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Contents

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Introducción

La mecánica cuántica es una teoría de la física que describe la dinámica de partículas microscópicas. Esta teoría ha tenido un gran impacto en el desarrollo de la física moderna y su tratamiento matemático riguroso ha sido una fuente de gran inspiración en el campo de la física matemática. La teoría difiere substancialmente de la mecánica clásica al no ser determinista. Esto significa que en general no se puede predecir el estado exacto de un sistema físico incluso si se conoce el estado completo del sistema en algún instante de tiempo. Solo se pueden asignar probabilidades a conjuntos de posibles estados o medidas. A cada sistema cuántico se le asigna un espacio de Hilbert H, donde los elementos con norma uno representan posibles estados físicos. La evolución temporal está dictada por la ecuación de Schrödinger:

$$i\hbar\frac{\partial\psi(t)}{\partial t} = H\psi(t),$$

Donde $\psi : [0, T] \to \mathsf{H}$ es una aplicación fuertemente diferenciable (con respecto a la topología del espacio de Hilbert) y H es un operador autoadjunto no acotado. Utilizando el cálculo funcional proporcionado por el teorema espectral aplicado a H, se puede resolver esta ecuación de la siguiente manera:

$$\psi(t) = \exp\left(-i\hbar^{-1}tH\right)\psi(0).$$

La familia de operadores $\exp(-i\hbar^{-1}tH)$ se conocen como el operador de evolución temporal en textos de física y forman una familia de operadores unitarios uniparamétricos, fuertemente continuos, que definen completamente la evolución temporal del sistema [35, 8, 36]. Se ha realizado mucho trabajo para estudiar esta formulación en un contexto matemático riguroso, y en esta línea, muchos resultados de la mecánica cuántica han sido rigurosamente justificados, y hay una gran cantidad de trabajo en estos temas [4, 32, 33, 30, 19].

En los textos de física, suele afirmarse que existe una formulación alternativa a la mecánica cuántica, conocida como la formulación de la integral de trayectoria de la mecánica cuántica, y está en pleno acuerdo con la formulación ortodoxa formulada a través de teoría de operadores. Esta formulación, desarrollada principalmente por Richard Feynman, ha sido crucial en muchos desarrollos teóricos de la teoría cuántica. En el contexto de la Electrodinámica Cuántica (QED), fue una herramienta fundamental para el análisis perturbativo de las amplitudes de dispersión de QED, lo que finalmente fue el trabajo por el cual R. Feynman recibió el Premio Nobel de Física. Esta formulación se basa en la construcción de una integral funcional sobre un espacio de trayectorias de dimensión infinita. La prescripción general se puede resumir con la siguiente relación formal utilizando la notación de braket:

$$\langle x_1, t_1 | x_0, t_0\rangle = \int_{C_{x_0,x_1}[t_0,t_1]} \exp(i\hbar^{-1}S(\gamma))D\gamma(t)$$

Aquí, en el lado izquierdo, tenemos la amplitud compleja de transición de una partícula que se encuentra en x_0 en el tiempo t_0 y viaja a x_1 en el tiempo t_1, y en el lado derecho tenemos la integral de trayectoria como una integral de la exponencial imaginaria de la acción clásica, sobre el espacio $C_{x_0,x_1}[t_0,t_1]$ de todas las trayectorias clásicas con condiciones de frontera $\gamma(t_i) = x_i$ para $i = 1, 2$. La acción clásica se define como

$$S(\gamma) = \int_{t_0}^{t_1} \mathcal{L}\left(\gamma(t), \dot{\gamma}(t), t\right) dt,$$

siendo $\mathcal{L}$ el Lagrangiano del sistema. En otras palabras, la prescripción nos dice que la función de Green para la ecuación de Schrödinger se puede calcular como una integral de trayectorias de la exponencial compleja de una acción clásica correspondiente [10, 11, 13]. Cuando estaba desarrollando esta herramienta, Feynman dio una definición provisional de esta integral como un límite de integrales sobre subespacios de dimensión finita de trayectorias poligonales, pero consideró que podría ser posible definir la integral utilizando las técnicas estándar de teoría de la medida. Esta afirmación no se desarrolló más y hasta el día de hoy todavía no hay una definición generalmente aceptada para la integral de trayectorias (ver [11] p. 34f).

