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The Arrovian Framework through the Lens of Combinatorial Topology

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Abstract

The Arrow's impossibility theorem, proposed by Kenneth J. Arrow in 1950, is a cornerstone of modern social choice theory that has sparked many proofs since its formulation. In this thesis, we present a generalization of this theorem to a class of preference domains that we call the *class of polarization and diversity over triples*, denoted $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$. Any domain in this class involves preference profiles in which there is *strong polarization* among some partition of the society (two coalitions of voters) over some triple of alternatives. That is, profiles in which society is divided into two coalitions of voters which agree on how to rank the alternatives in two pairs of alternatives, which belong to some triple of distinct alternatives $\{a, b, c\}$, and differ on how to rank the alternatives in the remaining pair of alternatives in $\{a, b, c\}$. Furthermore, when there are at least three voters, any domain in $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ also has at least one profile that is not *value-restricted*, a condition proposed by Sen in 1966. A profile that is not value-restricted is such that there is at least one triple of distinct alternatives, $\{a, b, c\}$, such that every alternative x in $\{a, b, c\}$ can occupy diverse places on the preference ranking of the voters restricted to $\{a, b, c\}$; that is, some voter says that x is the best alternative in $\{a, b, c\}$, other voter says that it is the second-best alternative in $\{a, b, c\}$, and a third distinct voter says that x is the worst alternative in $\{a, b, c\}$. An example of a domain in $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ that consists of group-separable profiles is presented. Group-separability (proposed by Inada in the 1960s) is a property of interest to the computational social choice literature. Following a paper published in the ACM Symposium on Principles of Distributed Computing (PODC) by Rajsbaum and Raventós-Pujol in 2022, we prove our results in an Arrovian combinatorial topology framework instead of the classical Arrovian framework. But in contrast to the work of these authors in PODC 2022, we do not restrict our analysis to the case of only two voters and only three alternatives, we allow for at least two voters and at least three alternatives. Furthermore, we prove that the classical and the combinatorial topology frameworks are equivalent, even in the context of domain restrictions.

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Contents

Chapter 1

Introduction

In 1950, Kenneth J. Arrow [1, 2], winner of the 1972 Nobel Prize in Economics, proved that certain desirable properties of certain type of voting methods are logically incompatible. This result, now known as Arrow's impossibility theorem (or just Arrow's theorem), gave birth to the academic field of Social Choice (see [3, 4] for a review on this field), a whole area of scientific inquiry which formally studies the aggregation of individual preferences to obtain collective outcomes. In recent decades, this area, traditionally of interest to economists, mathematicians and political scientist, has evolved to become also part of computer science in what is known as Computational Social Choice (see [14] for a review on this field).

According to Brandt et al. [15], Computational Social Choice is roughly about the two following endeavors:

1. Using tools and paradigms from computer science as new lens to study social choice mechanisms (mechanisms to aggregate individual information into collective outcomes) or to develop new ones.

2. Using social choice theory to guide the improvement or design of multi-agent systems in which agents' information needs to be aggregated to obtain collective outcomes.

In this thesis, following Rajsbaum and Raventós-Pujol [38], we use combinatorial topology to study Arrow's theorem. This area of mathematics has been very useful to study distributed computing [see 27]. Rajsbaum and Raventós-Pujol [38] establish some analogies between distributed computing and social choice from the point of view of combinatorial topology.

For further explaining the contents of this thesis, and in particular to present Arrow's theorem, we now need to introduce some notation and standard definitions, but we will revisit the topic more formally in Chapter 2 of this thesis[1]. We start with a finite set of alternatives, X, and a finite set of voters $\{1, \dots, n\}$, also denoted N. Each voter i has a *preference ranking* (sometimes just preference or just ranking) over the alternatives, which is a strict total order on X. If P_i is a preference ranking on X for some voter i and $x, y \in X$, we can write $xy \in P_i$ as xP_iy and read it as "voter i ranks alternative x above alternative y". We denote the set of all preferences on X as $W(X)$. A *preference profile* (on X) (from now on just a profile), denoted $\vec{P}$, is a n-tuple of the form $(P_1, \dots, P_n)$, where P_i is the

[1]Since we aim for Chapter 2 to be as self-contained as possible, we will repeat some definitions presented in this Introduction in Chapter 2. Apologies to the reader.

preference ranking for voter i. We denote the set of all profiles on X as $W(X)^n$ and call it the *unrestricted domain*. A *preference domain*, denoted D, is a non-empty subset of $W(X)^n$. For example, if $X = \{x, y, z\}$ and $n = 2$, then $D = \{(xyz, yzx), (zxy, zxy)\}$ is a domain of only two profiles out of the 36 profiles in the unrestricted domain for only two voters, i.e. $W(\{x, y, z\})^2$. To illustrate our notation (which generalizes in a straightforward manner to more voters and more alternatives), profile (xyz, yzx) should be read as "voter 1 ranks x over y, y over z, and x over z; and voter 2 ranks y over z, z over x, and y over x".

We follow [35], with slight adaptations, for the definitions in this paragraph. If $i \in N$, an *individual preference domain for voter* i, denoted D_i, is a non-empty subset of $W(X)$. We say that a preference domain D is Cartesian when it can be written as the Cartesian product of individual preference domains (including every individual). Formally, D is a *Cartesian preference domain* if $D = \prod_{i=1}^{n} D_i$. We say that D is *common* if it can be written as $D = D_c^n$, i.e. if it is Cartesian and $D_i = D_c$ for all $i \in N$.

A *social welfare function* (SWF or SWFs in plural) is a function of the form $f\colon D \to W(X)$, i.e. a function that associates with every profile in D a preference over the alternatives in X, which sometimes is referred to as the *social preference*.

Now we present some desirable properties for a SWF to have (how desirable these properties are is also a topic of inquiry and debate in the social choice literature, but we will not deal with that in this thesis). A SWF satisfies:

- *unanimity* if for all $\alpha, \beta \in X$, we have that every voter ranking α over β implies that α is ranked over β on the social preference.
- *Independence of Irrelevant Alternatives* (IIA) if the social ranking of any two alternatives depends solely on the ranking of those two alternatives in the individual rankings.
- *non-dictatorship* if there is no voter whose preference is always taken by the SWF as the social preference. If there is such a voter, he is called a *dictator* and the SWF in question is called a *dictatorship* or it is said to be *dictatorial*.

Arrow's impossibility theorem says that if there are at least three alternatives, then any SWF defined on the unrestricted domain that satisfies unanimity and IIA must be a dictatorship. In this thesis, following [38], we work with this strict total orders version of Arrow's theorem, which is one of some common versions in the social choice literature (the original version of this theorem [2] assumes weak orders). Arrow's theorem has been proven in many different ways. Arrow's proof [2] employed the concept of *decisive coalitions*, while Sen's [40] that of *almost-decisive coalitions*. Informally, a decisive coalition is a subset of voters G such that whenever they agree on ranking an alternative x over and alternative y, the social ranking places x over y, no matter the preferences of the other members of society, i.e. voters in G^c (the complement of G with respect to N). On the other hand, an almost-decisive coalition is a subset of voters G such that whenever every voter in G agrees on ranking x over y and every voter in G^c agrees on ranking y over x, then the preference of G prevails, i.e. the social ranking places x over y. Other proofs have employed the concept of *pivotal voters* [see 6, 22, 46]. There are also proofs that use almost-decisive coalitions and ultrafilters from set theory, like those of Kirman and Sondermann [32] and Hansson [24], and are of special interest to us because, in this thesis, we present a proof that falls into this category (but, as we will explain in a moment, our proof incorporates other elements like the use of combinatorial topology). Tang and Lin [43] presents an inductive proof and transforms the base case (the case of 2 voters and 3 alternatives) into a constraint satisfaction problem to prove the impossibility for this case with aid of a computer.

In 1993, Baryshnikov [10] presented an algebraic topology proof of Arrow's impossibility theorem. In order to do so, he used simplicial complexes to represent the set of all preferences and the set of all profiles. He also represented social welfare functions through chromatic simplicial maps. Rajsbaum and Raventós-Pujol [37, 38] used Baryshnikov's framework, but they provide some proofs of Arrow's impossibility theorem using combinatorial topology, instead of algebraic topology. These authors use the combinatorial topology approach for the case of 2 voters and 3 alternatives and then proceed by induction on the number of voters and alternatives to prove the general case.

In the social choice literature, Arrow's impossibility theorem has been circumvented in different ways. According to Barberà [7] this has been accomplished by relaxing one of the following assumptions: transitivity of social preferences, the unrestricted domain, IIA. In this thesis, we follow the second of these options. Therefore, we will work with non-empty sets of the unrestricted domain, i.e. $W(X)^n$. Following [35], we say that a domain D that does not escape Arrow's theorem is *Arrow-inconsistent*. Not escaping Arrow's theorem means that every SWF defined on D satisfying IIA and unanimity must be a dictatorship. If there is a SWF satisfying IIA, unanimity and non-dictatorship, then we would say that D escapes Arrow's theorem or, following [35], we say that D is *Arrow-consistent*. Following [20], we say that D is *super-Arrovian* if the next two conditions are satisfied:

1. D is Arrow-inconsistent
2. for every domain D' such that $D \subseteq D'$, we have that D' is Arrow-inconsistent.

In this thesis, we work with domain restrictions, and not only with the unrestricted domain. In particular, we will represent any preference domain $D \subseteq W(X)^n$ with a simplicial complex that we will denote N_D. Moreover, we will represent $W(X)$ with a simplicial complex $N_{W(X)}$ and any SWF satisfying IIA with a chromatic simplicial map. By using these combinatorial topology objects, we are working with a generalization of the combinatorial or algebraic topology framework that Baryshnikov introduced in [10] to prove Arrow's theorem (recall that this theorem assumes the unrestricted domain). Therefore, we will study Arrow's theorem under domain restrictions with aid of combinatorial topology objects. In the conclusions of this thesis, we comment on the advantages of doing so. In particular, we will see that this framework, in contrast to the classical one (the one that uses profiles of preferences, preference rankings and SWFs), provides geometric intuition to the study of domain restrictions. This combinatorial topology framework can also yield very simple proofs, as it was shown in [38], and as we aim to illustrate in this thesis.

Gaertner [21] says that efforts to study domains restrictions can be classified into the following two categories:

1. Study a particular method of preference aggregation like the simple majority rule and find out if there are domains in which such a method is a SWF satisfying IIA, unanimity and non-dictatorship.

2. Study particular domains and find out if there are SWFs that satisfy IIA, unanimity, and non-dictatorship for that domain.

This thesis falls in the second of these categories. We study domains and see if there are desirable functions that can be defined over them. But in the next paragraph, before presenting our contributions, we talk a little bit about the majority rule to provide a taste of the first of these two approaches.

The majority rule is not always a SWF. To understand it we first need to talk about the majority relation. Our exposition of the majority relation is based on [19]. Let $\vec{P}$ be a profile on X, the *majority relation* of $\vec{P} = (P_1, \ldots, P_n)$, $\geq_{\text{maj}}$, is a relation on X defined as follows, for any two different alternatives α and β in X:

$$\alpha \geq_{\text{maj}} \beta \text{ iff } |i \in N\colon \alpha P_i \beta| \geq |i \in N\colon \beta P_i \alpha|.$$

If $\alpha \geq_{\text{maj}} \beta$ and not $\beta \geq_{\text{maj}} \alpha$, then we write $\alpha >_{\text{maj}} \beta$. If D is a preference domain, the *majority rule* is a function that maps any profile $\vec{P}$ in D to the majority relation associated with $\vec{P}$. If for every profile $\vec{P}$ in D, the majority relation associated with $\vec{P}$ is a strict total order over X, then the majority rule over D is SWF. If the majority rule is a SWF over D it is easy to see that it escapes Arrow's impossibility theorem (except if D is such that there is a voter i that, for every profile in D, voter i is on the side of the majority for every pair of distinct alternatives). Therefore, efforts have been made to find domains in which the majority rule is a SWF. Notably, Sen [42] introduced a property called *value restriction* such that if the majority rule is defined over domains that consist of profiles satisfying this property and the number of voters, n, is odd, then this rule is a SWF (see also [41] and see Chapter 2 for a definition of this property). As it is reviewed by Elkind et al. [19], the following domain restrictions, which are popular in the computational social choice literature, are subsets of the value restriction domain when n is odd: *single-peaked*, *single-crossing on trees*, *single-crossing* (see [19] for definitions of these three domain restrictions). For n odd, also in these domains, the majority rule is a SWF.

1.1 Our Contributions

Our contributions are the following:

1. We present a detailed version of the generalization to domain restrictions of the algebraic topology framework that Baryshnikov used in [10] to prove Arrow's theorem. As we will see in the related work section of this thesis (Section 1.2), this generalization was suggested by Baryshnikov [10], but he did not get into the technical details of how and why would this generalization work (in the sense of providing a framework equivalent to the classical Arrovian framework to study possibility and impossibility results). In this thesis, we provide such details. By doing so, we hope to contribute to further formalize Baryshnikov's framework to domain restrictions (which we also call the combinatory or algebraic topology Arrovian framework).

2. We introduce a class of preference domains called the *class of polarization and diversity over triples*, denoted $\mathcal{D}^{\text{PT}} \cap \mathcal{D}^{\text{DT}}$, such that any domain in this class is super-Arrovian

(in particular, Arrow-inconsistent). The conditions that define this class are defined in terms of the combinatorial topology representation of domains instead of the classical (but equivalent) representation. That is, we present a generalization of Arrow's theorem through combinatorial topology. The class $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ has intuitive appeal because it has profiles that exhibit strong polarization over triples of alternatives, as well as profiles where there is diversity over triples of alternatives with respect to the places that the alternatives can take in the preferences of the voters. In Chapter 5, we provide and example of a domain in $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ that consists of *group-separable* profiles (the definition of such profiles appears in Chapter 2), a condition of interest to the computational social choice literature [see 19].

3. With the combinatorial topology approach started by [38], which uses Baryshnikov's constructions (but with simple combinatorial proofs instead of algebraic topology proofs), we prove that domains in $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ are super-Arrovian (Theorem 21). As part of this proof, we formalize and generalize an heuristic argument presented by them (we will be more specific in Chapter 4). Furthermore, to prove Theorem 21, we introduce a combinatorial topology version of the definition of almost-decisive coalitions and use it together with ultrafilters. To do so, we draw inspiration from [16, 32] which use ultrafilters to prove Arrow's theorem and a generalization of Arrow's theorem (in Section 1.2, we add more on the comparison between the use of ultrafilters in this thesis and in [16, 32]).

1.2 Related Work

We have already mentioned some of the broader literature relevant to this thesis, but in this section we will be more specific referring only to work closely related to ours. As we said before, Baryshnikov [10] proved Arrow's theorem for $n \geq 2$ voters and $|X| \geq 3$ alternatives in the context of the unrestricted domain. Moreover, he pointed out that his algebraic topology approach could be used to prove a generalization of Arrow's theorem that instead of the unrestricted domain assumes a domain with the *free triples property* (a domain $D \subseteq W(X)^n$ satisfies this property if for every $Y \subseteq X$, with $|Y| = 3$, and every profile $\vec{P} \in W(Y)^n$, there is a profile in D that restricted to Y equals $\vec{P}$. Such a Y is called a *free triple*.). He pointed out that the 2-skeleton of the simplicial complex that he uses for representing $W(X)^n$ provides relevant information for completing such a proof. Also, he suggested that "...domain restrictions have to be formulated in terms of the topology of N_W with some deleted simplices (forbidden orders) of maximal dimension" [10, p. 414], where N_W refers to a simplicial complex that he uses to represent the set of all strict total orders over the alternatives. He mentions an example of doing this for single-peaked domains with only three alternatives.

Rajsbaum and Raventós-Pujol [38] provide new combinatorial topology proofs within Baryshnikov's framework, but only for the base case of two voters and three alternatives. In particular, they use a generalization of the *index lemma* to prove the base case of Arrow's theorem (see [38] for more on the index lemma, which is in turn a generalization of Sperner's lemma). They also provide an heuristic argument to argue that the base case holds. As we said in Section 1.1, we formalize and generalize this heuristic argument. In particular, we show that it can be used even when there are $|X| \geq 3$ alternatives and $n \geq 2$ voters by restricting attention to the 2-skeleton of an input complex. Rajsbaum and Raventós-Pujol [37, 38] work with domain restrictions precisely by deleting simplices of maximal dimension from the input complex, but they do it only for the two voters and three alternatives case.

Neither in [10] nor in [37, 38] is formally specified (that is, dealing with the technical details) how the generalization of the combinatorial topology framework to allow for domain restrictions is defined nor why it is equivalent to the classical version of the Arrovian framework that deals with domain restrictions (the one that deals with sets of profiles/preferences and SWFs satisfying IIA, instead of simplicial complexes and chromatic simplicial maps) in a sense that we will make precise in Chapter 3. In particular, in our endeavor to formally define this generalized combinatorial topology framework that deals with domain restrictions, we allow for representing subprofiles as simplices and not only profiles as simplices of maximal dimension. The representation of the set of all strict total orders on X, i.e. $W(X)$, is the same as the one established in [10]. The representation of the SWFs satisfying IIA by chromatic simplicial complexes is a straightforward generalization of the one for the unrestricted domain established in [10]. Finally, via Theorem 7 and Corollary 8 in Chapter 3, we show that the classical and the combinatorial topology versions of the Arrovian framework (which allows for domain restrictions) are equivalent. In other words, we show that finding impossibility and possibility results is equivalent in both versions of the Arrovian framework.

In this thesis we work within the combinatorial topology Arrovian framework and use simple combinatorial arguments, like [37, 38] did, instead of algebraic topology, to prove our results. Hence, the results presented in this thesis build upon the combinatorial topology approach started in [38]. Since, we will prove a generalization of Arrow's theorem with this approach, namely that $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ is a class of Arrow-inconsistent domains (it will be clear that this class has as a member the unrestricted domain), we shall now mention some related work on impossibility and possibility results in the Arrovian framework.

Kalai and Muller [30] have characterized the domain restrictions which admit a SWF that satisfies IIA, unanimity, and non-dictatorship in the context of common domains. They show that a common domain D admits a SWF satisfying IIA, unanimity, and non-dictatorship for n voters if and only if D admits such a SWF for the case of 2 voters. Blair and Muller [11] characterize the domains that admit a SWF that satisfies IIA, unanimity, and *essentiality* (a stronger condition than non-dictatorship) in the context of cartisian preference domains. Kalai et al. [31] provide a sufficient condition for a common domain to be Arrow-inconsistent. They also proved their main results through a technique known as the local approach [see 35], which is similar in spirit to one we employ in our proof of Lemma 18. Rajsbaum and Raventós-Pujol [37] characterize the Arrow-consistent domains for the case of two voters and three alternatives and in the context of domains that satisfy the *free pairs property* (a domain $D \subseteq W(X)^n$ satisfies this property if for every pair of distinct alternatives $Y \subseteq X$ and every profile $\vec{P} \in W(Y)^n$, there is a profile in D that restricted to Y equals $\vec{P}$). In an unpublished working paper, Lara et al. [33] present an algorithm that for the case of only 2 voters and only 3 alternatives, decides if any domain given as input is Arrow-consistent and if so, it calculates all possible SWFs satisfying IIA, unanimity and non-dictatorship (hopefully this algorithm will be efficient). To the best of our knowledge, a complete characterization of the Arrow-consistent domains remains an open question.

Campbell and Kelly [16] provide a sufficient condition for a domain to be Arrow-inconsistent. This result is closer to our work than [11, 31, 30] in the sense that [16] allows for domains that are not cartesian. Therefore, we now define a condition called the *chain property* used by [16] to then present the generalization to Arrow's theorem that appears in [16].

A domain D has the *chain property* if $|X| \geq 3$ and for every two ordered pairs of alternatives (α, β) and (γ, δ) in X, there is a sequence $\alpha_1, \alpha_2 \ldots, \alpha_k$ of alternatives, where $k \geq 1$, such that $\{\alpha, \beta, \alpha_1\}, \{\beta, \alpha_1, \alpha_2\}, \ldots, \{\alpha_k, \gamma, \delta\}$ are free triples. Obviously, a domain with the free triple property satisfies the chain property, but as it is mentioned in [16], the implication in the other direction does not hold.

Now we want to state the generalized Arrow's theorem presented in [16]. However, these authors used weak orders, instead of strict total orders (as it is also the case for the original version of Arrow's theorem in [2]). Yet, reading the proof of [16] one can easily adapt it to the case with strict total orders so that it becomes comparable to our own generalization of Arrow's theorem. Hence, we present this adaptation, not the exact version that appears in [16].

Theorem 1 (Adapted from [16])**.** If $|X| \geq 3$ and D is a domain having the chain property, then every SWF with domain D that satisfies unanimity and IIA must be dictatorial.

As we will see, one way in which this generalization of Arrow's theorem (Theorem 1) differs from ours is that the chain property requires the existence of at least one free triple, while there are domains in $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ that do not any have free triple (we will see an example of such domains in example in Chapter 5).

We said in Section 1.1, that domains in $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ are super-Arrovian and not only Arrow-inconsistent. Hence, we now mention some work on super-Arrovian domains. Fishburn and Kelly [20] and Dasguptas [18] study super-Arrovian domains of minimal cardinality. In particular, Fisburn and Kelly [20] provide a characterization of super-Arrovian domains among Arrow-inconsistent domains. The domains of the generalization of Arrow's theorem that we present in Chapter 4 turn out to be super-Arrovian, but we did not prove this using this characterization. For the case of three alternatives and two voters, there is a minimal super-Arrovian domain presented in [20, Lemma 2 on p. 86] that belongs to the domains of our generalization of Arrow's theorem, we will be more specific about this in Chapter 4. For the case of three alternatives and three voters, there is a super-Arrovian domain presented in [20, Lemma 3 on p. 88] that is a subdomain of some domains of our generalization of Arrow's theorem.

