* $L_1(g)$ *is the set of an attracting fixed point, a repelling fixed point and a saddle fixed point.*

Proof. Observe that if we divide the element $\mathbf{g}_4^n$ by λ_3^n and $\mathbf{g}_4^{-n}$ by λ_1^{-n}, we get the following two sequences of transformations:

$$\lambda_3{}^{-n}\mathbf{g}_4^n = \begin{pmatrix} (\frac{\lambda_1}{\lambda_3})^n & 0 & 0 \\ 0 & (\frac{\lambda_2}{\lambda_3})^n & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \lambda_1^n\mathbf{g}_4^{-n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\frac{\lambda_1}{\lambda_2})^n & 0 \\ 0 & 0 & (\frac{\lambda_1}{\lambda_3})^n \end{pmatrix}.$$

The first sequence converge to the matrix $M_2 = \mathrm{Diag}(0,0,1)$, when $z \neq 0$. M_2 defines the constant transformation with value $\mathbf{e_3}$. The second sequence converges to the matrix $M_4 = \mathrm{Diag}(1,0,0)$ for $[\mathbf{x}] \in \mathbf{P}^2_{\mathbb{C}}$, such that $x \neq 0$.

In the case that $z = 0$ we divide $\mathbf{g}_4^n$ by λ_2^n, then we get the transformation $\mathrm{Diag}((\frac{\lambda_1}{\lambda_2})^n, 1, (\frac{\lambda_3}{\lambda_2})^n)$. For points in the line $\overleftrightarrow{\mathbf{e_1e_2}}$,, we have a sequence converging to the constant transformation $\mathbf{e_2}$. Also, if $\mathbf{g}_4^{-n}$ is multiplied by λ_2^n, the sequence $\mathrm{Diag}((\frac{\lambda_2}{\lambda_1})^n, 1, (\frac{\lambda_2}{\lambda_3})^n)$, for points in the line $\overleftrightarrow{\mathbf{e_2e_3}}$, converges to the constant transformation $M_5 = \mathrm{Diag}(0,1,0)$, which maps every point in $\overleftrightarrow{\mathbf{e_2e_3}}$ to $\mathbf{e_2}$. Then $L_1(g) = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$. □

Lemma 2.16. *If* $g \in \mathrm{PSL}(3,\mathbb{C})$ *is conjugate to a transformation with a lift in* $\mathrm{SL}(3,\mathbb{C})$ *as* $\mathbf{g}_4$ *in equation* (2.1)*, then* $L_2(g)$ *consist of an attracting and a repelling line.*

Proof. Consider a line $\ell \subset \mathbf{P}^2_{\mathbb{C}}$ such that $\ell \cap \overleftrightarrow{\mathbf{e_1e_2}} = p$ and $\ell \cap \overleftrightarrow{\mathbf{e_2e_3}} = q$, $p, q \neq \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$. If $p = [x_1 : y_1 : 0]$ and $q = [0 : y_2 : z_2]$ then the line that the two points define is given by $[y_1z_2 : -x_1z_2 : x_1y_2] \in (\mathbf{P}^2_{\mathbb{C}})^*$.

Take the sequence of lines $\lambda_1^n\mathbf{g}_4^n \cdot \ell$:

$$(y_1z_2 \quad -x_1z_2 \quad x_1y_2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & (\frac{\lambda_1}{\lambda_2})^n & 0 \\ 0 & 0 & (\frac{\lambda_1}{\lambda_3})^n \end{bmatrix} = (y_1z_2 \quad -\left(\frac{\lambda_1}{\lambda_2}\right)^n x_1z_2 \quad \left(\frac{\lambda_1}{\lambda_3}\right)^n x_1y_2), \tag{2.7}$$

converges to $[y_1z_2 : 0 : 0] = [\ell_1] \in (\mathbf{P}^2_{\mathbb{C}})^*$, when n tends to infinity, corresponding to the line $\overleftrightarrow{\mathbf{e_2e_3}} \subset \mathbf{P}^2_{\mathbb{C}}$.

So, for every $\mathbf{y} \in \overleftrightarrow{\mathbf{e_2e_3}}$ there is a convergent sequence $(\mathbf{x}_n) \subset \ell$ such that $g^n(\mathbf{x}_n) \longrightarrow \mathbf{y}$. The sequence $(\mathbf{x}_n)$ converges to q, on the contrary, suppose that $\mathbf{x}_n \longrightarrow \mathbf{x} \neq q$. Then $\mathbf{x} \notin \overleftrightarrow{\mathbf{e_2e_3}}$. So we would have:

$$g^n \cdot \mathbf{x}_n \longrightarrow g^n(\mathbf{x}) \longrightarrow \mathbf{e_3},$$

but we said that $g^n \cdot \mathbf{x}_n \longrightarrow \mathbf{y}$. The line $\overleftrightarrow{\mathbf{e_2e_3}}$ is an attracting line for g. Therefore $\overleftrightarrow{\mathbf{e_2e_3}} \subset L_2(g)$.

Analogously, using now the negative powers of g, g^{-n} we find that $\overleftrightarrow{\mathbf{e_1e_2}}$ is also contained in $L_2(g)$. The line $\overleftrightarrow{\mathbf{e_1e_2}}$ is a repelling line for g. So $\overleftrightarrow{\mathbf{e_1e_2}} \cup \overleftrightarrow{\mathbf{e_2e_3}} \subset L_2(g)$.

For the other contention, let C be the set $\overleftrightarrow{\mathbf{e_1e_2}} \cup \overleftrightarrow{\mathbf{e_2e_3}}$. Observe that for points $\mathbf{x} \in \mathbf{P}^2_{\mathbb{C}} - C$, the orbit of $\mathbf{x}$ accumulates on $L_0(g) \cup L_1(g)$, then by Lemma 1.17, $L_2(g) \subset C$. □

After Lemmas 2.14, 2.15 and 2.16, we have the next proposition.

Proposition 2.17. *If $g \in \mathrm{PSL}(3, \mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3, \mathbb{C})$ as $\mathbf{g}_4$ in equation (2.1), then $\Lambda_K(g)$ consist of an attracting and a repelling line.*

2.2 Limit sets of Parabolic Transformations

In this section, we make the analysis of the limit set of the cyclic group generated by a parabolic transformation.

Definition 2.18. The transformation $g \in \mathrm{PSL}(3, \mathbb{C})$ is called *parabolic* if there exists a family of $g-$invariant $3-$spheres T_r, $r \in \mathbb{R}$ and a fixed point $\mathbf{z}_g$ satisfying the following properties:

(i) For every pair of different elements $r_1, r_2 \in \mathbb{R}, \quad T_{r_1} \cap T_{r_2} = \mathbf{z}_g.$

(ii) $\Lambda_{\mathrm{K}}(g)$ is a complex line.

(iii) $\mathbf{P}^2_{\mathbb{C}} - \bigcup_{r\in\mathbb{R}} (T_r - \{\mathbf{z}_g\}) = \Lambda_{\mathrm{K}}(g).$

A transformation $g \in \mathrm{PSL}(3, \mathbb{C})$ is parabolic if and only if g has a lift $\mathbf{g} \in \mathrm{SL}(3, \mathbb{C})$ non-diagonalizable such that every eigenvalue has module one.

If the element is *parabolic*, then it has a lift whose Jordan canonical form is given by one of the following three matrices:

$$\mathbf{f}_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{f}_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{f}_3 = \begin{bmatrix} e^{2\pi it} & 1 & 0 \\ 0 & e^{2\pi it} & 0 \\ 0 & 0 & e^{-4\pi it} \end{bmatrix}, \tag{2.8}$$

where $t \in \mathbb{R}$.

The transformations $\mathbf{f}_1$ and $\mathbf{f}_2$ are called *unipotent transformations* of the first and second type respectively. $\mathbf{f}_3$ is named *ellipto-parabolic* transformation.

Kulkarni limit set of an unipotent type I element

Lemma 2.19. *If $g \in \mathrm{PSL}(3, \mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3, \mathbb{C})$ as $\mathbf{f}_1$ in equation (2.8), then $L_0(g)$ is the line of fixed points.*

Proof. Suppose that g is conjugate to $\mathbf{f}_1$. The elements of the group $G = \langle \mathbf{f}_1 \rangle$ have as general expression:

$$\mathbf{f}_1 = \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for all $n \in \mathbb{Z}$. Then a point $\mathbf{x} \in \mathbf{P}^2_{\mathbb{C}}$ belongs to $L_0(\mathbf{f}_1)$ only when for infinitely many values of $n \in \mathbb{Z}$, the next equation is satisfied:

$$\begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+ny \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{2.9}$$

So, if $y = 0$, for all $\mathbf{x} = [x : 0 : z]$, $\mathbf{x}$ is a global fixed point for G. □

Lemma 2.20. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{f}_1$ in equation (2.8), then $L_1(g)$ is a point lying in the line of fixed points.*

Proof. Consider the sequence $\frac{1}{n}\mathbf{f}_1^n$, $n \in \mathbb{Z}$. The orbits of points $\mathbf{x} \in \mathbf{P}^2_{\mathbb{C}} - L_0(\mathbf{f}_1)$ (or $\mathbf{x}$ such that $y = 0$) is formed by the points:

$$\frac{1}{n}\mathbf{f}_1^n(\mathbf{x}) = \begin{bmatrix} \frac{1}{n}x + y \\ \frac{1}{n}y \\ \frac{1}{n}z \end{bmatrix} \tag{2.10}$$

which converges to the point $[\mathbf{e}_1]$ when n tends to infinity. □

Lemma 2.21. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{f}_1$ in equation (2.8), then $L_2(g)$ is a point lying in the line of fixed points.*

Proof. Let K be a compact subset contained in $\mathbf{P}^2_{\mathbb{C}} - L_0(\mathbf{f}_1) \cup L_1(\mathbf{f}_1)$. That is $K \subset \overleftrightarrow{\mathbf{e_1 e_3}}$. The sequence $1/n\mathbf{f}_1$ converges to the transformation M_3 (equation (2.5)) for compact subsets of $\mathbf{P}^2_{\mathbb{C}} - \ker(M_3)$, according to Proposition 1.7. So for points in K, $(1/n)\mathbf{f}_1(\mathbf{x})$ converges to $\mathbf{e}_1$, but $\ker(M_3) = \overleftrightarrow{\mathbf{e_1 e_3}}$ and $K \subset \mathbf{P}^2_{\mathbb{C}} - \overleftrightarrow{\mathbf{e_1 e_3}}$. □

After Lemmas 2.19, 2.20 and 2.21, we have the next proposition.

Proposition 2.22. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{f}_1$ in equation (2.8), then $\Lambda_K(g)$ consist of a line of fixed points.*

Kulkarni limit set of an unipotent type II element

Lemma 2.23. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{f}_2$ in equation (2.8), then $L_0(g)$ is a fixed point.*

Proof. Suppose that g is conjugate to an element as $\mathbf{f}_2$. The general term of an element in $G = \langle \mathbf{f}_2 \rangle$ is either

$$\begin{bmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & -n & \frac{n(n+1)}{2} \\ 0 & 1 & -n \\ 0 & 0 & 1 \end{bmatrix}.$$

The equations that a point $\mathbf{x} \in \mathbf{P}^2_{\mathbb{C}}$ should satisfy in order to have infinite isotropy group are one of the following:

$$\begin{bmatrix} x + ny + \frac{n(n-1)}{2}z \\ y + nz \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x - ny + \frac{n(n+1)}{2}z \\ y - nz \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The equality in each case is satisfied in both cases, for each $n \in \mathbb{N}$, only by the point $\mathbf{e_1}$. ☐

Lemma 2.24. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{f}_2$ in equation (2.8), then $L_1(g)$ is a fixed point.*

Proof. To calculate the set $L_1(g)$, consider the sequences $\frac{2}{n(n-1)}\mathbf{f}_2^n$ and $\frac{2}{n(n+1)}\mathbf{f}_2^{-n}$. Both sequences converge to

$$M_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

when n tends to infinity. So for points $\mathbf{x} \in \mathbf{P}^2_{\mathbb{C}} - \overleftrightarrow{\mathbf{e_1 e_2}}$, the cluster points of the orbits is $\mathbf{e_1}$. For points in the line $\overleftrightarrow{\mathbf{e_1 e_2}}$, the transformations behaves as a classical parabolic transformation and the accumulation point is also $\mathbf{e_1}$. ☐

Lemma 2.25. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{f}_2$ in equation (2.8), then $L_2(g)$ is a $g-$invariant complex line where g acts as a classical parabolic transformation.*

Proof. Consider a line in $\mathbf{P}^2_{\mathbb{C}}$ passing trough $p = [p_1 : p_2 : 0] \in \overleftrightarrow{\mathbf{e_1 e_2}}$ and $\mathbf{e_3}$. This line has equation $p_2 x - p_1 y = 0$ and in $(\mathbf{P}^2_{\mathbb{C}})^*$ corresponds to the point $\ell = [p_2 : -p_1 : 0]$. The action of the group in the space of lines of the complex projective space is as in (1.7), and $\frac{2}{n(n-1)}\mathbf{f}_2^n \cdot \ell = \left[\frac{2}{n(n-1)}p_2 : \frac{-2}{n-1}p_2 - \frac{2}{n(n-1)}p_1 : p_2 + \frac{2}{n-1}p_1\right]$, this sequence of elements in $(\mathbf{P}^2_{\mathbb{C}})^*$ corresponding to lines of $\mathbf{P}^2_{\mathbb{C}}$ converge to $[0 : 0 : 1]$ when n tends to infinity, that is the line $\overleftrightarrow{\mathbf{e_1 e_2}}$, therefore $\overleftrightarrow{\mathbf{e_1 e_2}} \subset L_2(\mathbf{f}_2)$.

To prove the other contention, let $C = \overleftrightarrow{\mathbf{e_1 e_2}}$, and a compact subset $K \subset \mathbf{P}^2_{\mathbb{C}} - C$. As we have seen, all points outside the line $\overleftrightarrow{\mathbf{e_1 e_2}}$ converge to $\mathbf{e_1} = L_0(f_2) \cup L_1(f_2)$. By Lemma 1.17, $L_2(f_2) \subset C$. ☐

With Lemmas 2.23, 2.24 and 2.25, we have the next proposition.

Proposition 2.26. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{f}_2$ in equation (2.8), then $\Lambda_K(g)$ consist of a $g-$invariant complex line where g acts as a classical parabolic transformation.*

Kulkarni limit set of an ellipto-parabolic element

Lemma 2.27. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{f}_3$ in equation (2.8), then $L_0(g)$ is the set of three fixed points of g.*

Proof. Suppose that g is conjugate to an element as $\mathbf{f}_3$. The general term of an element in $G = \langle \mathbf{f}_3 \rangle$, for $n \in \mathbb{Z}$ is

$$\begin{bmatrix} e^{2\pi itn} & ne^{2\pi it(n-1)} & 0 \\ 0 & e^{2\pi itn} & 0 \\ 0 & 0 & e^{-4\pi itn} \end{bmatrix}$$

The equation that should be satisfied for a point $\mathbf{x} \in \mathbf{P}^2_{\mathbb{C}}$ to have infinite isotropy group is

$$\begin{bmatrix} e^{2\pi itn}x + ne^{2\pi it(n-1)}y \\ e^{2\pi itn}y \\ e^{-4\pi itn}z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Suppose $t \in \mathbb{Q}$, $t = r/s$ for $r, s \in \mathbb{N}$, then for all n multiple of q, $n = sm$, the transformation

$$\mathbf{f}_3^n = \mathbf{f}_3^{sm} = \begin{bmatrix} e^{2\pi irm} & sme^{2\pi ir/s(sm-1)} & 0 \\ 0 & e^{2\pi irm} & 0 \\ 0 & 0 & e^{-4\pi irm} \end{bmatrix},$$

and every point in the line $\overleftrightarrow{\mathbf{e_1 e_3}}$ is a fixed point of infinitely many elements in G.

If $t \in \mathbb{R} - \mathbb{Q}$, then the only points with infinite isotropy group are $\mathbf{e_1}$ and $\mathbf{e_3}$. □

Lemma 2.28. *If $g \in \mathrm{PSL}(3, \mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3, \mathbb{C})$ as $\mathbf{f}_3$ in equation* (2.8)*, then $L_1(g)$ is the set of three fixed points of g.*

Proof. To calculate the set $L_1(g)$, consider the sequence $\frac{1}{n}\mathbf{f}_3^n$ for $n \in \mathbb{Z}$. In any case, n positive or negative, the elements evaluated in points $\mathbf{x} \in \mathbf{P}^2_{\mathbb{C}}$ are:

$$\begin{bmatrix} \frac{1}{n}e^{2\pi itn} & e^{2\pi it(n-1)} & 0 \\ 0 & \frac{1}{n}e^{2\pi itn} & 0 \\ 0 & 0 & \frac{1}{n}e^{-4\pi itn} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{n}e^{2\pi itn}x + e^{2\pi it(n-1)}y \\ \frac{1}{n}e^{2\pi itn}y \\ \frac{1}{n}e^{-4\pi itn}z \end{bmatrix},$$

and when n tends to infinity, the orbits converges to $[e^{2\pi it(n-1)}y : 0 : 0]$.

If $t \in \mathbb{Q}$, we have to check what happens to $\mathbf{x} \in \mathbf{P}^2_{\mathbb{C}} - L_0(\mathbf{f}_3) = \mathbf{P}^2_{\mathbb{C}} - \overleftrightarrow{\mathbf{e_1 e_3}}$. The point $\mathbf{e_1}$ is an accumulation point of the sequence $\frac{1}{n}\mathbf{f}_3^n$, whenever $y \neq 0$.

If $t \in \mathbb{R} - \mathbb{Q}$, and if $y \neq 0$, the sequence $(\frac{1}{n}e^{2\pi itn}x + e^{2\pi it(n-1)}y, \frac{1}{n}e^{2\pi itn}y, \frac{1}{n}e^{-4\pi itn}z)^T$ converges to $\mathbf{e_1}$. But when $y = 0$, $\mathbf{f_3}$ is like an elliptic transformation on $\overleftrightarrow{\mathbf{e_1 e_3}}$, and there is a subsequence (n_k) such that $e^{2\pi itn_k} \to 1$, then the points in $\overleftrightarrow{\mathbf{e_1 e_3}}$ are limit points of orbits. □

Lemma 2.29. *If $g \in \mathrm{PSL}(3, \mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3, \mathbb{C})$ as $\mathbf{f}_3$ in equation* (2.8)*, then $L_2(g)$ is the line formed by the two fixed points of g.*

Proof. Consider a line in $\mathbf{P}^2_{\mathbb{C}}$ passing trough $p = [p_1 : p_2 : 0] \in \overleftrightarrow{\mathbf{e_1 e_2}}$ and $\mathbf{e_3}$. This line has equation $p_2 x - p_1 y = 0$ and in $(\mathbf{P}^2_{\mathbb{C}})^*$ corresponds to the point $\ell = [p_2 : p_1 : 0]$. The action of the group in the space of lines of the complex projective space is as in (1.7), and

$$\frac{1}{n}\mathbf{f}_3^n \cdot \ell = \big[\frac{1}{n}e^{-2\pi itn}p_2 : -e^{-2\pi it(n-1)}p_2 - \frac{1}{n}e^{-2\pi itn}p_1 : 0\big].$$

This sequence of elements in $(\mathbf{P}^2_{\mathbb{C}})^*$ corresponding to lines of $\mathbf{P}^2_{\mathbb{C}}$ converge to $[0:1:0]$ when n tends to infinity, that is the line $\overleftrightarrow{\mathbf{e_1 e_3}}$, therefore $\overleftrightarrow{\mathbf{e_1 e_3}} \subset L_2(\mathbf{f}_2)$.

To prove the other contention, let $C = \overleftrightarrow{\mathbf{e_1 e_3}}$, and a compact subset $K \subset \mathbf{P}^2_{\mathbb{C}} - C$. As we have seen, all points outside the line $\overleftrightarrow{\mathbf{e_1 e_3}}$ converge to $\mathbf{e_1} \in \overleftrightarrow{\mathbf{e_1 e_3}} = L_0(\mathbf{f}_3) \cup L_1(\mathbf{f}_3)$. By Lemma 1.17, $L_2(\mathbf{f}_3) \subset C$. □

With Lemmas 2.27, 2.28 and 2.29, we have the next proposition.

Proposition 2.30. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{f}_3$ in equation (2.8), then $\Lambda_K(g)$ consist of a $g-$invariant complex line where g acts as a classical elliptic transformation.*

2.3 Limit set of Elliptic Transformations

In this subsection we find the Kulkarni limit set for elliptic elements in $\mathrm{PSL}(3,\mathbb{C})$.

Definition 2.31. A transformation $g \in \mathrm{PSL}(3,\mathbb{C})$ is called *elliptic* if it preserves each one of the spheres $T(r)$, for every $r \in \mathbb{R}^+$ as in Remark 1.10. That is, if and only if there exists $h \in \mathrm{PSL}(3,\mathbb{C})$ such that $h^{-1}gh(T(r)) = T(r)$ for every $r \in \mathbb{R}^+$.

An element $g \in \mathrm{PSL}(3,\mathbb{C})$ is elliptic if and only if g has a lift $\mathbf{g} \in \mathrm{SL}(3,\mathbb{C})$ such that $\mathbf{g}$ is diagonalizable and every eigenvalue is a unitary complex number, [28, Corollary 4.4]. That is, the lift of the elliptic transformation is like the following matrix:

$$\mathbf{h} = \begin{bmatrix} e^{2\pi i\alpha} & 0 & 0 \\ 0 & e^{2\pi i\beta} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}. \tag{2.11}$$

Proposition 2.32. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{h}$ in equation (2.11), with $\alpha/\beta \in \mathbb{Q}$, then $L_0(g) = L_1(g) = L_2(g) = \Lambda_K(g) = \varnothing$.*

Proof. As α/β is rational, $\mathbf{h}$ has finite order, so any point in $\mathbf{P}^2_{\mathbb{C}}$ can have an infinite isotropy group, neither any point can be approximated by infinitely many different images of a point, nor infinitely many different images of a compact subset. Therefore, the Kulkarni limit set of g is empty. □

Proposition 2.33. *If $g \in \mathrm{PSL}(3,\mathbb{C})$ is conjugate to a transformation with a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{h}$ in equation (2.11), with $\alpha/\beta \in \mathbb{R} - \mathbb{Q}$, then $L_0(g) = \{\mathbf{x} \in \mathbf{P}^2_{\mathbb{C}} : \mathbf{x} \text{ is fixed point for } g\}$, $L_1(g) = \mathbf{P}^2_{\mathbb{C}}$ and $L_2(g) = \varnothing$.*

Proof. In this case, for every $n \in \mathbb{Z}$, $\mathbf{h}^n$ has as fixed point $\mathbf{e_3}$, so $\mathbf{e_3}$ has an infinite isotropy group. But as $\alpha/\beta \in \mathbb{R} - \mathbb{Q}$, we can find a subsequence (n_k) such that $e^{2\pi i\alpha n_k}$ and $e^{2\pi i\beta n_k}$ converge to 1, and

$$\mathbf{h}^{n_k}(\mathbf{x}) = \begin{pmatrix} e^{2\pi i\alpha n_k} & 0 & 0 \\ 0 & e^{2\pi i\beta n_k} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} e^{2\pi i\alpha n_k}x \\ e^{2\pi i\beta n_k}y \\ z \end{bmatrix} \to \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{2.12}$$

if k tends to infinity. Therefore $L_1(g) = \mathbf{P}^2_{\mathbb{C}}$ and consequently, $L_2(g) = \varnothing$. Finally, the Kulkarni limit set of g is $\Lambda_{\mathrm{K}}(g) = \mathbf{P}^2_{\mathbb{C}}$. □

Observe that if a subgroup $G \subset \mathrm{PSL}(3,\mathbb{C})$ has an elliptic element of infinite order, then G is not a Kleinian group.