Other works related to this thesis are those that employ (almost-)decisive coalitions with ultrafilters to prove Arrow's theorem or a generalization of this theorem to domains different than the unrestricted domain. For the context of the unrestricted domain and negatively transitive and asymmetric binary relations on X, Kirman and Sondermann [32] show that the set of all almost-decisive with respect to a an arbitrary SWF satisfying IIA and unanimity forms an ultrafilter of the set of all voters, N. Hansson [24] does the same but for preorders on X and decisive sets. Then, both show that if N is finite, their results imply that the SWF in question is a dictatorship.

Campbell and Kelly [16] use ultrafilters to prove the generalized version of Arrow's theorem stated as Theorem 1. Similar to [24, 32], but for domains satisfying the chain property (the unrestricted domain being one of them), Campbell and Kelly [16] show that the set of decisive coalitions with respect to (w.r.t.) an arbitrary SWF satisfying IIA and unanimity forms an ultrafilter of N. Like [24, 32], they show that this implies that there is a dictator for the SWF in question.

In contrast to [16, 24, 32], we work with the combinatorial topology approach, so we will prove that the set of all almost-decisive with respect to an arbitrary unanimous chromatic simplicial map of the form $f\colon N_D \to N_W(X)$ (we will be more specific what do we mean by chromatic and unanimous in Chapter 3), where $D \in \mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$, forms an ultrafilter. So we differ from [16, 24, 32] in that we use the combinatorial topology approach (instead of the classical) and in the domains that we use.

1.3 Organization

In Chapter 2, we present the basic definitions, results and notation that we will use in later chapters. Most of the material in this chapter are standard concepts from social choice theory, ultrafilters and combinatorial topology. In Chapter 3, we introduce the definitions to make the transition from the classical Arrovian framework, which uses sets of profiles/preferences and SWFs satisfying IIA, to the combinatorial topology version of the Arrovian framework, which uses simplicial complexes and chromatic simplicial maps. Additionally, this chapter, in conjunction with Appendix A, establishes the equivalence between the two frameworks. In Chapter 4, we present domain restrictions that have intuitive appeal and use them to prove a generalized version of Arrow's Theorem. In Chapter 5, we present an example of a domain that lives in a class introduced in Chapter 4. This domain consists of group-separable profiles. Finally, in Chapter 6, we will summarize our results and present connections of this thesis to certain topics in computational social choice and distributed computing. We will also discuss the nature of our proofs and present open research questions.

Chapter 2

Technical Background

In Section 2.1, we introduce the definition of domains and related definitions. In Section 2.2, we introduce the definition of social welfare functions and the relevant properties for their study within the Arrovian framework. In Section 2.3, we present the definition of ultrafilters and a useful theorem about ultrafilters that has been used to prove Arrow's impossibility theorem or generalizations of it. In Section 2.4 we provide a very brief introduction to basic concepts of combinatorial topology. In Section 2.5 we introduce the definition of value-restriction and group-separable preferences. We also give an intuitive example of a group-separable profile. Finally, in this section, we comment on the relation between group-separability, value-restriction and Arrow-consistency.

2.1 Preference Domains

Let X be a set of alternatives and $\{1, \ldots, n\}$ a set of voters, also denoted N. Throughout, we assume that $|X| \geq 3$ and $n \geq 2$. Let $Y \subseteq X$. A *strict total order* on Y is a binary relation P on Y that satisfies the following conditions for $x, y, z \in Y$:

1. If $x \neq y$, then xPy or yPx. (totality)

2. If xPy, then yPx does not hold. (asymmetry)

3. If xPy and yPz, then xPz. (transitivity)

If P is a strict total order on a set of alternatives $\{x, y, z\}$, and yPx and xPz and yPz hold (we are using infix notation), we will usually write P as yxz. Similarly, if (x, y) is an ordered pair of alternatives, we denote it as xy. Let $W(Y)$ be the set of all strict total orders on Y. Then, $W(X)$ is the set of all strict total orders on X.

A *preference profile* (or just *profile*) $\vec{P}$ on Y is an n-tuple of preferences $(P_1, \ldots, P_n)$, where P_i is a strict total order (on Y), interpreted as the stated preference of voter i at profile $\vec{P}$, for all $i \in N$. Let $W(Y)^n$ be the set of all preference profiles on Y. A *preference domain* D (or just *domain* if not confusion can arise) is a non-empty subset of $W(X)^n$.

Let $P \in W(X)$. The *restriction of P to Y*, denoted $P|_Y$, is a strict total order on Y defined in the following way: for all $x, y \in Y$, we have that $xP|_Y y$ iff xPy. If $\vec{P} \in W(X)^n$, the *restriction of $\vec{P}$ to Y*, denoted $\vec{P}|_Y$ is the profile $(P_1|_Y, \ldots, P_n|_Y) \in W(Y)^n$. Let D be a domain. We denote by $D|_Y$ the set $\{\vec{P}|_Y : \vec{P} \in D\}$. If $\vec{P}$ belongs to $D|Y$ we say that $\vec{P}$ is a *subprofile in* D. If $\vec{P} \in W(Y)^n$ and $\vec{P'} \in D$, we say that $\vec{P}$ is a *subprofile of* $\vec{P'}$ if $\vec{P} = \vec{P'}|_Y$, in which case $\vec{P}$ is a subprofile in D (w.r.t. Y). To consult more on the standard definitions of domains, preferences, profiles, and restricted profiles see [16, 35].

2.2 Social Welfare Functions

Let D be a preference domain. A *social welfare function* (*SWF* or *SWFs* for plural) is a function of the form $F : D \to W(X)$. In words, a SWF is a function that assigns to each profile of preferences in D a strict total order, which is commonly referred as the *social preference*.

Now that we have defined the concept of SWFs, we define some desirable properties that we would like SWFs to have. Let $x, y \in X$ and $\vec{P}, \vec{P'} \in D$. A SWF F satisfies:

- *unanimity* if we have the following: if for all $i \in N$ we have that xP_iy, then $xF(\vec{P})y$.
- *independende of irrelevant alternatives (IIA)* if $\vec{P}|_{\{x,y\}} = \vec{P'}|_{\{x,y\}}$ implies $F(\vec{P})|_{\{x,y\}} = F(\vec{P'})|_{\{x,y\}}$
- *non-dictatorship* if there is not a voter $i \in N$ such that $xF(\vec{P})y$ whenever xP_iy . Such a voter is called a *dictator*. So, a SWF satisfies non-dictatorship if there is no dictator. If there is a dictator for F, then F is a *dictatorship* or is said to be *dictatorial*.

The following proposition lists four well-known facts about dictatorships in the social choice literature.

Proposition 2. Let D be a preference domain and $F : D \to W(X)$ a dictatorship. The following holds:

1. F is well-defined.
2. F satisfies unanimity and IIA.
3. if $\vec{P} \in D$ implies that $P_i = P_j$ for all $i, j \in N$, then every voter is a dictator of F.
4. if $D = W(X)^n$, then F has only one dictator.

Proof. We start with 1. Let $i \in N$ be a dictator of F. Notice that F is in fact a SWF since it assigns to every profile in D the preference of i, which is a strict total order on X.

Now we proceed with 2. Let $x, y \in X$ and $\vec{P}, \vec{P'} \in D$ and $i \in N$ be a dictator of F. Notice that F satisfies unanimity because if xP_jy for all $j \in N$, it holds that xP_iy, which implies $xF(\vec{P})y$ since i is a dictator of F. Now let us see that F satisfies IIA. If $\vec{P}|_{\{x,y\}} = \vec{P'}|\{x, y\}$, then $P_i|_{\{x,y\}} = P'_i|_{\{x,y\}}$, but since i is a dictator of F, we have that $F(\vec{P})|_{\{x,y\}} = F(\vec{P'})|\{x, y\}$.

Part 3 follows directly from the definition of dictator.

Finally, for part 4, suppose $D = W(X)^n$. Let $i, j \in N$ be two dictators of F. To show: $i = j$. Let $\vec{P} \in W(X)^n$ such that xP_iy and yP_jx for some $x, y \in X$, $x \neq y$. Since i and j are dictators of F, we have that $xF(\vec{P})y$ and $yF(\vec{P})x$, a contradiction to the asymmetry of $F(\vec{P})$. $\square$

Corollary 3. If D is a preference domain, there exists a unanimous SWF with D as its domain and satisfying IIA.

Theorem 4 (Arrow's impossibility theorem)**.** If $|X| \geq 3$, every SWF with domain $W(X)^n$ that satisfies unanimity and IIA must be dictatorial.

As we mentioned in the introduction, one way of escaping Arrow's theorem is by allowing for SWF defined on preference domains other than the unrestricted one. We also mentioned that there are domain restrictions that do not escape Arrow's theorem. Following [35], we put a name to those domains that escape Arrow's theorem and those that do not. Let D be a domain and remember that $|X| \geq 3$. Following [35], we say that D is an *Arrow-inconsistent* domain if we have that any SWF defined on D satisfying unanimity and IIA must be a dictatorship. Also following [35], We say that D is an *Arrow-consistent* domain if there exists a SWF defined on D such that it satisfies unanimity, IIA and non-dictatorship. Following Fishburn and Kelly [20], D is *super-Arrovian* if it is Arrow-inconsistent and satisfies that for every domain D' such that $D \subseteq D' \subseteq W(X)^n$, D' is Arrow-inconsistent.

2.3 Ultrafilters

As we said in the introduction, some of the proofs of Arrow's theorem mix decisive or almost-decisive coalitions with ultrafilters. Since we plan to provide a proof that does this, we introduce the concept of ultrafilter.

Definition 1. An *ultrafilter* is a non-empty collection $\mathcal{U}$ of subsets of a set A that satisfies three conditions:

1. The empty set, $\varnothing$, does not belong to $\mathcal{U}$.
2. If $B \subseteq A$, then $B \in \mathcal{U}$ or $B^c \in \mathcal{U}$.
3. If $B, B' \in \mathcal{U}$, then $B \cap B' \in \mathcal{U}$.

The following result has been used in proofs that use ultrafilters to prove Arrow's theorem or some generalization of it, like those of [32] and [16], respectively. We will also use it to prove some of our results.

Theorem 5. If $\mathcal{U}$ is an ultrafilter of a finite set A, then there is some $a \in A$ such that $\mathcal{U} = \{B \subseteq A \colon a \in B\}$

The proof is omitted but can be found in [16]. This proof in [16] uses a fourth property of ultrafilters, but it is not hard to show that this property is implied by the three properties in Definition 1. For a reference on ultrafilters see [12].

2.4 Simplicial Complexes and Simplicial Maps

For the definitions in this section, we mostly follow [27] with some slight adaptations.

A collection K of finite and non-empty subsets of a set V is an (*abstract*) *simplicial complex* if the following condition is satisfied: if $s \in K$ and t is a non-empty subset of s, then $t \in S$.

Let K be a simplicial complex w.r.t. a set V. A *vertex* is an element of V. The set of all vertices of K, i.e. V, can also be denoted $V(K)$. A *simplex* is an element of K. The *dimension of a simplex* s, $\dim(s)$, is the number $|s| - 1$. A *k-simplex* is a simplex of dimension k. A simplex t is a *face* of s if $t \subseteq s$. A simplex s in K is a *facet* if it is maximal w.r.t. inclusion, i.e. if there is no simplex t of K such that s is strictly contained in t. The *dimension* of K, $\dim(K)$, is the maximum dimension among the dimensions of all its facets. The simplicial complex K is *pure* if all its facets are of the same dimension.

A simplicial complex C is a *subcomplex* of K if every simplex of C is a simplex of K. Let l be a non-negative integer. The *l-skeleton* of K, $\mathrm{skel}^l(K)$, is the set of simplices of K with dimension at most l. It is not hard to see that the $\mathrm{skel}^l(K)$ is a simplicial complex.

If K and C are simplicial complexes with sets of vertices $V(K)$ and $V(C)$, a *vertex map* is a function of the form $\mu \colon V(K) \to V(C)$. In words: a vertex map is a function that assigns to each vertex of K a vertex of C. A vertex map is called a simplicial map if it maps simplices to simplies. Formally, a vertex map $\mu \colon V(K) \to V(C)$ is a *simplicial map* if for all simplices s of K, we have that $\mu(s)$ is a simplex of C. If $\mu \colon V(K) \to V(C)$ is a simplicial map, we will always abuse notation and denote it $\mu \colon K \to C$. A simplicial map $\mu \colon K \to C$ is *rigid* if for each simplex $s \in K$ it holds that $|s| = |\mu(s)|$. Informally, simplicial map is rigid if it preserves the cardinality of simplices.

If K is a simplicial complex, a *m-labeling* (also *labeling*) is a function of the form $l \colon V(K) \to A$, where A is a set of cardinality m. An *m-coloring* (also *coloring*), denoted χ, is a m-labeling such that if u and v are two different vertices in some simplex t of K, then $\chi(u) \neq \chi(v)$. A *chromatic simplicial complex* is a simplicial complex K together with a coloring χ. If K and C are two chromatic simplicial complexes with m-colorings χ_K and χ_C, respectively, then a simplicial map $\phi \colon K \to C$ is *chromatic* if for every $v \in V(K)$, we have that $\chi_K(v) = \chi_C(\phi(v))$. Informally, a simplicial map is chromatic if it preserve colors.

2.5 Value-restriction and Group-separable Preferences

We follow [19] in this section. The domain restrictions we are about to introduce are relevant for our results in Chapter 4 and 5.

The notion of value-restricted preferences was introduced by Sen [42] in 1966.

Definition 2. A profile $\vec{P}$ on X is *value-restricted* over $Y \subseteq X$ if for every triple of distinct alternatives $\alpha, \beta, \gamma \in Y$, at least one of the three alternatives is never placed as the most-preferred, the middle-preferred or the least-preferred in the individual rankings of $\{\alpha, \beta, \gamma\}$ induced by $\vec{P}$.

We now aim to present the idea of group-separable preferences introduced by Inada [28, 29], but to do so, we first provide a piece of notation. If $\vec{P} = (P_1, \ldots, P_n)$ is a profile on a subset Y of X, and Y' and Y'' are non-empty subsets of Y, we write $Y' P_i Y''$ to denote that for all $\alpha \in Y'$ and all $\beta \in Y''$, we have $\alpha P_i \beta$. We read $Y' P_i Y''$ as "voter i ranks Y' over Y''".

Definition 3. A profile $\vec{P}$ on X is *group-separable* if for every $Y \subseteq X$ such that $|Y| \geq 2$, there exists a proper subset Z of Y such that $Z P_i (Y \setminus Z)$ or $(Y \setminus Z) P_i Y$ for all $i \in N$.

In words, $\vec{P}$ is group-separable if every subset Y of at least two alternatives can be partitioned in Z and $Y \setminus Z$ such that every voter ranks Z over $Y \setminus Z$ or viceversa (not all voters have to rank Z relative to $Y \setminus Z$ in the same way).

For example, let $X = \{\text{curry}, \text{pasta}, \text{cake}, \text{ice cream}\}$ and $n = 3$. Consider a profile $\vec{P}$ defined as follows:

- curryP_1pastaP_1cakeP_1ice cream
- pastaP_2curryP_2ice creamP_2cake
- ice creamP_3cakeP_3curryP_3pasta

Notice that voters 1 and 2 prefer any savory dish (curry and pasta) over any sweet dish (ice cream and cake), while voter 3 prefers any sweet dish over the savory ones. To see that $\vec{P}$ is group-separable fix $Y \subseteq X$ such that $|Y| \geq 2$. We need to find a partition $(Z, Y \setminus Z)$, where Z is a proper subset of Y, such that $Z P_i (Y \setminus Z)$ or $(Y \setminus Z) P_i Y$ for all $i \in N$. To do so just let Z be the subset of all the savory dishes in Y. Then $Y \setminus Z$ is the subset of all the sweet dishes in Y. Clearly, $Z P_1 (Y \setminus Z)$ for all $i \in \{1, 2\}$, and $(Y \setminus Z) P_3 Z$. Therefore, $\vec{P}$ is group-separable.

As it can be seen in [19], if a profile $\vec{P}$ is group-separable, then it is value-restricted. Hence, if n is odd and D is a domain that consists of group-separable profiles, the majority rule is a SWF that satisfies unanimity and IIA.

Chapter 3

The Combinatorial Topology Representation of the Arrovian Framework

In this chapter, we introduce simplicial complexes to represent any given domain D and the set of all preferences over X, $W(X)$, as well as chromatic simplicial maps that represent SWFs. In Appendix A, we present bijections that formally justify this translation from the classical version of the Arrovian framework to the combinatorial topology version. In Section 3.1, we define the simplicial complex that represents $W(X)$. In Section 3.2, for any given domain D, we define the simplicial complex that represents D. In Section 3.3, for any given SWF F we define the chromatic simplicial map that represents F. In this section, we also talk about the equivalence of using the classical and the combinatorial topology frameworks to study possibility and impossibility results.

Baryshnikov [10] used two simplicial complexes, denoted $N_{W(X)}$ and $N_{W(X)^n}$, to represent the set of all preferences, $W(X)$, and the unrestricted domain, $W(X)^n$, respectively. To do so, he established a bijection between the set of all facets of $N_{W(X)}$ and $W(X)$ and another bijection between the set of all facets of $N_{W(X)^n}$ and $W(X)^n$. Furthermore, he represented SWFs satisfying IIA with chromatic simplicial maps. In this thesis, we follow Barysnikov in representing $W(X)$ as $N_{W(X)}$, but work within a framework that allow us to represent any domain D (not only the unrestricted domain) as a simplicial complex, that we denote N_D, and any SWF satisfying IIA defined on D with a chromatic simplicial map. As we said in the introduction, this framework was suggested by Baryshnikov [10] (and it has also been used by [37], [38]), but we present a very detailed version of it.

In this chapter, we define $N_{W(X)}$, N_D and the chromatic simplicial maps that represent SWFs satisying IIA. In Appendix A, we prove that there are bijections between the subprofiles of D and the simplices of N_D, we also prove that the bijection from $W(X)$ to the facets of $N_{W(X)}$ is in fact a bijection, and that there is a bijection between the SWFs satisfying IIA and the chromatic simplicial maps from N_D to $N_{W(X)}$. With this last bijection, the equivalence between the classical and the combinatorial topology versions of the Arrovian framework is captured by Theorem 7 and Corollary 8.

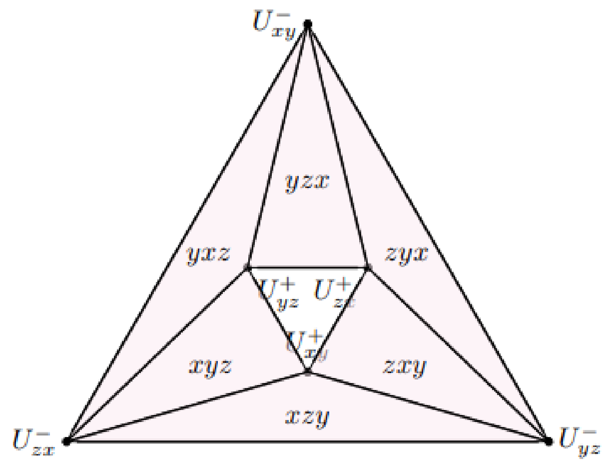

Figure 3.1: The simplicial complex $N_{W(\{x,y,z\})}$. This figure is adapted from Figure 1 in [38].

3.1 $W(X)$ as a Simplicial Complex

We introduce some notation needed to define the simplicial complex $N_{W(X)}$. Let $\sigma \in \{+,-\}$. We define $-\sigma \in \{+,-\}$ as follows: $-\sigma = +$ iff $\sigma = -$. Let $\alpha, \beta \in X$ and $U^{\sigma}_{\alpha\beta}$ be the set $\{P \in W(X)\colon\ \alpha P \beta \text{ iff } \sigma = +\}$. It is easy to see that $U^{+}_{\alpha\beta} = U^{-}_{\beta\alpha}$.

Definition 4. Let $N_{W(X)}$ be the simplicial complex defined as follows:

- its set of vertices, denoted $V(N_{W(X)})$, is

$$\bigcup_{\substack{\sigma \in \{+,-\} \\ \alpha,\beta \in X, \alpha \neq \beta}} \{U^{\sigma}_{\alpha\beta}\}$$

- a non-empty subset $S \subseteq V(N_{W(X)})$, where $S = \{v_1, \ldots, v_k\}$, is a $(k-1)$-simplex of $N_{W(X)}$ iff

$$\bigcap_{i=1}^{k} v_i \neq \varnothing$$

Checking that $N_{W(X)}$ is in fact a simplicial complex is easy.

If $X = \{x, y, z\}$, a depiction of $N_{W(X)}$ is shown in Figure 3.1. In this figure, the triangle (2-simplex) $\{U^{-}_{xy}, U^{+}_{yz}, U^{-}_{zy}\}$ represents the strict total order yxz. Notice that this triangle shares an edge (a 1-simplex) with the yzx triangle since yxz and yzx coincide in two pairwise comparisons of alternatives, i.e. they coincide on how they rank x relative to y and y relative to z, however they differ in how they rank x relative to z.