2.4 Relation between $L(G)$ and $\Lambda(G)$, with $G < \mathrm{PU}(2,1)$

Recall that the sequence $(g_m)_{m\in\mathbb{N}} \subset \mathrm{PSL}(3,\mathbb{C})$ converges to $g \in \mathrm{PS}(3,\mathbb{C})$ in the sense of pseudo-projective transformations if $g_m \longrightarrow g$, when $m \to \infty$ uniformly on compact subsets of $\mathbf{P}^2_{\mathbb{C}} - \ker(g)$.

We study the Lemma 4.2 of [9].

Lemma 2.34. *Let G be a subgroup of $\mathrm{PU}(2,1)$ a discrete subgroup, $(g_m)_{m\in\mathbb{N}} \subset G$ a sequence of distinct elements and $g \in PS(3,\mathbb{C}) - \mathrm{PSL}(3,\mathbb{C})$ such that $(g_m)_{m\in\mathbb{N}}$ converges to g in the sense of pseudo-projective transformations. Then:*

(i) The image $\mathrm{Im}(g)$ is a point in $\partial\mathbf{H}^2_{\mathbb{C}}$.

(ii) $\ker(g)^{\perp}$ is a point in $\partial\mathbf{H}^2_{\mathbb{C}}$.

Proof. For the proof of (i), we use Proposition 3.2 in [27], where it is asserted that given a sequence of distinct elements of a discrete subgroup $G \subset \mathrm{PU}(2,1)$, there is a subsequence and distinct elements $\mathbf{x}, \mathbf{y} \in \mathbf{L}(G)$ such that $g_m(\mathbf{z}) \to \mathbf{x}$ uniformly in compact sets of $\overline{\mathbf{H}^2_{\mathbb{C}}} - \{\mathbf{y}\}$.

As $\mathbf{H}^2_{\mathbb{C}}$ is an open subset of $\mathbf{P}^2_{\mathbb{C}}$, g is a holomorphic transformation, then $g(\mathbf{H}^2_{\mathbb{C}})$ is an open subset in the image of g. On the other hand, $g(\mathbf{H}^2_{\mathbb{C}}) = p$, is a closed subset of $\mathrm{Im}(g)$. The only set which is closed and open at the same time is either the total set or the empty set, follows that $\mathrm{Im}(g) = \{p\}$.

Having said this, $\mathrm{Im}(g)$ is a point in the boundary of the hyperbolic space. To prove the second part of the Lemma, recall that the sum of the dimension of the kernel of a transformation on a vector space plus the dimension of its image equals the dimension of the ambient vector space. If we consider that $g\colon \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ is a linear transformation and that $\dim_{\mathbb{C}}(\mathrm{Im}(g)) = 1$, we have $\dim_{\mathbb{C}}(\ker(g)) = 2$, and this implies that $[\ker(g)]$ is a projective line.

Then, the claims are $\mathbf{H}^2_{\mathbb{C}} \cap \ker(g) = \varnothing$ and $\partial\mathbf{H}^2_{\mathbb{C}} \cap \ker(g) \neq \varnothing$. Both claims are proven by contradiction. In the first case, let $x \in \mathbf{H}^2_{\mathbb{C}} \cap \ker(g)$, as the transformation g is not identically zero, we can take $\mathbf{x} \notin \mathrm{Im}(g)$. By Proposition 3.3 in [9], for (g_m), g, $\mathbf{x}$ and $\mathbf{H}^2_{\mathbb{C}}$, follows that there is line contained in $\mathbf{H}^2_{\mathbb{C}}$, which is a contradiction. Therefore $\mathbf{H}^2_{\mathbb{C}} \cap \ker(g) = \varnothing$.

For the second claim, assume that $\partial\mathbf{H}^2_{\mathbb{C}} \cap \ker(g) = \varnothing$. By (i) of this Lemma, it exists $p \in \partial\mathbf{H}^2_{\mathbb{C}}$ such that g_m converges uniformly to p, which is a constant transformation. Let $\mathbf{x}$ be an element in $\mathbf{H}^2_{\mathbb{C}}$ and U a neighborhood of p that satisfies $U \cap \mathbf{H}^2_{\mathbb{C}} \subset \mathbf{H}^2_{\mathbb{C}} - \{\mathbf{x}\}$. Then, there is $n_0 \in \mathbb{N}$ such that if $m > n_0$, $g_m(\mathbf{H}^2_{\mathbb{C}}) \subset U \cap \mathbf{H}^2_{\mathbb{C}} \subset \mathbf{H}^2_{\mathbb{C}} - \{\mathbf{x}\}$. But this is a contradiction, given that g_m is a homeomorphism of $\mathbf{H}^2_{\mathbb{C}}$. □

λ-Lemma

The next result is known as the λ-Lemma. We will use it in further arguments.

Lemma 2.35. *Let $g \in \mathrm{PU}(2,1)$ be a loxodromic element with fixed points $a, r \in \mathbf{P}^2_{\mathbb{C}}$; and let $\Omega \subset \mathbf{P}^2_{\mathbb{C}}$ be an open subset. Assume that $\langle g \rangle$ acts properly discontinuously in Ω. Then, $a^{\perp} \in \mathbf{P}^2_{\mathbb{C}} - \Omega$ or $r^{\perp} \in \mathbf{P}^2_{\mathbb{C}} - \Omega$.*

Theorem 2.36. *The Kulkarni limit set coincide with the perpendicular lines tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ in points of the Chen-Greenberg limit set.*

Proof. In [18], Kamiya shows that a non elementary discrete subgroup G, always has a loxodromic element g. By Lemma 2.35 we have that $a^{\perp}$ belongs to the Kulkarni limit set. As we saw in section 2.1, also $r^{\perp}$ belongs to $\Lambda_{\mathrm{K}}(G)$. Besides, $\Lambda_{\mathrm{K}}(G)$ is an invariant set and the action of G is transitive.

The fixed points of loxodromic elements are dense in the Chen-Greenberg limit set [10], then all the tangent lines to the ball in points of $\mathbf{L}(G)$ are, in fact in $\Lambda_{\mathrm{K}}(G)$. For transformations as $\mathbf{g}_3$ in (2.1), it happens that $\cup_{p \in \mathbf{L}(G)} \ell_p \subset \Lambda_{\mathrm{K}}(G)$.

To show the other contention, Lemma 1.17 is applied; then, if $C = \cup_{p \in \mathbf{L}(G)} \ell_p$, and we consider a compact set outside C, by Lemma 2.34, this compact set accumulates on some point of $\partial\mathbf{H}^2_{\mathbb{C}}$. That is $\Lambda_{\mathrm{K}}(G) \subset \bigcup_{p \in \mathbf{L}(G)} \ell_p$, and therefore

$$\Lambda_{\mathrm{K}}(G) = \bigcup_{p \in \mathbf{L}(G)} \ell_p. \tag{2.13}$$

□

Chapter 3

A limit set for the action of $G \subset \mathrm{PSL}(3, \mathbb{C})$ in $(\mathbf{P}^2_{\mathbb{C}})^*$

In this Chapter we give another definition of limit set for a subgroup of $\mathrm{PSL}(3, \mathbb{C})$ acting on $(\mathbf{P}^2_{\mathbb{C}})^*$; this new definition generalizes the one given in [11] in the sense that works even for subgroups that do not have proximal elements, which in $\mathrm{PSL}(3, \mathbb{C})$, according to the classification of elements, are only the strongly loxodromic elements of the group.

We use this definition to relate the Kulkarni limit set of a group G acting on $\mathbf{P}^2_{\mathbb{C}}$ and this new limit set for the same group G acting on $(\mathbf{P}^2_{\mathbb{C}})^*$.

In what follows we use $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$. The definitions and results where $\mathbb{F}$ is used are valid in both fields, however, the result is valid in $\mathbb{R}$.

3.1 The Conze and Guivarc'h limit set in $(\mathbf{P}^2_{\mathbb{C}})^*$

Definition 3.1. A matrix $A \in \mathrm{GL}(3, \mathbb{F})$ is said to be *proximal* if it has one and only one eigenvalue with modulus larger than the modulus of all the other eigenvalues. We will denote that eigenvalue by λ_A. For a proximal matrix A, the vector $v_A \in \mathbb{F}^3$ will denote the corresponding eigenvector to the eigenvalue λ_A, and is called the *dominant eigenvector of A*. A *proximal transformation* will be a transformation which has a proximal matrix as a lift.

Proposition 3.2. *Let A be a proximal transformation, being λ_A the eigenvalue of A with greater norm than the other eigenvalues. We define*

$$H_A^- = \{\omega \in \mathbb{C}^3 : \lambda_A^{-n} A^n \omega \to 0 \text{ as } n \to \infty\}.$$

If S be the pseudo-projective limit of the positive powers of A, then

$$\ker S = [H_A^-],$$

where $[H_A^-]$ denotes the projection of the vector subspace H_A^-.

Proof. There are three possibilities for A to be a proximal transformation, for example, if A is either as $\mathbf{g_1}, \mathbf{g_3}$ or $\mathbf{g_4}$ in equation (2.1) (with $|\lambda| < 1$, when corresponds).

- If A is $\mathbf{g_1} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}$, with $|\lambda| < 1$, then

$$(\lambda^{2n}\mathbf{g_1}^n) = \begin{pmatrix} \lambda^{3n} & 0 & 0 \\ 0 & \lambda^{3n} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \overrightarrow{n \to \infty} \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The kernel of S, is $\ker S = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{C} \right\}$. Now,

$$H_A^- = \left\{ \omega = \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \lambda^{2n} A^n \omega \to 0 \right\}.$$

$$\lambda^{2n} A^n \omega = \begin{bmatrix} \lambda^{3n} x \\ \lambda^{3n} y \\ z \end{bmatrix}.$$

The last sequence converges to $(0,0,0)$ if and only of $z = 0$. It is clear that

$$\ker S \subset H_A^-.$$

Because dim$(\ker S) =$ dim$(H_A^-) = 2$, we conclude that $\ker S = H_A^-$.

- If A is $\mathbf{g_3} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}$, then

$$(\lambda^{2n}\mathbf{g_3}^n) = \begin{pmatrix} \lambda^{3n} & n\lambda^{3n-1} & 0 \\ 0 & \lambda^{3n} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \overrightarrow{n \to \infty} \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The kernel of S, is $\ker S = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{C} \right\}$. Now,

$$H_A^- = \left\{ \omega = \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \lambda^{-2n} A^n \omega \to 0 \right\}.$$

$$\lambda^{-2n}A^n\omega = \begin{bmatrix} \lambda^{3n}x + n\lambda^{3n-1}y \\ \lambda^{3n}y \\ z \end{bmatrix}.$$

The last sequence converges to $(0,0,0)$ if and only of $z = 0$. It is clear that

$$\ker S \subset H_A^-.$$

We conclude that $\ker S = H_A^-$ because dim$(\ker S) =$ dim$(H_A^-) = 2$.

- If A is $\mathbf{g_4} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, then

$$(\lambda_3^{-n}\mathbf{g_4}^n) = \begin{pmatrix} (\lambda_1/\lambda_3)^n & 0 & 0 \\ 0 & (\lambda_2/\lambda_3)^n & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{n \to \infty} S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The kernel of S, is $\ker S = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{C} \right\}$. Now,

$$H_A^- = \left\{ \omega = \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \lambda_3^{-n}A^n\omega \to 0 \right\}.$$

$$\lambda_3^{-n}A^n\omega = \lambda_3^{-n} \begin{bmatrix} \lambda_1^n x \\ \lambda_2^n y \\ \lambda_3^n z \end{bmatrix} = \begin{bmatrix} \left(\frac{\lambda_1}{\lambda_3}\right)^n x \\ \left(\frac{\lambda_2}{\lambda_3}\right)^n y \\ z \end{bmatrix}.$$

It is clear that

$$\ker S \subset H_A^-.$$

And because dim$(\ker S) =$ dim$(H_A^-) = 2$, we conclude that $\ker S = H_A^-$.

There are other possibilities for A to be a proximal transformation, for example, if A is either as $\mathbf{g_1}$ or $\mathbf{g_3}$ in equation (2.1) with $|\lambda| < 1$. It is not hard to check the Proposition 3.2 is still true for different types of proximal elements. □

According to the classification of transformations of PSL$(3,\mathbb{C})$ given in [28], can be deduced that a proximal transformation is loxodromic, but the converse is not true: For example, if g has a lift conjugate to an element as $\mathbf{g_2}$ in equation (2.1) such that λ has norm greater than one, then g does not have a unique eigenvector greater than all the others.

Moreover, every proximal element has an attracting fixed point in $\mathbf{P}^2_{\mathbb{F}}$, then we have the following definition.

Definition 3.3 (Conze and Guivarc'h limit set)**.** Let G be a subgroup of $\mathrm{GL}(3, \mathbb{F})$ and consider its action on $\mathbf{P}^2_{\mathbb{F}}$. The *Conze and Guivarc'h limit set,* denoted by $L(G)$, is the closure of the subset of $\mathbf{P}^2_{\mathbb{F}}$ consisting of all the attracting fixed points of proximal elements of G.

We emphasize that $L(G)$ is always a $G-$invariant subset of $\mathbf{P}^2_{\mathbb{F}}$ and when the $G-$action is irreducible (i.e. it does not exist any proper subspace of $\mathbf{P}^2_{\mathbb{F}}$ invariant under the action of a subgroup of finite index in G) and when G has a proximal element, then $L(G)$ is a minimal subset for this $G-$action.

Example 3.4. Consider the strongly loxodromic transformation $\mathbf{g_4}$ in equation (2.1) acting on $\mathbf{P}^2_{\mathbb{C}}$. The Kulkarni limit set is $\Lambda_{\mathrm{K}}(\mathbf{g_4}) = \overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3}$, (Proposition 2.17). It is not hard to check that $L(\mathbf{g_4})$ is equal to $\{\mathbf{e_1}, \mathbf{e_3}\}$.

Observe that $L(\mathbf{g_4}) \subset L_0(\mathbf{g_4}) \subset \Lambda_{\mathrm{K}}(\mathbf{g_4})$. The action of G on $\mathbf{P}^2_{\mathbb{C}} - L(\mathbf{g_4})$ is not properly discontinuous, while the action of G on $\mathbf{P}^2_{\mathbb{C}} - \Lambda_{\mathrm{K}}(\mathbf{g_4})$ is.

3.1.1 Two sets of lines

In the article [3], the authors introduced the concept of *effective lines* of a discrete group G. First, if G' is the set $\{S$ pseudo-projective map of $\mathbf{P}^2_{\mathbb{C}} : S$ is a cluster point of G$\}$, then $\mathcal{E}(G) = \{\ell \subset \mathbf{P}^2_{\mathbb{C}} : \ell = \ker S,$ for some $S \in G'\} \subset (\mathbf{P}^2_{\mathbb{C}})^*$ is a subset of complex lines. The authors of that article prove in Proposition 4.2:

Proposition 3.5. *If $G \subset \mathrm{PSL}(3, \mathbb{C})$ is a discrete subgroup then $\mathcal{E}(G)$ is a closed subset of $(\mathbf{P}^2_{\mathbb{C}})^*$.*

As a corollary of the previous Proposition, the authors prove that under the same hypothesis, the union of lines in the set $\mathcal{E}(G)$, $\cup_{\ell\in\mathcal{E}}\ell$ is a closed set of $\mathbf{P}^2_{\mathbb{C}}$.

In another work [6], the authors introduce the set $E(G)$ as the subset of $(\mathbf{P}^2_{\mathbb{C}})^*$ consisting of all the complex lines ℓ for which there exists an element $g \in G$ such that $\ell \subset \Lambda_{\mathrm{K}}(g)$. This set has a property that the authors prove in their work in Theorem 1.3, and that is:

Theorem 3.6. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ an infinite discrete subgroup without fixed points nor invariant complex lines.*

(a) $Eq(G) = \Omega(G)$, is the maximal open set on which G acts properly and discontinuously. Moreover, if $E(G)$ contains more than three complex lines then every connected component of $\Omega(G)$ is complete Kobayashi hyperbolic.

(b) The set

$$\Lambda_K(G) = \overline{\bigcup_{\ell\in E(G)} \ell} = \bigcup_{\ell\in \overline{E(G)}} \ell = \overline{\bigcup_{g\in G} \Lambda_K(g)}.$$

is path-connected

(c) If $E(G)$ contains more than three complex lines then $\overline{E(G)} \subset (\mathbf{P}^2_{\mathbb{C}})^*$ is a perfect set. Also, it is the minimal closed $G-$invariant subset of $(\mathbf{P}^2_{\mathbb{C}})^*$.

We prove the following proposition:

Proposition 3.7. *If G be a discrete subgroup of $\mathrm{PSL}(3,\mathbb{C})$, with at least three lines in general position in $E(G)$, then*

$$\overline{E(G)} = \mathcal{E}(G). \tag{3.1}$$

Proof. First we prove $E(G) \subset \mathcal{E}(G)$. Let ℓ be a line in $E(G)$, there is an element $g \in G$ such that $\ell \subset \Lambda_{\mathrm{K}}(g)$. Each line in the Kulkarni limit set is the kernel of the pseudo-projective transformation obtained as the limit of g^n or g^{-n}, with $n \in \mathbb{N}$, as can be concluded by [6, Lemma 3.2]. It follows that $\overline{E(G)} \subset \mathcal{E}(G)$ because $\mathcal{E}(G)$ is closed, 3.5.

Conversely, if $\ell \in \mathcal{E}(G)$, then $\ell = \ker S$ where $S = \lim_{n\to\infty} g_n$, for some sequence $(g_n) \subset G$. Take $\ell_0 \subset \Lambda_K(g_0)$ a line in $E(G)$ not passing through the point $\mathrm{Im}(S)$. By [6, Lemma 3.2(3)], the sequence $g_n^{-1} \cdot \ell_0$ converges to $\ker S = \ell$, where for each $n \in \mathbb{N}$, $g_n^{-1} \cdot \ell_0 \subset \Lambda_{\mathrm{K}}(g_n^{-1} g_0 g_n)$ is in $E(G)$. □

3.2 Who is the Kulkarni limit set in $(\mathbf{P}^2_{\mathbb{C}})^*$?

In this section we extend the Definition 3.1 to work with every type of elements in $\mathrm{PSL}(3,\mathbb{C})$. We propose the following definition.

Definition 3.8. Let us consider $G \subset \mathrm{PSL}(3,\mathbb{C})$ acting on $(\mathbf{P}^2_{\mathbb{C}})^*$. We say that $\mathbf{q} \in (\mathbf{P}^2_{\mathbb{C}})^*$ is a limit point of G if there exists an open subset $U \subset (\mathbf{P}^2_{\mathbb{C}})^*$ and there exists a sequence $\{g_n\} \subset G$, $g_n \neq g_m$ if $n \neq m$, such that for every $\mathbf{p} \in U$

$$\lim_{n\to\infty} g_n \cdot \mathbf{p} = \mathbf{q} \tag{3.2}$$

The set of limit points will be called the limit set, denoted by $\hat{L}(G)$.

The limit set $\hat{L}(G)$ has as subset the Conze and Guivarc'h limit set, this, because not every element in G is a proximal element. However, with the new definition, we consider all the elements in the group G to get $\hat{L}(G)$.

Example 3.9. If $g \in \mathrm{PSL}(3,\mathbb{C})$ is a strongly loxodromic element, then without loss of generality we can assume that g is induced by the matrix $\mathbf{g_4}$ in equation (2.1).

When we consider the element g acting on $(\mathbf{P}^2_{\mathbb{C}})^*$ we notice that the complex lines

$$\ell_1 = \{[x:y:z] \in \mathbf{P}^2_{\mathbb{C}} : x = 0\},$$

$$\ell_2 = \{[x:y:z] \in \mathbf{P}^2_{\mathbb{C}} : y = 0\}$$

$$\ell_3 = \{[x:y:z] \in \mathbf{P}^2_{\mathbb{C}} : z = 0\}$$

correspond to the eigenvectors of

$$(\mathbf{g}^{-1})^T = \begin{pmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_3} \end{pmatrix}.$$

Indeed, if $[A : B : C] \in (\mathbf{P}^2_{\mathbb{C}})^*$ represents a complex line in $\mathbf{P}^2_{\mathbb{C}}$, then

$$(A\,B\,C)(\mathbf{g}^{-1})^T = (A\,B\,C)\begin{pmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_3} \end{pmatrix} = \left(\frac{A}{\lambda_1}\,\frac{B}{\lambda_2}\,\frac{C}{\lambda_3}\right).$$

and this is equal to $\lambda(A, B, C)$ if and only if two of the entries of the vector are zero. In which case, corresponds to one of the lines ℓ_1, ℓ_2 or ℓ_3. Hence, ℓ_1, ℓ_2, ℓ_3 are the fixed points for the action of g on $(\mathbf{P}^2_{\mathbb{C}})^*$.

In fact, ℓ_1 is an attracting fixed point, because for every $\eta \in U_1 = (\mathbf{P}^2_{\mathbb{C}})^* \backslash \overleftrightarrow{\ell_2, \ell_3}$, $g^n \cdot \eta \to \ell_1$ as $n \to \infty$; and ℓ_3 is a repelling fixed point because for every $\eta \in U_3 = (\mathbf{P}^2_{\mathbb{C}})^* \backslash \overleftrightarrow{\ell_1, \ell_2}$, $g^{-n} \cdot \eta \to \ell_1$ as $n \to \infty$. Where $\overleftrightarrow{\ell_j, \ell_k}$ denotes the projective line passing through the points $\ell_j, \ell_k \in (\mathbf{P}^2_{\mathbb{C}})^*$.

Therefore, ℓ_1 and ℓ_3 are the only limit points, according to Definition 3.8, for the cyclic group generated by g. For the element ℓ_1 the open subset needed is U_1 and the sequence is g^n, meanwhile for the element ℓ_3, the open subset is U_3 and the sequence is g^{-n}, the Kulkarni limit set is:

$$\Lambda_K(g) = \ell_1 \cup \ell_3 = \bigcup_{\ell \in \hat{L}(g)} \ell.$$

In what follows, we can know the limit set $\hat{L}(g)$ for the cyclic groups generated by the different type of elements in $\mathrm{PSL}(3, \mathbb{C})$.

Lemma 3.10. *If $G = \langle g \rangle \subset \mathrm{PSL}(3, \mathbb{C})$ is a cyclic subgroup then:*

i) $\Lambda_K(G) = \bigcup_{\ell \in \hat{L}(G)} \ell$ whenever g is neither a complex homothety nor a screw.

ii) If g is either a complex homothety or a screw then $\Lambda_K(G) \supsetneq \bigcup_{\ell \in \hat{L}(G)} \ell$.

Proof. It is enough to verify the Lemma for the elements of different type presented in Chapter 2.

i) • If $g \in \mathrm{PSL}(3, \mathbb{C})$ is a loxoparabolic transformation, g has a lift in $\mathrm{SL}(3, \mathbb{C})$ whose Jordan canonical form is given by the matrix $\mathbf{g_3}$ in the equation (2.1), g acts on $(\mathbf{P}^2_{\mathbb{C}})^*$ as we said in equation

$$(\alpha', \beta', \gamma') = (\alpha, \beta, \gamma)\mathbf{g}^{-1}. \tag{3.3}$$

.

For any $[A : B : C]$ in the open subset U_1 of $(\mathbf{P}^2_{\mathbb{C}})^*$, where

$$U_1 = \{[A : B : C] \in (\mathbf{P}^2_{\mathbb{C}})^* : A \neq 0\}, \tag{3.4}$$

and for the sequence $\{g^n\}_{n \in \mathbb{N}}$, the sequence of lines in $(\mathbf{P}^2_{\mathbb{C}})^*$ given by

$$\left(\mathbf{g}^{-n}\right)^T \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{pmatrix} \lambda^{-n} & 0 & 0 \\ -n\lambda^{-(n+1)} & \lambda^{-n} & 0 \\ 0 & 0 & \lambda^{2n} \end{pmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} \tag{3.5}$$

is projectively the same as the sequence

$$\frac{\lambda^{n+1}}{n}\left(\mathbf{g}^{-n}\right)^T \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{pmatrix} \frac{\lambda}{n}A \\ -A+\frac{\lambda}{n}B \\ \frac{\lambda^{3n+1}}{n}C \end{pmatrix}, \tag{3.6}$$

and this last sequence converges to the line given by $[0:1:0]$.

Now, take the action of g^{-1} in $(\mathbf{P}^2_{\mathbb{C}})^*$ and let U_3 be the open subset of $(\mathbf{P}^2_{\mathbb{C}})^*$ defined by $\{[A:B:C] \in (\mathbf{P}^2_{\mathbb{C}})^* : C \neq 0\}$. The sequence of lines in $(\mathbf{P}^2_{\mathbb{C}})^*$ given by:

$$\left(\mathbf{g}^{n}\right)^T \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{pmatrix} \lambda^n & 0 & 0 \\ n\lambda^{n-1} & \lambda^n & 0 \\ 0 & 0 & \lambda^{-2n} \end{pmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} \tag{3.7}$$

is projectively equivalent to the sequence:

$$\lambda^{2n}\left(\mathbf{g}^{n}\right)^T \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \lambda^n A \\ n\lambda^{n-1}A+\lambda^{3n}B \\ C \end{bmatrix}, \tag{3.8}$$

converges to the line $[0:0:1]$, whenever $[A:B:C]$ is in U_3.

So $\hat{L}(G) = \{\ell_2, \ell_3\}$, therefore, the lemma is true for loxoparabolic elements.

- If g is conjugate to an element as $\mathbf{g}_4$, the limit set is calculated in Example 3.9.

We have to explore now the elements conjugate to some parabolic element.

- When g is conjugate to an element as $\mathbf{f}_1$, a type I unipotent element, for any $[A:B:C] \in U_1 \subset (\mathbf{P}^2_{\mathbb{C}})^*$, and for any of the sequences (g^n) or (g^{-n}), the orbits of the line defined by $[A:B:C] \in (\mathbf{P}^2_{\mathbb{C}})^*$ given in the first case by

$$(\mathbf{f_1}^{-n})^T \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -n & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

is projectively equivalent to the sequence:

$$\frac{1}{n}(\mathbf{f_1}^{-n})^T \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{A}{n} \\ \frac{B}{n} - A \\ \frac{C}{n} \end{bmatrix},$$

and the sequence converges to the line given by $[0:1:0] \in (\mathbf{P}^2_{\mathbb{C}})^*$ when n tends to infinity, that is $\overleftrightarrow{e_1 e_3} \subset \mathbf{P}^2_{\mathbb{C}}$.

- If g is conjugate to an element as $\mathbf{f}_2$, a type II unipotent element, for any $[A:B:C] \in U_1 \subset (\mathbf{P}^2_{\mathbb{C}})^*$, and for any of the sequences (g^n) or (g^{-n}), the orbits of the line defined by $[A:B:C] \in (\mathbf{P}^2_{\mathbb{C}})^*$ given in the first case by

$$(\mathbf{f_2}^{-n})^T \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ \frac{n(n-1)}{2} & n & 1 \end{pmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

is projectively equivalent to the sequence:

$$\frac{2}{n(n-1)}(\mathbf{f_2}^{-n})^T \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{2A}{n(n-1)} \\ \frac{2A}{n-1} + \frac{2B}{n(n-1)} \\ A + \frac{2B}{n-1} + \frac{2C}{n(n-1)} \end{bmatrix},$$

and when n tends to infinity the sequence converges to $[0:0:1] \in (\mathbf{P}^2_{\mathbb{C}})^*$, corresponding to the line $\overleftrightarrow{e_1 e_2} \subset \mathbf{P}^2_{\mathbb{C}}$.

- For g conjugate to $\mathbf{f_3}$, consider $[A:B:C] \in U_1$. Then for any sequence: either (g^n) or (g^{-n}), the sequences of lines in $(\mathbf{P}^2_{\mathbb{C}})^*$ given in the case of (g^n) by:

$$(\mathbf{f_3}^{-n})^T \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{pmatrix} e^{-2\pi int} & 0 & 0 \\ -ne^{-2\pi i(n-1)t} & e^{-2\pi int} & 0 \\ 0 & 0 & e^{4\pi int} \end{pmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix},$$

which is projectively equivalent to

$$\frac{1}{n}(\mathbf{f_3}^{-n})^T \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{e^{-2\pi int}}{n} x \\ e^{-2\pi i(n-1)t} x + \frac{e^{-2\pi int}}{n} y \\ \frac{e^{4\pi int}}{n} z \end{bmatrix}.$$

When n tends to infinity the sequence converges to $[0:1:0] \in (\mathbf{P}^2_{\mathbb{C}})^*$, corresponding to the line $\overleftrightarrow{e_1 e_3} \subset \mathbf{P}^2_{\mathbb{C}}$.

- Finally, we check the elliptic element. If g is conjugate to an element as $\mathbf{h}$ in equation (2.11). If $\mathbf{h}$ has finite order, that is $t \in \mathbb{Q}$, then the limit set is empty. On the contrary, if $\mathbf{h}$ has infinite order, then the limit set is $(\mathbf{P}^2_{\mathbb{C}})^*$.

ii) • If $g \in \mathrm{PSL}(3, \mathbb{C})$ is a complex homothety as in $\mathbf{g_1}$ of equation (2.1), for any line $\ell = [A : B : C] \in U_3 \subset (\mathbf{P}^2_{\mathbb{C}})^*$, the sequence

$$g^n \cdot \ell = (\mathbf{g_1}^{-n})^T \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \lambda^{-n} A \\ \lambda^{-n} B \\ \lambda^{2n} C \end{bmatrix},$$

this is projectively equivalent to the sequence

$$\lambda^{2n}(\mathbf{g_1}^{-n})^T \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{A}{\lambda^{3n}} \\ \frac{B}{\lambda^{3n}} \\ C \end{bmatrix},$$

which converges to $\ell_3 = [0 : 0 : 1]$ as $n \to \infty$, so $\hat{L}(g) = \{\ell_3\}$. It follows, from Proposition 2.5, that $\Lambda_K(g) = \overleftrightarrow{e_1, e_2} \cup \{\mathbf{e_3}\}$. Hence

$$\bigcup_{\ell \in \hat{L}(g)} \ell \subsetneq \Lambda_K(G).$$

- The case when $g \in \mathrm{PSL}(3, \mathbb{C})$ is a screw element, that is g is induced by a matrix of the form $\mathbf{g_2}$ in equation (2.1), is very similar to the previous case:
  For any line $\ell = [A : B : C] \in U_3 \subset (\mathbf{P}^2_{\mathbb{C}})^*$, the sequence

$$g^n \cdot \ell = (\mathbf{g_2}^{-n})^T \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \lambda^{-n} A \\ \mu^{-n} B \\ (\lambda\mu)^n C \end{bmatrix},$$

this is projectively equivalent to the sequence

$$(\lambda\mu)^n(\mathbf{g_2}^{-n})^T \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{A}{(\lambda^2\mu)^n} \\ \frac{B}{(\lambda\mu^2)^n} \\ C \end{bmatrix},$$

which converges to $\ell_3 = [0 : 0 : 1]$ as $n \to \infty$, so $\hat{L}(g) = \{\ell_3\}$. It follows, from Proposition 2.9, that $\Lambda_K(g) = \overleftrightarrow{e_1, e_2} \cup \{\mathbf{e_3}\}$. Hence

$$\bigcup_{\ell \in \hat{L}(g)} \ell \subsetneq \Lambda_K(G).$$

□

We summarize the previous proof in the following Table:

Table 3.1: Limit set for the action in $\mathbf{P}^2_{\mathbb{C}}$ and $(\mathbf{P}^2_{\mathbb{C}})^*$

Loxodromic elements				
g acting in $\mathbf{P}^2_{\mathbb{C}}$	$\Lambda_{\mathrm{K}}(g)$	g acting in $(\mathbf{P}^2_{\mathbb{C}})^*$	$\hat{L}(G)$	Open subset
Complex Homothety $\mathbf{g_1} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}$ $\lvert\lambda\rvert > 1$	$\{e_3\} \cup \overleftrightarrow{e_1 e_2}$	$g^* = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}$	$\{\ell_3\}$	U_3
Screw $\mathbf{g_2} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & (\lambda\mu)^{-1} \end{pmatrix}$ $\lambda \neq \mu$ $\lvert\lambda\rvert = \lvert\mu\rvert > 1$	$\{e_3\} \cup \overleftrightarrow{e_1 e_2}$	$g^* = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \mu^{-1} & 0 \\ 0 & 0 & \lambda\mu \end{pmatrix}$	$\{\ell_3\}$	U_3
Loxoparabolic $\mathbf{g_3} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}$ $\lvert\lambda\rvert > 1$	$\overleftrightarrow{e_1 e_2} \cup \overleftrightarrow{e_1 e_3}$	$g^{-1} = \begin{pmatrix} \lambda^{-1} & -\lambda^{-2} & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}$	$\{\ell_2, \ell_3\}$	U_1 and U_3 resp.
Strongly loxodromic $\mathbf{g_4} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ $\lvert\lambda_1\rvert < \lvert\lambda_2\rvert < \lvert\lambda_3\rvert$	$\overleftrightarrow{e_1 e_2} \cup \overleftrightarrow{e_2 e_3}$	$g^{-1} = \begin{pmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_3^{-1} \end{pmatrix}$	$\{\ell_1, \ell_3\}$	U_1 and U_3 resp.
Parabolic elements				
Type I.a $\mathbf{f_1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\overleftrightarrow{e_1 e_3}$	$g^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\{\ell_2\}$	U_1
Type II $\mathbf{f_2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\overleftrightarrow{e_1 e_2}$	$g^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$	$\{\ell_3\}$	U_1
$\mathbf{f_3} = \begin{pmatrix} e^{2\pi it} & 1 & 0 \\ 0 & e^{2\pi it} & 0 \\ 0 & 0 & e^{-4\pi it} \end{pmatrix}$ $e^{-2\pi it} \neq 1$	$\overleftrightarrow{e_1 e_3}$	$g^{-1} = \begin{pmatrix} e^{-2\pi it} & -e^{-4\pi it} & 0 \\ 0 & e^{-2\pi it} & 0 \\ 0 & 0 & e^{4\pi it} \end{pmatrix}$	$\{\ell_2\}$	U_1
Elliptic elements				
$\mathbf{h} = \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix}$ $\lvert g\rvert < \infty$	$\varnothing$	$g^{-1} = \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix}$	$\varnothing$	
$\mathbf{h} = \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix}$ $\lvert g\rvert = \infty$	$\mathbf{P}^2_{\mathbb{C}}$	$g^{-1} = \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix}$	$(\mathbf{P}^2_{\mathbb{C}})^*$	$(\mathbf{P}^2_{\mathbb{C}})^*$

If G is a discrete subgroup of $\mathrm{PSL}(3,\mathbb{C})$, we recall that $\mathcal{E}(G)$ denotes the set of complex lines, ℓ, for which there exists a sequence $(g_n) \subset G$ of distinct elements such that g_n converges to the pseudo-projective transformation S as $n \to \infty$, and $\ell = \ker S$.

We will use the following lemma proved in [6], as Lemma 3.2.

Lemma 3.11. *Let (g_n) be a sequence of elements in $\mathrm{PSL}(3,\mathbb{C})$. There exists a subsequence, still denoted (g_n), and a pseudo-projective transformation S such that:*

- *The sequence (g_n) converges uniformly to S on compact subsets of $\mathbf{P}^2_{\mathbb{C}} - \ker S$;*
- *if $\mathrm{Im}S$ is a complex line, then there exists a pseudo-projective map T such that $g_n^{-1} \to T$, when $n \to \infty$ uniformly on compact subsets of $\mathbf{P}^2_{\mathbb{C}} - \ker T$. Moreover, $\mathrm{Im}S = \ker T$ and $\ker S = \mathrm{Im}T$.*
- *if $\ker S$ is a complex line, then there exists a pseudo-projective map T such that $g_n^{-1} \to T$, when $n \to \infty$ uniformly on compact subsets of $\mathbf{P}^2_{\mathbb{C}} - \ker T$ and $\mathrm{Im}S \subset \ker T$. Moreover, if ℓ is a complex line not passing through $\mathrm{Im}S$ then the sequence of complex lines $g_n^{-1}(\ell)$ goes to the complex line $\ker S$ as $n \to \infty$.*

Proposition 3.12. *If $G \subset \mathrm{PSL}(3,\mathbb{C})$ is a discrete subgroup then*

$$\mathcal{E}(G) = \hat{L}(G).$$

Proof. Let ℓ be in $\mathcal{E}(G)$, thus there exists a sequence of distinct elements $(g_n) \subset G$ and a pseudo-projective transformation S, such that $g_n \to S$ as $n \to \infty$ uniformly on compact subsets of $\mathbf{P}^2_{\mathbb{C}} \setminus \ker S = \mathbf{P}^2_{\mathbb{C}} \setminus \ell$. By Lemma 3.11, we can assume that there exists R pseudo-projective transformation, such that $g_n^{-1} \to R$ as $n \to \infty$ uniformly on compact subsets of $\mathbf{P}^2_{\mathbb{C}} \setminus \ker R$. Moreover, if η is a complex line in the open set $U = \{\eta \in (\mathbf{P}^2_{\mathbb{C}})^* : \mathrm{Im}(S) \text{ does not lie on } \eta\}$ then $g_n^{-1} \cdot \eta \to \ker S$ as $n \to \infty$.

Conversely, let $\ell = [A : B : C] \in \hat{L}(G)$, so there is a non-empty open set $U \subset (\mathbf{P}^2_{\mathbb{C}})^*$ such that $g_n \cdot \eta \to \ell$ as $n \to \infty$ for every $\eta \in U$. If we use Lemma 3.11 for the sequence of projective transformations $[(\mathbf{g}_n^{-1})^T]$, we obtain a pseudo-projective transformation S such that

$$[(\mathbf{g}_n^{-1})^T] \to S \text{ as } n \to \infty \text{ uniformly on compact subsets of } \mathbf{P}^2_{\mathbb{C}} \setminus \ker S. \tag{3.9}$$

Moreover, the hypothesis that all lines η in the non-empty open set U satisfy that $g_n \cdot \eta \to \ell$ as $n \to \infty$ imply that $\mathrm{Im}(S)$ consists of one point. In fact, $\mathrm{Im}(S) = \{[A : B : C]\}$, so we can write $S = [\mathbf{s}]$, where

$$\mathbf{s} = \begin{pmatrix} \lambda A & \mu A & \nu A \\ \lambda B & \mu B & \nu B \\ \lambda C & \mu C & \nu C \end{pmatrix}, \text{ where } |\lambda| + |\mu| + |\nu| \neq 0.$$

It follows from (3.9) that

$$g_n^{-1} = [\mathbf{g}_n^{-1}] \to S' = [\mathbf{s}^T] \text{ as } n \to \infty \text{ uniformly on compact subsets of } \mathbf{P}^2_{\mathbb{C}} \setminus \ker S'.$$

Moreover, $\ker S' = [\ker \mathbf{s}^T] = \{[x : y : z] | Ax + By + Cz = 0\} = \ell$.

□

Corolary 3.13. *If $G \subset \mathrm{PSL}(3,\mathbb{C})$ is a discrete subgroup then $\hat{L}(G) \subset (\mathbf{P}^2_{\mathbb{C}})^*$ is a closed set.*

Proof. The Proposition 3.12 implies that $\hat{L}(G) = \mathcal{E}(G)$, and 3.5 states that $\mathcal{E}(G) \subset (\mathbf{P}^2_{\mathbb{C}})^*$ is closed. □

Corolary 3.14. *If G be a discrete subgroup of $\mathrm{PSL}(3,\mathbb{C})$, and H subgroup of G, with $[G:H] < \infty$, then $\hat{L}(H) = \hat{L}(G)$.*

Proof. It is not hard to check that $\hat{L}(H) \subset \hat{L}(G)$. Let $\ell \in \hat{L}(G)$. Since $\hat{L}(G) = \mathcal{E}(G)$, there exists a sequence $(g_n) \subset G$ such that $g_n \to S$, S a pseudo-projective transformation, and $\ker S = \ell$. As $[G:H] < \infty$, there exists $a \in G$ and $(h_n) \subset H$, with $h_n \neq h_m$ whenever $n \neq m$. Without loss of generality $g_n = ah_n$. If $R = \lim_{n\to\infty} h_n$, then $\ker R = \ker S$, this implies that $\mathcal{E}(H) = \hat{L}(H)$. □

Corolary 3.15 (Properties of limit set $\hat{L}(G)$)**.** *Let G be a discrete subgroup of $\mathrm{PSL}(3,\mathbb{C})$. Assume that G acts in $\mathbf{P}^2_{\mathbb{C}}$ without global fixed points nor invariant lines, and $\hat{L}(G)$ contains at least four elements, then:*

(i) $\hat{L}(G)$ is a perfect set and it is the minimal closed set for the action of G on $(\mathbf{P}^2_{\mathbb{C}})^*$.

(ii) The G-orbit of any $\eta \in \hat{L}(G)$ is dense in $\hat{L}(G)$.

(iii) $\hat{L}(G)$ is the closure of the set of loxodromic fixed points, and if there are parabolic elements in G, then $\hat{L}(G)$ is the closure of the set of parabolic fixed points as well.

(iv) $\hat{L}(G) = (\mathbf{P}^2_{\mathbb{C}})^*$ or it has empty interior.

Proof. First, we prove (i). The Proposition 3.12 implies $\hat{L}(G) = \mathcal{E}(G)$, and Proposition 3.7 together with item (c) from Theorem 3.6 implies the result.

The proof of (ii) and (iii) is a consequence of the minimality of $\hat{L}(G)$.

Now, we prove (iv). We notice that

$$\mathrm{Eq}(G) = \mathbf{P}^2_{\mathbb{C}} \setminus \bigcup_{\ell \in \mathcal{E}(G)} \ell = \mathbf{P}^2_{\mathbb{C}} \setminus \bigcup_{\ell \in \hat{L}(G)} \ell,$$

where the first equality is directly obtained from [3, Corollary 4.5], and the second is obtained by Proposition 3.12 above. As $\hat{L}(G)$ contains at least four elements, by [3, Proposition 4.10], there exists a loxodromic element in G.

Let us assume that U is a non-empty open subset of $\hat{L}(G)$. By (iii), there exists $\ell \in U$ where ℓ is an attracting fixed line for a loxodromic $g_0 \in G$. If $\varnothing \neq W$ is an open set contained in $(\mathbf{P}^2_{\mathbb{C}})^* \setminus \hat{L}(G)$ then there is $\eta \in W$ such that $g_0^n \cdot \eta \in U$ for all n large enough. This is a contradiction to the fact that $\hat{L}(G)$ is G-invariant.

□

Having proved the properties of $\hat{L}(G)$ we can state the next theorem.

Theorem 3.16. *Let $G \leqslant \mathrm{PSL}(3, \mathbb{C})$ be an infinite discrete subgroup acting on $\mathbf{P}^2_{\mathbb{C}}$ without fixed points nor invariant lines. Let $\hat{L}(G)$ be the limit set of G acting on $(\mathbf{P}^2_{\mathbb{C}})^*$, then*

$$\Lambda_K(G) = \bigcup_{\ell \in \hat{L}(G)} \ell.$$

Proof. First we observe that $\Lambda_{\mathrm{K}}(G) = \mathbf{P}^2_{\mathbb{C}} \setminus \mathrm{Eq}(G)$ by item (a) from Theorem 3.6. Then $\mathbf{P}^2_{\mathbb{C}} \setminus \mathrm{Eq}(G) = \bigcup_{\ell \in \mathcal{E}(G)} \ell$, as [3, Corollary 4.5] states. Finally, by Proposition 3.12, $\bigcup_{\ell \in \mathcal{E}(G)} \ell = \bigcup_{\ell \in \hat{L}(G)} \ell$. □

In the following example we present a group Γ with parabolic elements in which $\hat{L}(\Gamma)$ is identified with the classical limit set Λ for discrete subgroups of PU$(2,1)$ acting on $\mathbf{H}^2_{\mathbb{C}}$.

Example 3.17. In [13] N. Gusevskii and J. R. Parker give a type-preserving representation ρ of the group PSL$(2, \mathbb{Z})$ in PU$(2,1)$. The image under ρ of the two generators of $\Gamma = $ PSL$(2, \mathbb{Z})$ generate a discrete subgroup in PU$(2,1)$, $\rho(\Gamma)$. In [27], the author shows that the Kulkarni limit set of $\rho(\Gamma)$ is the set:

$$\Lambda_K(\Gamma) = \bigcup_{x \in \Lambda} \ell_x,$$

where ℓ_x is a tangent line to $\partial \mathbf{H}^2_{\mathbb{C}}$ in x.

Now, by Theorem 3.16, we show that $\hat{L}(\Gamma) = \{\ell_x \in (\mathbf{P}^2_{\mathbb{C}})^* : x \in \Lambda\}$.

Chapter 4

Examples of Schottky type groups

Schottky groups have been widely used in the theory of classical Kleinian groups to prove important results.

In 1999, J. Seade and A. Verjovsky [31] defined Schottky groups in the context of complex Kleinian groups for complex projective spaces of odd dimension. In 2008, in his Ph. D thesis, A. Cano [7] proved that Schottky groups cannot act on $\mathbf{P}^{2n}_{\mathbb{C}}$ as subgroups of $\mathrm{PSL}(2n+1, \mathbb{C})$.