3.2 D as a Simplicial Complex

We want to represent each profile (or even better, any subprofile) with a simplex of a simplicial complex that we denote N_D. To illustrate how this representation works, suppose

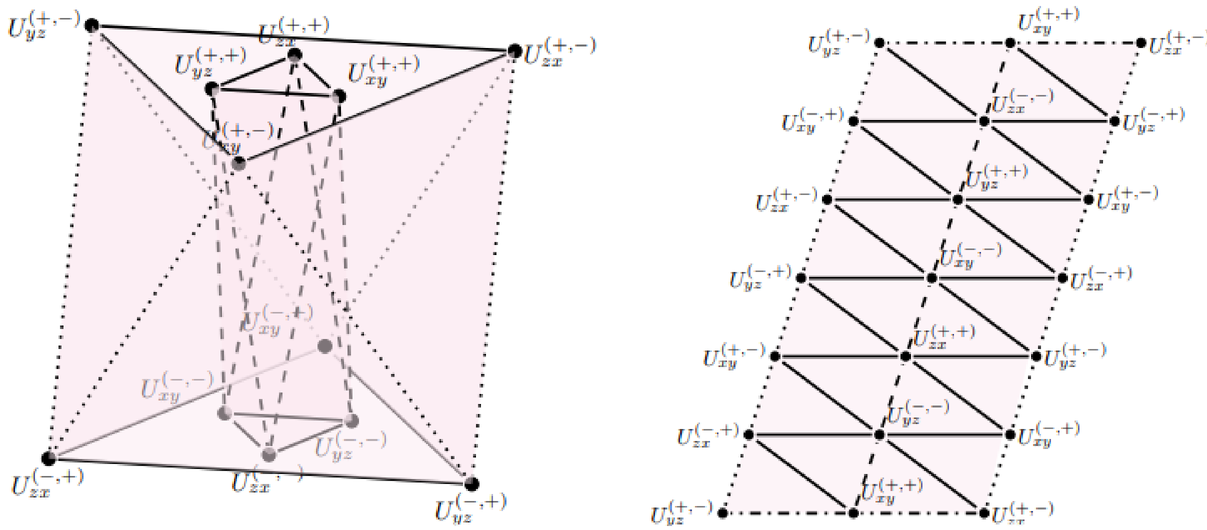

Figure 3.2: The simplicial complex $N_{W(\{x,y,z\})^2}$. The torus (the drawing on the right side of this figure) is obtained by identifying the vertices according to the patterns of the edges. The cylinders (the drawing on the left side of this figure) are connected by the torus by identifying vertices according to the patterns of the edges. This figure and its description are adapted from Figure 3 in [38].

$X = \{w, x, y, z\}$ and D is a domain that has (xyz, xzy) as a subprofile. It is not hard to see that the profile (xyz, xzy) is a subprofile of at most 16 profiles in D (one of them $(wxyz, xwzy)$ if this profile is in D). We represent (xyz, xzy) with a triangle (a 2-simplex) which is a face of at most 16 different tetrahedra, one for each of the 16 profiles possibly in D having (xyz, xzy) as a subprofile. This way of representing subprofiles and simplices defines bijections between collections of these objects, to see these bijections and the proofs that they work, see Appendix A.

Before giving the formal definition of N_D, let's see how this simplicial complex looks if $X = \{x, y, z\}$, $n = 2$ and $D = W(X)^n$. This is depicted in Figure 3.2. The complex $N_{W(\{x,y,z\})^2}$ has a triangle representing each of the $(3!)^2 = 36$ profiles in $N_{W(\{w,y,z\})^2}$. It consists of two cylinders and a torus glued to the cylinders in a certain way (see the figure's description). One of these cylinders consists of all the profiles of complete agreement (w.r.t. the pairwise comparisons of the alternatives), and the other one of those of complete disagreement. The torus consists of all the profiles that has some degree of disagreement (but not complete). A closer look to some of these triangles is provided by Figure 3.3. Before proceeding with the formal definition of N_D, we introduce some notation (similar, but not exactly analogous, to the one introduced for $N_{W(X)}$).

Let $\vec{\sigma} \in \{+, -\}^n$, i.e. $\vec{\sigma}$ is an n-tuple whose components are $+$ or $-$ signs. For example, $\vec{\sigma} = (-, +, -, -)$. The i-component of $\vec{\sigma}$ is denoted $\vec{\sigma}_i$. We define $-\vec{\sigma}$ as follows: for all $i \in N$, we have that $(-\vec{\sigma})_i = +$ iff $\vec{\sigma}_i = -$. For example if $\vec{\sigma} = (+, -, +)$, then $-\vec{\sigma} = (-, +, -)$.

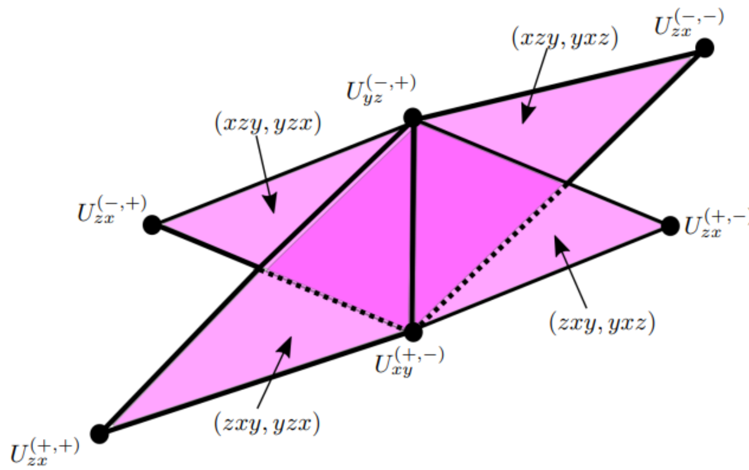

Figure 3.3: A closer look to some 2-simplices in $N_{W(\{x,y,z\})^2}$. This figure is inspired by Figure 2 in [38].

If D is a preference domain, let L denote the following set of labels:

$$\bigcup_{\substack{\vec{\sigma}\in\{+,-\}^n \\ \alpha,\beta\in X,\alpha\neq\beta}} \{U_{\alpha\beta}^{\vec{\sigma}}\}.$$

For every label $U_{\alpha\beta}^{\vec{\sigma}} \in L$, let $s_D(U_{\alpha\beta}^{\vec{\sigma}})$ denote the set

$$\{\vec{P} \in D\colon \text{for all } i \in N,\ \alpha P_i \beta \text{ iff } \vec{\sigma}_i = +\}.$$

Notice that for every $\vec{\sigma}$ and every $\alpha, \beta \in X$, $\alpha \neq \beta$, we have $s_D(U_{\beta\alpha}^{-\vec{\sigma}}) = s_D(U_{\alpha\beta}^{\vec{\sigma}})$. For our purposes, this fact allows us to treat the element $U_{\alpha\beta}^{\vec{\sigma}}$ of L and the element $U_{\beta\alpha}^{-\vec{\sigma}}$ of L as if they were the same element and write $U_{\alpha\beta}^{\vec{\sigma}} = U_{\beta\alpha}^{-\vec{\sigma}}$. For a formal justification of this, see Appendix A.

Definition 5. Let N_D denote the simplicial complex defined as follows:

- its set of vertices, denoted $V(N_D)$, is

$$\{u \in L\colon s_D(u) \neq \varnothing\}$$

- a non-empty subset $S \subseteq V(N_D)$, where $S = \{v_1, \ldots, v_k\}$, is a $(k-1)$-simplex of N_D iff

$$\bigcap_{i=1}^{k} s_D(v_i) \neq \varnothing$$

As with $N_{W(X)}$, checking that N_D is in fact a simplicial complex is easy. The construction of N_D is a generalization of the way $N_{W(X)^n}$ is constructed in [10], but for technical reasons related to allowing for domain restrictions, we introduced a distinction between a label $U_{\alpha\beta}^{\vec{\sigma}}$ and the set $s_D(U_{\alpha\beta}^{\vec{\sigma}})$.

3.3 Social Welfare Functions as Chromatic Simplicial Maps

We want to represent any social welfare function defined on a given domain D and satisfying IIA with simplicial maps that are chromatic w.r.t. the labels involving the alternatives. To be more precise, whenever we say a simplicial map of the form $f\colon N_D \to N_{W(X)}$ is chromatic we mean that $f(U^{\vec{\sigma}}_{\alpha\beta}) = U^{\sigma}_{\alpha\beta}$. Informally, f preserves the $\alpha\beta$'s labels. Let $\mathcal{F}_D$ be the set of all social welfare functions defined on D satisfying IIA and let $\mathcal{M}_D$ be the set of all chromatic simplicial maps of the form $f\colon N_D \to N_{W(X)}$. The following construction is a straightforward generalization of the bijection from $\mathcal{F}_{N(W)^n}$ and to $\mathcal{M}_{N(W)^n}$ established by Baryshnikov [10].

Definition 6. Let $\mathcal{B}\colon \mathcal{F}_{\mathcal{D}} \to \mathcal{M}_D$ such that $\mathcal{B}(F)$ is the chromatic simplicial map defined as follows: $\mathcal{B}(F)$ assigns any vertex $U^{\vec{\sigma}}_{\alpha\beta}$ of N_D to the vertex $U^{\sigma}_{\alpha\beta}$ of $N_{W(X)}$, where $\sigma = +$ iff we have the following: $\alpha F(\vec{P})\beta$, for every $\vec{P} \in s_D(U^{\vec{\sigma}}_{\alpha\beta})$.

Proposition 6. If $F \in \mathcal{F}_D$, then $\mathcal{B}(F)$ is well-defined.

The proof is an easy generalization of the arguments that appears in [38] for the case of the unrestricted domain, but we write it for completeness.

Proof. Let $U^{\vec{\sigma}}_{\alpha\beta}$ be a vertex of N_D. Let $\vec{P}, \vec{P}' \in s_D(U^{\vec{\sigma}}_{\alpha\beta})$. To show: $\alpha F(\vec{P})\beta$ iff $\alpha F(\vec{P}')\beta$. Since $\vec{P}, \vec{P}' \in s_D(U^{\vec{\sigma}}_{\alpha\beta})$, for all $i \in N$, $\alpha P_i \beta$ iff $\sigma_i = +$ and $\alpha P'_i \beta$ iff $\sigma_i = +$. Hence, for all $i \in N$, $\alpha P_i \beta$ iff $\alpha P'_i \beta$. Then, since F satisfies IIA, we get $\alpha F(\vec{P})\beta$ iff $\alpha F(\vec{P}')\beta$. $\square$

It is not hard to show that $\mathcal{B}$ is in fact a chromatic simplicial map. Its existence allows us to talk interchangeably about SWFs satisfying IIA and their corresponding chromatic simplicial maps.

We introduce some useful definitions and notation. A *coalition* is a subset of N. If G is a coalition, let $\vec{\sigma}^G$ denote the element of $\{+,-\}^n$ such that $\sigma^G_i = +$ iff $i \in G$. For instance, if $N = \{1,2,3\}$ and $G = \{1,3\}$, then $\vec{\sigma}^G$ denotes $(+,-,+)$. In particular, $\vec{\sigma}^N$ denotes the element of $\{+,-\}^n$ such that $\sigma^N_i = +$ for all $i \in N$. Analogously, $\vec{\sigma}^{\varnothing}$ denotes the element of $\{+,-\}^n$ such that $\sigma^{\varnothing}_i = -$ for all $i \in N$.

Now we define unanimity and dictatorship in the context our chromatic simplicial maps.

Definition 7. Let $f\colon N_D \to N_{W(X)}$ be a chromatic simplicial map. We say that f satisfies *unanimity* if, for all $\alpha, \beta \in X$, we have that if $U^{\vec{\sigma}^N}_{\alpha\beta}$ is a vertex of N_D, then $f(U^{\vec{\sigma}^N}_{\alpha\beta}) = U^{+}_{\alpha\beta}$. We say that f is *dictatorial* if there is a voter $i \in N$ such that: for all $\alpha, \beta \in X$, if $U^{\vec{\sigma}}_{\alpha\beta}$ is a vertex of N_D, then $f(U^{\vec{\sigma}}_{\alpha\beta}) = U^{\vec{\sigma}_i}_{\alpha\beta}$. Such a voter is called a *dictator for f.*

Theorem 7. Let $F \in \mathcal{F}_{\mathrm{D}}$ and $f \in \mathcal{M}_D$. Let $\mathcal{B}^{-1}$ be the inverse function of the bijection $\mathcal{B}$. The following hold:

1. $\mathcal{B}^{-1}(f)$ satisfies IIA.

2. $\mathcal{B}(F)$ is unanimous iff F is unanimous.

3. $\mathcal{B}(F)$ is dictatorial iff F is dictatorial.

The proof of Theorem 7 is in Appendix A.

Corollary 8. A domain D is Arrow-inconsistent iff any chromatic simplicial map of the form $f\colon N_D \to N_{W(x)}$ satisfying unanimity is dictatorial.

The proof of Corollary 8 is in Appendix A.

Corollary 8 can be intuitively interpreted as saying that finding possibility and impossibility results in the combinatorial topology framework is equivalent to finding them in the classical framework.

Chapter 4

A Generalization of Arrow's Impossibility Theorem

In Section 4.1, we define a combinatorial topolgy version of almost-decisiveness in the context of domain restrictions. Our goal in this chapter is to define a class of domains $\mathcal{D}$ such that if $f\colon N_D \to N_{W(X)}$ is a unanimous chromatic simplicial map with $D \in \mathcal{D}$, then the set of all almost-decisive coalitions w.r.t. f is an ultrafilter w.r.t. N. In Sections 4.2, 4.3, and 4.4 we find classes of domains that guarantee property 1, 2 and 3 of the definition of ultrafilters, respectively. Finally, in Section 4.3, by using the class of domains that guarantees property 2 we prove a generalized version of Arrow's theorem.

4.1 Almost-decisiveness in the Combinatorial Topology Framework and Ultrafilters

The following definition is the combinatorial topology version of the notion of almost-decisiveness (to consult definition of almost-decisiveness within the classical framework see [16]).

Definition 8. Let $f\colon N_D \to N_W$ be a chromatic simplicial map, $Y \subseteq X$, and G a coalition. If ab is a ordered pair of distinct alternatives a and b in X, we say that G is *almost-decisive over ab w.r.t. f* if $f(U_{ab}^{\vec{\sigma}^G}) = U_{ab}^{+}$ whenever $U_{ab}^{\vec{\sigma}^G}$ is a vertex of N_D. We say that G is *almost-decisive over Y w.r.t. f* if for all $a, b \in Y$ such that $U_{ab}^{\vec{\sigma}^G}$ is a vertex of N_D, we have that $f(U_{ab}^{\vec{\sigma}^G}) = U_{ab}^{+}$. If G is almost-decisive over X w.r.t. f, we just say that it is *almost-decisive w.r.t. f*.

In words, if G is almost-decisive then when everyone in G agrees on ranking a over b and everyone not in G agrees on ranking b over a, then society ranks a over b.

Now we present a useful lemma that follow easily from Definition 8.

Lemma 9. Let G be an almost-decisive coalition over $Y \subseteq X$ and $\beta, \alpha \in X$, where $\alpha \neq \beta$. If $U_{\alpha\beta}^{\vec{\sigma}^{G^c}}$ is a vertex of N_D, then $f(U_{\alpha\beta}^{\vec{\sigma}^{G^c}}) = U_{\alpha\beta}^{-}$.

Proof. Observe that:

- $U^{\vec{\sigma}^G}_{\beta\alpha} = U^{\vec{\sigma}^{G^c}}_{\alpha\beta}$.
- $U^{+}_{\beta\alpha} = U^{-}_{\alpha\beta}$.
- The almost-decisiveness of G over Y implies that

$$f(U^{\vec{\sigma}^G}_{\beta\alpha}) = U^{+}_{\beta\alpha}.$$

Taking these observations together yields the desired result. □

The following lemma will be used in Section 4.5 to prove our generalization of Arrow's theorem.

Lemma 10. Let $f\colon N_D \to N_{W(X)}$ be a chromatic simplicial map and $\mathcal{G}$ the set of all almost-decisive coalitions w.r.t. f. If $\mathcal{G}$ is an ultrafilter of the set of all voters N, then f is dictatorial.

Proof. Suppose $\mathcal{G}$ is an ultrafilter of N. Since N is finite, by Theorem 5 there exists a voter, call it d, such that $\mathcal{G} = \{B \subseteq N : d \in B\}$.

Let $U^{\vec{\sigma}}_{\alpha\beta}$ be a vertex of N_D. Since $U^{\vec{\sigma}}_{\alpha\beta}$ is an arbitrary vertex of N_D, by Definition 7 we have that d is a dictator for f if $f(U^{\vec{\sigma}}_{\alpha\beta}) = U^{\vec{\sigma}_d}_{\alpha\beta}$. Clearly, there exists a coalition G of N such that $\vec{\sigma} = \vec{\sigma}^G$. Therefore, it suffices to show that $f(U^{\vec{\sigma}^G}_{\alpha\beta}) = U^{\vec{\sigma}^G_d}_{\alpha\beta}$.

By property 3 of the definition of an ultrafilter, G or G^c is an element of $\mathcal{G}$, i.e. one of them is an almost-decisive. We proceed by checking the two possible cases.

Case 1: $G \in \mathcal{G}$. Then by definition of almost-decisiveness $f(U^{\vec{\sigma}^G}_{\alpha\beta}) = U^{+}_{\alpha\beta}$. Also, since $G \in \mathcal{G}$, voter d is in G, so $\vec{\sigma}^G_d = +$. Therefore, we have $f(U^{\vec{\sigma}^G}_{\alpha\beta}) = U^{\vec{\sigma}^G_d}_{\alpha\beta}$.

Case 2: $G^c \in \mathcal{G}$. Then by Lemma 9, we have $f(U^{\vec{\sigma}^G}_{\alpha\beta}) = U^{-}_{\alpha\beta}$. Also, since $G \in \mathcal{G}$, voter d is in G^c, we have that $\vec{\sigma}^G_d = -$. Therefore, we have $f(U^{\vec{\sigma}^G}_{\alpha\beta}) = U^{\vec{\sigma}^G_d}_{\alpha\beta}$.

Therefore, d is a dictator for f, so f is dictatorial. □

4.2 Unanimity Vertices and the First Ultrafilter Property

Definition 9. If G is a coalition and $Y \subseteq X$, then let $\mathcal{D}^{GY}$ be the class of domains defined as follows: $D \in \mathcal{D}^{GY}$ iff there exist $\alpha, \beta \in Y$ such that $U^{\vec{\sigma}^G}_{\alpha\beta}$ is a vertex of N_D.

In particular, if $G = N$ and $Y = X$, $\mathcal{D}^{NX}$ denotes the class of domains that consists of all domains D for which there exist $\alpha, \beta \in X$ such that $U^{\vec{\sigma}^N}_{\alpha\beta}$ is a vertex of N_D. In words, $\mathcal{D}^{NX}$ consists of the domains that have at least one unanimity vertex in their associated simplicial complex.

Proposition 11. Let $f\colon N_D \to N_{W(X)}$ be a chromatic simplicial map and $Y \subseteq X$ such that $|Y| \geq 2$. We have that $D \in D^{GY}$ iff G or G^c is not almost-decisive over Y w.r.t. f.

Proof. We start with the $\Rightarrow$ direction. Suppose $D \in D^{GY}$. We proceed by contradiction assuming that G and G^c are almost-decisive over Y (w.r.t. f). Since $D \in D^{GY}$, there exist $\alpha, \beta \in Y$ such that $U^{\vec{\sigma}^G}_{\alpha\beta}$ is a vertex of N_D. Since G is almost-decisive over Y, we have that $f(U^{\vec{\sigma}^G}_{\alpha\beta}) = U^+_{\alpha\beta}$, but since G^c is also almost-decisive over Y, by Lemma 9 we have that $f(U^{\vec{\sigma}^G}_{\alpha\beta}) = U^-_{\alpha\beta}$, a contradiction.

Now we prove the $\Leftarrow$ direction. We show that the contrapositive statement holds. Suppose $D \notin D^{GY}$. Then, for every $\alpha, \beta \in Y$, the element $U^{\vec{\sigma}^G}_{\alpha\beta}$ of L is not a vertex of N_D.Then G and G^c are almost-decisive over Y w.r.t. f by vacuity. $\square$

Corollary 12. Let $f \colon N_D \to N_{W(X)}$ be a chromatic simplicial map and $Y \subseteq X$ such that $|Y| \geq 2$. We have that $D \in D^{NX}$ iff G or G^c is not almost-decisive w.r.t. f.

Proposition 13. Let $f \colon N_D \to N_{W(X)}$ be a chromatic and unanimous simplicial map and $\mathcal{G}$ the set of all almost-decisive coalitions (over X) w.r.t. f. We have that $\varnothing \notin \mathcal{G}$ iff $D \in \mathcal{D}^{NX}$.

Proof. First, we prove the $\Rightarrow$ direction. Suppose $\varnothing \notin \mathcal{G}$. For contradiction, suppose $D \notin \mathcal{D}^{NX}$. Then, by Corollary 12, N and $\varnothing$ are almost-decisive (w.r.t. f). But then $\varnothing \in \mathcal{G}$, a contradiction.

Finally, we prove the $\Leftarrow$ direction. Suppose $D \in \mathcal{D}^{NX}$, by Corollary 12, N or $\varnothing$ is not almost-decisive. But by unanimity of f, N is almost-decisive, therefore $\varnothing$ is not almost-decisive, i.e. $\varnothing \notin \mathcal{G}$. $\square$

4.3 Polarization and the Third Ultrafilter Property

Our objective in this section is the following: given a unanimous chromatic simplicial map $f \colon N_D \to N_{W(X)}$, and denoting the set of all almost-decisive coalitions w.r.t. f by $\mathcal{G}$, we want to define a class of preference domains $\mathcal{D}$ such that if $D \in \mathcal{D}$, then $\mathcal{G}$ satisfies the third property of the ultrafilter definition w.r.t. the set of all voters, N. That is, we want a $\mathcal{D}$ such that if $D \in \mathcal{D}$, the following holds:

$$\text{if } G \text{ is a coalition, then } G \in \mathcal{G} \text{ or } G^c \in \mathcal{G}$$

We will introduce a class of domains that we call the *class of polarized over triples*, denoted $\mathcal{D}^{\mathrm{PT}}$ that achieves the objective stated in the previous paragraph. In order to define this class and prove that it guarantees that the third ultrafilter property holds, we introduce some definitions as well as some lemmas.