After these results, there has been the need to find some groups with the type of dynamic that Schottky groups have. We work with a type of groups introduced by J. Tits in [35]. J.-P. Conze and Y. Guivarc'h [11] pick up the definition of Schottky type group; the main difference with the classical Schottky groups is that in this case the transformations does not pair the exterior of a compact set exactly to the interior of another compact subset, it is enough for the image to be contained in the interior of the other compact subset.

Their work is done in real projective spaces, however we can use some of their results in the complex case. When the results we state are valid using the fields $\mathbb{R}$ or $\mathbb{C}$, we write $\mathbb{F}$.

In this Chapter, we show that given a closed subset of $(\mathbf{P}^2_{\mathbb{R}})^*$ with empty interior, and at least three points in general position, it is possible to find a complex Kleinian group such that its limit set is very close to the closed subset considering the Hausdorff distance.

Also we build a family of subgroups of PSL$(3, \mathbb{R})$ acting on the complex projective plane. Each group G of the family is a Schottky type group as in Definition 4.1. It will be a free group generated by two loxodromic transformations g and f. The group in this example will be a discrete group which do not satisfy that the elements g and f share a flag in its Kulkarni limit set, contrary to the hypothesis of the Proposition 5.9 where the generators of the group share a flag in its Kulkarni limit set.

The work done in this Chapter was done with W. Barrera and J. P. Navarrete in the beautiful city: Mérida.

4.1 Schottky type groups and some properties

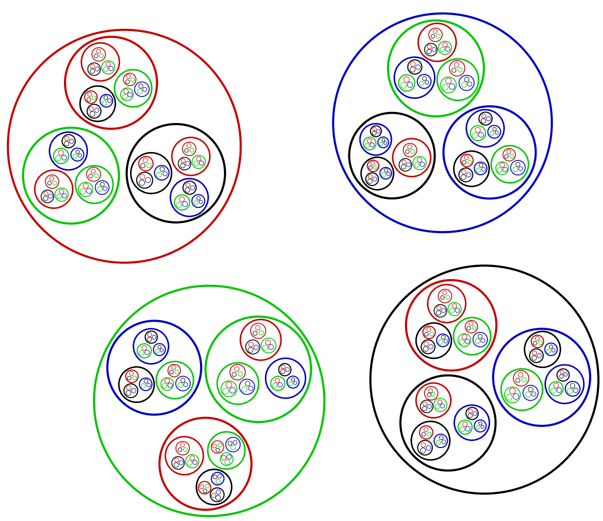
First, we write the definition of Schottky type group that Conze and Guivarc'h present in their paper [11].

Definition 4.1. Let (X, δ) be a complete metric space. A group Γ of homeomorphisms of X, generated by a finite symmetric set $\Sigma \subset Isom(X)$ (namely, $a^{-1} \in \Sigma$ for all $a \in \Sigma$) is called a *group of Schottky type* if there exists $\{C_a\}_{a\in\Sigma}$ a family of compact subsets of X, and a point $p \in X$ such that $p \notin \bigcup_{a\in\Sigma} C_a$ and $a(p) \in C_a$ for all $a \in \Sigma$, and the following conditions are satisfied:

(1) for $a, b \in \Sigma$, $C_a \cap C_b = \varnothing$ if $a \neq b$;

(2) for $a, b \in \Sigma$, $a(C_b) \subset Int(C_a)$, except when $ab = e$;

(3) for all sequences $\{a_n\}$ such that $a_n \neq a_{n+1}^{-1}$ for all $n > 1$, the diameter of $a_1 a_2 \cdots a_n(C_{a_{n+1}})$ tends to zero, when n tends to infinity.



The fact that there is a point p outside every compact set together with property (2) of the previous definition guarantees that the group generated by Σ is free and discrete [11, Proposition 5.2].

In the context of Schottky type groups there is a definition of a convex set, used in [11]:

Definition 4.2. A closed subset C of $\mathbf{P}^2_{\mathbb{C}}$ is said to be *convex* if it is contained in the complement of a projective hyperplane H and it is convex as a subset of the affine space $\mathbf{P}^2_{\mathbb{C}} - H$.

Following [11], and for making the notation easier, we introduce the next definition.

Notation 4.3. We will say that a set of homeomorphisms of a metric space X satisfies condition (S^+) if items (1) and (2) of Definition 4.1 are satisfied.

For the sake of completeness we give a brief introduction to the limit set that Conze and Guivarc'h present in [11]. Also, we provide some examples in order to compare the limit set of Conze and Guivarc'h and the limit set in the sense of Kulkarni.

Recall the following Proposition, proved in Chapter 3:

Proposition 3.2. Let A be a proximal transformation, being λ_A the eigenvalue of A with greater norm than the other eigenvalues. We define

$$H_A^- = \{\omega \in \mathbb{C}^3 : \lambda_A^{-n} A^n \omega \to 0 \text{ as } n \to \infty\}.$$

If S be the pseudo-projective limit of the positive powers of A, then

$$\ker S = [H_A^-],$$

where $[H_A^-]$ denotes the projection of the vector subspace H_A^-.

We restate Propositions 5.9 and 5.10 that Conze and Guivarc'h proved in their work [11].

Proposition 4.4. *Let Σ be a set of projective transformations. For each $a \in \Sigma$, let C_a be disjoint compact convex sets in the projective space such that $[H_b^-] \cap C_a = \varnothing$ if $b \neq a^{-1}$ and $a \in \Sigma$ with eigenvector $a^+ \in C_a$. Then for all sufficiently large n the family $\widehat{\Sigma}_n = \{(a^n, C_a) | a \in \Sigma\}$ satisfies condition (S^+).*

Moreover, under the same hypothesis if the family $\hat{\Sigma}$ satisfies condition (S^+), then condition (3) as in Definition 4.1 also holds.

Remark 4.5. If b is a proximal element and S is a pseudo-projective limit of the positive powers of b, then $[H_b^-] = [\ker S]$. And the condition $[H_b^-] \cap C_a \neq \varnothing$ in the Proposition 4.4 can be restated as $[\ker S] \cap C_a \neq \varnothing$.

4.2 First example: a subgroup acting on $(\mathbf{P}^2_{\mathbb{C}})^*$ whose limit set $\hat{L}$ is close to a given closed subset

In this section we build a group G acting on $(\mathbf{P}^2_{\mathbb{C}})^*$ with the particularity that given a closed subset $C \subset (\mathbf{P}^2_{\mathbb{C}})^*$ with empty interior, the Hausdorff distance between the limit set $\hat{L}(G)$ and C is smaller than any given positive real number.

We will prove the result with the help of four lemmas. First we need to guarantee that given three points in $(\mathbf{P}^2_{\mathbb{C}})^*$, there is a transformation g, strongly loxodromic, that has one point as attracting, the second as repelling and the third a saddle point.

Then, for a closed subset $C \subset (\mathbf{P}^2_{\mathbb{R}})^*$ consisting of four points, we carefully construct the group so the hypotheses of Proposition 4.4 are satisfied, because the generators of the group will be, for some $N \in \mathbb{N}$ the N-th power, g_i^N, of the original strongly loxodromic transformations g_i. Then, we have the result for a set of a finite number of points.

Given (X, d) a metric space, it is well known that the collection of compact subsets of X has a distance called the *Hausdorff distance*. We recall the definition of this distance. If A and B are compact subsets of X and $A_r = \{x \in X : d(x, A) < r\}$ is the $r-$neighborhood of A, then the Hausdorff distance between A and B is

$$d_H(A, B) = \inf\{r > 0 | A \subset B_r \text{ and } B \subset A_r\}. \tag{4.1}$$

The lemmas that will guide us to the proof of Theorem 4.11 start now.

Lemma 4.6. *Given η, $\mu, \nu \in (\mathbf{P}^2_{\mathbb{F}})^*$, there exists $g \in PSL(3, \mathbb{F})$ strongly loxodromic transformation satisfying the following:*

(i) η, μ and ν are fixed lines for g and $\hat{L}(g) = \{\eta, \mu\}$.

(ii) For all neighborhood W such that $\overline{W} \subset \mathbf{P}^2_{\mathbb{F}} \setminus \overleftrightarrow{\mu, \nu}$, and any neighborhood U of η, there there exists $N \in \mathbb{N}$ such that $g^n \cdot W \subset U$ for $n > N$.

(iii) For all neighborhood W such that $\overline{W} \subset \mathbf{P}^2_{\mathbb{F}} \setminus \overleftrightarrow{\eta, \nu}$, and any neighborhood V of μ, there exists $N \in \mathbb{N}$ such that $g^{-n} \cdot W \subset V$ for $n > N$.

The proof of this Lemma follows from Lemma 3.11

Remark 4.7. Let η and μ be elements in $(\mathbf{P}^2_{\mathbb{F}})^*$, and let F be a finite subset of $(\mathbf{P}^2_{\mathbb{F}})^*$. Then, there exists $\epsilon > 0$ and a $\nu \in (\mathbf{P}^2_{\mathbb{F}})^*$ such that $\overleftrightarrow{\eta, \nu}$ does not intersect the balls with radius ϵ and center in $F \cup \{\mu\}$. And $\overleftrightarrow{\nu, \mu}$ does not intersect the closure of the balls with radius ϵ and center in $F \cup \{\eta\}$.

The next lemma illustrates the construction of the group of G_ϵ of Theorem 4.11, for the particular case when the closed subset $C \subset (\mathbf{P}^2_{\mathbb{R}})^*$ consists of four points.

Lemma 4.8. *Given $F = \{\eta_1, \mu_1, \eta_2, \mu_2\} \subset (\mathbf{P}^2_{\mathbb{R}})^*$, and $\epsilon > 0$, there exists a Schottky type group G_ϵ such that*

$$d_H(\hat{L}(G_\epsilon), F) < \epsilon \tag{4.2}$$

Proof. Consider $\epsilon > 0$. Let U_i and V_i balls with center η_i and μ_i, respectively, and radius $0 < \epsilon' \leqslant \epsilon$ such that the U_1, U_2, V_1, V_2 are pairwise disjoint. Using Remark 4.7 there exists $\epsilon_1 > 0$ and a $\nu_1 \in (\mathbf{P}^2_{\mathbb{R}})^*$ such that $\overleftrightarrow{\eta_1, \nu_1}$ does not intersect the closure of the balls with radius ϵ_1 and center in $\{\eta_2, \mu_2\} \cup \{\mu_1\}$. And $\overleftrightarrow{\nu_1, \mu_1}$ does not intersect the closure of the balls with radius ϵ_1 and center in $\{\eta_2, \mu_2\} \cup \{\eta_1\}$.

Analogously, there exists $\epsilon_2 > 0$ and a $\nu_2 \in (\mathbf{P}^2_{\mathbb{R}})^*$ such that $\overleftrightarrow{\eta_2, \nu_2}$ does not intersect the balls with radius ϵ_2 and center in $\{\eta_1, \mu_1\} \cup \mu_2$. And $\overleftrightarrow{\nu_2, \mu_2}$ does not intersect the closure of the balls with radius ϵ_2 and center in $\{\eta_1, \mu_1\} \cup \{\eta_2\}$. We take $\epsilon_3 = \min\{\epsilon', \epsilon_1, \epsilon_2\}$.

Applying Lemma 4.6, there are strongly loxodromic transformations $g_1, g_2 \in \mathrm{PSL}(3, \mathbb{R})$ such that η_i, μ_i, ν_i are fixed points for the transformation g_i for $i = 1, 2$ and $\hat{L}(g_i) = \{\eta_i, \mu_i\}$. Also, for every open subset W such that $\overline{W} \subset (\mathbf{P}^2_{\mathbb{R}})^* \setminus \overleftrightarrow{\mu_i, \nu_i}$ and any neighborhood U_i of η_i there exists $N_i \in \mathbb{N}$ such that for $n > N_i$, $g_i^n \cdot W \subset U_i$. And for every open subset W such that $\overline{W} \subset (\mathbf{P}^2_{\mathbb{R}})^* \setminus \overleftrightarrow{\eta_i, \nu_i}$ and any neighborhood V_i if μ_i there exists $M_i \in \mathbb{N}$ such that for $n > M_i$, $g_i^{-n} \cdot W \subset V_i$, $i = 1, 2$. In particular, if we take U_i and V_i as balls with radius ϵ_3, we have the hypothesis of Proposition 4.4. So, for $N = \max\{N_1, N_2, M_1, M_2\}$, the group $G_\epsilon = \langle g_1^N, g_2^N \rangle$ is a Schottky type group. As $F \subset \hat{L}(G_\epsilon)$ and $\hat{L}(G_\epsilon) \subset \bigcup_{f \in F} B(f, \epsilon_3)$, it is not hard to check that $d_H(\hat{L}(G_\epsilon), F) < \epsilon_3 < \epsilon$.

□

Lemma 4.9. *Given F a finite subset of points in $(\mathbf{P}^2_{\mathbb{F}})^*$ and $\epsilon > 0$, there exists a Schottky type group G_ϵ such that*

$$d_H(\hat{L}(G_\epsilon), F) < \epsilon \tag{4.3}$$

The proof of this lemma is analogous to the proof of Lemma 4.8.

Lemma 4.10. *Let C be a closed subset of $(\mathbf{P}^2_{\mathbb{F}})^*$ such that $\bigcup_{\ell\in C} \ell \neq \mathbf{P}^2_{\mathbb{F}}$. Then there exists $\epsilon > 0$ such that $\overline{C}_\epsilon = \{\ell \in (\mathbf{P}^2_{\mathbb{F}})^* : d(\ell, C) \leqslant \epsilon\}$, satisfies $\bigcup_{\ell\in\overline{C}_\epsilon} \ell \neq \mathbf{P}^2_{\mathbb{F}}$.*

Proof. Let p be a point in $\mathbf{P}^2_{\mathbb{F}} \setminus \bigcup_{\ell\in C} \ell$, then there is a line $\mathcal{L}$ in $(\mathbf{P}^2_{\mathbb{F}})^*$ such that $\mathcal{L} \cap C = \varnothing$ and $p \in \bigcup_{\ell\in\mathcal{L}} \ell$. Then there exists $\epsilon > 0$ satisfying $\overline{N_\epsilon(C)} \cap \mathcal{L} = \varnothing$. Therefore, p is in $\mathbf{P}^2_{\mathbb{F}} \setminus \bigcup_{\ell\in\overline{N_\epsilon(C)}} \ell$. □

So, we can state the following theorem.

Theorem 4.11. *Given $\epsilon > 0$ and a closed subset $C \subset (\mathbf{P}^2_{\mathbb{R}})^*$ such that C has at least three points in general position and $\bigcup_{\ell\in C} \ell \neq \mathbf{P}^2_{\mathbb{C}}$, there is a complex Kleinian group G_ϵ, such that the Hausdorff distance between $\hat{L}(G_\epsilon)$ and C is smaller than ϵ.*

Proof. Consider $\epsilon > 0$. With out loss of generality, we choose F finite subset of C such that F has three points in general position and

$$d_H(F, C) < \epsilon/2. \tag{4.4}$$

By Lemma 4.9 there exists a Schottky type group G_ϵ such that

$$d_H(\hat{L}(G_\epsilon), F) < \epsilon/2. \tag{4.5}$$

From equations (4.4) and (4.5) we have

$$d_H(\hat{L}(G_\epsilon), C) < \epsilon.$$

By Lemma 4.10, for any ϵ small enough, we have the equality

$$\mathbf{P}^2_{\mathbb{C}} \neq \bigcup_{\ell\in\hat{L}(G_\epsilon)} \ell.$$

And by Theorem 3.16, $\Lambda_{\mathrm{K}}(G_\epsilon) = \bigcup_{\ell\in\hat{L}(G_\epsilon)} \ell$. □

4.3 Family of Schottky type groups

In this second example, we build a family of Schottky type groups acting on $\mathbf{P}^2_{\mathbb{R}}$. The groups of the family satisfy, in particular the second property of the definition of Schottky type groups.

Unlike the previous example, the generators of these groups are not the power of some element, instead they are defined according to their attracting, repelling and saddle points. However the saddle points of the second generator is chosen to assure that the group is not an affine group. A group of this family does not leave invariant any $3-$sphere contained in $\mathbf{P}^2_{\mathbb{C}}$, and therefore it is not a subgroup pf $PU(2,1)$.

The first transformation g we will consider is given in terms of its fixed points and the corresponding eigenvalues. The attracting and repelling points will be in the affine chart $\{z = 1\} \subset \mathbf{P}^2_{\mathbb{R}}$.

Around each of these points we take ellipsoidal neighborhoods. We consider a bilineal matrix E and get the equation $\mathbf{x}^T E\mathbf{x}$, whose zeros will define the ellipsoids. E will be a matrix whose entries are related to the length of the axes of the conic.

As well, the second transformation f will be given in terms of its fixed points, only that the geometric place where these points can stay, is restricted. They should stay in the preimage of the interior of the neighborhoods of the fixed points of g.

Given that we will be using conics to determine the neighborhoods of the fixed points of the transformations, in the first Section we will recall some results on conics.

4.3.1 Brief review on conics

Consider the equation of an ellipse centered in (h, k):

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \tag{4.6}$$

where a,$b \in \mathbb{R}$. The equation (4.6) is equivalent to:

$$b^2x^2 + a^2y^2 - 2b^2hx - 2a^2ky + b^2h^2 + a^2k^2 - a^2b^2 = 0. \tag{4.7}$$

On the other hand, quadrics are built through a symmetric matrix E, that is, $E = E^T$.

To find the bilinear form that induce it, consider the following equation:

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0,$$

with A, B, C, D, E and $F \in \mathbb{R}$. Moreover, in the affine chart $\{z = 1\}$ the equation is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{4.8}$$

Comparing equations (4.7) and (4.8), we can find the coefficients of the matrix in terms of the point where the ellipse is centered $[h : k : 1]$ and the length of its axes is given by a and $b \in \mathbb{R}$. We call this matrix:

$$E(h, k, a, b) := \begin{pmatrix} b^2 & 0 & -b^2\,h \\ 0 & a^2 & -a^2\,k \\ -b^2h & -a^2\,k & b^2\,h^2 + a^2\,k^2 - a^2\,b^2 \end{pmatrix}, \tag{4.9}$$

We will denote by $\mathbf{B}$, the ellipse

$$\mathbf{B} := \{[x : y : z] \in \mathbf{P}^2_{\mathbb{R}} : b^2x^2 + a^2y^2 - 2b^2hx - 2a^2ky + b^2h^2 + a^2k^2 - a^2b^2 = 0\}.$$

The points $(x, y) \in \mathbb{R}^2$ satisfying equation (4.8) may draw an ellipse, a parabola or a hyperbola depending on the discriminant $B^2 - 4AC$. This happens because if we think in $\mathbb{R}^2$ as the affine chart $\{z = 1\}$ of the real projective plane $\mathbf{P}^2_{\mathbb{R}}$, the corresponding equation of the quadric in the homogeneous coordinates $[x : y : z]$ would be as equation 4.3.1 and the curve would touch the line at infinity $\{z = 0\}$ in either zero, one or two points, as the following calculations show.

The points in the line at infinity are of the form $[x : y : 0]$, considering these points, equation (4.3.1) reduces to:

$$Ax^2 + Bxy + Cy^2 = 0.$$

If there are points (x, y) satisfying this equation, we can find them making

$$y = x\frac{-B \pm \sqrt{B^2 - 4AC}}{2C}.$$

So the points at infinity which satisfy the equation are:

$$\left[1 : \frac{-B + \sqrt{B^2 - 4AC}}{2C} : 0\right] \quad \text{and} \quad \left[1 : \frac{-B - \sqrt{B^2 - 4AC}}{2C} : 0\right],$$

and if $B^2 - 4AC = 0$, there is only one point at infinity, so we have a parabola. If $B^2 - 4AC > 0$ then the quadric is a hyperbola, but if $B^2 - 4AC < 0$ then there are no points at infinity, and the quadric is an ellipse.

Remark 4.12. The points around which we will construct the neighborhoods are attracting and repelling points for a transformation g.

Once the ellipse $\mathbf{B}$ around an attracting point is given by the equation (4.8), we will be interested in the points $\mathbf{p} = [x : y : z] \in \mathbf{P}^2_{\mathbb{R}}$ such that $g(\mathbf{p}) = w \in \mathbf{B}$, that is $g(\mathbf{p})$ satisfies $(g \cdot \mathbf{p})^T\, E\,(g \cdot \mathbf{p}) = 0$, which is the same as:

$$\mathbf{p}^T\, \mathbf{g}^T\, E\, \mathbf{g}\, \mathbf{p} = 0. \tag{4.10}$$

And when talking about the points in the ball around the repelling point of the transformation g, the points we want to define, satisfy:

$$\mathbf{p}^T\, (\mathbf{g}^{-1})^T\, E\, (\mathbf{g}^{-1})\, \mathbf{p} = 0. \tag{4.11}$$

4.3.2 First transformation: g

In the proof of the next proposition, it is explained how to get the first transformation g and the neighborhoods around the fixed points of g.

Proposition 4.13. *Given $h_1 \in \mathbb{R}$, there exists a loxodromic transformation g with attracting point $[h_1 : 0 : 1]$ and repelling point $[-h_1 : 0 : 1]$, each point with an ellipsoidal neighborhood $\mathbf{B_1}$ and $\mathbf{B_2}$ respectively, given in terms of the length of their axes a_i, b_i, $i = 1, 2$, such that $g^{-1}(\mathbf{B_1})$ and $g(\mathbf{B_2})$ are* *hyperbolas with equations:*

$$\Big(\frac{1}{16h_1^2}\Big)\big((4h_1^2b_1^2 - 9a_1^2b_1^2)x^2 - 16a_1^2h_1^2y^2 - (30a_1^2b_1^2h_1 + 8h_1^3b_1^2)x + 4h_1^4b_1^2 - 25a_1^2b_1^2h_1^2\big) = 0$$

and

$$\Big(\frac{1}{16h_1^2}\Big)\big((4b_2^2h_1^2 - 9a_2^2b_2^2)x^2 + 16a_2^2h_1^2y^2 + (8b_2^2h_1^3 + 30a_2^2b_2^2h_1)x - 25a_2^2b_2^2h_1^2 + 4b_2^2h_1^4\big) = 0,$$

respectively.

Proof. Let h_1 be any real number. We build a strongly loxodromic transformation g that has as attracting and repelling points $[h_1 : 0 : 1]$, $[-h_1 : 0 : 1]$ and $[0 : 1 : 0]$ as saddle point, with eigenvalues 2, $1/2$, and 1 respectively. The transformation is built conjugating the diagonal matrix $D = \mathrm{Diag}(2, 1, 1/2)$ by the matrix of change of coordinates:

$$P = \begin{pmatrix} h_1 & 0 & -h_1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

then the transformation g has a lift in $\mathrm{GL}(3, \mathbb{C})$ given by:

$$P \cdot D \cdot P^{-1} = \mathbf{g} = \begin{pmatrix} 5/4 & 0 & \frac{3h_1}{4} \\ 0 & 1 & 0 \\ \frac{3}{4h_1} & 0 & 5/4 \end{pmatrix} \tag{4.12}$$

In the equation of the matrix defining the bilinear form, we substitute the corresponding values of the center of the ellipse B_1: $[h_1 : 0 : 1]$ and the length of its axes by: a_1 and $b_1 \in \mathbb{R}$

$$E_1 := E(h_1, 0, a_1, b_1) = \begin{pmatrix} b_1^2 & 0 & -b_1^2h_1 \\ 0 & a_1^2 & 0 \\ -b_1^2h_1 & 0 & b_1^2h_1^2 - a_1^2b_1^2 \end{pmatrix}$$

Then:

$$\begin{aligned} \mathbf{B_1} &:= \{\mathbf{p} \in \mathbf{P}^2_{\mathbb{R}} : \mathbf{p}^T E_1 \mathbf{p} = 0\} \\ &= \{[x : y : 1] \in \mathbf{P}^2_{\mathbb{C}} : -a_1^2b_1^2 + a_1^2y^2 + b_1^2h_1^2 - 2b_1^2h_1x + b_1^2x^2 = 0\}, \end{aligned}$$

where $\mathbf{p} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

To know which is the ellipse around $[-h_1 : 0 : 1]$ we substitute in (4.9) and get the matrix $E_2 := E(-h_1, 0, a_2, b_2)$, the ellipse will be:

$$\begin{aligned} \mathbf{B_2} &:= \{\mathbf{p} \in \mathbf{P}^2_{\mathbb{R}} : \mathbf{p}^T E_2 \mathbf{p} = 0\} \\ &= \{[x : y : 1] \in \mathbf{P}^2_{\mathbb{C}} : -a_2^2b_2^2 + a_2^2y^2 + b_2^2h_1^2 + 2b_2^2h_1x + b_2^2x^2 = 0\}. \end{aligned}$$

So, we have the ellipse $\mathbf{B_1}$ around $[h_1 : 0 : 1]$, and $\mathbf{B_2}$ around $[-h_1 : 0 : 1]$, each one with length of their axes a_i and b_i for $i = 1, 2$. To have an image of what is happening in the chart $\{z = 1\} \subset \mathbf{P}^2_{\mathbb{C}}$, we have the Figure 4.1.

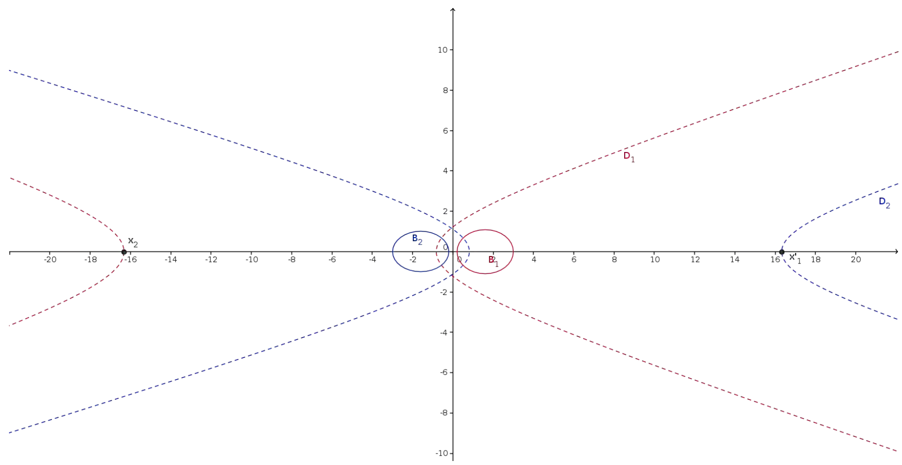


Figure 4.1: The neighborhoods $\mathbf{B_1}$, $\mathbf{B_2}$, $\mathbf{D_1}$ and $\mathbf{D_2}$.

The interior part of $\mathbf{B_i}$, $\mathrm{Int}(\mathbf{B_i})$, will be the compact region bounded by $\mathbf{B_i}$, and these will be the neighborhoods of the fixed points.

Now, we would like to find regions $\mathbf{D_1}$ and $\mathbf{D_2}$ in $\mathbf{P}^2_{\mathbb{R}}$ such that whenever we evaluate g in points $\mathbf{p} \in \mathrm{Int}(\mathbf{D_1})$, the image $g(\mathbf{p}) \in \mathrm{Int}(\mathbf{B_1})$ and if $\mathbf{p} \in \mathrm{Int}(\mathbf{D_2})$, then $g^{-1}(\mathbf{p}) \in \mathrm{Int}(\mathbf{B_2})$. This is $\mathbf{D_1} = g^{-1}(\mathbf{B_1})$ and $\mathbf{D_2} = g(\mathbf{B_2})$.

As we said in Remark 4.12, $\mathbf{D_1}$ will be the ellipse defined by equation: (4.10), with the correct parameters.

$$\mathbf{D_1} := \{\mathbf{p} \in \mathbf{P}^2_{\mathbb{R}} : \mathbf{p}^T\, \mathbf{g}^T\, E_1\, \mathbf{g}\, \mathbf{p} = 0\},$$

which is the same as

$$\begin{aligned} \mathbf{D_1} := \quad & \{[x : y : 1] \in \mathbf{P}^2_{\mathbb{R}} : \big(\tfrac{1}{16h_1^2}\big)\big((4h_1^2b_1^2 - 9a_1^2b_1^2)x^2 - 16a_1^2h_1^2y^2 \\ & -(30a_1^2b_1^2h_1 + 8h_1^3b_1^2)x + 4h_1^4b_1^2 - 25a_1^2b_1^2h_1^2\big) = 0\} \end{aligned} \tag{4.13}$$

Remark 4.14. When $y = 0$ the point x which satisfies the equation can be found solving the following equation:

$$(4h_1^2b_1^2 - 9a_1^2b_1^2)x^2 - (30a_1^2b_1^2h_1 + 8h_1^3b_1^2)x + 4h_1^4b_1^2 - 25a_1^2b_1^2h_1^2 = 0$$

We need the quantity $B^2 - 4AC$ to be less than zero, for $\mathbf{D_1}$ to have points in the line at infinity. Where $A, B,$ and C are coefficients of x^2, xy and y^2, respectively, the equation describing the pre-image under g of $\mathbf{B_1}$, which is $\mathbf{D_1}$.

In this case $B = 0$, and we have:

$$\begin{aligned} & -4(4b_1^2h_1^2 - 9a_1^2b_1^2)(-16a_1^2h_1^2) > 0 \\ \Leftrightarrow \quad & 4a_1^2b_1^2h_1^4 - 9a_1^4b_1^2h_1^2 > 0 \\ \Leftrightarrow \quad & 4h_1^2 > 9a_1^2 \\ \Leftrightarrow \quad & \frac{2h_1}{3} > a_1. \end{aligned} \tag{4.14}$$

Analogously, $\mathbf{D_2}$ will be defined by the equation (4.11) with the correct parameters: that is:

$$\mathbf{D_2} := \{\mathbf{p} \in \mathbf{P}^2_{\mathbb{R}} : \mathbf{p}^T\, \mathbf{g^{-1}}^T\, E_2\, \mathbf{g^{-1}}\, \mathbf{p} = 0\},$$

which is the same as

$$\begin{aligned} \mathbf{D_2} = \quad & \{[x:y:1] \in \mathbf{P}^2_{\mathbb{R}} : \left(\tfrac{1}{16h_1^2}\right)\big((4b_2^2h_1^2 - 9a_2^2b_2^2)x^2 + 16a_2^2h_1^2y^2 \\ & +(8b_2^2h_1^3 + 30a_2^2b_2^2h_1)x - 25a_2^2b_2^2h_1^2 + 4b_2^2h_1^4\big) = 0\} \end{aligned} \tag{4.15}$$

Again, $\mathbf{D_2}$ will have two points at infinity, whenever $B^2 - 4AC > 0$, in this case, if and only if

$$\begin{aligned} & -4(4b_2^2h_1^2 - 9a_2^2b_2^2)(16a_2^2h_1^2) > 0 \\ \Leftrightarrow \quad & 4h_1^2 < 9a_2^2 \\ \Leftrightarrow \quad & \frac{2h_1}{3} < a_2. \end{aligned} \tag{4.16}$$

We will choose a_1 smaller than $(2/3)h_1$ and a_2 greater than $(2/3)h_1$ to have in the affine chart $\{z = 1\}$ two components of the quadric. Both $\mathbf{D_1}$ and $\mathbf{D_2}$ separate the projective plane in two components, one of them is homeomorphic to the interior of a sphere and the other is the neighborhood of a projective line.

□

4.3.3 Second transformation: f

In order to find the second strongly loxodromic transformation f we need to find suitable attracting and repelling points, to achieve this we should know the subset of $\mathbf{P}^2_{\mathbb{R}}$ where these points can be, so we find the x coordinate of the quadrics $\mathbf{D_1}$ and $\mathbf{D_2}$ when $y = 0$. When substituting $y = 0$ in equations (4.13) and (4.15) we have two quadratic equations with variable x :

$$(4h_1^2b_1^2 - 9a_1^2b_1^2)x^2 - (30a_1^2b_1^2h_1 + 8h_1^3b_1^2)x + 4h_1^4b_1^2 - 25a_1^2b_1^2h_1^2 = 0, \tag{4.17}$$

and

$$(4b_2^2h_1^2 - 9a_2^2b_2^2)x^2 + (30a_2^2b_2^2h_1 + 8b_2^2h_1^3)x + 4b_2^2h_1^4 - 25a_2^2b_2^2h_1^2 = 0. \tag{4.18}$$

The roots of the equation (4.17) are:

$$x_1 = \frac{h_1(15a_1^2 + 4h_1^2) + 4h_1^2\sqrt{16a_1^2 + h_1^2 - 1}}{4h_1^2 - 9a_1^2} \quad \text{and}$$
$$x_2 = \frac{h_1(15a_1^2 + 4h_1^2) - 4h_1^2\sqrt{16a_1^2 + h_1^2 - 1}}{4h_1^2 - 9a_1^2}. \quad (4.19)$$

And the roots for equation (4.18) are:

$$x_1' = \frac{-h_1(4h_1^2 + 15a_2^2) + 16h_1^2a_2}{4h_1^2 - 9a_2^2} \quad \text{and} \quad x_2' = \frac{-h_1(4h_1^2 + 15a_2^2) - 16h_1^2a_2}{4h_1^2 - 9a_2^2} \quad (4.20)$$

We will call the component which is homeomorphic to the interior of a sphere, the interior of the quadric. In the intersection of the interior of the quadrics $\mathbf{D_1}$ and $\mathbf{D_2}$, we are going to place the attracting and repelling points of another transformation f, as well as some neighborhoods $\mathbf{B_3}$ and $\mathbf{B_4}$ of each point.

To find the second transformation f, we chose the points $[h_2 : k_2 : 1]$, $[-h_2 : k_2 : 1]$ and $[e : 1 : 0]$, with $e \neq 0$, as attracting, repelling and saddle points, with eigenvalues $2, 1/2$ and 1, respectively, and h_2, k_2 and $e \in \mathbb{R}$.

Remark 4.15. Let $[h_2 : k_2 : 1]$ and $[-h_2 : k_2 : 1]$ be points such that they lie in the interior of $\mathbf{D_1} \cap \mathbf{D_2}$. That is, we ask the first coordinate of the center to have a large norm, and the second coordinate to have norm different from zero:

$$\begin{aligned} |h_2| &>> \max\{|x_2|, |x_1'|\} \\ k_2 &\neq 0; \end{aligned}$$

The first coordinate of the saddle point e should be different form zero, so the group generated by f and g is not the suspension of a group.

Proposition 4.16. *Given $[h_2 : k_2 : 1]$ and $[-h_2 : k_2 : 1]$ as in Remark 4.15 and g as in Proposition 4.13, there exists a transformation $f \in \mathrm{PSL}(3, \mathbb{R})$ with $[h_2 : k_2 : 1]$ as attracting point, $[-h_2 : k_2 : 1]$ as repelling point with disjoint ellipsoidal neighborhoods $\mathbf{B_3}$ and $\mathbf{B_4}$, respectively, given in terms of the length of its axes a_i, b_i, $i = 3, 4$ such that $f^{-1}(\mathbf{B_3})$ and $f(\mathbf{B_4})$, together with $\mathbf{B_1}$ and $\mathbf{B_2}$ satisfy:*

(i) $\mathbf{D_3} := f^{-1}(\mathbf{B_3})$ *is given by equation*

$$\begin{aligned} \Big(\frac{1}{16|h_2|^2}\Big)\big(&(4b_3^2h_2^2 - 9a_3^2b_3^2)x^2 + (16a_3^2h_2^2 + 4h_2^2b_3^2e^2 - 9a_3^2b_3^2e^2)y^2 \\ &+ (8b_3^2h_2^2e + 18a_3^2b_3^2e)xy - (8b_3^2h_2^3 + 30a_3^2b_3^2h_2 + 8b_3^2h_2^2k_2e + 18a_3^2b_3^2k_2e)x \\ &- (8b_3^2h_2^3e - 8b_3^2h_2^2k_2e^2 - 30a_3^2b_3^2h_2e + 32a_3^2h_2^2k_2)y + 4b_3^2h_2^4 + 16a_3^2h_2^2k_2^2 \\ &- 25a_3^2b_3^2h_2^2 - 9a_3^2b_3^2(ek)^2 + 4b_3^2h_2^2k_2^2e^2 + 8b_3^2h_2^3k_2e - 30a_3^2b_3^2h_2k_2e\big) = 0 \quad (4.21) \end{aligned}$$

and $\mathbf{D_4} := f(\mathbf{B_4})$ *is given by equation*

$$\left(\frac{1}{16{h_2}^2}\right)\big((4b_4^2h_2^2 - 9a_4^2b_4^2)x^2 + (4b_4^2h_2^2e^2 + 16a_4^2h_2^2 - 9a_4^2b_4^2e^2)y^2 \\ + (8b_4^2h_2^2e + 18a_4^2b_4^2e)xy + (8b_4^2h_2^3 + 30a_4^2b_4^2h_2 - 8b_4^2h_2^2k_2e - 18a_4^2b_4^2k_2e)x \\ + (8b_4^2h_2^3e - 30a_4^2b_4^2h_2e + 18a_4^2b_4^2e^2k_2 - 32a_4^2h_2^2k_2 - 8b_4^2h_2^2k_2e^2)y \\ - 8b_4^2h_2^3k_2e + 30a_4^2b_4^2h_2k_2e + 4b_4^2h_2^4 - 25a_4^2b_4^2h_2^2 + 16a_4^2(h_2k_2)^2 \\ + 4b_4^2(h_2k_2e)^2 - 9a_4^2b_4^2(ek_2)^2\big) = 0 \quad (4.22)$$

(ii) $\mathbf{B_3}, \mathbf{B_4} \subset \mathbf{D_1} \cap \mathbf{D_2}$,

(iii) $\mathbf{B_1}$ *and* $\mathbf{B_2} \subset f^{-1}(\mathbf{B_3})$,

(iv) $\mathbf{B_1}$ *and* $\mathbf{B_2} \subset f(\mathbf{B_4})$.

Proof. The transformation is:

$$\mathbf{f} = \begin{pmatrix} \frac{5}{4} & -\frac{e}{4} & \frac{ek_2+3h_2}{4} \\ \frac{3k_2}{4h_2} & \frac{4h_2-3ek_2}{4h_2} & \frac{3ek_2{}^2+h_2k_2}{4h_2} \\ \frac{3}{4h_2} & -\frac{3e}{4h_2} & \frac{3ek_2+5h_2}{4h_2} \end{pmatrix}. \quad (4.23)$$

Following the same idea of transformation g, we use equation (4.9) to build the neighborhood of the point $[h_2 : k_2 : 1]$ and the length of its axes: a_3, b_3. The bilinear form is:

$$E_3 = \begin{pmatrix} b_3^2 & 0 & -b_3^2h_2 \\ 0 & a_3^2 & -a_3^2k_2 \\ -b_3^2h_2 & -a_3^2k_2 & a_3^2k_2^2 + b_3^2h_2^2 - {a_3}^2{b_3}^2 \end{pmatrix}$$

The neighborhood $\mathbf{B_3}$ of $[h_2 : k_2 : 1]$ is given by the equation:

$$\begin{aligned} \mathbf{B_3} := & \ \{\mathbf{p} \in \mathbf{P}^2_{\mathbb{R}} : \ \mathbf{p}^T E_3\, \mathbf{p} = 0\} \\ = & \ \{[x : y : 1] \in \mathbf{P}^2_{\mathbb{R}} : b_3^2x^2 + a_3^2y^2 - 2b_3^2h_2x - 2a_3^2k_2y + b_3^2h_2^2 \\ & +a_3^2k_2^2 - a_3^3b_3^3 = 0\} \end{aligned}$$

The parameter b_3 should satisfy that $h_2 - a_3 > x_1'$, so the ball $\mathbf{B_3}$ is contained in the interior of $\mathbf{D_1} \cap \mathbf{D_2}$.

Now, let us present the neighborhood $\mathbf{B_4}$ of $[-h_2 : k_2 : 1]$. This will be given as the set:

$$\begin{aligned} \mathbf{B_4} := & \ \{\mathbf{p} \in \mathbf{P}^2_{\mathbb{R}} : \mathbf{p}^T E_4\, \mathbf{p} = 0\} \\ = & \ \{[x : y : 1] \in \mathbf{P}^2_{\mathbb{C}} : b_4^2x^2 + a_4^2y^2 + 2b_4^2h_2x - 2a_4^2k_2y + b^2h_2^2 \\ & +a_4^2k_2^2 - a_4^2b_4^2 = 0\}. \end{aligned}$$

The parameter b_4 should satisfy that $-h_2 + a_4 < x_2$, so the ball $\mathbf{B_4}$ is contained in the interior of $\mathbf{D_1} \cap \mathbf{D_2}$.

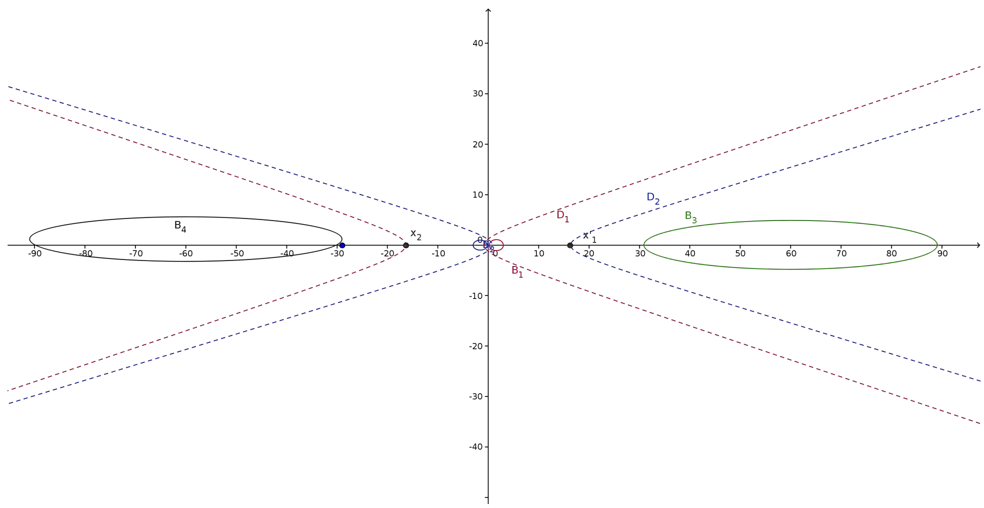


Figure 4.2: The neighborhoods $\mathbf{B_3}$ and $\mathbf{B_4}$.

After defining the neighborhoods $\mathbf{B_3}$ and $\mathbf{B_4}$ of the attracting and repelling points of f, we would like to find regions $\mathbf{D_3}$ and $\mathbf{D_4}$ in $\mathbf{P}^2_{\mathbb{R}}$ such that whenever we evaluate f in points $\mathbf{p} \in \mathbf{D_3}$, the image $f(\mathbf{p}) \in \mathbf{B_3}$, and if $\mathbf{p} \in \mathbf{D_4}$ then $f^{-1}(\mathbf{p}) \in \mathbf{B_4}$.