At this point, we want to introduce the notion of polarized profiles. Such profiles are explicitly used in a proof by [18, Lemma 7 on p. 527], although not with that name.

Definition 10. A profile $\vec{P}$ on $Y \subseteq X$ is *polarized* if there exist $P, P' \in W(Y)$, and a non-empty coalition G distinct from N, such that $P_i = P$ for all $i \in G$ and $P_j = P'$ for all $j \in G^c$. We denote such a $\vec{P}$ as $(G \colon P, G^c \colon P')$.

For example, if $n = 5$, $X = \{x, y, z\}$ and $G = \{1, 4\}$, the profile $(xyz, yzx, yzx, xyz, yzx)$ is a polarized profile and can be denoted as $(G \colon xyz, G^c \colon yzx)$, the idea being to communicate that every voter in G has xyz as their ranking and every voter outside G has yzx as their ranking.

Certain polarized profiles over triples of alternatives are relevant to our results. These profiles are called *critical profiles* by [37], but we will call them *strongly polarized* profiles.

Definition 11. Let $Y \subseteq X$, such that $|Y| = 3$, and $\vec{P} = (G\colon P, G^c\colon P')$ a polarized profile on Y. The profile $\vec{P}$ is *strongly polarized* if P and P' differ on how they rank two different pairs of alternatives and coincide on how they rank the remaining pair of alternatives.

Remark 1. For a given coalition G and set $Y \subseteq X$, such that $|Y| = 3$, there are exactly 12 strongly polarized profiles on Y.

Now we define two sets of profiles that are going to be the basis to construct the class $\mathcal{D}^{\mathrm{PT}}$ of domains. These sets appeared in [20, Lemma 2 on p. 87] for the case of 3 alternatives and $n \in \{2, 3\}$ voters.

Definition 12. Let G be a non-empty coalition distinct from N and $\{\alpha, \beta, \gamma\} \subseteq X$, $\alpha \neq \beta \neq \gamma \neq \alpha$. Let $D_1(G, \{\alpha, \beta, \gamma\})$ denote the set of preferences

$$\{(G\colon \beta\gamma\alpha, G^c\colon \alpha\beta\gamma), (G\colon \beta\alpha\gamma, G^c\colon \alpha\gamma\beta), (G\colon \alpha\beta\gamma, G^c\colon \gamma\alpha\beta),$$
$$(G\colon \alpha\gamma\beta G^c\colon \gamma\beta\alpha), (G\colon \gamma\alpha\beta, G^c\colon \beta\gamma\alpha), (G\colon \gamma\beta\alpha, G^c\colon \beta\alpha\gamma)\}.$$

Let us comment on $D_1(G, \{\alpha, \beta, \gamma\})$. It is easy to check that each of the six profiles in $D_1(G, \{\alpha, \beta, \gamma\})$ is strongly polarized. Also, observe that for every strict total order P on Y, there exists a unique profile in $D_1(G, \{\alpha, \beta, \gamma\})$ such that every voter in G has P as her preference. Denoting the simplicial complex associated with $D_1(G, \{\alpha, \beta, \gamma\})$ as $N_{D_1(G,\{\alpha,\beta,\gamma\})}$ is quite cumbersome, so let us denote it as $B_1(G, \{\alpha, \beta, \gamma\})$. This simplicial complex is depicted in Figure 4.1.

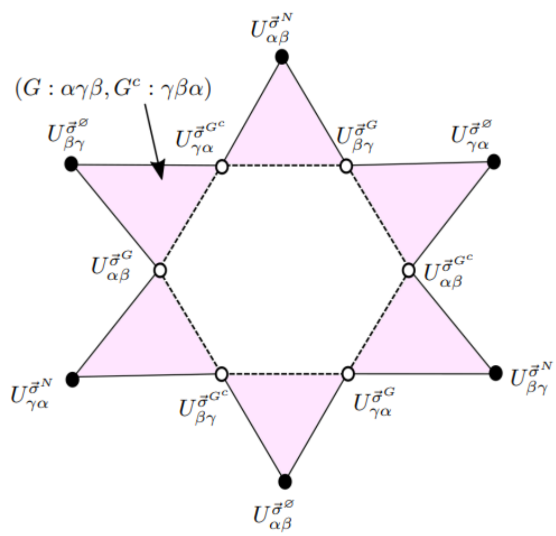

Figure 4.1: The simplicial complex $B_1(G, \{\alpha, \beta, \gamma\})$.

Remark 2. Given a polarized profile $(G\colon P, G^c\colon P')$ w.r.t. a coalition G and $Y \subseteq X$, such that $|Y| = 3$, we have that $(G\colon P', G^c\colon P)$ is a polarized profile w.r.t. G and Y.

Observe that we can take each of the polarized profiles in $D_1(G, \{\alpha, \beta, \gamma\})$ and apply Remark 2 to obtain another polarized profile. We define a set of profiles whose members are those profiles obtained in this manner.

Definition 13. Let G be a non-empty coalition distinct from N and $\{\alpha, \beta, \gamma\} \subseteq X$. Let $D_2(G, \{\alpha, \beta, \gamma\})$ denote the domain

$$\{(G\colon \alpha\beta\gamma, G^c\colon \beta\gamma\alpha), (G\colon \alpha\gamma\beta, G^c\colon \beta\alpha\gamma), (G\colon \gamma\alpha\beta, G^c\colon \alpha\beta\gamma),$$
$$(G\colon \gamma\beta\alpha G^c\colon \alpha\gamma\beta), (G\colon \beta\gamma\alpha, G^c\colon \gamma\alpha\beta), (G\colon \beta\alpha\gamma, G^c\colon \gamma\beta\alpha)\}.$$

Clearly, like in the case of $D_1(G, \{\alpha, \beta, \gamma\})$, for every strict total order P on Y, there exists a unique profile in $D_2(G, \{\alpha, \beta, \gamma\})$ such that every voter in G has P as her preference. The simplicial complex $N_{D_2(G,\{\alpha,\beta,\gamma\})}$, also denoted $B_2(G, \{\alpha, \beta, \gamma\})$, is depicted in Figure 4.2. Notice that $D_1(G, \{\alpha, \beta, \gamma\}) \cap D_2(G, \{\alpha, \beta, \gamma\}) = \varnothing$, hence by Remark 1, we have that $D_1(G, \{\alpha, \beta, \gamma\}) \cup D_2(G, \{\alpha, \beta, \gamma\})$ consists of the total 12 strongly polarized profiles w.r.t. G and $\{\alpha, \beta, \gamma\}$.

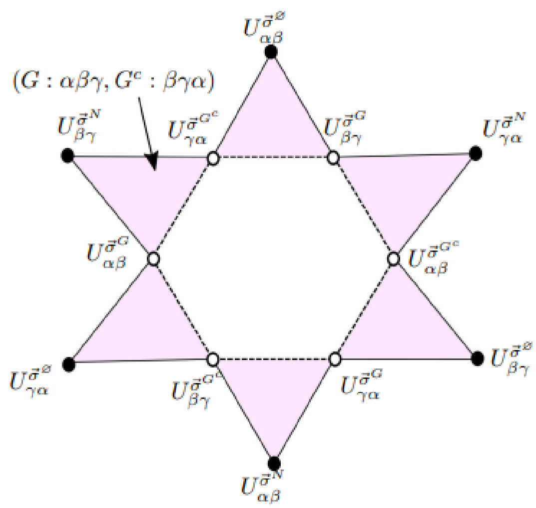

Figure 4.2: The simplicial complex $B_2(G, \{\alpha, \beta, \gamma\})$

Remark 3. The simplicial complex $B_i(G, \{\alpha, \beta, \gamma\})$, for all $i \in \{1, 2\}$, contains all the edges of the form $\{U^{\vec{\sigma}^G}_{ab}, U^{\vec{\sigma}^{G^c}}_{ca}\}$ for some $a, b, c \in \{\alpha, \beta, \gamma\}$.

To provide additional details, if a profile $\vec{P}$ in a domain D, i.e. a facet of N_D, has an edge of the form $\{U^{\vec{\sigma}^G}_{ab}, U^{\vec{\sigma}^{G^c}}_{ca}\}$ as a face, that means that in that profile any voter in G disagrees with any voter in G^c on at least two pairs of alternatives: $\{a, b\}$ and $\{a, c\}$. Notice, in Figures 4.1 and 4.2, that every profile of $B_i(G, \{\alpha, \beta, \gamma\})$, for all $i \in \{1, 2\}$, has an edge of this form. Moreover, every profile of $B_i(G, \{\alpha, \beta, \gamma\})$ has a unanimity vertex, i.e. a vertex for which everyone at N agrees on the pair in question. Therefore, as we already said before, every triangle in $B_i(G, \{\alpha, \beta, \gamma\})$ represents a profile in which G and G^c disagree on two pairs of alternatives and agree on the remaining pair.

Definition 14 below is relevant to prove some subsequent lemmas. Lemmas 15, 16 and 17 formalize and generalize an heuristic argument made by [37, 38]. We will explain more about this once we present the proofs of the three lemmas and the geometric intuition of Lemmas 16 and 17. Furthermore, in the context of only 3 alternatives and only $n \in \{2, 3\}$ voters, the proof of these Lemmas taken together is very similar to a proof carried out by Fishburn and Kelly [20, Lemma 2 and Lemma 3 on pp. 87–88] to show that certain domain is super-Arrovian. However, the proofs in [20] use the classical approach instead of the combinatorial topology approach and decisive coalitions instead of almost-decisive coalitions.

Definition 14. An edge of $N_{W(X)}$ is called a *determined by transitivity* edge (*DbT* edge, for short) if it is of the form $\{U^+_{\alpha\beta}, U^+_{\beta\gamma}\}$ for some $\alpha, \beta, \gamma \in X$. An edge of $N_{W(X)}$ is a *non-DbT* edge if it is not a DbT edge.

To motivate our definition, notice that a DbT edge $\{U^+_{\alpha\beta}, U^+_{\beta\gamma}\}$ represents the strict total orders on X ranking α over β and β over γ. Then $\{U^+_{\alpha\beta}, U^+_{\beta\gamma}\}$ is a face of exactly one 2-simplex of $N_{W(X)}$ among the 2-simplices of $N_{W(X)}$ that only involve alternatives in $\{\alpha, \beta, \gamma\}$, namely, it is a face of the 2-simplex $\{U^+_{\alpha\beta}, U^+_{\beta\gamma}, U^+_{\alpha\gamma}\}$ (in contrast, notice that $\{U^+_{\alpha\beta}, U^+_{\beta\gamma}, U^+_{\gamma\alpha}\}$ is not a 2-simplex since it represents the intransitive ranking $\alpha\beta\gamma\alpha$)[1].

Notice that an edge of $N_{W(X)}$ of the form $\{U^-_{\alpha\beta}, U^-_{\gamma\alpha}\}$ is a DbT edge since it can be rewritten as $\{U^+_{\beta\alpha}, U^+_{\alpha\gamma}\}$. Of course, DbT edges live in the 1-skeleton of $N_{W(X)}$. Fix three different alternatives, $\alpha, \beta, \gamma \in X$. Figure 4.3 depicts the part of the 1-skeleton of $N_{W(X)}$ that involves only alternatives in $\{\alpha, \beta, \gamma\}$. In this figure, DbT edges are represented with dashed-lines and non-DbT edges with solid-lines.

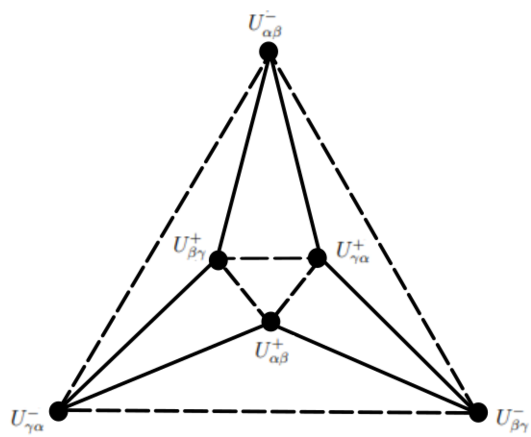

Figure 4.3: DbT edges (in dashed-lines) and non-DbT edges (in solid-lines) living in the part of the 1-skeleton of $N_{W(X)}$ that only involves alternatives in $\{\alpha, \beta, \gamma\}$.

[1]When $|X| = 3$, it can be shown that an edge of $N_{W(X)}$ is a DbT edge iff it is a 1-simplex in the boundary of $N_{W(X)}$. The *boundary* of a pure simplicial complex K is the simplicial complex induced by the $(\dim(K) - 1)$-simplices that each is the face of a unique facet of $N_{W(X)}$.

Lemma 14. Let $f\colon N_D \to N_{W(X)}$ be a unanimous chromatic simplicial map. If $\{U_{bc}^{\vec{\sigma}}, U_{ca}^{\vec{\sigma}'}, U_{ab}^{\vec{\sigma}^N}\}$ is a triangle of N_D, then the edge $\{U_{bc}^{\vec{\sigma}}, U_{ca}^{\vec{\sigma}'}\}$ cannot by mapped by f to $\{U_{bc}^{+}, U_{ca}^{+}\}$.

Proof. Suppose $\{U_{bc}^{\vec{\sigma}}, U_{ca}^{\vec{\sigma}'}, U_{ab}^{\vec{\sigma}^N}\}$ is a triangle of N_D and denote it T. We proceed by contradiction: suppose $\{U_{bc}^{\vec{\sigma}}, U_{ca}^{\vec{\sigma}'}\}$ is mapped by f to $\{U_{bc}^{+}, U_{ca}^{+}\}$. By unanimity, $f(U_{ab}^{\vec{\sigma}^N}) = U_{ab}^{+}$. Then T is mapped by f to $\{U_{ab}^{+}, U_{bc}^{+}, U_{ca}^{+}\}$, which is not a simplex (since it corresponds to the intransitive ranking $abca$). □

Lemma 15. Let G be a non-empty coalition distinct from N; $Y \subseteq X$, such that $|Y| = 3$; and $f\colon N_D \to N_{W(X)}$ a unanimous chromatic simplicial map. If $B_1(G, Y)$ (resp. $B_2(G, Y)$) is a subcomplex of N_D, then

1. Any edge of the form $\{U_{ac}^{\vec{\sigma}^G}, U_{ba}^{\vec{\sigma}^{G^c}}\}$, for some $a, b, c \in Y$, cannot be mapped to $\{U_{ac}^{-}, U_{ba}^{-}\}$ (resp. $\{U_{ac}^{+}, U_{ba}^{+}\}$).
2. Any edge of the form $\{U_{ac}^{\vec{\sigma}^{G^c}}, U_{ba}^{\vec{\sigma}^G}\}$, for some $a, b, c \in Y$, cannot be mapped to $\{U_{ac}^{+}, U_{ba}^{+}\}$ (resp. $\{U_{ac}^{-}, U_{ba}^{-}\}$).

Proof. Suppose $B_1(G, Y)$ (resp. $B_2(G, Y)$) is a subcomplex of N_D. Then the triangles $\{U_{ac}^{\vec{\sigma}^G}, U_{ba}^{\vec{\sigma}^{G^c}}, U_{cb}^{\vec{\sigma}^\varnothing}\}$ and $\{U_{ac}^{\vec{\sigma}^{G^c}}, U_{ba}^{\vec{\sigma}^G}, U_{cb}^{\vec{\sigma}^N}\}$ (resp. $\{U_{ac}^{\vec{\sigma}^G}, U_{ba}^{\vec{\sigma}^{G^c}}, U_{cb}^{\vec{\sigma}^N}\}$ and $\{U_{ac}^{\vec{\sigma}^{G^c}}, U_{ba}^{\vec{\sigma}^G}, U_{cb}^{\vec{\sigma}^\varnothing}\}$) are triangles of N_D. The desired results follow by applying Lemma 14. □

Lemma 16. Let G be a non-empty coalition distinct from N; $Y \subseteq X$, such that $|Y| = 3$; and $f\colon N_D \to N_{W(X)}$ a unanimous chromatic simplicial map. If $B_i(G, Y)$ is a subcomplex of N_D for some $i \in \{1, 2\}$, then any edge of the form $\{U_{\beta\gamma}^{\vec{\sigma}^G}, U_{\alpha\beta}^{\vec{\sigma}^{G^c}}\}$ or $\{U_{\beta\gamma}^{\vec{\sigma}^{G^c}}, U_{\alpha\beta}^{\vec{\sigma}^G}\}$, for some $\alpha, \beta, \gamma \in Y$, is mapped by f to an non-DbT edge; in particular, to $\{U_{\beta\gamma}^{+}, U_{\alpha\beta}^{-}\}$ or $\{U_{\beta\gamma}^{-}, U_{\alpha\beta}^{+}\}$.

Proof. W.l.o.g. suppose $B_1(G, Y)$ is a subcomplex of N_D (the other case is analogous). Let $\alpha, \beta, \gamma \in Y$, $\alpha \neq \beta \neq \gamma \neq \alpha$. We will only prove the case of an edge of the form $\{U_{\beta\gamma}^{\vec{\sigma}^G}, U_{\alpha\beta}^{\vec{\sigma}^{G^c}}\}$ since the other case is analogous. Denote $\{U_{\beta\gamma}^{\vec{\sigma}^G}, U_{\alpha\beta}^{\vec{\sigma}^{G^c}}\}$ by e. Since f is a chromatic simplicial map, to get the desired result it suffices to show that e cannot be mapped to a DbT edge.

By part 1 of Lemma 15, the edge e cannot be mapped to $\{U_{\beta\gamma}^{-}, U_{\alpha\beta}^{-}\}$ under f, so let us show that it cannot be mapped to the other DbT edge: $\{U_{\beta\gamma}^{+}, U_{\alpha\beta}^{+}\}$.

We proceed by contradiction: suppose e is mapped by f to $\{U_{\beta\gamma}^{+}, U_{\alpha\beta}^{+}\}$. Since f is chromatic,

$$f(U_{\beta\gamma}^{\vec{\sigma}^{G^c}}) = U_{\beta\gamma}^{+}. \tag{4.1}$$

Observe the following three things:

- By chromaticity of f, we have that $f(U_{\alpha\beta}^{\vec{\sigma}^{G^c}}) = U_{\alpha\beta}^{+}$.
- By Remark 3, we have that $\{U_{\alpha\beta}^{\vec{\sigma}^{G^c}}, U_{\gamma\alpha}^{\vec{\sigma}^G}\}$ is an edge in $B_1(G, Y)$.
- By part 2 of Lemma 15, we have that $\{U_{\alpha\beta}^{\vec{\sigma}^{G^c}}, U_{\gamma\alpha}^{\vec{\sigma}^G}\}$ cannot be mapped to $\{U_{\alpha\beta}^{+}, U_{\gamma\alpha}^{+}\}$ under f.

Taking these three observations together as well as the chromaticy of f, we get that $\{U_{\alpha\beta}^{\vec{\sigma}^{G^c}}, U_{\gamma\alpha}^{\vec{\sigma}^G}\}$ is mapped to $\{U_{\alpha\beta}^+, U_{\gamma\alpha}^-\}$. Now observe the following three things:

- By chromaticity of f, we have that $f(U_{\gamma\alpha}^{\vec{\sigma}^G}) = U_{\gamma\alpha}^-$.
- By Remark 3, we have that $\{U_{\gamma\alpha}^{\vec{\sigma}^G}, U_{\beta\gamma}^{\vec{\sigma}^{G^c}}\}$ is an edge in $B_1(G, Y)$.
- By part 1 of Lemma 15, we have that $\{U_{\gamma\alpha}^{\vec{\sigma}^G}, U_{\beta\gamma}^{\vec{\sigma}^{G^c}}\}$ cannot be mapped to $\{U_{\gamma\alpha}^-, U_{\beta\gamma}^-\}$ under f.

Taking these three observations together as well as the chromaticy of f, we get that $\{U_{\gamma\alpha}^{\vec{\sigma}^G}, U_{\beta\gamma}^{\vec{\sigma}^{G^c}}\}$ is mapped to $\{U_{\gamma\alpha}^-, U_{\beta\gamma}^+\}$. Now observe the following three things:

- By chromaticity of f, we have that $f(U_{\beta\gamma}^{\vec{\sigma}^{G^c}}) = U_{\beta\gamma}^+$.
- By Remark 3, we have that $\{U_{\beta\gamma}^{\vec{\sigma}^{G^c}}, U_{\alpha\beta}^{\vec{\sigma}^G}\}$ is an edge in $B_1(G, Y)$.
- By part 2 of Lemma 15, we have that $\{U_{\beta\gamma}^{\vec{\sigma}^{G^c}}, U_{\alpha\beta}^{\vec{\sigma}^G}\}$ cannot be mapped to $\{U_{\beta\gamma}^+, U_{\alpha\beta}^+\}$ under f.

Taking these three observations together as well as the chromaticy of f, we get that $\{U_{\beta\gamma}^{\vec{\sigma}^{G^c}}, U_{\alpha\beta}^{\vec{\sigma}^G}\}$ is mapped to $\{U_{\beta\gamma}^+, U_{\alpha\beta}^-\}$. Now observe the following three things:

- By chromaticity of f, we have that $f(U_{\alpha\beta}^{\vec{\sigma}^G}) = U_{\alpha\beta}^-$.
- By Remark 3, we have that $\{U_{\alpha\beta}^{\vec{\sigma}^G}, U_{\gamma\alpha}^{\vec{\sigma}^{G^c}}\}$ is an edge in $B_1(G, Y)$.
- By part 1 of Lemma 15, we have that $\{U_{\alpha\beta}^{\vec{\sigma}^G}, U_{\gamma\alpha}^{\vec{\sigma}^{G^c}}\}$ cannot be mapped to $\{U_{\alpha\beta}^-, U_{\gamma\alpha}^-\}$ under f.