As we said in Remark 4.12, $\mathbf{D_3}$ will be delimited by the ball defined by equation (4.10), with the correct parameters; we would also like $\mathbf{D_3}$ and $\mathbf{D_4}$ to have in its interior the neighborhoods $\mathbf{B_1}$ and $\mathbf{B_2}$.

$$\mathbf{D_3} := \{\mathbf{p} \in \mathbf{P}^2_{\mathbb{R}} : \mathbf{p}^T\, \mathbf{f}^T\, E_3\, \mathbf{f}\, \mathbf{p} = 0\},$$

which is the same as

$$\begin{aligned} \mathbf{D_3} := \quad & \{[x:y:1] \in \mathbf{P}^2_{\mathbb{R}} : \left(\tfrac{1}{16|h_2|^2}\right)\big((4b_3^2h_2^2 - 9a_3^2b_3^2)x^2 + (16a_3^2h_2^2 + 4h_2^2b_3^2e^2 - 9a_3^2b_3^2e^2)y^2 \\ & +(8b_3^2h_2^2e + 18a_3^2b_3^2e)xy - (8b_3^2h_2^3 + 30a_3^2b_3^2h_2 + 8b_3^2h_2^2k_2e + 18a_3^2b_3^2k_2e)x \\ & -(8b_3^2h_2^3e - 8b_3^2h_2^2k_2e^2 - 30a_3^2b_3^2h_2e + 32a_3^2h_2^2k_2)y + 4b_3^2h_2^4 + 16a_3^2h_2^2k_2^2 \\ & -25a_3^2b_3^2h_2^2 - 9a_3^2b_3^2(ek)^2 + 4b_3^2h_2^2k_2^2e^2 + 8b_3^2h_2^3k_2e - 30a_3^2b_3^2h_2k_2e\big) = 0\} \end{aligned}$$

As we show in the Appendix A.1 given an equation as (4.8), the angle of the axis of the conic is obtained by the equality A.2, which in this case is:

$$\cot 2\theta = \frac{(1-e^2)(4b_3^2h_2^2 - 9a_3^2b_3^2) - 16a_3^2h_2^2}{2eb_3^2(4h_2^2 + 9a_3^2)}$$

We choose a_3 and b_3 so θ is such that the first coordinate of the points in $\mathbf{D_3}$, $a_3\cos\theta$ is smaller than $-h_1 - a_2$:

$$a_3 \cos\theta < -h_1 - a_2,$$

and $\mathbf{B_1}$ and $\mathbf{B_2}$ lie in the interior of $\mathbf{D_3}$

Analogously, $\mathbf{D_4}$ will be defined by the equation (4.11) with parameters a_4 and b_4 such that $\mathbf{B_1}$ and $\mathbf{B_2}$ are contained in the interior of $\mathbf{D_4}$.

$$\mathbf{D_4} := \{\mathbf{p} \in \mathbf{P}^2_{\mathbb{R}} : \mathbf{p}^T\, \mathbf{f^{-1}}^T\, E_4\, \mathbf{f^{-1}}\, \mathbf{p} = 0\},$$

which is the same as

$$\begin{aligned}\mathbf{D_4} := \quad & \{[x:y:1] \in \mathbf{P}^2_{\mathbb{R}} : \{\left(\tfrac{1}{16{h_2}^2}\right)\big((4b_4^2h_2^2 - 9a_4^2b_4^2)x^2 + (4b_4^2h_2^2e^2 + 16a_4^2h_2^2 - 9a_4^2b_4^2e^2)y^2 \\ & +(8b_4^2h_2^2e + 18a_4^2b_4^2e)xy + (8b_4^2h_2^3 + 30a_4^2b_4^2h_2 - 8b_4^2h_2^2k_2e - 18a_4^2b_4^2k_2e)x \\ & +(8b_4^2h_2^3e - 30a_4^2b_4^2h_2e + 18a_4^2b_4^2e^2k_2 - 32a_4^2h_2^2k_2 - 8b_4^2h_2^2k_2e^2)y \\ & -8b_4^2h_2^3k_2e + 30a_4^2b_4^2h_2k_2e + 4b_4^2h_2^4 - 25a_4^2b_4^2h_2^2 + 16a_4^2(h_2k_2)^2 \\ & +4b_4^2(h_2k_2e)^2 - 9a_4^2b_4^2(ek_2)^2\big) = 0\}\end{aligned}$$

The angle of the axis of the conic $\mathbf{D_4}$ is obtained by the equality A.2, which in this case is:

$$\cot 2\theta = \frac{(1-e^2)(4b_4^2h_2^2 - 9a_4^2b_4^2) - 16a_4^2h_2^2}{2eb_4^2(4h_2^2 + 9a_4^2)}$$

We choose a_4 and b_4 so that the angle θ is such that the first coordinate of the points in $\mathbf{D_4}$, $a_4 \cos\theta$ is greater than $h_1 + a_1$:

$$a_4 \cos\theta > h_1 + a_1.$$

□

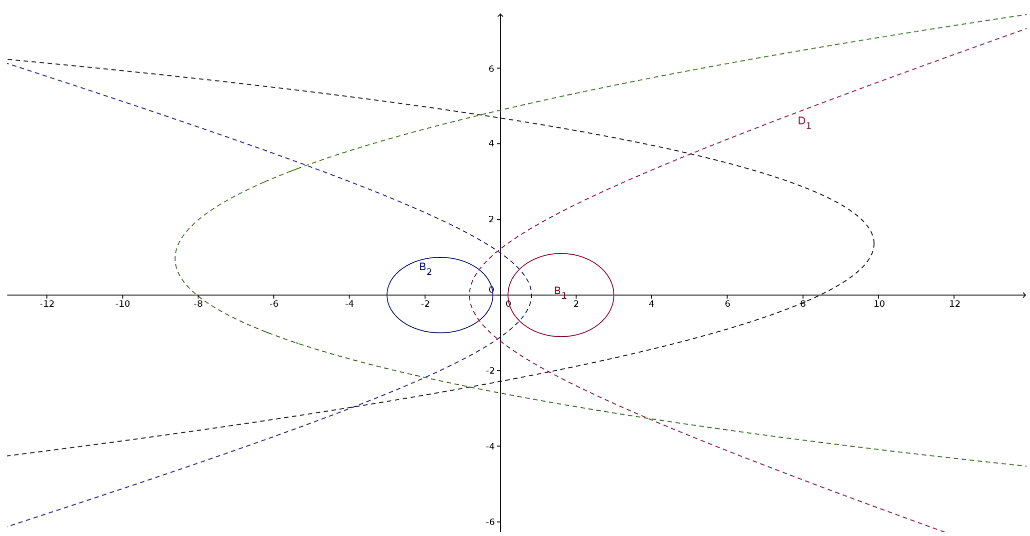


Figure 4.3: The neighborhoods $\mathbf{D_3}$ and $\mathbf{D_4}$.

4.3.4 The group

Theorem 4.17. *If g is a transformation as in Proposition 4.13 and if f is a transformation as in Proposition 4.16, then $G = \langle f, g\rangle$ is a Schottky type group acting on $\mathbf{P}^2_{\mathbb{R}}$ with nonempty region of discontinuity.*

Proof. The generators f and g are proximal elements. The sets $\mathrm{Int}(\mathbf{B}_i)$, for $i = 1, 2, 3, 4$, are convex compact sets, and they have in its interior the attracting point for the transformation g, g^{-1}, f and f^{-1}, respectively.

So, f and g together with $\mathrm{Int}(\mathbf{B}_i)$, $i = 1, 2, 3, 4$, satisfy conditions (1) and (2) of Definition 4.1. Also the $\mathrm{Int}(\mathbf{B}_i)'s$ are convex subsets. By Proposition 4.4 we can say that condition (3) in Definition 4.1 is also satisfied, so $\langle f, g\rangle$ is a Schottky type group acting in $\mathbf{P}^2_{\mathbb{R}}$.

Observe that $\mathrm{Int}(\mathbf{B_i}) \cap \mathrm{Int}(\mathbf{B_j}) = \varnothing$ if $i \neq j$; also

$$
\begin{gathered}
g(\mathrm{Int}(\mathbf{B_3})) \text{ and } g(\mathrm{Int}(\mathbf{B_4})) \text{ are contained in } \mathrm{Int}(\mathbf{B_1}), \\
g^{-1}(\mathrm{Int}(\mathbf{B_3})) \text{ and } g^{-1}(\mathrm{Int}(\mathbf{B_4})) \text{ are contained in } \mathrm{Int}(\mathbf{B_2}), \\
f(\mathrm{Int}(\mathbf{B_1})) \text{ and } f(\mathrm{Int}(\mathbf{B_2})) \text{ are contained in } \mathrm{Int}(\mathbf{B_3}), \text{ and} \\
f^{-1}(\mathrm{Int}(\mathbf{B_1})) \text{ and } f^{-1}(\mathrm{Int}(\mathbf{B_2})) \text{ are contained in } \mathrm{Int}(\mathbf{B_4}).
\end{gathered}
$$

□

Chapter 5

Properties of groups with two generators

In many results on the theory of complex Kleinian groups is frequently asked to the group to be discrete, however it is not easy to determine if a group is discrete or not. For subgroups of $\mathrm{PSL}(2,\mathbb{C})$ it took a lot of time to get necessary and sufficient conditions to know if a group generated by two elements is discrete. Many mathematicians worked in understanding the conditions for the necessity and frequently what they studied was the trace of an element and the trace of the commutator of both generators. In the context of subgroups of $\mathrm{PSL}(2,\mathbb{C})$ the famous criterion for discreteness is Jørgensen's inequality. Later, in [34], Delin Tan makes a generalization of the previous equation, giving some results on particular values of any of terms concerning the trace of the generators of a group. Also B. Maskit in [25] studied subgroups G of $\mathrm{PSL}(2,\mathbb{C})$, generated by two elements A and B, where A has two fixed points and B maps one fixed point of A onto the other, and finds when G is discrete. On another paper, L. Baribeau and T. Ransford [1] study the subset $D \subset \mathbb{C}^3$ of numbers $(\mathrm{tr}^2(f)-4, \mathrm{tr}^2(g)-4, \mathrm{tr}(fgf^{-1}g^{-1}))$, which arise from discrete subgroups. In dimension two, we would like to have similiar results that can help us to know if a group is discrete. The need of results on the subject have appeared continuously when working on different problems. We iniciate the study of discreteness for subgroups generated by two elements of $\mathrm{PSL}(3,\mathbb{C})$. We begin an analogous work, inspired on an initial results in dimension one, [24, pag.19]:

Proposition 5.1. *If g and f are non trivial elements of $\mathrm{PSL}(2,\mathbb{C})$, where f is loxodromic and f and g have exactly one fixed point in common, then $\langle f, g\rangle$ is not discrete.*

In Propositon 5.1 it is required that the elements have one fixed point in common. However for subgroups acting on $\mathbf{P}^2_{\mathbb{C}}$, where the limit set of the transformations contains lines, sounds more accurate if the elements $f, g \in \mathrm{PSL}(3,\mathbb{C})$ have in common not only points; in this case the elements will have a flag (Definition **??**) in common.

In the first section of this Chapter we find propositions that tell us when a group is not discrete, depending on the properties of the elements generating the group and their limit set. Specifically we take $f \in \mathrm{PSL}(3,\mathbb{C})$ a loxodromic element, and g another transformation in $\mathrm{PSL}(3,\mathbb{C})$ such that some dynamical conditions are satisfied, then we take either $f^{-m} \circ g \circ f^m$ or $f^m \circ g \circ f^{-m}$ to find a convergent sequence.

The second transformation g will be previously determined in Lemmas 5.5, 5.6, 5.7 and 5.8, each lemma considers the transformation f as one type of loxodromic element (strongly loxodromic, complex homothety, screw or loxoparabolic, respectively), the characteristics of g depend on the type of f. We use those Lemmas to prove the corresponding following Propositions; first we write a definition:

Definition 5.2. Let f and g be two different transformations in $\mathrm{PSL}(3,\mathbb{C})$. We say that *$f$ and g have a flag $p \in \ell$ in common* if whenever $\ell \subset \Lambda_{\mathrm{K}}(f)$ and p is a fixed point for $\langle f \rangle$, then the line ℓ is invariant under the action of g and p is a global fixed point, where *flag in $\mathbf{P}^2_{\mathbb{C}}$ is a complex projective line ℓ and a point $\mathbf{p}$ in ℓ. The flag will be denoted as $\mathbf{p} \in \ell$.*

Proposition 5.9. Let $f \in \mathrm{PSL}(3,\mathbb{C})$ be a strongly loxodromic transformation with a lift $\mathbf{f} \in \mathrm{SL}(3,\mathbb{C})$, $\mathbf{f} = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3)$ with $|\lambda_1| < |\lambda_2| < |\lambda_3|$. For $g \in \mathrm{PSL}(3,\mathbb{C})$ such that f and g have a flag in common, the Kulkarni limit set $\Lambda_{\mathrm{K}}(f)$ is not invariant under g and g is triangular the group $\langle \mathbf{f}, \mathbf{g} \rangle$ is not discrete.

Proposition 5.11. Let $f \in \mathrm{PSL}(3,\mathbb{C})$ be a complex homothety with a lift $\mathbf{f} \in \mathrm{SL}(3,\mathbb{C})$, $\mathbf{f} = \mathrm{Diag}(\lambda, \lambda, \lambda^{-2})$ and $|\lambda| > 1$. For $g \in \mathrm{PSL}(3,\mathbb{C})$ such that f and g have a flag in common, the Kulkarni limit set $\Lambda_{\mathrm{K}}(f)$ is not invariant under g, the group $\langle \mathbf{f}, \mathbf{g} \rangle$ is not discrete.

Proposition 5.12. Let f and g be transformations in $\mathrm{PSL}(3,\mathbb{C})$ such that f is a screw transformation with a lift $\mathbf{f} \in \mathrm{SL}(3,\mathbb{C})$, $\mathbf{f} = \mathrm{Diag}(\lambda, \mu, (\lambda\mu)^{-1})$ and $|\lambda| = |\mu| > 1$. If $g \in \mathrm{PSL}(3,\mathbb{C})$ is such that f and g have a flag in common, the Kulkarni limit set $\Lambda_{\mathrm{K}}(f)$ is not invariant under g, the group $\langle \mathbf{f}, \mathbf{g} \rangle$ is not discrete.

Proposition 5.13. Let f and g be transformations in $\mathrm{PSL}(3,\mathbb{C})$ such that f is a loxoparabolic transformation with a lift $\mathbf{f} \in \mathrm{SL}(3,\mathbb{C})$,

$$\mathbf{f} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}$$

where $|\lambda| > 1$. If $g \in \mathrm{PSL}(3,\mathbb{C})$ is such that f and g have the flag in common: $e_1 \in \overleftrightarrow{e_1 e_2}$, and the Kulkarni limit set $\Lambda_{\mathrm{K}}(f)$ is not invariant under g, then $\langle \mathbf{f}, \mathbf{g} \rangle$ is not discrete.

The commutator of two elements can be relevant to find properties of the group that they generate, for example in [4] and in [5], the commutator is widely used. Also inspired by the next result for transformations of the sphere $\mathbb{S}^2$, we analyze the commutator of two elements in $\mathrm{PSL}(3,\mathbb{C})$;

Proposition 5.3. *[24, pag.12] If $f \in \mathrm{PSL}(2,\mathbb{C})$ has exactly two fixed points and f and $g \in \mathrm{PSL}(2,\mathbb{C})$ share exactly one fixed point, then the commutator $[f,g]$ is parabolic.*

A loxodromic element f in $\mathrm{PSL}(3,\mathbb{C})$ has either one or two invariant lines on its Kulkarni limit set and if we consider a second element $g \in \mathrm{PSL}(3,\mathbb{C})$ its Kulkarni limit set, which can contain also a line, could coincide with one line of $\Lambda_K(f)$. A more suitable requirement would be that the two transformations have a flag in common. However, another dynamical condition arises to guarantee that the commutator is parabolic, that is, that the Kulkarni limit set of the first transformation is not invariant under the second. In symbols, $\mathbf{g}(\Lambda_K(\mathbf{f})) \neq \Lambda_K(\mathbf{f})$.

Proposition 5.16. Let f and g be two transformations in $\mathrm{PSL}(3,\mathbb{C})$ such that f is a loxodromic element, and $\mathbf{g}(\Lambda_K(\mathbf{f})) \neq \Lambda_K(\mathbf{f})$. Suppose that $\mathbf{f}$ and $\mathbf{g}$ have one flag in common, then $[\mathbf{f},\mathbf{g}]$ is parabolic.

The proof comprises fifteen cases, depending on the type of loxodromic element f we take first and the different flags that f and g have in common. In each case, we look at the commutator $[f,g]$ (which is not the identity), its trace, and the trace of $[f,g]^{-1}$, just to see that these traces are equal to 3, and satisfy the hypothesis of Theorem 5.15, and thus, we conclude that the commutator is parabolic.

We would like to understand the Kulkarni limit set of the elementary subgroups of $\mathrm{PSL}(3,\mathbb{C})$; one crucial step in this understanding is to characterize the metabelian groups; to achieve this we propose a program presented in the third Section of this Chapter. In the program, the results presented in Sections 5.1 and 5.2 are going to be essencial.

5.1 Non-discrete subgroups with two generators

Remark 5.4. Observe that if $\mathbf{p}$ and $\mathbf{q}$ are two different points in the line ℓ, then the flag $\mathbf{p} \in \ell$ is different from the flag $\mathbf{q} \in \ell$.

The lemmas that present the form of the transformation g are stated

Lemma 5.5. *Let $f \in \mathrm{PSL}(3,\mathbb{C})$ be a strongly loxodromic transformation with a lift $\mathbf{f} \in \mathrm{SL}(3,\mathbb{C})$, $\mathbf{f} = \mathrm{Diag}(\lambda_1,\lambda_2,\lambda_3)$ with $|\lambda_1| < |\lambda_2| < |\lambda_3|$. There exists $g \in \mathrm{PSL}(3,\mathbb{C})$ such that f and g have a flag in common and the Kulkarni limit set $\Lambda_K(f)$ is not invariant under g. Moreover, the form of g depends on the flag in the following way:*

- *If the flag is $e_1 \in \overleftrightarrow{e_1e_2}$, then* $\mathbf{g} = \mathbf{g_1} = \begin{pmatrix} a & b & c \\ 0 & e & t \\ 0 & 0 & j \end{pmatrix}$, *with* $|b| + |c| \neq 0$,
- *If the flag is $e_2 \in \overleftrightarrow{e_1e_2}$, then* $\mathbf{g} = \mathbf{g_2} = \begin{pmatrix} a & 0 & c \\ d & e & t \\ 0 & 0 & j \end{pmatrix}$, *with* $c \neq 0$.
- *If the flag is $e_2 \in \overleftrightarrow{e_2e_3}$, then* $\mathbf{g} = \mathbf{g_3} = \begin{pmatrix} a & 0 & 0 \\ d & e & t \\ s & 0 & j \end{pmatrix}$, *with* $s \neq 0$.
- *If the flag is $e_3 \in \overleftrightarrow{e_2e_3}$, then* $\mathbf{g} = \mathbf{g_4} = \begin{pmatrix} a & 0 & 0 \\ d & e & 0 \\ s & h & j \end{pmatrix}$, *with* $|s| + |h| \neq 0$.

Proof. Suppose that f is conjugate to a transformation that has a lift:

$$\mathbf{f} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

then, the Kulkarni limit set is $\Lambda_K(f) = \overleftrightarrow{e_1e_2} \cup \overleftrightarrow{e_2e_3}$. The possible flags that f and g have in common are: $e_1 \in \overleftrightarrow{e_1e_2}$, $e_2 \in \overleftrightarrow{e_1e_2}$, $e_2 \in \overleftrightarrow{e_2e_3}$ and $e_3 \in \overleftrightarrow{e_2e_3}$. We will prove the result for each of these flags.

The flag is $e_1 \in \overleftrightarrow{e_1e_2}$. First consider g having a lift given by a matrix

$$\mathbf{g} = \begin{pmatrix} a & b & c \\ d & e & t \\ s & h & j \end{pmatrix}, \tag{5.1}$$

This matrix when multiplied by $(x, y, 0)^T$ is equal to $(ax + by, dx + ey, sx + hy)^T$, and the line $\overleftrightarrow{e_1e_2}$ is invariant under g if and only if $s = h = 0$. Moreover, we are supposing that the point e_1 is a fixed point of g; this happens if and only if $d = 0$. The line $\overleftrightarrow{e_2e_3}$ is invariant under g :

$$\mathbf{g} = \begin{pmatrix} a & b & c \\ 0 & e & t \\ 0 & 0 & j \end{pmatrix} \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \neq \begin{pmatrix} 0 \\ y' \\ z' \end{pmatrix},$$

then $|b| + |c| \neq 0$. So the transformation is given by the matrix:

$$\mathbf{g_1} = \begin{pmatrix} a & b & c \\ 0 & e & t \\ 0 & 0 & j \end{pmatrix}, \text{ with } |b| + |c| \neq 0. \tag{5.2}$$

The flag is $e_2 \in \overleftrightarrow{e_1e_2}$. In this case, unlike the previous, e_2 is the fixed point. Then, besides $s = h = 0$, also $b = 0$. Then the line $\overleftrightarrow{e_2e_3}$ is not invariant under g if and only if $c \neq 0$. And the transformation is:

$$\mathbf{g_2} = \begin{pmatrix} a & 0 & c \\ d & e & t \\ 0 & 0 & j \end{pmatrix}, \text{ with } c \neq 0. \tag{5.3}$$

The flag is $e_2 \in \overleftrightarrow{e_2e_3}$. If g is as in equation (5.1), suppose the line $\overleftrightarrow{e_2e_3}$ is invariant under g. When the general expression of g is multiplied by $(0, y, z)^T$ the result is $(by + cz, ey + tz, hy + jz)^T$, and this point is in the line $\overleftrightarrow{e_2e_3}$ if and only if $b = c = 0$. Then, e_2 is fixed point if and only if $h = 0$. Finally, if $a \neq 0$, then $g(\overleftrightarrow{e_1e_2}) \neq \overleftrightarrow{e_1e_2}$. The transformation we are looking for is:

$$\mathbf{g_3} = \begin{pmatrix} a & 0 & 0 \\ d & e & t \\ s & 0 & j \end{pmatrix}, \text{ with } s \neq 0. \tag{5.4}$$

The flag is $e_3 \in \overleftrightarrow{e_2e_3}$. In this case, if g besides having $b = c = 0$, also has $t = 0$ then e_3 is a fixed point for the transformation. For g to move the line $\overleftrightarrow{e_1e_2}$, it is necessary that the following equation is satisfied: $|s| + |h| \neq 0$. The transformation is:

$$\mathbf{g_4} = \begin{pmatrix} a & 0 & 0 \\ d & e & 0 \\ s & h & j \end{pmatrix}, \text{ with } |s| + |h| \neq 0. \tag{5.5}$$

□

Lemma 5.6. *Let $f \in \mathrm{PSL}(3, \mathbb{C})$ be a complex homothety with a lift $\mathbf{f} \in \mathrm{SL}(3, \mathbb{C})$, $\mathbf{f} = \mathrm{Diag}(\lambda, \lambda, \lambda^{-2})$ where $|\lambda| > 1$. Then, there exists $g \in \mathrm{PSL}(3, \mathbb{C})$ such that f and g have a flag in common, and the Kulkarni limit set $\Lambda_K(f)$ is not invariant under g. Moreover, the form of g is:*

$$\mathbf{g} = \mathbf{g_5} = \begin{pmatrix} a & \frac{(\mu-a)x_0}{y_0} & c \\ \frac{(\mu-e)y_0}{x_0} & e & t \\ 0 & 0 & j \end{pmatrix}, \textit{ with } |c| + |t| \neq 0.$$

where μ a positive non-zero complex number.

Proof. The Kulkarni limit set of a complex homothety is $\Lambda_{\mathrm{K}}(f) = \overleftrightarrow{e_1e_2} \cup \{e_3\}$. And every point $p \in \overleftrightarrow{e_1e_2}$ is a fixed point. Then f and g have in common any of the flags $p \in \overleftrightarrow{e_1e_2}$.

As we have seen, the line $\overleftrightarrow{e_1e_2}$ is invariant if and only if $s = h = 0$. If the transformation g has as fixed point $(x_0, y_0, 0)^T \in \overleftrightarrow{e_1e_2}$, the expression of g is:

$$\mathbf{g_5} = \begin{pmatrix} a & \frac{(\mu-a)x_0}{y_0} & c \\ \frac{(\mu-e)y_0}{x_0} & e & t \\ 0 & 0 & j \end{pmatrix},$$

for some $\mu \neq 0$. Also the point e_3 should not be invariant under this matrix. To achieve this, the requirement is $|c| + |t| \neq 0$. In summary, $\mathbf{g}$ is like this:

$$\mathbf{g_5} = \begin{pmatrix} a & \frac{(\mu-a)x_0}{y_0} & c \\ \frac{(\mu-e)y_0}{x_0} & e & t \\ 0 & 0 & j \end{pmatrix}, \text{ with } |c| + |t| \neq 0. \tag{5.6}$$

□

Lemma 5.7. *Let $f \in \mathrm{PSL}(3, \mathbb{C})$ be a screw transformation with a lift $\mathbf{f} \in \mathrm{SL}(3, \mathbb{C})$, $\mathbf{f} = \mathrm{Diag}(\lambda, \mu, (\lambda\mu)^{-1})$ and $|\lambda| = |\mu| > 1$. Then, there exists $g \in \mathrm{PSL}(3, \mathbb{C})$ such that f and g have a flag in common, and the Kulkarni limit set $\Lambda_K(f)$ is not invariant under g. Moreover, the form of g is:*

- *If the flag is $e_1 \in \overleftrightarrow{e_1e_2}$, then $\mathbf{g} = \mathbf{g_6} = \begin{pmatrix} a & b & c \\ 0 & e & t \\ 0 & 0 & j \end{pmatrix}$, with $|c| + |t| \neq 0$.*

- *If the flag is* $e_2 \in \overleftrightarrow{e_1e_2}$, *then* $\mathbf{g} = \mathbf{g_7} = \begin{pmatrix} a & 0 & c \\ d & e & t \\ 0 & 0 & j \end{pmatrix}$, *with* $|c| + |t| \neq 0$.