Taking these three observations together as well as the chromaticy of f, we get that $\{U_{\alpha\beta}^{\vec{\sigma}^G}, U_{\gamma\alpha}^{\vec{\sigma}^{G^c}}\}$ is mapped to $\{U_{\alpha\beta}^-, U_{\gamma\alpha}^+\}$. Now observe the following three things:

- By chromaticity of f, we have that $f(U_{\gamma\alpha}^{\vec{\sigma}^{G^c}}) = U_{\gamma\alpha}^+$.
- By Remark 3, we have that $\{U_{\gamma\alpha}^{\vec{\sigma}^{G^c}}, U_{\beta\gamma}^{\vec{\sigma}^G}\}$ is an edge in $B_1(G, Y)$.
- By part 2 of Lemma 15, we have that $\{U_{\gamma\alpha}^{\vec{\sigma}^{G^c}}, U_{\beta\gamma}^{\vec{\sigma}^G}\}$ cannot be mapped to $\{U_{\gamma\alpha}^+, U_{\beta\gamma}^+\}$ under f.

Taking these three observations together as well as the chromaticy of f, we get that $\{U_{\gamma\alpha}^{\vec{\sigma}^{G^c}}, U_{\beta\gamma}^{\vec{\sigma}^G}\}$ is mapped to $\{U_{\gamma\alpha}^+, U_{\beta\gamma}^-\}$. Since f is chromatic, we have that $f(U_{\beta\gamma}^{\vec{\sigma}^G}) = U_{\beta\gamma}^-$, a contradiction to equation 4.1. $\square$

Next we present the geometric intuition behind this proof. By hypothesis, $B_1(G,Y)$ is a subcomplex of N_D. Let $\alpha,\beta,\gamma \in Y$ such that $\alpha \neq \beta \neq \gamma \neq \alpha$. Then we can represent $B_1(G,Y)$ as it is depicted in Figure 4.1. Let C_1 denote the dashed line cycle of $B_1(G,\{\alpha,\beta,\gamma\})$ in this figure [2]. With the goal of reaching a contradiction, we assume that the edge $\{U^{\vec{\sigma}^G}_{\beta\gamma}, U^{\vec{\sigma}^{G^c}}_{\alpha\beta}\} \in C_1$ is mapped by f to the DbT edge $\{U^+_{\beta\gamma}, U^+_{\alpha\beta}\}$. This is indicated in Figure 4.4 by labelling edge $\{U^{\vec{\sigma}^G}_{\beta\gamma}, U^{\vec{\sigma}^{G^c}}_{\alpha\beta}\}$ in C_1 and labelling with the same number the edge of $N_{W(X)}$ to which $\{U^{\vec{\sigma}^G}_{\beta\gamma}, U^{\vec{\sigma}^{G^c}}_{\alpha\beta}\}$ is mapped under f. By chromaticity of f we know that every edge in C_1 is mapped to some edge in the part of $skel^1(N_{W(X)})$ only involving alternatives in $\{\alpha,\beta,\gamma\}$. This is why only this subcomplex of $N_{W(X)}$ is depicted in Figure 4.4. Applying, succesively, chromaticity of f and the relevant part of Lemma 15 implies that C_1 has to be mapped over $skel^1(N_{W(X)})$ as indicated by the numbers that act as labels. As it can be seen, $f(U^{\vec{\sigma}^G}_{\beta\gamma}) = U^+_{\beta\gamma}$ and $f(U^{\vec{\sigma}^G}_{\beta\gamma}) = U^-_{\beta\gamma}$, a contradiction.

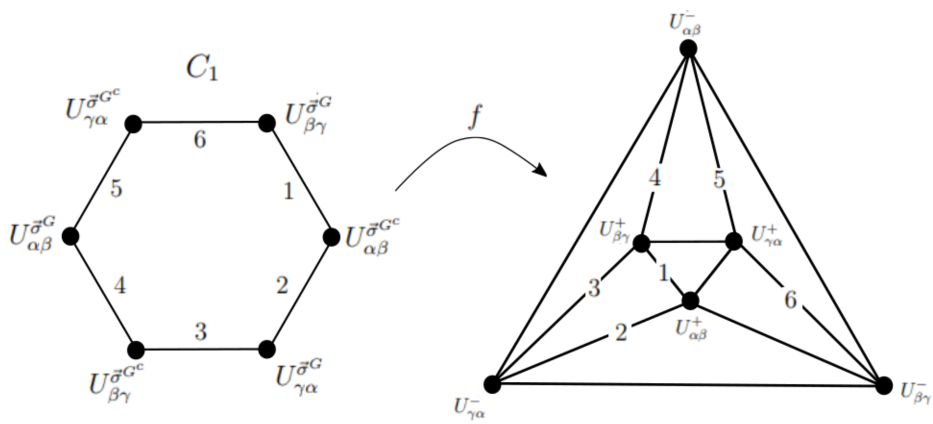

Figure 4.4: Geometric intuition behind the proof of Lemma 16

Lemma 17. Let G be a coalition; $Y = \{\alpha,\beta,\gamma\} \subseteq X$, such that $|Y| = 3$; and $f\colon N_D \to N_{W(X)}$ a unanimous chromatic simplicial map. If $B_i(G,Y)$ is a subcomplex of N_D for some $i \in \{1,2\}$ whenever G is non-empty and distinct from N, then (either) G or G^c is almost-decisive over Y.

Proof. Suppose $B_i(G,Y)$ is a subcomplex of N_D for some $i \in \{1,2\}$ whenever G is non-empty and distinct from N.

If $G = \varnothing$ or $G = N$, then G or G^c equals N, but then by unanimity of f, G or G^c is almost-decisive over Y.

Suppose then $G \neq \varnothing$ and $G \neq N$. Hence, there is $i \in \{1,2\}$ such that $B_i(G,Y)$ is a subcomplex of N_D. Let $\alpha,\beta,\gamma \in Y$ with $\alpha \neq \beta \neq \gamma \neq \alpha$. We begin by asking: where could f map edge $\{U^{\vec{\sigma}^G}_{\beta\gamma}, U^{\vec{\sigma}^{G^c}}_{\alpha\beta}\}$ of $B_i(G,Y)$? By Lemma 16, there are only two options: $\{U^+_{\beta\gamma}, U^-_{\alpha\beta}\}$ or $\{U^-_{\beta\gamma}, U^+_{\alpha\beta}\}$. We proceed by cases.

Case 1: $f(\{U^{\vec{\sigma}^G}_{\beta\gamma}, U^{\vec{\sigma}^{G^c}}_{\alpha\beta}\}) = \{U^+_{\beta\gamma}, U^-_{\alpha\beta}\}$. Then, by chromaticity of f, it holds that

$$f(U^{\vec{\sigma}^G}_{\beta\gamma}) = U^+_{\beta\gamma} \text{ and } f(U^{\vec{\sigma}^{G^c}}_{\alpha\beta}) = U^-_{\alpha\beta}. \tag{4.2}$$

[2]Formally, C_1 is subcomplex of $B_1(G,\{\alpha,\beta,\gamma\})$ of dimension 1

But then by Lemma 16, we have that $f(\{U_{\alpha\beta}^{\vec{\sigma}^{G^c}}, U_{\gamma\alpha}^{\vec{\sigma}^G}\}) = \{U_{\alpha\beta}^-, U_{\gamma\alpha}^+\}$. Then, by chromaticity of f, it holds that

$$f(U_{\gamma\alpha}^{\vec{\sigma}^G}) = U_{\gamma\alpha}^+. \tag{4.3}$$

But then by Lemma 16, we have that $f(\{U_{\gamma\alpha}^{\vec{\sigma}^G}, U_{\beta\gamma}^{\vec{\sigma}^{G^c}}\}) = \{U_{\gamma\alpha}^+, U_{\beta\gamma}^-\}$. Then, by chromaticity of f, it holds that

$$f(U_{\beta\gamma}^{\vec{\sigma}^{G^c}}) = U_{\beta\gamma}^-. \tag{4.4}$$

But then by Lemma 16, we have that $f(\{U_{\beta\gamma}^{\vec{\sigma}^{G^c}}, U_{\alpha\beta}^{\vec{\sigma}^G}\}) = \{U_{\beta\gamma}^-, U_{\alpha\beta}^+\}$. Then, by chromaticity of f, it holds that

$$f(U_{\alpha\beta}^{\vec{\sigma}^G}) = U_{\alpha\beta}^+. \tag{4.5}$$

But then by Lemma 16, we have that $f(\{U_{\alpha\beta}^{\vec{\sigma}^G}, U_{\gamma\alpha}^{\vec{\sigma}^{G^c}}\}) = \{U_{\alpha\beta}^+, U_{\gamma\alpha}^-\}$. Then, by chromaticity of f, it holds that

$$f(U_{\gamma\alpha}^{\vec{\sigma}^{G^c}}) = U_{\gamma\alpha}^-. \tag{4.6}$$

Taking 4.2 to 4.6, we obtain that G is almost-decisive over Y.

Case 2: $f(\{U_{\beta\gamma}^{\vec{\sigma}^G}, U_{\alpha\beta}^{\vec{\sigma}^{G^c}}\}) = \{U_{\beta\gamma}^-, U_{\alpha\beta}^+\}$. Analogously to case 1, successively applying chromaticity of f and Lemma 16 we get:

$$f(U_{\beta\gamma}^{\vec{\sigma}^G}) = U_{\beta\gamma}^-, f(U_{\alpha\beta}^{\vec{\sigma}^{G^c}}) = U_{\alpha\beta}^+, f(U_{\gamma\alpha}^{\vec{\sigma}^G}) = U_{\gamma\alpha}^-, \tag{4.7}$$

$$f(U_{\beta\gamma}^{\vec{\sigma}^{G^c}}) = U_{\beta\gamma}^+, f(U_{\alpha\beta}^{\vec{\sigma}^G}) = U_{\alpha\beta}^-, \text{ and } f(U_{\gamma\alpha}^{\vec{\sigma}^{G^c}}) = U_{\gamma\alpha}^+. \tag{4.8}$$

Therefore, for case 2, it holds that G^c is almost-decisive over Y. □

Let us proceed with the geometric intuition behind the proof of Lemma 16. Let C_2 denote the cycle that consists of the non-DbT edges that are represented in solid-lines in Figure 4.3. Cycle C_2 is represented on the right of Figure 4.5, along with cycle C_1 represented on the left of this figure. By Lemma 17, C_1 has to be mapped over C_2 under f. So we can start by asking where could edge $\{U_{\beta\gamma}^{\vec{\sigma}^G}, U_{\alpha\beta}^{\vec{\sigma}^{G^c}}\}$ by mapped under f. There are two options: $\{U_{\beta\gamma}^+, U_{\alpha\beta}^-\}$ or $\{U_{\beta\gamma}^-, U_{\alpha\beta}^+\}$. So let us see both cases.

Case 1 (resp. 2): $f(\{U_{\beta\gamma}^{\vec{\sigma}^G}, U_{\alpha\beta}^{\vec{\sigma}^{G^c}}\}) = \{U_{\beta\gamma}^+, U_{\alpha\beta}^-\}$ (resp. $f(\{U_{\beta\gamma}^{\vec{\sigma}^G}, U_{\alpha\beta}^{\vec{\sigma}^{G^c}}\}) = \{U_{\beta\gamma}^-, U_{\alpha\beta}^+\}$) . In this case, by chromaticity, C_1 has to be mapped as follows: if e is an edge of C_2 and x is the number that acts as a label for e, then f maps e to the edge in C_1 that has x as the first (resp. second) number appearing in its label. For example, edge $\{U_{\beta\gamma}^{\vec{\sigma}^{G^c}}, U_{\gamma\alpha}^{\vec{\sigma}^G}\}$, with label 3, gets mapped to $\{U_{\beta\gamma}^-, U_{\gamma\alpha}^+\}$ (resp. $\{U_{\beta\gamma}^+, U_{\gamma\alpha}^-\}$), with label $3, 6$ (resp. $6, 3$). Looking at how the vertices are mapped, we can see that G (resp. G^c) is almost-decisive over Y.

For the case of only two voters and three alternatives, Rajsbaum and Raventós-Pujol [37, 38] say that the cycle that we call C_1 has to be mapped over the cycle C_2 due to the unanimity edges in dashed-lines in Figure 3.2, but they do not go into the details of why. We formalized this via 15 and 16. Furthermore, we generalized their argument because we do not need the unanimity edges, only the unanimity vertices. Moreover, we showed that this same argument can be applied when there are $n \geq 2$ voters and $|X| \geq 3$ alternatives if we have the structure provided by the $B_i(\cdot, \cdot)$'s and focus on the relevant part of the 2-skeleton of N_D. Finally, [37, 38], like us, say that C_1 can be mapped over C_2 in two ways, one of which makes voter 1 the dictator and the other makes voter the dictator. In our case, since we are dealing $n \geq 2$ voters and $|X| \geq 3$ alternatives, we can only conclude that G or G^c is almost-decisive over the triple of alternatives Y.

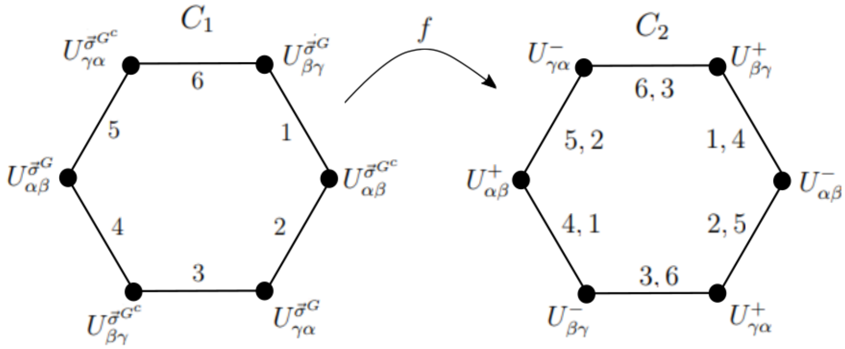

Figure 4.5: Geometric intuition behind the proof of Lemma 17.

We are now ready to define the class D^{PT} of domains.

Definition 15. The class of preference domains of *polarized over triples*, denoted $\mathcal{D}^{\mathrm{PT}}$, is defined as follows: $\mathcal{D}^{\mathrm{PT}}$ iff for every coalition G that is non-empty and distinct from N, and every triple $\{\alpha, \beta, \gamma\} \subseteq X$, $\alpha \neq \beta \neq \gamma \neq \alpha$, we have that $B_1(G, \{\alpha, \beta, \gamma\})$ is a subcomplex of N_D or $B_2(G, \{\alpha, \beta, \gamma\})$ is a subcomplex of N_D (or equivalently: $D_1(G, \{\alpha, \beta, \gamma\})$ is a subset of $D|_{\{\alpha,\beta,\gamma\}}$ or $D_2(G, \{\alpha, \beta, \gamma\})$ is a subset of $D|_{\{\alpha,\beta,\gamma\}}$).

The following lemma says that, given a domain in $\mathcal{D}^{\mathrm{PT}}$, the almost-decisiveness of a coalition over an ordered pair of alternatives spreads to all ordered pairs of alternatives. This sort of "contagion" result has been used in other ultrafilter proofs. For instance, for the case of the unrestricted domain, [32] shows that this contagion of almost-decisiveness occurs. As another example, [16] has a contagion lemma for the case of domains satisfying the chain property.

Lemma 18. Let $f\colon N_D \to N_W$ be a unanimous chromatic simplicial map, where $D \in \mathcal{D}^{\mathrm{PT}}$, let G be a non-empty coalition distinct from N, and α and β two different alternatives in X such that $f(U_{\alpha\beta}^{\vec{\sigma}^G}) = U_{\alpha\beta}^{+}$, then G is almost-decisive.

Proof. Suppose there are two different alternatives α and β in X such that $f(U_{\alpha\beta}^{\vec{\sigma}^G}) = U_{\alpha\beta}^{+}$. Let $\gamma, \delta \in X \setminus \{\alpha, \beta\}$ with $\gamma \neq \delta$. To show:

1. $f(U_{\gamma\delta}^{\vec{\sigma}^G}) = U_{\gamma\delta}^{+}$

2. $f(U_{\alpha\gamma}^{\vec{\sigma}^G}) = U_{\alpha\gamma}^{+}$

3. $f(U_{\gamma\alpha}^{\vec{\sigma}^G}) = U_{\gamma\alpha}^{+}$

4. $f(U_{\beta\gamma}^{\vec{\sigma}^G}) = U_{\beta\gamma}^{+}$

5. $f(U_{\gamma\beta}^{\vec{\sigma}^G}) = U_{\gamma\beta}^{+}$

6. $f(U_{\beta\alpha}^{\vec{\sigma}^G}) = U_{\beta\alpha}^{+}$

We focus on proving 1 and we will prove 2-6 along the way.

Let $Y_1 = \{\alpha, \beta, \gamma\}$. Since $f(U_{\alpha\beta}^{\vec{\sigma}^G}) = U_{\alpha\beta}^{+}$, by Lemma 9 G^c cannot be almost-decisive over Y_1. Then by Lemma 17, G is almost-decisive over Y_1. Therefore, 2-6 hold. In particular, $f(U_{\alpha\gamma}^{\vec{\sigma}^G}) = U_{\alpha\gamma}^{+}$.

Let $Y_2 = \{\alpha, \gamma, \delta\}$. Since $f(U_{\alpha\gamma}^{\vec{\sigma}^G}) = U_{\alpha\gamma}^{+}$, by Lemma 9 G^c cannot be almost-decisive over Y_2. Then by Lemma 17, G is almost-decisive over Y_2. Then $f(U_{\gamma\delta}^{\vec{\sigma}^G}) = U_{\gamma\delta}^{+}$, i.e. 1 holds. □

Now we give the geometric intuition of this proof. Observe that Y_1 and Y_2 are triples that share exactly two alternatives, i.e. α and γ. Moreover, since we are working with an arbitrary domain in $\mathcal{D}^{\mathrm{PT}}$, we have that:

- $B_1(G, Y_1)$ or $B_2(G, Y_1)$ exist as a subcomplex of N_D, and
- $B_1(G, Y_2)$ or $B_2(G, Y_2)$ exist as a subcomplex of N_D

Let $B(G, Y_1) = B_i(G, Y_1)$, where $i \in \{1, 2\}$, and such that $B_i(G, Y_1)$ exist in N_D. Let $B(G, Y_2) = B_i(G, Y_2)$, where $i \in \{1, 2\}$, and such that $B_i(G, Y_2)$ exist in N_D. It is easy to see that $B(G, Y_1)$ and $B(G, Y_2)$ share exactly two non-unanimous vertices, namely $U_{\alpha\gamma}^{\vec{\sigma}^G}$ and $U_{\gamma\alpha}^{\vec{\sigma}^G}$, as depicted in Figure 4.8. So intuitively, the fact $f(U_{\alpha\beta}^{\vec{\sigma}^G}) = U_{\alpha\beta}^{+}$ is spreading almost-decisiviness along the cycles depicted in Figure 4.8 until it reaches all the target vertices.

This argument is similar in spirit to the local approach of Kalai et al. [31]. To read more on the local approach check [35].

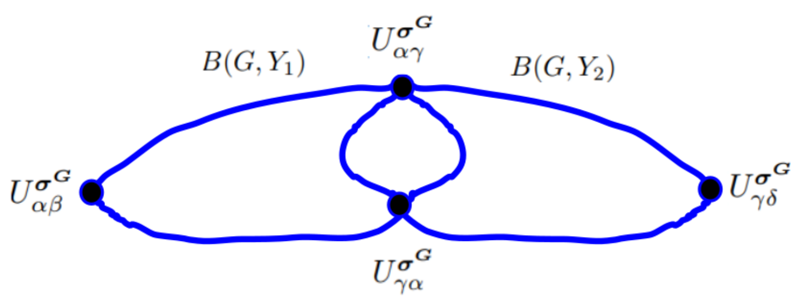

Figure 4.6: Geometric ntuition behind the proof of Lemma 18.

The subsequent theorem states that membership to $\mathcal{D}^{\mathrm{PT}}$ provides a sufficient condition for the set of all almost-decisive coalitions (w.r.t. a given f) to satisfy the third property of ultrafilters.

Theorem 19. Let $f\colon N_D \to N_W$ be a unanimous chromatic simplicial map, where $D \in \mathcal{D}^{\mathrm{PT}}$ and let G be a coalition. We have that G or G^c is almost-decisive.

Proof. If $G = \varnothing$ or $G = N$, then G or G^c equals N, but then by unanimity of f, G or G^c is almost-decisive. Suppose then $G \neq \varnothing$ and $G \neq N$.

Let $\alpha, \beta \in X$, $\alpha, \beta \in X$. Then $f(U_{\alpha\beta}^{\vec{\sigma}^G}) = U_{\alpha\beta}^{+}$ or $f(U_{\alpha\beta}^{\vec{\sigma}^G}) = U_{\alpha\beta}^{-}$ (equivalently, $f(U_{\beta\alpha}^{\vec{\sigma}^{G^c}}) = U_{\beta\alpha}^{+}$). In the first case, Lemma 18 implies that G is almost-decisive. In the second case, Lemma 18 implies that G^c is almost-decisive. $\square$

4.4 Diversity and the Second Ultrafilter Property

In this section, given a unanimous chromatic simplicial map $f\colon N_D \to N_{W(X)}$ we want to define a class $\mathcal{D}$ such that if $D \in \mathcal{D}$, then the set of all almost-decisive coalitions $\mathcal{G}$ satisfies the second property of ultrafilters w.r.t. N, i.e.

$$\text{if } G, G' \in \mathcal{G}, \text{ then } (G \cap G') \in \mathcal{G}.$$

We will present a class of preference domains that we will call the *class of diversity over triples*, denoted $\mathcal{D}^{\mathrm{DT}}$, that is going to fulfill the requirement stated in the previous paragraph. Then we will use $D \in \mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$, which we call the *class of polarization and diversity over triples*, to obtain the second property of ultrafilters, allowing us to obtain a generalized version of Arrow's theorem.