Proof. The Kulkarni limit set of a screw is $\Lambda_K(f) = \overleftrightarrow{e_1e_2} \cup e_3$. The possible flags that f and g can have in common are $e_1 \in \overleftrightarrow{e_1e_2}$ and $e_2 \in \overleftrightarrow{e_1e_2}$. As we have seen, the line $\overleftrightarrow{e_1e_2}$ is invariant under g if and only if $s = h = 0$.

The flag is $e_1 \in \overleftrightarrow{e_1e_2}$. For e_1 to be fixed, d should be zero. And for e_3 to be moved by g, then $|c| + |t| \neq 0$. The transformation is:

$$\mathbf{g_6} = \begin{pmatrix} a & b & c \\ 0 & e & t \\ 0 & 0 & j \end{pmatrix}, \text{ with } |c| + |t| \neq 0. \tag{5.7}$$

The flag is $e_2 \in \overleftrightarrow{e_1e_2}$. For e_2 to be fixed, b should be zero. And for e_3 to be moved by g, then $|c| + |t| \neq 0$. The transformation is:

$$\mathbf{g_7} = \begin{pmatrix} a & 0 & c \\ d & e & t \\ 0 & 0 & j \end{pmatrix}, \text{ with } |c| + |t| \neq 0. \tag{5.8}$$

□

Lemma 5.8. *Let* $f \in \mathrm{PSL}(3, \mathbb{C})$ *be a loxoparabolic transformation with a lift* $\mathbf{f} \in \mathrm{SL}(3, \mathbb{C})$,

$$\mathbf{f} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}$$

where $|\lambda| > 1$. *Then, there exists* $g \in \mathrm{PSL}(3, \mathbb{C})$ *such that* f *and* g *have a flag in common, and the Kulkarni limit set* $\Lambda_K(f)$ *is not invariant under* g. *Moreover, the form of* g *is:*

- *If the flag is* $e_1 \in \overleftrightarrow{e_1e_2}$, *then* $\mathbf{g} = \mathbf{g_8} = \begin{pmatrix} a & b & c \\ 0 & e & t \\ 0 & 0 & j \end{pmatrix}$, *with* $t \neq 0$.

- *If the flag is* $e_1 \in \overleftrightarrow{e_1e_3}$, *then* $\mathbf{g} = \mathbf{g_9} = \begin{pmatrix} a & b & c \\ 0 & e & 0 \\ 0 & h & j \end{pmatrix}$, *with* $h \neq 0$.

- *If the flag is* $e_3 \in \overleftrightarrow{e_1e_3}$, *the* $\mathbf{g} = \mathbf{g_{10}} = \begin{pmatrix} a & b & 0 \\ 0 & e & 0 \\ s & h & j \end{pmatrix}$, *with* $|s| + |h| \neq 0$.

Proof. The Kulkarni limit set of a loxoparabolic transformation is $\Lambda_K(f) = \overleftrightarrow{e_1e_2} \cup \overleftrightarrow{e_1e_3}$. The possible flags that f and g can have in common are $e_1 \in \overleftrightarrow{e_1e_2}$ and $e_1 \in \overleftrightarrow{e_1e_3}$ and $e_3 \in \overleftrightarrow{e_1e_3}$.

The flag is $e_1 \in \overleftrightarrow{e_1e_2}$. The line $\overleftrightarrow{e_1e_2}$ is invariant under g if and only if $s = h = 0$. Also, e_1 is fixed if $d = 0$. The line $\overleftrightarrow{e_1e_3}$ should not be invariant under the transformation g, then:

$$\mathbf{g_8} = \begin{pmatrix} a & b & c \\ 0 & e & t \\ 0 & 0 & j \end{pmatrix} \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} ax + cz \\ tz \\ jz \end{pmatrix}$$

Then, $t \neq 0$. And $\mathbf{g_8}$ with the condition $t \neq 0$ is the transformation that makes the Lemma be true.

The flag is $e_1 \in \overleftrightarrow{e_1e_3}$. The coefficients of the general expression of transformation $g \in \mathrm{PSL}(3, \mathbb{C})$, as indicated in equation (5.1) must be $d = t = 0$ so the line $\overleftrightarrow{e_1e_3}$ is invariant. For e_1 to be a fixed point is necessary to ask $s = 0$. Finally, for g to move the line $\overleftrightarrow{e_1e_2}$, the coefficient h must be different from zero. Then

$$\mathbf{g_9} = \begin{pmatrix} a & b & c \\ 0 & e & 0 \\ 0 & h & j \end{pmatrix}, \text{ with } h \neq 0. \tag{5.9}$$

The flag is $e_3 \in \overleftrightarrow{e_1e_3}$. The coefficients of the general expression of transformation $g \in \mathrm{PSL}(3, \mathbb{C})$, as indicated in equation (5.1) must be $d = t = 0$ so the line $\overleftrightarrow{e_1e_3}$ is invariant. For e_3 to be a fixed point is necessary to ask $c = 0$. Finally, for g to move the line $\overleftrightarrow{e_1e_2}$, the coefficient h must be different from zero. Then

$$\mathbf{g_{10}} = \begin{pmatrix} a & b & 0 \\ 0 & e & 0 \\ s & h & j \end{pmatrix}, \text{ with } h \neq 0. \tag{5.10}$$

□

Therefore, with an exhaustive method, we prove Propositions 5.9, 5.11, 5.12 and 5.13:

Proposition 5.9. *Let $f \in \mathrm{PSL}(3, \mathbb{C})$ be a strongly loxodromic transformation with a lift $\mathbf{f} \in \mathrm{SL}(3, \mathbb{C})$, $\mathbf{f} = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3)$ with $|\lambda_1| < |\lambda_2| < |\lambda_3|$. For $g \in \mathrm{PSL}(3, \mathbb{C})$ such that f and g have a flag in common and and the Kulkarni limit set $\Lambda_K(f)$ is not invariant under g and g is triangular, $\langle \mathbf{f}, \mathbf{g} \rangle$ is not discrete.*

Proof. The transformations f and g can have in common any of the flags mentioned in Lemma 5.5. For each flag, there is a transformation $\mathbf{g_i}$, $\mathbf{i} = 1, \ldots 4$, satisfying the hypothesis of this Proposition. Then, we can consider the following sequences of elements in the group $\langle f, g \rangle$:

$$\mathbf{f}^m \circ \mathbf{g}_1 \circ \mathbf{f}^{-m} = \begin{pmatrix} a & b\left(\frac{\lambda_1}{\lambda_2}\right)^m & c\left(\frac{\lambda_1}{\lambda_3}\right)^m \\ 0 & e & t\left(\frac{\lambda_2}{\lambda_3}\right)^m \\ 0 & 0 & j \end{pmatrix}, \quad \mathbf{f}^m \circ \mathbf{g}_2 \circ \mathbf{f}^{-m} = \begin{pmatrix} a & 0 & c\left(\frac{\lambda_1}{\lambda_3}\right)^m \\ 0 & e & t\left(\frac{\lambda_2}{\lambda_3}\right)^m \\ 0 & 0 & j \end{pmatrix},$$

$$\mathbf{f}^{-m}\circ\mathbf{g}_3\circ\mathbf{f}^m=\begin{pmatrix} a & 0 & 0\\ d\left(\frac{\lambda_1}{\lambda_2}\right)^m & e & 0\\ s\left(\frac{\lambda_1}{\lambda_3}\right)^m & 0 & j\end{pmatrix},\quad \mathbf{f}^{-m}\circ\mathbf{g}_4\circ\mathbf{f}^m=\begin{pmatrix} a & 0 & 0\\ d\left(\frac{\lambda_1}{\lambda_2}\right)^m & e & 0\\ s\left(\frac{\lambda_1}{\lambda_3}\right)^m & h\left(\frac{\lambda_2}{\lambda_3}\right)^m & j\end{pmatrix},$$

In each case, the compositions represent the $m-th$ element of a convergent sequence of different elements in the group $\langle\mathbf{f},\mathbf{g}\rangle$; the group generated by $\mathbf{f}$ and $\mathbf{g}$ is not discrete. □

Remark 5.10. The loxodromic transformation f could also have one of the following forms $\mathrm{Diag}(\lambda_3,\lambda_2,\lambda_1)$ or $\mathrm{Diag}(\lambda_2,\lambda_1,\lambda_3)$ or $\mathrm{Diag}(\lambda_2,\lambda_3,\lambda_1)$, with $|\lambda_1|<|\lambda_2|<|\lambda_3|$. In the case that f is $\mathrm{Diag}(\lambda_3,\lambda_2,\lambda_1)$ then the result is still true, while if f has the second or third form, with an analogous analysis, we can conclude that whenever the fixed point of both transformations is an attracting or repelling point for the action, then $\langle f,g\rangle$ is not discrete. When the fixed point of both transformations is the saddle point, then we can not know.

There are examples, that some times, the groups can be discrete, for example if the group is the fundamental group of an Inoue Surface, that is Sol_0^4, Sol_1^4, $Sol_1^{'4}$.

Proposition 5.11. *Let* $\mathbf{f}\in\mathrm{PSL}(3,\mathbb{C})$ *be a complex homothety. For* $g\in\mathrm{PSL}(3,\mathbb{C})$ *such that* f *and* g *have a flag in common, and the Kulkarni limit set* $\Lambda_K(f)$ *is not invariant under* g, $\langle\mathbf{f},\mathbf{g}\rangle$ *is not discrete.*

Proof. We can suppose that $\mathbf{f}$ is the matrix $\mathrm{Diag}(\lambda,\lambda,\lambda^{-2})$ with $|\lambda|>1$. The points in the line $\overleftrightarrow{e_1e_2}$ are all fixed points for $\mathbf{f}$. Then the flag that the transformations $\mathbf{f}$ and $\mathbf{g}$ have in common can be $\mathbf{p}\in\overleftrightarrow{e_1e_2}$ for every $\mathbf{p}$. Suppose that $\mathbf{p}=[x_0:y_0:0]$ is the fixed point for $\mathbf{g}$. In that case, $\mathbf{g}$ has the form $\mathbf{g_5}$ as in equation (5.6) of Lemma 5.6 and the sequence

$$\mathbf{f}^{-m}\circ\mathbf{g}_5\circ\mathbf{f}^m=\begin{pmatrix} a & \frac{(\mu-a)x_0}{y_0} & \frac{c}{\lambda^{3m}}\\ \frac{(\mu-e)y_0}{x_0} & e & \frac{t}{\lambda^{3m}}\\ 0 & 0 & j\end{pmatrix}$$

converges to an element in $\mathrm{PSL}(3,\mathbb{C})$. So, the group generated by $\mathbf{f}$ and $\mathbf{g}$ is not discrete.

In the case that f has a lift $\mathbf{f}=\mathrm{Diag}(\lambda,\lambda^{-2},\lambda)$ with $|\lambda|>1$, the Kulkarni limit set of this transformation is $\Lambda_{\mathrm{K}}(\mathbf{f})=\overleftrightarrow{e_1e_3}\cup\{e_2\}$. And the possible flags that f and g can have in common are $\mathbf{p}\in\overleftrightarrow{e_1e_3}$, for any $\mathbf{p}=[x_0:0:z_0]$, fixed point for g. In that case, g has the form:

$$\begin{pmatrix} a & b & \frac{(\mu-a)x_0}{z_0}\\ 0 & e & 0\\ \frac{(\mu-j)z_0}{x_0} & h & j\end{pmatrix},\ \text{for some } \mu\neq 0.$$

The composition $f^{-m}\circ g\circ f^m$ is equal to:

$$\begin{pmatrix} a & b\lambda^{-3m} & \frac{(\mu-a)x_0}{z_0}\\ 0 & e & 0\\ \frac{(\mu-j)z_0}{x_0} & h\lambda^{-3m} & j\end{pmatrix},$$

it converges to an element in $\mathrm{PSL}(3,\mathbb{C})$. So, the group generated by $\mathbf{f}$ and $\mathbf{g}$ in this way is not discrete. □

Proposition 5.12. *Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be transformations in* $\mathrm{PSL}(3,\mathbb{C})$ *such that* $\mathbf{f}$ *is a screw transformation. If* $g \in \mathrm{PSL}(3,\mathbb{C})$ *is such that* f *and* g *have a flag in common, and the Kulkarni limit set* $\Lambda_K(f)$ *is not invariant under* g, *then* $\langle \mathbf{f}, \mathbf{g}\rangle$ *is not discrete.*

Proof. If f has a lift $\mathbf{f} \in \mathrm{SL}(3,\mathbb{C})$, $\mathbf{f} = \mathrm{Diag}(\lambda, \mu, (\lambda\mu)^{-1})$ where $|\lambda| = |\mu| > 1$, the flag that the Kulkarni limit set of f and g can be either $e_1 \in \overleftrightarrow{e_1 e_2}$ or $e_2 \in \overleftrightarrow{e_1 e_2}$.

In the first case, $\mathbf{g}$ is like $\mathbf{g_6}$ in equation (5.7) and in the second case is like $\mathbf{g_7}$ in equation (5.8). The conjugates $\mathbf{f}^m \circ \mathbf{g}_6 \circ \mathbf{f}^{-m}$ and $\mathbf{f}^m \circ \mathbf{g}_7 \circ \mathbf{f}^{-m}$ are

$$\begin{pmatrix} a & b\left(\frac{\mu}{\lambda}\right)^m & \frac{c\lambda}{(\lambda\mu)^m} \\ 0 & e & \frac{t\lambda}{\mu^{2m}} \\ 0 & 0 & j \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & 0 & \frac{c\lambda}{(\lambda\mu)^m} \\ d\left(\frac{\lambda}{\mu}\right)^m & e & \frac{t\lambda}{\mu^{2m}} \\ 0 & 0 & j \end{pmatrix}.$$

So we can find a convergent sequence to an element in $\mathrm{PSL}(3,\mathbb{C})$. So, the group generated by $\mathbf{f}$ and $\mathbf{g}$ is not discrete.

If f has a lift $\mathbf{f} \in \mathrm{SL}(3,\mathbb{C})$, $\mathbf{f} = \mathrm{Diag}(\lambda, (\lambda\mu)^{-1}, \mu)$ where $|\lambda| = |\mu| > 1$, the flag that the Kulkarni limit set of f and g can be either $e_1 \in \overleftrightarrow{e_1 e_3}$ or $e_3 \in \overleftrightarrow{e_1 e_3}$. In these cases, for an appropiate g, the conjugates $f^{-m} \circ g \circ f^m$ are:

$$\begin{pmatrix} a & b(\lambda^2\mu)^{-m} & c\left(\frac{\mu}{\lambda}\right)^m \\ 0 & e & 0 \\ 0 & h(\lambda\mu^2)^{-m} & j \end{pmatrix}, \text{ and } \begin{pmatrix} a & b(\lambda^2\mu)^{-m} & 0 \\ 0 & e & 0 \\ s\left(\frac{\lambda}{\mu}\right)^m & h(\lambda\mu^2)^{-m} & j \end{pmatrix};$$

in both cases, we can also find a convergent sequence to an element in $\mathrm{PSL}(3,\mathbb{C})$. So, the group generated by $\mathbf{f}$ and $\mathbf{g}$ is not discrete. □

The study of the loxoparabolic elements is not that straightforward. In Lemma 5.8 we saw that there are three possible flags that f and g can have in common.

Proposition 5.13. *Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be transformations in* $\mathrm{PSL}(3,\mathbb{C})$ *such that* $\mathbf{f}$ *is a loxoparabolic transformation with a lift* $\mathbf{f} \in \mathrm{SL}(3,\mathbb{C})$,

$$\mathbf{f} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}$$

where $|\lambda| > 1$. *If* $g \in \mathrm{PSL}(3,\mathbb{C})$ *is such that* f *and* g *have the flag in common is* $e_1 \in \overleftrightarrow{e_1 e_2}$, *and the Kulkarni limit set* $\Lambda_K(f)$ *is not invariant under* g, *then* $\langle \mathbf{f}, \mathbf{g}\rangle$ *is not discrete.*

Proof. We can find a sequence of different elements which converges:

$$\mathbf{f}^{-m} \circ \mathbf{g}_8 \circ \mathbf{f}^m = \begin{pmatrix} a & (a-e)\lambda^{-m} + b & \lambda^{-3m}(c - t\lambda^{-m}) \\ 0 & e & t\lambda^{-3m} \\ 0 & 0 & j \end{pmatrix}.$$

Therefore, the group generated by $\langle f, g\rangle$ is not discrete. □

However, when either $e_1 \in \overleftrightarrow{e_1e_3}$, or $e_3 \in \overleftrightarrow{e_1e_3}$ are the flags in common, we need to add another condition to g to be sure that the group generated by f and g is not discrete. This condition is merely algebraic. In the case that the flag is $e_1 \in \overleftrightarrow{e_1e_3}$, the condition is that the elements of the diagonal of the lift of g are equal and the coefficient in the third column and firt row of the matrix, is zero. Then, the following sequence converges:

$$\mathbf{f}^m \circ \mathbf{g}_9 \circ \mathbf{f}^{-m} = \begin{pmatrix} a & b & 0 \\ 0 & e & 0 \\ 0 & h\lambda^{-3m} & j \end{pmatrix}$$

In the case that the flag is $e_3 \in \overleftrightarrow{e_1e_3}$, it is only needed that the elements of the diagonal of the lift of g are equal. And we have the sequence

$$\mathbf{f}^m \circ \mathbf{g}_{10} \circ \mathbf{f}^{-m} = \begin{pmatrix} a & b & 0 \\ 0 & e & 0 \\ s\lambda^{-3m} & h\lambda^{-3m} + s\lambda^{-2m} & j \end{pmatrix},$$

which is also convergent. And the group $\langle f, g \rangle$ is not discrete.

Remark 5.14. If $\mathbf{f} = \begin{pmatrix} r^m & 0 & 0 \\ 0 & \frac{1}{(r\,s)^m} & 0 \\ 0 & 0 & s^m \end{pmatrix}$, then the element $\mathbf{g}^m = \mathbf{f}^{-m} \circ \mathbf{g} \circ \mathbf{f}^m$ would be as follows:

$$\mathbf{g}_m = \begin{pmatrix} a & \frac{b}{r^m\,(r\,s)^m} & \frac{c\,s^m}{r^m} \\ 0 & d & e\,s^m\,(r\,s)^m \\ 0 & 0 & j \end{pmatrix},$$

and in this way, it is not possible to conclude that the group generated by such $\mathbf{f}$ and $\mathbf{g}$ is not discrete.

The family of groups of Chapter 4 shows that there are groups generated by loxodromic elements (in particular, proximal elements) they do not have any flag in common, and they are discrete complex Kleinian groups.

5.2 Parabolic Commutator

We mention first a Theorem on which the proof of Proposition 5.16 relies.

Theorem 5.15. *[28, Theorem 7.3 (v)] Let* $F(x, y) = x^2y^2 - 4(x^3 + y^3) + 18xy - 27 \in \mathbb{C}[x, y]$, *and* $\mathbf{g} \in \mathrm{SL}(3, \mathbb{C})$. *Assume that* g *is the transformation in* $\mathrm{PSL}(3, \mathbb{C})$ *induced by* $\mathbf{g}$.

(v) g *is a unipotent parabolic transformation if and only if* $\mathrm{tr}(\mathbf{g}) \in 3C_3$, $\mathrm{tr}(\mathbf{g}^1) = \mathrm{tr}(\mathbf{g})$ *and* g *is not the identity element.*

Then, we begin with the proof of Proposition 5.16.

Proposition 5.16. *Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be two transformations in* $\mathrm{PSL}(3,\mathbb{C})$ *such that* $\mathbf{f}$ *is a loxodromic element, and* $\mathbf{g}(\Lambda_K(\mathbf{f})) \neq \Lambda_K(\mathbf{f})$. *Suppose that* $\mathbf{f}$ *and* $\mathbf{g}$ *have one flag in common, then* $[\mathbf{f},\mathbf{g}]$ *is parabolic.*

Proof. **Case A.** Consider $\mathbf{f}$ as a strongly loxodromic element. The transformation $\mathbf{f}$ is conjugated to a matrix of the form:

$$\mathbf{f} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \text{ with } |\lambda_1| < |\lambda_2| < |\lambda_3|. \tag{5.11}$$

The Kulkarni limit set of $\mathbf{f}$ is given by $\Lambda_{\mathrm{K}}(f) = \overleftrightarrow{e_1 e_2} \cup \overleftrightarrow{e_2 e_3}$.

Subcase A.1 The invariant line for $\mathbf{g}$ is $\overleftrightarrow{e_1 e_2}$ and e_1 is the fixed point of $\mathbf{g}$. If g has a lift $\mathbf{g} \in \mathrm{SL}(3,\mathbb{C})$ as $\mathbf{g_1}$ in Lemma 5.5, then the commutator is

$$[\mathbf{f},\mathbf{g}] = \begin{pmatrix} 1 & -\frac{b(\lambda_2-\lambda_1)}{e\lambda_2} & \frac{bt(\lambda_2-\lambda_1)}{ej\lambda_2} + \frac{c(\lambda_1-\lambda_3)}{j\lambda_3} \\ 0 & 1 & -\frac{t(\lambda_3-\lambda_2)}{j\lambda_3} \\ 0 & 0 & 1 \end{pmatrix}$$

The trace of the commutator $[\mathbf{f},\mathbf{g}]$ and its inverse is 3. And $[\mathbf{f},\mathbf{g}]$ is not the identity because b and c are not zero simultaneously.

Subcase A.2 The invariant line for $\mathbf{g}$ is $\overleftrightarrow{e_1 e_2}$ and e_2 is the fixed point of $\mathbf{g}$. If the transformation $\mathbf{g}$ is as $\mathbf{g_2}$ in Lemma 5.5, then the commutator is:

$$[\mathbf{f},\mathbf{g}] = \begin{pmatrix} 1 & 0 & -\frac{c(\lambda_3-\lambda_1)}{j\lambda_3} \\ \frac{d(\lambda_2-\lambda_1)}{a\lambda_1} & 1 & \frac{t(\lambda_3-\lambda_2)}{j\lambda_3} + \frac{cd(\lambda_2-\lambda_1)}{aj\lambda_1} \\ 0 & 0 & 1 \end{pmatrix}$$

The trace of the commutator $[\mathbf{f},\mathbf{g}]$ and its inverse is 3. As $-\frac{c(\lambda_3-\lambda_1)}{j\lambda_3} \neq 0$ and $[\mathbf{f},\mathbf{g}]$ is not the identity.

Subcase A.3 The invariant line for $\mathbf{g}$ is $\overleftrightarrow{e_2 e_3}$ and e_2 is the fixed point of $\mathbf{g}$. If g has a lift in $\mathrm{SL}(3,\mathbb{C})$ as $\mathbf{g_3}$ in Lemma 5.5, then the commutator is:

$$[\mathbf{f},\mathbf{g}] = \begin{pmatrix} 1 & 0 & 0 \\ \frac{st(\lambda_3-\lambda_2)}{aj\lambda_3} + \frac{d(\lambda_2-\lambda_1)}{a\lambda_1} & 1 & \frac{t(\lambda_2-\lambda_3)}{j\lambda_3} \\ \frac{s(\lambda_3-\lambda_1)}{a\lambda_1} & 0 & 1 \end{pmatrix}$$

The traces of the commutator $[\mathbf{f},\mathbf{g}]$ and its inverse is 3. As at least $\frac{s(\lambda_3-\lambda_1)}{a\lambda_1} \neq 0$, and $[\mathbf{f},\mathbf{g}]$ is not the identity.