Definition 16. The *class of diversity over triples*, denoted $\mathcal{D}^{\mathrm{DT}}$, is a class of preference domains defined as follows: $D \in \mathcal{D}^{\mathrm{DT}}$ iff for every two coalitions G and G' such that $G \not\subseteq G'$ and $G' \not\subseteq G$, there exists three alternatives $\alpha, \beta, \gamma \in X$ such that $\{U_{\alpha\beta}^{\vec{\sigma}^G}, U_{\beta\gamma}^{\vec{\sigma}^{G'}}, U_{\gamma\alpha}^{\vec{\sigma}^{(G \cap G')^c}}\}$ is a 2-simplex of N_D.

To reflect upon Definition 16, consider the following equivalent way to define $\mathcal{D}^{\mathrm{DT}}$: $D \in \mathcal{D}^{\mathrm{DT}}$ iff for every two coalitions G and G' such that $G \not\subseteq G'$ and $G' \not\subseteq G$, there exists three alternatives $\alpha, \beta, \gamma \in X$ such that there is a profile $P \in D$ such that:

- if $i \in G \setminus G'$, then $P_i = \gamma\alpha\beta$;
- if $i \in G \cap G'$, then $P_i = \alpha\beta\gamma$; and
- if $i \in G' \setminus G$, then $P_i = \beta\gamma\alpha$.
- if $i \in N \setminus (G \cup G')$, then $P_i = \gamma\beta\alpha$

If $G \cap G'$ is non-empty, then then such a profile $\vec{P}$ has the property that there is a voter whose preference in $\vec{P}$ restricted to $\{\alpha, \beta, \gamma\}$ is $\gamma\alpha\beta$; another voter with $\alpha\beta\gamma$; and a third voter $\beta\gamma\alpha$. In other words, for any alternatives $a, b, c \in \{\alpha, \beta, \gamma\}$, $a \neq b \neq c$, there exists a voter in N that, in $\vec{P}$, ranks a on top of b and c; another voter that ranks a in the middle of b and c; an yet a third voter that ranks a below b and c. Therefore such a profile $\vec{P}$ is not value-restricted.

If $n \geq 3$, there exists coalitions G and G' satisfying $G \not\subseteq G'$ and $G' \not\subseteq G$ such that $G \cap G' \neq \varnothing$. For example, $G = \{1, 2\}$ and $G' = \{2, 3\}$. Therefore, for $n \geq 3$, if $D \in \mathcal{D}^{\mathrm{DT}}$, then D has at least a profile that is not value-restricted.

Observe that if $n = 2$, then the only pair of coalitions satisfying $G \nsubseteq G'$ and $G' \nsubseteq G$ is $\{\{1\},\{2\}\}$. In that case, Definition 16 requires the existence of a profile with a subprofile of the form $(\gamma\alpha\beta, \beta\gamma\alpha)$ or $(\beta\gamma\alpha, \gamma\alpha\beta)$, which are strongly polarized profiles on $\{\alpha, \beta, \gamma\}$ (see Definition 11).

Now we show that $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ is sufficient to induce the second property of ultrafilters.

Theorem 20. Let $f\colon N_D \to N_{W(X)}$ be a unanimous chromatic simplicial map such that $D \in \mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$. If G and G' are two almost-decisive coalitions (w.r.t. f), then the coalition $G \cap G'$ is almost-decisive.

Proof. Suppose G and G' are two almost-decisive coalitions (w.r.t f). If G or G' is empty. Then $G \cap G'$ is empty and we are done. Suppose then G and G' are non-empty.

If $G \subseteq G'$ or $G' \subseteq G$ then $G \cap G'$ is G or G' and we are done. Suppose then $G \nsubseteq G'$ and $G' \nsubseteq G$.

So let us assume that $G \nsubseteq G'$ and $G' \nsubseteq G$. We proceed by contradiction, suppose $G \cap G'$ is not almost-decisive. Then, since $D \in \mathcal{D}^{\mathrm{PT}}$, by Theorem 19, $(G \cap G')^c$ is almost-decisive. Since $D \in \mathcal{D}^{\mathrm{DT}}$, $G \nsubseteq G'$ and $G \nsubseteq G'$, there exists alternatives $\alpha, \beta, \gamma \in X$ such that $\{U_{\alpha\beta}^{\vec{\sigma}^G}, U_{\beta\gamma}^{\vec{\sigma}^{G'}}, U_{\gamma\alpha}^{\vec{\sigma}^{(G\cap G')^c}}\}$ is a 2-simplex of N_D, denote it T.

Since, G, G' and $(G \cap G')^c$ are almost-decisive, T is mapped to $U^+_{\alpha\beta}, U^+_{\beta\gamma}, U^+_{\gamma\alpha}$, but this is not a simplex of $N_{W(X)}$ (since it corresponds to the intransitive ranking $\alpha\beta\gamma\alpha$), a contradiction. □

4.5 Arrow-inconsistency on $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$

Now we present the generalized version of Arrow's theorem:

Theorem 21. If $D \in \mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$, then D is Arrow-inconsistent.

Proof. Let $D \in \mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$, $f\colon N_D \to N_{W(X)}$ be a unanimous chromatic simplicial map, and $\mathcal{G}$ the set of all almost-decisive coalitions w.r.t. f.

By Lemma 10, if we show that $\mathcal{G}$ is an ultrafilter w.r.t. N, we are done.

Since $D \in \mathcal{D}^{\mathrm{PT}}$, we have that N_D has a unanimity vertex. Therefore, by Proposition 13, $\varnothing \notin \mathcal{G}$. Hence, property 1 of ultrafilters hold.

Also, since $D \in \mathcal{D}^{\mathrm{PT}}$, Theorem 20 guarantees that property 3 of ultrafilters holds.

Finally, having $D \in \mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ guarantees, by Theorem 19, that property 2 of ultrafilters hold.

Therefore, $\mathcal{G}$ is an ultrafilter w.r.t. N. □

We finalize this chapter proving that the property of belonging to $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ is closed upward under inclusion.

Proposition 22. Let D and D' are domains such that $D \in \mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ and $D \subseteq D'$. We have that $D' \in \mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$

Proof. Since $D \subseteq D'$, it is easy to see that N_D is a subcomplex of $N_{D'}$, we denote this fact as $N_D \subseteq N_{D'}$.

Firstly, let us see that $D' \in \mathcal{D}^{\mathrm{PT}}$. To see this, let G be a non-empty coalition distinct from N and $Y \subseteq X$ such that $|Y| = 3$. Since $D \in \mathcal{D}^{\mathrm{PT}}$, there exists an $i \in \{1, 2\}$ such that $B_i(G, Y) \subseteq N_D$. Since $N_D \subseteq N_{D'}$, we have that $B_i(G, Y) \subseteq N_{D'}$. Hence, $D' \in \mathcal{D}^{\mathrm{PT}}$.

Secondly, let us prove that $D' \in \mathcal{D}^{\mathrm{DT}}$. To see this, let G and G' coalitions such that $G \setminus G'$ and $G' \setminus G$ are non-empty. Since $D \in \mathcal{D}^{\mathrm{DT}}$, there exists $\alpha, \beta, \gamma \in X$, all different from each other, such that $\{U^{\vec{\sigma}^G}_{\alpha\beta}, U^{\vec{\sigma}^{G'}}_{\beta\gamma}, U^{\vec{\sigma}^{(G\cap G')^c}}_{\gamma\alpha}\}$ is a 2-simplex of N_D. Since $N_D \subseteq N_{D'}$, the set $\{U^{\vec{\sigma}^G}_{\alpha\beta}, U^{\vec{\sigma}^{G'}}_{\beta\gamma}, U^{\vec{\sigma}^{(G\cap G')^c}}_{\gamma\alpha}\}$ is a 2-simplex of N'_D. Hence, $D' \in \mathcal{D}^{\mathrm{DT}}$. Therefore, $D' \in \mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$. □

Proposition 23. If $D \in \mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$, then D is super-Arrovian.

Proof. Follows from combining Theorem 7 and Proposition 22. □

For the case of 3 alternatives and 2 voters, super-Arrovian domain "D^*" in the proof of Lemma 2 in [20, p. 87] is clearly a member of $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$. For the case of 3 alternatives and 3 voters, it is easy to see that the super-Arrovian domain that appears in the proof of Lemma 3 in [20, pp. 88–89] is a subdomain of some domains in $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$. For the case of $|X| \geq 3$ alternatives and $n \geq 2$ voters, the unrestricted domain clearly belongs to $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$. For the case of 4 alternatives and 2 voters, in the next chapter, we will present a domain in $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ that consists of group-separable profiles.

Chapter 5

An Example of a Domain in $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$

In this chapter, we work under the assumption that there are only two voters and only four alternatives. In Section 5.1, we present a domain, called $D_{B,4}$, different from the unrestricted domain, that belongs to the class $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ defined in Chapter 4. In Section 5.2, we show that this domain in fact belongs to this class and prove that every profile in $D_{B,4}$ is group-separable.

5.1 The Domain $D_{B,4}$

Suppose there are only two voters, $n = 2$, and only four alternatives, $X = \{w, x, y, z\}$. Let $D_{B,4}$ be the domain defined as follows: $\vec{P} \in D_{B,4}$ iff $\vec{P}$ is of the form $(G : \alpha\beta\gamma\delta, G^c : \gamma\delta\alpha\beta)$ for some $\alpha, \beta, \gamma, \delta \in X$, all distinct from each other. For example: $(xywz, wzxy)$, $(yxwz, wzyx)$, and $(zwxy, xyzw)$ are all members of $D_{B,4}$. Notice that once you fix the preference ranking of everyone in G or everyone in G^c to be a ranking of the form $\alpha\beta\gamma\delta$ for some $\alpha, \beta, \gamma, \delta \in X$, then the rankings of the complementary coalition must be $\gamma\delta\alpha\beta$ (and this is a also a sufficient condition for the resulting profile to be in $D_{B,4}$). Therefore, it is not hard to see that there are 4! profiles satisfying this property, so $|D_{B,4}| = 4! = 24$.

In order to study $D_{B,4}$ we are going to analyze its 2-skeleton. We start doing that in the next proposition.

Proposition 24. The 2-skeleton of $N_{D_{B,4}}$, denoted $skel^2(N_{D_{B,4}})$, is

$$\bigcup_{\substack{j\in\{1,2\} \\ G\subseteq N \\ \{\alpha,\beta,\gamma\}\subseteq X}} B_j(G, \{\alpha, \beta, \gamma\}),$$

where $B_j(G, \{\alpha, \beta, \gamma\})$ is the simplicial complex associated with the domain in Definition 12 if $j = 1$ and Definition 13 if $j = 2$.

Proof. Let G be a coalition. To show:

1. If $\vec{P} \in D_{B,4}$ and $Y \subseteq X$, with $|Y| = 3$, then any subprofile of $\vec{P}$ on Y is a strongly polarized profile $(G : P, G^c : P')$ for some $P, P' \in W(\{\alpha, \beta, \gamma\})$. Proving this, shows that

$$skel^2(N_{D_{B,4}}) \subseteq \bigcup_{\substack{j\in\{1,2\}\\ G\subseteq N\\ \{\alpha,\beta,\gamma\}\subseteq X}} B_j(G, \{\alpha, \beta, \gamma\}).$$

2. Let $Y = \{\alpha, \beta, \gamma\} \subseteq X$. Any profile strongly polarized profile $(G : R, G^c : P')$ for some $P, P' \in W(\{\alpha, \beta, \gamma\})$ is a subprofile of some profile $\vec{P} \in D_{B,4}$. Due to Definitions 12 and 13, this shows that

$$\bigcup_{\substack{j\in\{1,2\}\\ G\subseteq N\\ \{\alpha,\beta,\gamma\}\subseteq X}} B_j(G, \{\alpha, \beta, \gamma\}) \subseteq skel^2(N_{D_{B,4}}).$$

Let us start with 1. Let $\vec{P} \in D_{B,4}$. Then $\vec{P}$ can be written as $(G : abcd, G^c : cdab)$ for some $a, b, c, d \in \{w, x, y, x\} = X$, all different from each other. We have that $(G : abcd, G^c : cdab)$

- restricted to $\{a, b, c\}$ is $(G : abc, G^c : cab)$, note that abc and cab coincide in how they rank exactly one pair of alternatives;
- restricted to $\{a, b, d\}$ is $(G : abd, G^c : dab)$, note that abd and dab coincide in how they rank exactly one pair of alternatives;
- restricted to $\{a, c, d\}$ is $(G : acd, G^c : cda)$, note that acd and cda coincide in how they rank exactly one pair of alternatives;
- restricted to $\{b, c, d\}$ is $(G : bcd, G^c : cdb)$, note that bcd and cdb coincide in how they rank exactly one pair of alternatives.

Clearly, these are all the possible restrictions of $(G : abcd, G^c : cdab)$ to a subset of three different alternatives (all different from each other) of $\{w, x, y, z\}$. This completes the proof of 1.

Now let us prove 2. It is not hard to see that there are 12 profiles $W(\{\alpha, \beta, \gamma\})^n$ satisfying the required condition. Do to symmetries it suffices to show that $(\alpha\beta\gamma, \gamma\alpha\beta)$, $(\alpha\beta\gamma, \beta\gamma\alpha)$, $(\alpha\gamma\beta, \beta\alpha\gamma)$ and $(\alpha\gamma\beta, \gamma\beta\alpha)$ are subprofiles of some profile $\vec{P} \in D_{B,4}$. Let $\delta \in \{w, x, y, z\} \setminus \{\alpha, \beta, \gamma\}$. We have the following:

- $(\alpha\beta\gamma, \gamma\alpha\beta)$ is a subprofile of $(\alpha\beta\gamma\delta, \delta\gamma\alpha\beta) \in D_{B,4}$.
- $(\alpha\beta\gamma, \beta\gamma\alpha)$ is a subprofile of $(\alpha\delta\beta\gamma, \beta\gamma\alpha\delta) \in D_{B,4}$.
- $(\alpha\gamma\beta, \beta\alpha\gamma)$ is a subprofile of $(\alpha\gamma\delta\beta, \delta\beta\alpha\gamma) \in D_{B,4}$.
- $(\alpha\gamma\beta, \gamma\beta\alpha)$ is a subprofile of $(\delta\alpha\gamma\beta, \gamma\beta\delta\alpha) \in D_{B,4}$.

$\square$

Remark 4. Clearly, $D_{B,4} \in \mathcal{D}^{\mathrm{PT}}$.

Remark 5. $D_{B,4}$ does not satisfy the chain property mentioned in Chapter 1 because there are no free triples in X w.r.t. $D_{B,4}$.

By remark 4, at least for the case of $n = 2$ and four alternatives, the unrestricted domain is not the only domain in $\mathcal{D}^{\mathrm{PT}}$.

In Figure 5.1, we have depicted $skel^2(N_{D_{B,4}})$ schematically (omitting making explicit what vertices and edges are shared between the $B_i(G, \cdot)$'s).

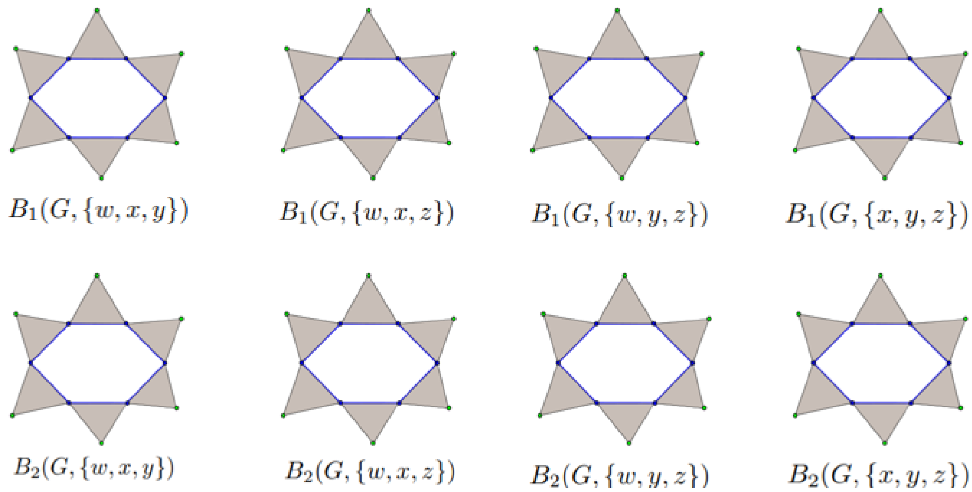

Figure 5.1: Schematic (not accurate) drawing of $skel^2(N_{D_{B,4}})$.

5.2 Arrow-inconsistency and Group-separability of $D_{B,4}$

In this section, we will prove important properties of $D_{B,4}$ using Proposition 24.

Proposition 25. $D_{B,4}$ belongs to $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$.

Proof. Clearly, by Proposition 24, $D_{B,4} \in D^{\mathrm{PT}}$.

We now prove that $D_{B,4} \in D^{\mathrm{DT}}$. As we said in our discussion of the definition of D^{DT} in Section 4.4, when $n = 2$, a domain belongs to D^{DT} iff it has a subprofile of the form $(\gamma\alpha\beta, \beta\gamma\alpha)$ or $(\beta\gamma\alpha, \gamma\alpha\beta)$. But as we also said in the referred section, these profiles are strongly polarized profiles, and therefore, by Proposition 24, they live in

$$\bigcup_{\substack{j\in\{1,2\}\\ \{\alpha,\beta,\gamma\}\subseteq X}} B_j(1, \{\alpha, \beta, \gamma\}).$$

. Therefore, by Proposition 24, $D_{B,4} \in D^{\mathrm{DT}}$. $\square$

Corollary 26. $D_{B,4}$ is Arrow-inconsistent.

This corollary is true by our generalization of Arrow's theorem, i.e. Theorem 21.

Another thing that we can prove about $D_{B,4}$ is that it is group-separable (see Chapter 2 for a definition).

Proposition 27. Every profile in $D_{B,4}$ is group-separable.

Proof. Fix a profile $\vec{P} \in D_{B,4}$. To show: $\vec{P}$ is group-separable. By definition of $D_{B,4}$ can rewrite $\vec{P}$ as $(\alpha\beta\gamma\delta, \gamma\delta\alpha\beta)$.

Let $Y \subseteq X$ such that $|Y| \geq 2$. We want to exhibit a proper subset Z of Y such that $ZP_i(Y \setminus Z)$ or $(Y \setminus Z)P_iY$ for all $i \in N$.

We define Z as follows: for all $x \in Y$, $x \in \{\alpha, \beta\}$.

We will show that $ZP_1(Y|Z)$ and $(Y \setminus Z)P_2Z$ (to understand this notation, see Chapter 2).

Let $z \in Z$ and $z' \in Y \setminus Z$. By definition, $z \in Z$ implies $z \in \{\alpha, \beta\}$, and $z' \in Z'$ implies $z \in \{\gamma, \delta\}$.

Since $P_1 = \alpha\beta\gamma\delta$ and $P_2 = \gamma\delta\alpha\beta$, we have zP_1z' and zP_2z'. But then, since z and z' were chosen arbitrarily, $ZP_1(Y|Z)$ and $(Y \setminus Z)P_2Z$. $\square$

Chapter 6

Conclusion

In this thesis, we generalized, allowing for $n \geq 2$ voters and $|X| \geq 3$ alternatives, the combinatorial topology approach of [37, 38], which only dealt with two voters and three alternatives. Using this approach, we showed that if a preference domain belongs to the class of polarization and diversity over triples, $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$, it is Arrow-inconsistent, i.e. $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ provides a generalization of Arrow's theorem. Also, our proofs show that the ultrafilter technique can be used in yet another domain restriction. Finally, we provided $D_{B,4}$ as an example of a domain in $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ (other than the unrestricted domain), which has the interesting property of being group-separable.

6.1 Connecting our Contributions to some Topics in Computational Social Choice and Distributed Computing

To explain how our contributions might be relevant to some topics in computational social choice and distributed computing, we should start by saying that SWFs are not the only voting rules that are important in social choice. In particular, a very relevant kind of voting rule is the so-called social choice function that we define now. If $D \subseteq W(X)^n$ is a preference domain, a *social choice function* (*SCF* and *SCFs* for plural) is a function of the form $f : D \to \mathcal{P}(X) \setminus \{\varnothing\}$, where $\mathcal{P}(X)$ is the power set of the set of alternatives X. In other words, a SCF maps a profile in D to a non-empty subset of the alternatives. Following [47], if f is a SCF such that $|f(\vec{P})| = 1$, for all $\vec{P}$ in its domain, then f is called *resolute*. For resolute SCFs defined on the unrestricted domain $W(X)^n$, there is a very famous impossibility result, the Gibbard-Satterthwaite theorem, by Gibbard [23] and Satterthwaite [39]. Informally, this theorem says that when there are $|X| \geq 3$ alternatives, a surjective and resolute SCF whose outcome is immune to misrepresentation of preferences on behalf of voters must be a dictatorship (see [47] for a formal version of this statement).