Subcase A.4 The invariant line for $\mathbf{g}$ is $\overleftrightarrow{e_2 e_3}$ and e_3 is the fixed point of $\mathbf{g}$. If $\mathbf{g}$ is as $\mathbf{g_4}$ in Lemma

5.5, then the commutator is:

$$[\mathbf{f},\mathbf{g}] = \begin{pmatrix} 1 & 0 & 0 \\ \frac{d(\lambda_2-\lambda_1)}{a\lambda_1} & 1 & 0 \\ \frac{s(\lambda_3-\lambda_1)}{a\lambda_1} + \frac{dh(\lambda_3-\lambda_2)}{ae\lambda_2} & \frac{h(\lambda_3-\lambda_2)}{e\lambda_2} & 1 \end{pmatrix}$$

The traces of the commutator $[\mathbf{f},\mathbf{g}]$ and its inverse is 3. The sum $\left|\frac{s(\lambda_3-\lambda_1)}{a\lambda_1}\right| + \left|\frac{h(\lambda_3-\lambda_2)}{e\lambda_2}\right|$ is different form zero and $[\mathbf{f},\mathbf{g}]$ is not the identity.

Case B. Consider $\mathbf{f}$ as a loxoparabolic element. The transformation $\mathbf{f}$ is conjugated to a matrix of the form:

$$\mathbf{f} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix} \text{ with } |\lambda| > 1. \tag{5.12}$$

The Kulkarni limit set of $\mathbf{f}$ is given by $\Lambda_{\mathrm{K}}(\langle f \rangle) = \overleftrightarrow{e_1 e_2} \cup \overleftrightarrow{e_1 e_3}$.

Subcase B.1 The invariant line for $\mathbf{g}$ is $\overleftrightarrow{e_1 e_2}$ and e_1 is the fixed point of $\mathbf{g}$. Then $\mathbf{g}$ is as $\mathbf{g_8}$ in Lemma 5.8, and the commutator is:

$$[\mathbf{f},\mathbf{g}] = \begin{pmatrix} 1 & \frac{e-a}{e\lambda} & \frac{ce\lambda(\lambda^3-1)}{j} + \frac{t(e\lambda^3-e+a)}{ej\lambda} \\ 0 & 1 & \frac{t(\lambda^3-1)}{j} \\ 0 & 0 & 1 \end{pmatrix}$$

The traces of the commutator $[\mathbf{f},\mathbf{g}]$ and its inverse is 3. As $\frac{t(\lambda^3-1)}{j} \neq 0$ and $[\mathbf{f},\mathbf{g}]$ is not the identity.

Subcase B.2 The invariant line for $\mathbf{g}$ is $\overleftrightarrow{e_1 e_3}$ and e_1 is the fixed point of $\mathbf{g}$. Then $\mathbf{g}$ is as $\mathbf{g_9}$ in Lemma 5.8 and the commutator is:

$$[\mathbf{f},\mathbf{g}] = \begin{pmatrix} 1 & -\frac{c\,h\,\lambda^4 - c\,h\,\lambda + (a-e)\,j}{e\,j\,\lambda} & \frac{c\,\lambda^3 - c}{j} \\ 0 & 1 & 0 \\ 0 & -\frac{h\,\lambda^3 - h}{e\,\lambda^3} & 1 \end{pmatrix}$$

The traces of the commutator $[\mathbf{f},\mathbf{g}]$ and its inverse is 3. As $-\frac{h\,\lambda^3-h}{e\,\lambda^3} \neq 0$ and $[\mathbf{f},\mathbf{g}]$ is not the identity.

Subcase B.3 The invariant line for $\mathbf{g}$ is $\overleftrightarrow{e_1 e_3}$ and e_3 is the fixed point of $\mathbf{g}$. Then for $\mathbf{g}$ as $\mathbf{g_{10}}$ in Lemma 5.8, the commutator is:

$$[\mathbf{f},\mathbf{g}] = \begin{pmatrix} 1 & \frac{e-a}{e\,\lambda} & 0 \\ 0 & 1 & 0 \\ -\frac{\left(\lambda^3-1\right)s}{a\,\lambda^3} & \frac{\left(b\,\lambda^4 - b\,\lambda - a\right)s - a\,h\,\lambda^4 + a\,h\,\lambda}{a\,e\,\lambda^4} & 1 \end{pmatrix}$$

The traces of the commutator $[\mathbf{f},\mathbf{g}]$ and its inverse is 3. As $\left|\frac{\left(\lambda^3-1\right)s}{a\,\lambda^3}\right| + \left|\frac{h(\lambda^3-1)}{e\lambda^3}\right| \neq 0$, $[\mathbf{f},\mathbf{g}]$ is not the identity.

Case C. Consider **f** as a complex homothety. The transformation **f** is conjugated to a matrix of the form:

$$\mathbf{f} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix} \text{ with } |\lambda| > 1. \tag{5.13}$$

The Kulkarni limit set of **f** is given by $\Lambda_{\mathrm{K}}(\langle f \rangle) = \overleftrightarrow{e_1 e_2} \cup \{e_3\}$, where $\overleftrightarrow{e_1 e_2}$ is a line of fixed points for f. For any point $\mathbf{P}^2_{\mathbb{C}}$ in $\overleftrightarrow{e_1 e_2}$, the flag $\mathbf{P}^2_{\mathbb{C}} \in \overleftrightarrow{e_1 e_2}$ can be the flag that f and g have in common. Then g can be as in $\mathbf{g_5}$ in Lemma 5.6. So the commutator is:

$$[\mathbf{f}, \mathbf{g}] = \begin{pmatrix} 1 & 0 & \frac{c(\lambda^3-1)}{j} \\ 0 & 1 & \frac{t(\lambda^3-1)}{j} \\ 0 & 0 & 1 \end{pmatrix} \tag{5.14}$$

The traces of the commutator $[\mathbf{f}, \mathbf{g}]$ and its inverse is 3. The sum $\left|\frac{c(\lambda^3-1)}{j}\right| + \left|\frac{t(\lambda^3-1)}{j}\right|$ is different form zero and $[\mathbf{f}, \mathbf{g}]$ is not the identity.

Case D Consider **f** as screw element. The transformation **f** is conjugated to a matrix of the form:

$$\mathbf{f} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & (\lambda\mu)^{-1} \end{pmatrix} \text{ with } |\lambda| = |\mu| > 1. \tag{5.15}$$

The Kulkarni limit set of **f** is given by $\Lambda_{\mathrm{K}}(\langle f \rangle) = \overleftrightarrow{e_1 e_2} \cup \{e_3\}$.

Subcase D.1 The invariant line is $\overleftrightarrow{e_1 e_2}$ and e_1 is the fixed point for **g**. Then **g** is as $\mathbf{g_6}$ in Lemma 5.7. So the commutator is:

$$[\mathbf{f}, \mathbf{g}] = \begin{pmatrix} 1 & \frac{-b(\mu-\lambda)}{e\mu} & \frac{bt(\mu-\lambda)}{ej\mu} + \frac{c(\lambda^2\mu-1)}{j} \\ 0 & 1 & \frac{t(\lambda\mu^2-1)}{j} \\ 0 & 0 & 1 \end{pmatrix}$$

The traces of the commutator $[\mathbf{f}, \mathbf{g}]$ and its inverse is 3. The sum $\left|\frac{c(\lambda^2\mu-1)}{j}\right| + \left|\frac{t(\lambda\mu^2-1)}{j}\right|$ does not vanish and $[\mathbf{f}, \mathbf{g}]$ is not the identity.

Subcase D.2 The invariant line is $\overleftrightarrow{e_1 e_2}$ and e_2 is the fixed point for **g**. Then **g** is as $\mathbf{g_7}$ in Lemma 5.7. So the commutator is:

$$[\mathbf{f}, \mathbf{g}] = \begin{pmatrix} 1 & 0 & \frac{c(\lambda^2\mu-1)}{j} \\ \frac{d(\mu-\lambda)}{a\lambda} & 1 & \frac{t(\lambda\mu^2-1)}{j} - \frac{dc(\mu-\lambda)}{aj\lambda} \\ 0 & 0 & 1 \end{pmatrix}$$

The traces of the commutator $[\mathbf{f},\mathbf{g}]$ and its inverse is 3. The sum $\left|\frac{c(\lambda^2\mu-1)}{j}\right| + \left|\frac{t(\lambda\mu^2-1)}{j}\right|$ is different from zero and $[\mathbf{f},\mathbf{g}]$ is not the identity.

All the commutators, fulfill the requirements of Theorem 5.15, so it is parabolic. □

Example 5.17. However, we can find two elements in $\mathrm{PGL}(3,\mathbb{C})$ such that their commutator is parabolic, one of those elements $\mathbf{f}$ has two complex projective lines $\overleftrightarrow{e_1e_2}$ and $\overleftrightarrow{e_2e_3}$ in the Kulkarni limit set and the other element $\mathbf{g}$ has a different line in its Kulkarni limit set, that is, f and g do not have a flag in common. Let $\mathbf{f}$ and $\mathbf{g}$ be the following matrices:

$$\mathbf{f} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \quad \text{and} \quad \mathbf{g} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.16}$$

The commutator is the same as the matrix $\mathbf{g}$. Yet the limit set of the cyclic group generated by $\mathbf{g}$ is the line $\overleftrightarrow{e_1e_3}$.

5.3 Applications and Future research

There are groups that have been fully classified according to their limit set, in particular, in terms of the number of lines lying in the Kulkarni limit set. For example, in [4], the authors provide a presentation of the group if its Kulkarni limit set is one complex line. Also, in [2] the authors provide an algebraic characterization of the subgroups of $\mathrm{PSL}(3,\mathbb{C})$ for which the maximum number of complex lines in general position contained in its limit set, according to Kulkarni, is equal to four. And it is proved that if the group has five or more lines in general position in its Kulkarni limit set, then the group has an infinity of lines in general position. Then it is only missing to find a description of the subgroups of $\mathrm{PSL}(3,\mathbb{C})$ with exactly two lines on its Kulkarni limit set. In this Section we propose a program to describe those groups.

An important case is to consider the purely parabolic groups, and this work it is done in [5]. The case that has not been studied is the dynamical and algebraic properties of the metabelian groups. The solution of this problem is essencial to have a complete description of the "elementary" subgroups of $\mathrm{PSL}(3,\mathbb{C})$; the study of the metabelian groups will help to better understand which is the "correct" notion of the limit set for subgroups of $\mathrm{PSL}(3,\mathbb{C})$ acting on $\mathbf{P}^2_{\mathbb{C}}$.

We have the following:

Definition 5.18. Let G be a group. G is metabelian if and only if $[[f_1,g_1],[f_2,g_2]]$ is the identity for every $f_1,g_1,f_2,g_2 \in G$.

In order to achieve a characterization of metabelian groups we propose the following program:

Step 1.

Prove that if G is a strongly irreducible subgroup of $\mathrm{PSL}(3,\mathbb{C})$, then there is a strongly loxodromic element in G.

Step 2.

Prove that if p, q, r, s are different points in $\mathbf{P}^2_{\mathbb{C}}$ and g_1 and $g_2 \in \mathrm{PSL}(3, \mathbb{C})$ are strongly loxodromic transformations satisfying that $p(q)$ is the attracting (repelling) point of g_1 and $r(s)$ is the attracting (repelling) point of g_2, then for some $N \in \mathbb{N}$, the group $\langle g_1^N, g_2^N \rangle$ is a Schottky-like group as defined in [8, Definition 2.1].

Definition 5.19. Let $\Sigma \subset \mathrm{PSL}(n+1, \mathbb{C})$ be a finite set which is symmetric (i.e. $a^{-1} \in \Sigma$ for all $a \in \Sigma$) and $A_\sigma = \{A_a\}_{a \in \Sigma}$ a family of compact non-empty pairwise disjoint subsets of $\mathbf{P}^n_{\mathbb{C}}$ such that for each $a \in \Sigma$ we have

$$\bigcup_{b \in \Sigma - \{a^{-1}\}} a(A_b) \subset A_a.$$

The group generated by Σ is called a Schottky-like group; and it is free, finitely generated and discrete.

Step 3.

Prove that if $G \subset \mathrm{PSL}(3, \mathbb{C})$ is irreducible, then there is a Schottky-like subgroup $H < G$.

Step 4.

When the previous statements are true, we can conclude that if $G \subset \mathrm{PSL}(3, \mathbb{C})$ is metabelian, then G is virtually reducible.

Step 5.

Prove that if $G \subset \mathrm{PSL}(3, \mathbb{C})$ is metabelian, then G is virtually triangularizable.

Step 6.

Prove that if $G \subset \mathrm{PSL}(3, \mathbb{C})$ is metabelian then G is finitely generated, finitely presented and an HNN-extension. Moreover

$$G = \langle \mathrm{PP}(G), g_1, g_2, g_3 \rangle,$$

where g_i is a loxodromic transformation for $i = 1, 2, 3$.

Step 7.

Prove that if $G \subset \mathrm{PSL}(3, \mathbb{C})$ is metabelian, then G is as one of the following forms:

1. $G < \mathrm{Sol}^4_0$ or $G < \mathrm{Sol}^4_1$ or $G < \mathrm{Sol'}^4_0$.
2. G is a subgroup of a hyperbolic toral group.
3. G is a subgroup whose all elements are parabolic.

Step 8.

Prove that if $G \subset \mathrm{PSL}(3, \mathbb{C})$ is a metabelian group, then the Kulkarni limit set $\Lambda_{\mathrm{K}}(G)$ is one of the following options:

- a complex projective line,
- a cone of complex projective lines over a circle,
- two cones of lines over circles, such that there are four lines in general position,
- a complex projective line and a point outside the line or
- three complex projective lines in general position.

The region of discontinuity $\Omega_K(G)$ is the maximal open subset where the action is properly discontinuous and coincide with the equicontinuity region.

The results in Sections 5.1 and 5.2 will be fundamental for the proof of Theorem 5.3, and Corollaries 5.3 and 5.3. Compare with the results in [5].

Appendix A

Conics

A.1 Rotation of a conic

To determine the angle of the conic, given by the equation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad \text{where} \quad B \neq 0,$$

introduce the variables:

$$x = \hat{x}\cos\theta - \hat{y}\sin\theta \quad \text{and} \quad y = \hat{x}\sin\theta + \hat{y}\cos\theta$$

and substitute for x and y in the original equation. This gives us then new equation in $\hat{x}$ and $\hat{y}$:

$$\begin{aligned} A(\hat{x}\cos\theta - \hat{y}\sin\theta)^2 + B(\hat{x}\cos\theta - \hat{y}\sin\theta)(\hat{x}\sin\theta + \hat{y}\cos\theta) + C(\hat{x}\sin\theta + \hat{y}\cos\theta)^2 \\ + D(\hat{x}\cos\theta - \hat{y}\sin\theta) + E(\hat{x}\sin\theta + \hat{y}\cos\theta) + F = 0. \end{aligned} \tag{A.1}$$

Performing the multiplication and collecting the similar terms gives:

$$\begin{aligned} &\hat{x}(A(\cos\theta)^2 + B(\cos\theta\sin\theta) + C(\sin\theta)^2) \\ &\quad + \hat{x}\hat{y}[-2A\cos\theta\sin\theta + B((\cos\theta)^2 - (\sin\theta)^2) + 2C\sin\theta\cos\theta] \\ &\quad + \hat{y}(A(\sin\theta)^2 - B\sin\theta\cos\theta + C(\cos\theta)^2) \\ &\quad + \hat{x}(D\cos\theta + E\sin\theta) + \hat{y}(-D\sin\theta + E\cos\theta) + F = 0 \end{aligned}$$

To eliminate the $\hat{x}\hat{y}-$term from this equation, choose θ so that the coefficient of this term is zero, that is, so that

$$-2A\cos\theta\sin\theta + B((\cos\theta)^2 - (\sin\theta)^2) + 2C\sin\theta\cos\theta = 0.$$

Simplifying this equation we have:

$$B((\cos\theta)^2 - (\sin\theta)^2) = 2(A-C)\cos\theta\sin\theta$$

and

$$\frac{(\cos\theta)^2 - (\sin\theta)^2}{2\cos\theta\sin\theta} = \frac{A-C}{B}.$$

Using the double angle formulas for sine and cosine gives

$$\cot 2\theta = \frac{A-C}{B}. \tag{A.2}$$

Bibliography

[1] Line Baribeau and Thomas Ransford. On the set of discrete two-generator groups. *Math. Proc. Cambridge Philos. Soc.*, 128(2):245–255, 2000.

[2] Waldemar Barrera, Angel Cano, and Juán Navarrete. Subgroups of $\mathrm{PSL}(3,\mathbb{C})$ with four lines in general position in its limit set. *Conformal Geometry and Dynamics.*, 15:160–176, 2011.

[3] Waldemar Barrera, Angel Cano, and Juán Navarrete. On the number of lines in the limit set for discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$. *Pacific J. Math.*, 281(1):17–49, 2016.

[4] Waldemar Barrera, Angel Cano, and Juan Pablo Navarrete. One line complex Kleinian groups. *Pacific J. Math.*, 272(2):275–303, 2014.

[5] Waldemar Barrera, Angel Cano, Juan Pablo Navarrete, and José Seade. Towards a Sullivan Dictionary in dimension two, Part I: Purely parabolic complex Kleinian groups. https://arxiv.org/pdf/1802.08360.pdf.

[6] Waldemar del Jesús Barrera Vargas, Angel Cano Cordero, and Juan Pablo Navarrete Carrillo. The limit set of discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$. *Math. Proc. Cambridge Philos. Soc.*, 150(1):129–146, 2011.

[7] Angel Cano. Schottky groups can not act on $\mathbb{P}^{2n}_{\mathbb{C}}$ as subgroups of $\mathrm{PSL}_{2n+1}(\mathbb{C})$. *Bull. Braz. Math. Soc. (N.S.)*, 39(4):573–586, 2008.

[8] Angel Cano, Luis Loaeza, and Alejandro Ucan-Puc. On Classical Uniformization Theorems for Higher Dimensional Complex Kleinian Groups. *Bulletin of the Brazilian Mathematical Society*, April 2017.

[9] Angel Cano, Juan Pablo Navarrete, and José Seade. *Complex Kleinian groups*, volume 303 of *Progress in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2013.

[10] S. S. Chen and L. Greenberg. Hyperbolic spaces. pages 49–87, 1974.

[11] J.-P. Conze and Y. Guivarc'h. Limit sets of groups of linear transformations. *Sankhyā Ser. A*, 62(3):367–385, 2000. Ergodic theory and harmonic analysis (Mumbai, 1999).

[12] William M. Goldman. *Complex hyperbolic geometry*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1999. Oxford Science Publications.

[13] Nikolay Gusevskii and John R. Parker. Complex hyperbolic quasi-Fuchsian groups and Toledo's invariant. *Geom. Dedicata*, 97:151–185, 2003. Special volume dedicated to the memory of Hanna Miriam Sandler (1960–1999).

[14] Joachim Hilgert and Karl-Hermann Neeb. *Structure and geometry of Lie groups*. Springer Monographs in Mathematics. Springer, New York, 2012.

[15] Jeremy Gray. *Linear Differential Equations and Group Theory from Riemann to Poincaré*. Modern Birkhäuser Classics. Birkhäuser Basel, 2 edition, 2008.

[16] Yueping Jiang, Shigeyasu Kamiya, and John R. Parker. Jørgensen's inequality for complex hyperbolic space. *Geom. Dedicata*, 97:55–80, 2003. Special volume dedicated to the memory of Hanna Miriam Sandler (1960–1999).

[17] Troels Jørgensen. On discrete groups of Möbius transformations. *Amer. J. Math.*, 98:739–749, 1976.

[18] Shigeyasu Kamiya. Notes on elements of $\mathrm{U}(1, n; \mathbf{C})$. *Hiroshima Math. J.*, 21(1):23–45, 1991.

[19] Linda Keen and Caroline Series. The Riley slice of Schottky space. *Proc. London Math. Soc. (3)*, 69(1):72–90, 1994.

[20] Yohei Komori and Caroline Series. The Riley slice revisited. In *The Epstein birthday schrift*, volume 1 of *Geom. Topol. Monogr.*, pages 303–316. Geom. Topol. Publ., Coventry, 1998.

[21] R. S. Kulkarni. Groups with domains of discontinuity. *Math. Ann.*, 237(3):253–272, 1978.

[22] Joseph Lehner. *Discontinuous groups and automorphic functions*. Mathematical Surveys, No. VIII. American Mathematical Society, Providence, R.I., 1964.

[23] M. Yu. Lyubich and V. V. Suvorov. Free subgroups of $\mathrm{SL}_2(\mathbf{C})$ with two parabolic generators. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 155(Differentsial'naya Geometriya, Gruppy Li i Mekh. VIII):150–155, 195, 1986.

[24] Bernard Maskit. *Kleinian groups*, volume 287 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1988.

[25] Bernard Maskit. Some special 2-generator Kleinian groups. *Proc. Amer. Math. Soc.*, 106(1):175–186, 1989.

[26] A. V. Masleĭ. Sufficient conditions for the discreteness for 2-generator subgroups in $\mathrm{PSL}(2, \mathbb{C})$. *Sibirsk. Mat. Zh.*, 54(5):1069–1086, 2013.

[27] J.-P. Navarrete. On the limit set of discrete subgroups of $\mathrm{PU}(2, 1)$. *Geom. Dedicata*, 122:1–13, 2006.

[28] Juan-Pablo Navarrete. The trace function and complex Kleinian groups in $\mathbb{P}^2_{\mathbb{C}}$. *Internat. J. Math.*, 19(7):865–890, 2008.

[29] John R. Parker and Pierre Will. A complex hyperbolic Riley slice. *Geom. Topol.*, 21(6):3391–3451, 2017.

[30] H. Poincaré. Sur les groupes kleins. *Acta Math.*, 3(1):49–92, 1883. Mir.

[31] José Seade and Alberto Verjovsky. Actions of discrete groups on complex projective spaces. In *Laminations and foliations in dynamics, geometry and topology (Stony Brook, NY, 1998)*, volume 269 of *Contemp. Math.*, pages 155–178. Amer. Math. Soc., Providence, RI, 2001.

[32] José Seade and Alberto Verjovsky. Higher dimensional complex Kleinian groups. *Math. Ann.*, 322(2):279–300, 2002.

[33] José Seade and Alberto Verjovsky. Complex Schottky groups. *Astérisque*, (287):xx, 251–272, 2003. Geometric methods in dynamics. II.

[34] De Lin Tan. On two-generator discrete groups of Möbius transformations. *Proc. Amer. Math. Soc.*, 106(3):763–770, 1989.

[35] J. Tits. Free subgroups in linear groups. *J. Algebra*, 20:250–270, 1972.

[36] Kenji Ueno. *An introduction to algebraic geometry*, volume 166 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1997. Translated from the 1995 Japanese original by Katsumi Nomizu.

Index