6.1.1 Complexity Aspects of Voting Rules

According to Brandt et al. [15], prior to the late 1980s, the social choice literature was mainly concerned with normative appealing impossibility and possibility results, like Arrow's and Gibbard-Satterthwaite's impossibility theorems, and the computational aspects related to voting rules were largely ignored. In particular, the computational cost of determining the winner of an election (the *winner determination problem* as it is called in [19]) was not taken into account. A rule that takes too long to compute a winner might be unusable in real applications. In 1989, Bartholdi III et al. [8] showed that, for the *Dodgson voting rule* and the *Kemeny winner* rule (both SCFs), the problem of determining if a given alternative is the winner of the election is NP-hard. These authors also showed that, for the *Kemeny ranking* rule (a SWF), determining the winner ranking is also NP-hard. In 1997, Hemaspaandra et al. [25] showed that the winner determination problem for the Dodgson voting rule is parallel access to NP complete, Θ_2^p-complete. In 2005, Hemmaspaandra et al. [26] proved that this same problem but for the Kemeny winner and ranking rules is also Θ_2^p-complete.

Our goal now is to explain how the topic of this thesis relates to the winner determination problem. Elkind et al. [19] say that domain restrictions that help escape impossibility theorems like those of Arrow and Gibbard-Satterthwaite tend to be useful in making hard winner determination problems computationally easier on the restricted inputs. According to [19], the idea of relating complexity problems with domain restrictions that work for escaping impossibility theorems was started by Walsh [45]. As it is reviewed in Elkind et al. [19], for preferences satisfying being *single-peaked on a tree* (see [19] for a definition), *single-crossing on a tree* (see [19] for a definition), and group-separable preferences, the winner determination problem of the Kemeny winner rule and *Young rule* (a SCF whose winner determination problem is also hard for the unrestricted domain) can be decided in polynomial time [see also 13]. As it is also reviewed in Elkind et al. [19], preferences single-crossing on a tree and group-separable preferences make the strict part of the majority relation transitive, for n odd (so they can help us escape Arrow's theorem). However, as it is also said in [19], preferences single-peaked on a tree does not necessarily make the majority relation transitive. Since in this thesis we propose new techniques to discover domain restrictions for Arrow's theorem such techniques can eventually lead to finding new domain restrictions that make hard complexity problems, related to voting rules, easier.

6.1.2 Combinatorial Topology and Distributed Computing

The contributions of this thesis also relate to the intersection of computational social choice and distributed computing. This latter field studies how n computing processes can communicate with each other (for instance, by writing in a shared memory or by message passing) to solve tasks (for precise definitions of the basic concepts of distributed computing see [5, 27]). Combinatorial topology has been very useful in distributed computing [see 27]. Rajsbaum and Raventós-Pujol [38] draw interesting analogies in the way that combinatorial topology has been used in distributed computing and how it is starting to be used in social choice. A future research question would be to see whether this thesis can contribute to establishing new analogies between the use of combinatorial topology in these fields.

We now explain a second way in which this thesis might relate to distributed computing.

So far we have discussed scenarios where there is an implicit trusted central entity that takes a profile of preferences and computes the result of the vote according to a voting rule. In 2013, Chauhan and Garg [17] moved beyond centralized voting procedures to study distributed ones, i.e., procedures in which n computational processes (the voters in this context), each starting with a preference ranking, have to communicate with each other to reach an agreement over the outcome of the election. In particular, these authors study how to implement SWFs and SCFs in a distributed manner, in a *synchronous system* (a system in which processes' massages take at most certain constant units of time to reach other processes) and allowing for *byzantine failures* (arbitrary failures of some processes). They require *termination* and *agreement* among the correct voters (that is, every non-byzantine voter has to output some winner, a ranking for distributed SWFs and an alternative for distributed SCFs, and any two non-byzantine voters have to have the same output), and they tried different validity requirements. So the problem of a distributed election is very similar to *byzantine agreement* (sometimes the word "consensus" is used instead of "agreement"), except maybe for some different *validity* requirements (for a reference on byzantine agreement and on a standard validity requirement see [5]). For instance, Chauhan and Garg [17] considered the following validity requirement: if an alternative x is the top-ranked alternative for a majority of correct voters, then x must be the outcome of the election. They show that when there are $n \geq 2$ voters, $|X| \geq 2$ alternatives, and $f \geq \frac{n}{4}$, where f is the maximum number of byzantine failures that can occur in any run, implementing a distributed SCF satisfying this validity condition is impossible. See [17] for additional impossibility and possibility by these authors.

Melnyk et al. [36] focus on distributed SWFs. In particular, they assume that the inputs of each process are preference rankings and the output is also a ranking. These authors look for protocols that solve agreement and termination, plus a validity requirement tailored to the context of voting. For a synchronous system, they propose two distributed algorithms to solve this problem satisfying the following validity requirement called *Pareto-validity*: for any pair of alternatives x and y, if all correct voters rank x over y, then the consensus ranking has to rank x over y. Furthermore, one of these algorithms is also optimal with respect to approximating some fairness condition.

In [36], it is argued that distributed preferential voting has applications in distributed machine learning. Moreover, Tseng [44] says that distributed voting is also relevant for masking faults in safety-critical systems. It would be interesting to see how different domain restrictions (for the correct processes) provide impossibility and possibility results in different distributed voting environments.

6.2 A Note on the Nature of our Proofs

Notice that, in contrast to [10] (which uses algebraic topology) and the generalized index lemma proof by [38] for Arrow's theorem, we obtained the proof of our main theorem, i.e. Theorem 21, without a topology result or topological reasoning beyond the use of the simplicial complexes and simplicial maps. So a question is, to what extent is algebraic or combinatorial topology playing a role in Arrow's theorem or its generalizations? Related to this question is the fact that Rajsbaum and Raventós-Pujol [38] showed that contractability of N_D does not characterize Arrow-consitency as it was suggested in the topological social choice literature [see 34]. But independently of the answer to this question, the combinatorial topology objects we used were useful in at least two ways.

Firstly, our results contribute to the idea that one can successfully learn about domains through their 2-skeleton. This was already illustrated by Baryshnikov [10] for the case of the unrestricted domain and domains that satisfy the free triples property. The 2-skeleton corresponds to subprofiles over triples, and studying domains or profiles from the subprofiles is standard in the social choice literature. However, the combinatorial topology objects give us a geometric tool to do that, i.e. the 2-skeleton. We believe that this geometric view can be useful in designing algorithms like the one in [33].

Secondly, when working with the combinatorial topology objects we do not have to invoke IIA, since it is already somehow embedded in the construction of the chromatic simplicial maps. This leads to very simple proofs by contradiction by invoking the simplicial nature of the map.

6.3 Additional Future Work

Now we mention some additional questions for future research. By Remark 5 in Chapter 5, $D_{B,4}$ does not belong to the domain restriction assumed in Theorem 1. Actually, in contrast to the chain property, domains in $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ not necessarily have a free triple. It would also be interesting to compare all the domains that appear in [18, 20] with domains in $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$. More broadly, a future line of research might be to study the relation of domains in $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ with other domains in the social choice literature.

A major part of the social choice literature deals with SCFs (instead of SWFs). As we said before, the Gibbard-Satterthwaite theorem is a very important impossibility theorem for these functions. It could be interesting to mention that Gibbard proved his theorem as a corollary to Arrow's theorem. An algebraic topology proof of the Gibbard-Satterthwaite theorem by Baryshnikov and Root is already available in [9]. An open problem is to prove this theorem with combinatorial topology. Another question is how does the Gibbard-Satterthwaite impossibility behave over $\mathcal{D}^{\mathrm{PT}} \cap \mathcal{D}^{\mathrm{DT}}$ or over other domain restrictions but through a combinatorial topology framework.

Finally, as we said in the Introduction, Lara et al. [33] present an algorithm to compute all the SWF that escape Arrow for any given domain for the case of 2 voters and 3 alternatives. This algorithm was also designed under the combinatorial topology approach, so can we combine it with the techniques exposed in this thesis to get an algorithm that help us obtain unanimous and non-dictatorial SWFs satisfying IIA for higher dimensions through the use of the 2-skeletons of the input domains? Can this get us closer to a complete characterization of Arrow's theorem in the context of strict total orders?

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Appendix A

Equivalence between the Classical and the Combinatorial Topology Approaches

In Section A.1, given a domain D, we formally construct the chromatic simplicial complex N_D. In Section A.2, we show that for all $Y \subseteq X$, such that $|Y| \geq 2$, there exist a bijection between the set of all subpreferences of D on Y, denoted $D|_Y$, and the set of all $(\binom{|Y|}{2} - 1)$-simplices that only involve alternatives in Y, denotes $S(Y)$. These bijections allow us to talk about subprofiles and their corresponding simplices interchangeably. In Section A.3, we prove that there is a bijection between $W(X)$ to the set of facets of $N_{W(X)}$. This, together with any other result in Section A.3, were already proven by Baryshnikov [10], but we present them in this appendix for it to be a self-contained reference for the equivalence between the classical and combinatorial topology approaches. In Section A.4, we prove that the bijection $\mathcal{B}\colon \mathcal{F}_D \to \mathcal{M}_D$ introduced in Chapter 3 is in fact a bijection. Finally, in Section A.5 we provide the missing proofs of Theorem 7 and Corollary 8.

A.1 Constructing N_D from Scratch

Let $Y \subseteq X$ such that $|Y| \geq 2$. Let L' be the following set:

$$\bigcup_{\substack{\vec{\sigma} \in \{+,-\}^n \\ \alpha,\beta \in X, \alpha \neq \beta}} \{U^{\vec{\sigma}}_{\alpha\beta}\}.$$

Let $D \subseteq W(X)^n$ such that $D \neq \varnothing$. If $U^{\vec{\sigma}}_{\alpha\beta} \in L'$, we define

$$s_D(U^{\vec{\sigma}}_{\alpha\beta}) = \{\vec{P} \in D\colon \text{for all } \ i \in N,\ \alpha P_i \beta \text{ iff } \sigma_i = +\}.$$

Let $\sim_{s_D}$ be a binary relation on L' defined as follows: $U^{\vec{\sigma}}_{\alpha\beta} \sim_{s_D} U^{\vec{\sigma}'}_{\gamma\delta}$ iff $s_D(U^{\vec{\sigma}}_{\alpha\beta}) = s_D(U^{\vec{\sigma}'}_{\gamma\delta})$. Clearly, $\sim_{s_D}$ is an equivalence relation. It is not hard to see that if $U^{\vec{\sigma}}_{\alpha\beta} \in L'$ the equivalence class $[U^{\vec{\sigma}}_{\alpha\beta}]$ induced by $\sim_{s_D}$ is $\{U^{\vec{\sigma}}_{\alpha\beta}, U^{-\vec{\sigma}}_{\beta\alpha}\}$. Abusing notation, we drop the brackets from $[U^{\vec{\sigma}}_{\alpha\beta}]$

and just write $U^{\vec{\sigma}}_{\alpha\beta}$ or $U^{-\vec{\sigma}}_{\beta\alpha}$ to refer to this equivalence class. Hence, if L is the partition of L' induced by $\sim_{s_D}$, we simply write L as

$$\bigcup_{\substack{\vec{\sigma}\in\{+,-\}^n \\ \alpha,\beta\in X,\alpha\neq\beta}} \{U^{\vec{\sigma}}_{\alpha\beta}\}.$$

Therefore, it makes sense to write $U^{\vec{\sigma}}_{\alpha\beta} = U^{-\vec{\sigma}}_{\beta\alpha}$. We also abuse notation in defining $s_D(U^{\vec{\sigma}}_{\alpha\beta})$ as

$$\{\vec{P} \in D \colon \text{for all } \ i \in N,\ \alpha P_i \beta \text{ iff } \sigma_i = +\}.$$

where $U^{\vec{\sigma}}_{\alpha\beta}$ is interpreted as an equivalence class in L (instead of an element of L').

Let N_D is a simplicial complex defined as follows:

- Its set of vertices, denoted $V(N_D)$ is

$$\{u \in L \colon s_D(u) \neq \varnothing\}.$$

- a non-empty subset $S \subseteq V(N_D)$, where $S = \{v_1, \dots, v_k\}$, is a $(k-1)$-simplex of N_D iff

$$\bigcap_{i=1}^{k} s_D(v_i) \neq \varnothing$$

Proposition 28. If D is a domain, the simplicial complex N_D together with a labeling $\chi \colon V(N_D) \to \{Y \subseteq X \colon |Y| = 2\}$ defined as

$$\chi(U^{\vec{\sigma}}_{\alpha\beta}) = \{\alpha, \beta\}$$

is a chromatic simplicial complex.

Proof. It is easy to show that N_D is a simplicial complex, so we only have to prove that the χ labeling is a coloring, i.e. we have to show that if t is a simplex of N_D, for all $U^{\vec{\sigma}}_{\alpha\beta}, U^{\vec{\sigma}'}_{\gamma\delta} \in t$ such that $U^{\vec{\sigma}}_{\alpha\beta} \neq U^{\vec{\sigma}'}_{\gamma\delta}$, we have that $\chi(U^{\vec{\sigma}}_{\alpha\beta}) \neq \chi(U^{\vec{\sigma}'}_{\gamma\delta})$.

By definition of χ, notice that is suffices to show that ($\alpha \neq \gamma$ and $\alpha \neq \delta$) or ($\beta \neq \gamma$ and $\beta \neq \delta$).

Since t is a simplex of N_D there exists $\vec{P} \in D$ such that $\vec{P} \in s_D(U^{\vec{\sigma}}_{\alpha\beta}) \cap s_D(U^{\vec{\sigma}'}_{\gamma\delta})$.

Suppose $\alpha = \gamma$ or $\alpha = \delta$. To show: $\beta \neq \gamma$ and $\beta \neq \delta$. We proceed by contradiction supposing that $\beta = \gamma$ or $\beta = \delta$. We proceed by cases.

Case 1: $\beta = \gamma$. Then $\alpha = \delta$. This leads to $U^{\vec{\sigma}'}_{\gamma\delta} = U^{\vec{\sigma}'}_{\beta\alpha} = U^{-\vec{\sigma}'}_{\alpha\beta}$. We proceed by subcases.

Subcase 1.1: $\vec{\sigma} = -\vec{\sigma}'$. Then $U^{\vec{\sigma}}_{\alpha\beta} = U^{\vec{\sigma}'}_{\gamma\delta}$, a contradiction.

Subcase 1.2: $\vec{\sigma} \neq -\vec{\sigma}'$. Then since $\vec{P} \in s_D(U^{\vec{\sigma}}_{\alpha\beta}) \cap s_D(U^{\vec{\sigma}'}_{\gamma\delta}) = s_D(U^{\vec{\sigma}}_{\alpha\beta}) \cap s_D(U^{-\vec{\sigma}'}_{\alpha\beta})$, there is a voter $i \in N$ such that $\alpha P_i \beta$ and $\beta P_i \alpha$, a contradiction to the asymmetry of P_i.

Case 2: $\beta = \delta$. Then $\alpha = \gamma$. This leads to $U^{\vec{\sigma}'}_{\gamma\delta} = U^{\vec{\sigma}'}_{\alpha\beta}$. We proceed by subcases.

Subcase 2.1: $\vec{\sigma} = \vec{\sigma}'$. Then $U^{\vec{\sigma}}_{\alpha\beta} = U^{\vec{\sigma}'}_{\gamma\delta}$, a contradiction.

Subcase 2.2: $\vec{\sigma} \neq \vec{\sigma}'$. Then since $\vec{P} \in s_D(U^{\vec{\sigma}}_{\alpha\beta}) \cap s_D(U^{\vec{\sigma}'}_{\gamma\delta}) = s_D(U^{\vec{\sigma}}_{\alpha\beta}) \cap s_D(U^{\vec{\sigma}'}_{\alpha\beta})$, there is a voter $i \in N$ such that $\alpha P_i \beta$ and $\beta P_i \alpha$, a contradiction to the asymmetry of P_i.

$\square$

A.2 A Bijection from $D|_Y$ to $S(Y)$

Let $S(Y)$ be the set of all $(\binom{|Y|}{2}-1)$-simplex of N_D that only involve alternatives of Y, i.e. if $U^{\vec{\sigma}}_{\alpha\beta} \in S(Y)$, then $\alpha, \beta \in Y$.

Our objective is to define a bijection between $D|_Y$ and $S(Y)$ to talk about subprofiles of D and simplices in an interchangeably way. Consider then a function $g_Y \colon D|_Y \to S(Y)$ defined as follows:

$$\text{for all } \vec{P} \in D|_Y, g_Y(\vec{P}) = \{U^{\vec{\sigma}}_{xy} \in L \colon x, y \in Y; x \neq y; \text{for all } i,\ \vec{\sigma}_i = + \text{ iff } xy \in P_i\}.$$

Proposition 29. For all $\vec{P} \in D|_Y$, we have that $g_Y(\vec{P})$ is indeed in $S(Y)$.

Proof. Let $\vec{P} \in D|_Y$. By construction,

$$\vec{P} \in \bigcap_{g_Y(\vec{P})} s_D(v).$$

Therefore, $g_Y(\vec{P})$ is a simplex of N_D. Also by construction, it only involves alternatives in Y, i.e. if $U^{\vec{\sigma}}_{\alpha\beta} \in g_Y(\vec{P})$, then $\alpha, \beta \in Y$. We still have to prove that $g_Y(\vec{P})$ is a $(\binom{|Y|}{2}-1)$-simplex, i.e. that it has cardinality $\binom{|Y|}{2}$.

Observe that $\vec{P}$ is an n-tuple of strict total orders on Y, which in particular are asymmetric and total. Then for all $x, y \in Y$, $x \neq y$ there exists a unique $\vec{\sigma} \in \{+, -\}^n$ such that for all $i \in N$, $\vec{\sigma}_i = +$ iff $xy \in P_i$. Since there are $\binom{|Y|}{2}$ different pairs of alternatives in Y, we have that $g_Y(\vec{P})$ is of cardinality $\binom{|Y|}{2}$. Therefore, $g_Y(\vec{P}) \in S(Y)$. $\square$

We will show that the function that we define next is the inverse function of g_Y. Let $h_Y \colon S(Y) \to D|_Y$ defined as follows:

$$\text{for all } \{v_1, v_2, \ldots, v_{\binom{|Y|}{2}}\} \in S(Y),\ h_Y(\{v_1, v_2, \ldots, v_{\binom{|Y|}{2}}\}) = \vec{P}|_Y \text{ for all } \vec{P} \in \bigcap_{i=1}^{\binom{|Y|}{2}} s_D(v_i).$$

Proposition 30. h is well-defined.

Proof. Firstly, notice that $\bigcap_{i=1}^{\binom{|Y|}{2}} s_D(v_i)$ is non-empty since $\{v_1, v_2, \ldots, v_{\binom{|Y|}{2}}\} \in S(Y)$.

Finally, notice that if $\vec{P}, \vec{P}' \in \bigcap_{i=1}^{\binom{|Y|}{2}} s_D(v_i)$, then $\vec{P}|_Y = \vec{P}'|_Y$. $\square$

Proposition 31. g_Y is a bijection with inverse function h_Y.

Proof. To show: $h_Y(g_Y(x)) = x$ for all $x \in D|_Y$ and $g_Y(h_Y(x)) = x$ for all $x \in S(Y)$.

Let $\vec{P} \in D|_Y$. We have that

$$h_Y(g_Y(\vec{P})) = \vec{P}'|_Y \text{ for all } \vec{P}' \in \bigcap_{u \in g_Y(\vec{P})} s_D(u).$$

To show: $\vec{P} = \vec{P}'|_Y$. Let $i \in N$. Since P_i and $P'_i|_Y$ are strict total orders on Y, it suffices to show that for all $x, y \in Y$, if $xy \in P_i$, then $xy \in P'_i|_Y$. Let $xy \in P_i$. Since $g_Y(\vec{P}) \in S(Y)$

and N_D is chromatic, there exists $U^{\vec{\sigma}}_{xy}$ such that for all $j \in N$, $\vec{\sigma}_j = +$ iff $xy \in P_j$. Hence, the fact that $xy \in P_i$ implies $\vec{\sigma}_i = +$. On the other hand, notice that $\vec{P'} \in s_D(U^{\vec{\sigma}}_{xy})$. Hence, $xy \in \vec{P}'_i$. Finally we get $xy \in P'_i|_Y$ due to the fact that $x, y \in Y$.

Let $\{v_1, v_2, \ldots, v_{\binom{|Y|}{2}}\} \in S(Y)$. We have that

$$g_Y(h_Y(\{v_1, v_2, \ldots, v_{\binom{|Y|}{2}}\})) = \{U^{\vec{\sigma}}_{xy} \in L\colon x, y \in Y; x \neq y; \text{for all } i,\ \sigma_i = + \text{ iff } xy \in P_i|_Y\}$$

for all $\vec{P} \in \bigcap_{i=1}^{\binom{|Y|}{2}} s_D(v_i)$.

To show: $\{v_1, v_2, \ldots, v_{\binom{|Y|}{2}}\} = g(h(\{v_1, v_2, \ldots, v_{\binom{|Y|}{2}}\}))$.

Notice that both $\{v_1, v_2, \ldots, v_{\binom{|Y|}{2}}\}$ and $g_Y(h_Y(\{v_1, v_2, \ldots, v_{\binom{|Y|}{2}}\}))$ belong to $S(Y)$ and remember that N_D is chromatic. Therefore, for all distinct x and y in Y, there exists a unique $\vec{\sigma} \in \{+, -\}^n$ and a unique $\vec{\sigma}' \in \{+, -\}^n$ such that $U^{\vec{\sigma}}_{xy} \in \{v_1, v_2, \ldots, v_{\binom{|Y|}{2}}\}$ and $U^{\vec{\sigma}'}_{xy} \in g_Y(h_Y(\{v_1, v_2, \ldots, v_{\binom{|Y|}{2}}\}))$. Clearly, if we show that $\vec{\sigma} = \vec{\sigma}'$ we are done. Fix $\vec{P} \in \bigcap_{i=1}^{\binom{|Y|}{2}} s_D(v_i)$ and observe the following three things:

1. Since $U^{\vec{\sigma}'}_{xy} \in g_Y(h_Y(\{v_1, v_2, \ldots, v_{\binom{|Y|}{2}}\}))$, we have that, for all $i \in N$, $\vec{\sigma}'_i = +$ iff $xy \in P_i|Y$.

2. Since $\vec{P} \in \bigcap_{i=1}^{\binom{Y}{2}} s_D(v_i)$ and $U^{\vec{\sigma}}_{xy} \in \{v_1, v_2, \ldots, v_{\binom{|Y|}{2}}\}$, we have $\vec{P} \in s_D(U^{\vec{\sigma}}_{xy})$. Then, for all $i \in N$, $\vec{\sigma}_i = +$ iff $xy \in P_i$.

3. Since $x, y \in Y$, for all $i \in N$, $xy \in P_i$ iff $xy \in P_i|_Y$.

Taking these three facts together, we obtain, for all $i \in N$, $\vec{\sigma}_i = +$ iff $\vec{\sigma}'_i = +$. Therefore, $\vec{\sigma} = \vec{\sigma}'$. □

A.3 A Bijection from $W(X)$ to the facets of $N_{W(X)}$

All the results in this section were already proven (or implicitly implied in an obvious way) in [10], but we will be more explicit about certain details and omit others. We as well use some different terminology than [10].

Proposition 32. The simplicial complex $N_{W(X)}$ together with a labeling $\chi\colon V(N_{W(X)}) \to \{Y \subseteq X\colon |Y| = 2\}$ defined as

$$\chi(U^{\sigma}_{\alpha\beta}) = \{\alpha, \beta\}$$

is a chromatic simplicial complex.

The proof of this proposition can be easily adapted from the proof of Proposition 28, hence it is omitted.

Corollary 33. If t is a simplex of $N_{W(X)}$, then $\dim(t) \leq \binom{|X|}{2} - 1$.

Let $A(N_W(X))$ be the set of simplices of maximum dimension among the simplices of $N_{W(X)}$ (we will later show that $N_{W(X)}$ is pure, so we will see that $A(N_W(X))$ coincide with the set of facets of $N_{W(X)}$). We introduce a function between $W(X)$ and the set of all facets of $A(N_{W(X)})$. Let $\bar{g} \colon W(X) \to A(N_{W(X)})$ defined as follows:

$$\bar{g}(P) = \{U^+_{xy} \colon xy \in P\}.$$

Proposition 34. $\bar{g}$ is well-defined and $\bar{g}(P)$ is a $(\binom{|X|}{2} - 1)$-simplex for all $\vec{P} \in W(X)$

Proof. Let $P \in W(X)$. By totality and asymmetry of P, we have $|P| = \binom{|X|}{2} - 1$. Then $|g(P)| = \binom{|X|}{2}$. Then by construction, P belongs to the intersection of the elements of $g(P)$. So $g(P)$ is a $(\binom{|X|}{2} - 1)$-simplex. Finally, by Corollary 33, $g(P)$ is of maximum dimension among the simplices of $N_{W(X)}$. $\square$

Consider a function $\bar{h} \colon A(N_{W(X)}) \to W(X)$ defined as:

$$\bar{h}(\{v_1, \ldots, v_{\binom{|X|}{2}}\}) = P \text{ such that } P \in \bigcap_{i=1}^{\binom{|X|}{2}} v_i.$$

It is not hard to see that $\bar{h}$ is well-defined (observe that $\bigcap_{i=1}^{\binom{|X|}{2}} v_i$ has to have a unique element).

We want to show the following:

Proposition 35. $\bar{g}$ is a bijection with inverse function $\bar{h}$.

Proof. Let $P \in W(X)$. We have that $\bar{h}(\bar{g}(P))) = \bar{h}(\{U^+_{xy} \colon xy \in P\}) = P'$ such that $P' \in \bigcap_{u \in \bar{g}(P)} u$, then, it is easy to see that that $P = P'$.

Let $\{v_1, \ldots, v_{\binom{|X|}{2}}\} \in A(N_{W(X)})$. We have that

$$\bar{g}(\bar{h}(\{v_1, \ldots, v_{\binom{|X|}{2}}\})) = \{U^+_{xy} \colon xy \in P\}$$

such that $P \in \bigcap_{i=1}^{\binom{|X|}{2}} v_i$. Invoking chromaticity of $N_{W(X)}$, it is not hard to see that the desired result holds. $\square$

So we have shown that there is a bijection between the strict total orders on X and the set of simplices of maximum dimension of $N_{W(X)}$. Finally, if we show that $N_{W(X)}$ is pure, $A(N_{W(X)})$ is also the set of facets of $N_{W(X)}$, and so g could then be thought as a bijection between the strict total orders on X and the set of facets of $N_{W(X)}$. So let us prove it.

Proposition 36. The simplicial complex $N_{W(X)}$ is pure.

Proof. Let t be a facet of $N_{W(X)}$. Since this facet was chosen in an arbitrary way, to prove that the simplicial complex $N_{W(X)}$ is pure, it suffices to show that $\dim(t) = \binom{|X|}{2} - 1$.

Observe that we can write t as $\{u_1, \dots, u_{\dim(t)+1}\}$. Also, since t is a simplex of $N_{W(X)}$, there exists a strict total order P on X in $\cap_{i=1}^{\dim(t)+1} u_i$. Clearly, P is of cardinality $\binom{|X|}{2}$.

Since we have seen that simplices of maximum dimension of $N_{W(X)}$ are of dimension $\binom{|X|}{2} - 1$, we have that $\dim(t) \leq \binom{|X|}{2} - 1$. Hence, if we show that $\dim(t) < \binom{|X|}{2} - 1$ leads to contradiction we are done. Let us do it. Suppose that $\dim(t) < \binom{|X|}{2} - 1$. Since $|P| = \binom{|X|}{2}$, the hypothesis $\dim(t) < \binom{|X|}{2} - 1$ implies the existence of a pair of alternatives $x, y \in X$ such that $xy \in P$ and $U^+_{xy} \notin t$. Observe that $P \in (\cap_{i=1}^{\dim(t)+1} u_i) \cap U^+_{xy}$. But then $t' = t \cup \{U^+_{xy}\}$ is a simplex of $N_{W(X)}$ with t as a face. But then t is not a facet of $N_{W(X)}$, a contradiction. $\square$

A.4 A Bijection from $\mathcal{F}_D$ to $\mathcal{M}_D$

We already introduced $\mathcal{B}\colon \mathcal{F}_D \to \mathcal{M}_D$ in Chapter 3. Here we prove that it is in fact a bijection, whose inverse function,

$$\mathcal{B}^{-1}\colon M_D \to F_D,$$

is defined as follows: for every $f \in \mathcal{M}_D$, we have that $\mathcal{B}^{-1}(f)\colon D \to W(X)$ is a function defined as $(\mathcal{B}^{-1}(f))(\vec{P}) = \bar{h}(f(g_X(\vec{P})))$, where the functions $g_X\colon D \to S(X)$ and $\bar{h}\colon A(N_{W(X)}) \to W(X)$ are defined in Sections A.2 and A.3, respectively. Notice that $g_X(p)$ lives in $S(X)$, and since f is rigid and N_D is of dimension $\binom{|X|}{2} - 1$ (these things are easy to check), $f(g_X(p))$ is a facet of $N_{W(X)}$, i.e. it belongs to $A(N_{W(X)})$.

Before showing that $\mathcal{B}$ is a bijection, we first show the following:

Proposition 37. Let $f \in \mathcal{M}_D$. The SWF $\mathcal{B}^{-1}(f)$ satisfies IIA.

Proof. Let $\alpha, \beta \in X$, $\alpha \neq \beta$, and $\vec{P}', \vec{P}'' \in D$ such that for all $i \in N$, $\alpha\beta \in P'_i$ iff $\alpha\beta \in P''_i$. To show: $\alpha\beta \in (\mathcal{B}^{-1}(f))(\vec{P}')$ iff $\alpha\beta \in (\mathcal{B}^{-1}(f))(\vec{P}'')$.

By definition,

$$(\mathcal{B}^{-1}(f))(\vec{P}') = \bar{h}(f(g_X(\vec{P}'))) = P' \text{ such that } P' \in \bigcap_{v \in f(g_X(\vec{P}'))} v,$$

where $f(g_X(\vec{P}')) = \{f(U^{\vec{\sigma}}_{xy})\colon x, y \in X; x \neq y; \text{for all } i, \vec{\sigma}_i = + \text{ iff } xy \in P'_i\}$. Also,

$$(\mathcal{B}^{-1}(f))(\vec{P}'') = \bar{h}(f(g_X(\vec{P}''))) = P'' \text{ such that } P'' \in \bigcap_{v \in f(g_X(\vec{P}''))} v,$$

where $f(g_X(\vec{P}'')) = \{f(U^{\vec{\sigma}}_{xy})\colon x, y \in X; x \neq y; \text{for all } i, \vec{\sigma}_i = + \text{ iff } xy \in P''_i\}$.

By chromaticity of $N_{W(X)}$ and f, there exist $U^{\vec{\sigma}'}_{\alpha\beta} \in g_X(\vec{P}')$ and $U^{\vec{\sigma}''}_{\alpha\beta} \in g_X(\vec{P}'')$ such that (for all $i \in N$, $\vec{\sigma}'_i = +$ iff $\alpha\beta \in P'_i$) and (for all $i \in N$, $\vec{\sigma}''_i = +$ iff $\alpha\beta \in P''_i$). But by hypothesis, for all $i \in N$, $\alpha\beta \in P'_i$ iff $\alpha\beta \in P''_i$. But then, for all $i \in N$, $\vec{\sigma}'_i = +$ iff $\vec{\sigma}''_i = +$. Therefore, $U^{\vec{\sigma}'}_{\alpha\beta} = U^{\vec{\sigma}''}_{\alpha\beta}$. Hence, $f(U^{\vec{\sigma}'}_{\alpha\beta}) = f(U^{\vec{\sigma}''}_{\alpha\beta})$. Therefore, $\alpha\beta \in (\mathcal{B}^{-1}(f))(\vec{P}')$ iff $\alpha\beta \in (\mathcal{B}^{-1}(f))(\vec{P}'')$. $\square$

Proposition 38. $\mathcal{B}$ is a bijection with inverse function $\mathcal{B}^{-1}$.

Proof. First, we have to show that for all $f \in M_D$, it holds that $\mathcal{B}(\mathcal{B}^{-1}(f))\colon N_D \to N_{W(X)}$ is such that $\mathcal{B}(\mathcal{B}^{-1}(f)) = f$.

Let $U^{\vec{\sigma}}_{\alpha\beta} \in V(N_D)$. To show: $\mathcal{B}(\mathcal{B}^{-1}(f))(U^{\vec{\sigma}}_{\alpha\beta}) = f(U^{\vec{\sigma}}_{\alpha\beta})$.

By definition of $\mathcal{B}$, we have that $\mathcal{B}(\mathcal{B}^{-1}(f))(U^{\vec{\sigma}}_{\alpha\beta}) = U^{\sigma}_{\alpha\beta}$ such that $\sigma = +$ iff $\alpha\beta \in (\mathcal{B}^{-1}(f))(\vec{P})$ for all $\vec{P} \in s_D(U^{\vec{\sigma}}_{\alpha\beta})$.

This is well-defined since $\mathcal{B}^{-1}(f)$ satisfies IIA by Proposition 37. So we can fix $\vec{P} \in s_D(U^{\vec{\sigma}}_{\alpha\beta})$ and we have that

$$(\mathcal{B}^{-1}(f))(U^{\vec{\sigma}}_{\alpha\beta}) = U^{\sigma}_{\alpha\beta} \text{ such that } (\sigma = + \text{ iff } \alpha\beta \in (\mathcal{B}^{-1}(f))(\vec{P})).$$

Then,

$$(\mathcal{B}^{-1}(f))(U^{\vec{\sigma}}_{\alpha\beta}) = U^{\sigma}_{\alpha\beta} \text{ such that } (\sigma = + \text{ iff } \alpha\beta \in P' \text{ such that } P' \in \bigcap_{v \in f(g_X(\vec{P}))} v), \tag{A.1}$$

where $f(g_X(\vec{P})) = \{f(U^{\vec{\sigma}}_{xy})\colon x, y \in X; X \neq y; \text{for all } i, \vec{\sigma}_i = + \text{ iff } xy \in P_i\}$ On the other hand, if $f(U^{\vec{\sigma}}_{\alpha\beta}) = U^{\sigma'}_{\alpha\beta}$, since $\vec{P} \in s_D(U^{\vec{\sigma}}_{\alpha\beta})$, we have the following:

$$f(U^{\vec{\sigma}}_{\alpha\beta}) = U^{\sigma'}_{\alpha\beta} \in f(g_X(\vec{P})). \tag{A.2}$$

We proceed by contradiction supposing $\sigma \neq \sigma'$. We now proceed by cases.

Case 1: $\sigma = +$ and $\sigma' = -$. Since $\sigma = +$, by A.1, $\alpha\beta \in P'$. Also notice that by A.2, $U^{-}_{\alpha\beta} \in f(g_X(\vec{P}))$, then, by definition of $(\mathcal{B}^{-1}(f))(\vec{P})$, we have $\beta\alpha \in P'$, a contradiction to the asymmetry of P'.

Case 2: $\sigma = -$ and $\sigma' = +$. This case leads to a contradiction in an analogous way.

The second part of this proof consists on proving that

$$\mathcal{B}^{-1}(\mathcal{B}(F))\colon D \to W(X)$$

is such that $\mathcal{B}^{-1}(\mathcal{B}(F)) = F$, for all $F \in \mathcal{F}_D$.

Let $\vec{P} \in D$. To show: $(\mathcal{B}^{-1}(\mathcal{B}(F)))(\vec{P}) = F(\vec{P})$.

By definition of our functions,

$$(\mathcal{B}^{-1}(\mathcal{B}(F)))(\vec{P}) = \bar{h}((\mathcal{B}(F))(g_X(\vec{P}))) = P \text{ such that } P \in \bigcap_{v \in (\mathcal{B}(\mathcal{F}))(g_X(\vec{P}))} v,$$

where $(\mathcal{B}(\mathcal{F}))(g_X(\vec{P})) = \{(\mathcal{B}(F))(U^{\vec{\sigma}}_{xy}\colon \text{for all } i, \vec{\sigma}_i = + \text{ iff } xy \in P_i\}$.

So, we want to prove that $P = F(\vec{P})$. Let $\alpha\beta \in X$, $\alpha \neq \beta$. To show: $\alpha\beta \in P$ iff $\alpha\beta \in F(\vec{P})$.

Since $(\mathcal{B}(F))(g_Y(\vec{P}))$ is a facet of the chromatic simplicial complex $N_{W(X)}$ and since $\mathcal{B}(F)$ is chromatic, there exists a unique $\vec{\sigma}' \in \{+, -\}^n$ such that $U^{\vec{\sigma}}_{\alpha\beta} \in g_Y(\vec{P})$. Then,

$$\vec{P} \in s_D(U^{\vec{\sigma}}_{\alpha\beta}). \tag{A.3}$$

On the other hand, by definition of $\mathcal{B}$,

$$(\mathcal{B}(F))(U_{\alpha\beta}^{\vec{\sigma}'}) = U_{\alpha\beta}^{\sigma} \text{ such that } (\sigma = + \text{ iff } \alpha\beta \in F(\vec{P'}) \text{ for all } \vec{P'} \in s_D(U_{\alpha\beta}^{\vec{\sigma}'})). \tag{A.4}$$

Combining A.3 and A.4, we get that

$$(\mathcal{B}(F))(U_{\alpha\beta}^{\vec{\sigma}'}) = U_{\alpha\beta}^{\sigma} \text{ such that } (\sigma = + \text{ iff } \alpha\beta \in F(\vec{P})). \tag{A.5}$$

First we show that $P \subseteq F(\vec{P})$. Suppose $\alpha\beta \in P$. Then since $P \in U_{\alpha\beta}^{\sigma}$, we have that $\sigma = +$ (otherwise, we would contradict the asymmetry of P). But then, by A.5, $\alpha\beta \in F(\vec{P})$.

Finally, we show that $F(\vec{P}) \subseteq P$. Suppose $\alpha\beta \in F(\vec{P})$. Then by A.5, we have $\sigma = +$, so $(\mathcal{B}(F))(U_{\alpha\beta}^{\vec{\sigma}'}) = U_{\alpha\beta}^{+}$. Then $P \in U_{\alpha\beta}^{+}$. But then $\alpha\beta \in P$. □

A.5 Missing Proofs

A.5.1 Proof of Theorem 7

Proof. Part (1) was already proven as Proposition 37.

Let us prove (2). By definition of $\mathcal{B}$, for every vertex of N_D of the form $U_{\alpha\beta}^{\vec{\sigma}^N}$, we have that $(\mathcal{B}(F))(U_{\alpha\beta}^{\vec{\sigma}^N}) = U_{\alpha\beta}^{+}$ iff $\alpha F(\vec{P})\beta$ for all $\vec{P} \in s_D(U_{\alpha\beta}^{\vec{\sigma}^N})$.

We start with the $\Rightarrow$ direction. Suppose $\mathcal{B}(F)$ is unanimous. Then $(\mathcal{B}(F))(U_{\alpha\beta}^{\vec{\sigma}^N}) = U_{\alpha\beta}^{+}$. Hence, $\alpha F(\vec{P})\beta$ for all $\vec{P} \in s_D(U_{\alpha\beta}^{\vec{\sigma}^N})$. Hence, F is unanimous.

Now we prove the $\Leftarrow$ direction. Suppose F is unanimous. Then for all $\vec{P} \in s_D(U_{\alpha\beta}^{\vec{\sigma}^N})$, we have that $\alpha F(\vec{P})\beta$, then $(\mathcal{B}(F))(U_{\alpha\beta}^{\vec{\sigma}^N}) = U_{\alpha\beta}^{+}$. Therefore, $\mathcal{B}(F)$ is unanimous.

Finally, we want to prove (3). Remember that for every vertex $U_{\alpha\beta}^{\vec{\sigma}}$ of N_D, we have that $(\mathcal{B}(F))(U_{\alpha\beta}^{\vec{\sigma}}) = U_{\alpha\beta}^{+}$ iff $\alpha F(\vec{P})\beta$ for all $\vec{P} \in s_D(U_{\alpha\beta}^{\vec{\sigma}})$.

We begin with the $\Rightarrow$ direction. Suppose $\mathcal{B}(F)$ is dictatorial. Let $i \in N$ be a dictator. If $\vec{P} \in D$, we want to show that $\alpha P_i\beta$ implies $\alpha F(\vec{P})\beta$. Suppose $\vec{P} \in D$. Let $\vec{\sigma}_{\vec{P}} \in \{+,-\}^n$ be such that for all $j \in N$, $(\vec{\sigma}_{\vec{P}})_j = +$ iff $\alpha P_j\beta$. Then $\vec{P} \in s_D(U_{\alpha\beta}^{\vec{\sigma}_{\vec{P}}})$. On the other hand, since $\mathcal{B}(F)$ is dictatorial, $(\mathcal{B}(F))(U_{\alpha\beta}^{\vec{\sigma}_{\vec{P}}}) = U_{\alpha\beta}^{+}$. Hence, since $\vec{P} \in s_D(U_{\alpha\beta}^{\vec{\sigma}_{\vec{P}}})$, it holds that $\alpha F(\vec{P})\beta$.

We now show that $\Leftarrow$ direction. Suppose F is dictatorial. Let $i \in N$ be a dictator. Let $U_{\alpha\beta}^{\vec{\sigma}}$ be a vertex of N_D. To show: $f(U_{\alpha\beta}^{\vec{\sigma}}) = U_{\alpha\beta}^{\vec{\sigma}_i}$. For every $\vec{P} \in s_D(U_{\alpha\beta}^{\vec{\sigma}})$, we have that F being dictatorial implies that $\alpha F(\vec{P})\beta$ iff $\vec{\sigma}_i = +$. But then $f(U_{\alpha\beta}^{\vec{\sigma}}) = U_{\alpha\beta}^{\vec{\sigma}_i}$.

□

A.5.2 Proof of Corollary 8

Proof. We start with the $\Rightarrow$ direction. Suppose D is Arrow-inconsistent. Let $f\colon N_D \to N_W(X)$ be a chromatic simplicial map, i.e. $f \in \mathcal{M}_D$, satisfying unanimity. To show: f is dictatorial.

Notice that by part 1 of Theorem 7, we have that $\mathcal{B}^{-1}(f) \in \mathcal{F}_D$. Also, since $\mathcal{B}^{-1}$ is the inverse function of the bijection $\mathcal{B}$, we have $\mathcal{B}(\mathcal{B}^{-1}(f)) = f$. Therefore, by part 2 of Theorem

7, f being unanimous implies that $\mathcal{B}^{-1}(f)$ is unanimous. Then, since D is Arrow-inconsistent, $\mathcal{B}^{-1}(f)$ is dictatorial. Then, by part 3 of Theorem 7, f is dictatorial.

Let us prove the $\Leftarrow$ direction. Suppose that for all $f \in \mathcal{M}_D$, f unanimous implies that f is dictatorial. We want to show that D is Arrow-inconsistent. Let $F \in \mathcal{F}_D$ satisfying unanimity. To show: F is dictatorial.

Observe that $\mathcal{B}(F) \in \mathcal{M}_D$. Since F is unanimous, by part 2 of Theorem 7 we have that $\mathcal{B}(F)$ is unanimous. Then applying our hypothesis, $\mathcal{B}(F)$ is dictatorial. Then by part 3 of Theorem 7, F is dictatorial. $\square$