Un primer acercamiento al problema de dar una definición matemáticamente rigurosa de esta integral, planteado por Gel'fand y Yaglom, fue utilizar la medida de Wiener de la teoría de procesos estocásticos, donde ya se había realizado una construcción análoga para la ecuación del calor. Gel'fand y Yaglom propusieron en [16] definir la integral como un límite de integrales de Wiener con varianza compleja. Se creía que este procedimiento podría utilizarse para definir adecuadamente una medida en un espacio de trayectorias, sin embargo, en [6], Richard Cameron demostró que esta técnica tenía deficiencias fundamentales y no podía utilizarse para definir una medida. Este resultado negativo nos indica de cierta manera que se necesita una nueva definición matemática, y desde entonces han aparecido muchas propuestas. En [29], Nelson presentó una definición que utiliza una generalización de la fórmula del producto de Lie aplicada a operadores autoadjuntos de Schrödinger, dando un acercamiento de "abajo hacia arriba", pero también dio una definición que utiliza integrales de Wiener estudiando la convergencia puntual sobre un parámetro complejo. En [26], Ito propuso definir la integral de trayectorias utilizando medidas complejas generalizadas sobre espacios de Hilbert. Utilizando este enfoque, fue posible considerar todos los potenciales que son la transformada de Fourier de una medida compleja acotada y también polinomios de segundo orden. S. Albeverio, Krohn y Mazzuchi extendieron esta definición al caso del oscilador anarmónico y el campo escalar bosónico libre; este enfoque tiene la ventaja de poder considerar la expansión semi-clásica utilizando el método de fase estacionaria [2]. En [20], Hida definió la integral de Feynman utilizando cálculo de ruido blanco, que luego fue utilizado en [1] por Albeverio, Hahn y Sengupta para definir la integral de Feynman para teorías topológicas de Chern-Simons en tres dimensiones. Fujiwara Kumano-go, Tsuchida e Ichinose desarrollaron el enfoque de cortes temporales, el cual se utilizará en este texto. El método de cortes temporales está inspirado en la prescripción de Feynman, en la que la integral se considera como el límite de integrales sobre espacios de curvas poligonales. En esta dirección, existen resultados para la ecuación de Schrödinger magnética dependiente del tiempo con límites sobre crecimiento de los potenciales, para la ecuación de Dirac e incluso en el contexto de la electrodinámica cuántica no relativista [13, 14, 15, 22, 24].

El enfoque de cortes temporales contiene dos formulaciones diferentes. Una de ellas fue desarrollada principalmente por Fujiwara, Kumano-go, entre otros, en la cual el propagador se construye como una sucesión de integrales oscilatorias sobre espacios de trayectorias localmente clásicas [13, 14]. La otra formulación fue desarrollada por Ichinose, en la cual el operador de evolución temporal se construye como el límite fuerte de una sucesión de operadores integrales oscilatorios definidos en espacios de trayectorias poligonales [22, 24]. A lo largo de este texto, nos centraremos en el método de Ichinose. En este método, para una partición fija de un intervalo temporal $\Delta = \{t_0, t_1, \ldots, t_\nu\}$ y puntos arbitrarios $x_0, x_1 \ldots, x_\nu = x \in \mathbb{R}^n$, si consideramos γ_{Δ,x_i} como la trayectoria poligonal que se define de tal manera que en el intervalo $[t_i, t_{i+1}]$ sea el segmento de recta que une a x_i y x_{i+1}, entonces definimos el operador integral oscilatorio (ver [27, 31]) de la siguiente manera:

$$C_\Delta f(x_\nu) = K_\Delta \mathrm{Os} \int \exp(i\hbar^{-1} S(\gamma_{\Delta,x_i})) f(x_0) dx_0 \ldots dx_{\nu-1},$$

donde f es una función de Schwartz, y K_Δ es una constante de normalización. El método se ocupa en establecer varias propiedades de esta familia de operadores para luego dar con una prueba de que converge fuertemente al operador de evolución temporal de un Hamiltoniano cuántico correspondiente. Esto se hace generalmente en cuatro pasos: En primer lugar, se comienza estudiando la regularidad y suavidad del operador fundamental, que es la integral de camino sobre segmentos de línea. En segundo lugar, se estudian las extensiones enL^2 y en ciertos espacios de Sobolev del operador fundamental. Luego, se obtienen resultados de estabilidad aplicando la ecuación de Schrödinger al operador fundamental. Finalmente, la sucesión de operadores en trayectorias poligonales se construye como una composición de operadores fundamentales y se demuestra la convergencia.

El enfoque de cortes temporales también ha tenido éxito en la construcción de la versión en espacio de fases de la integral de trayectorias utilizando ambos métodos [23, 28]. En particular, Ichinose ha tenido éxito al aplicarlo a la ecuación de Dirac, incluso al considerar trayectorias poligonales no causales, lo que es esencialmente un nuevo acercamiento para la construcción de la integral de trayectoria para esta ecuación [25]. En este enfoque, se comienza con la función Hamiltoniana clásica $H(x, p, t)$ y para trayectorias generales $(x(t), p(t))$ en el espacio de fases, se comienza con el funcional de acción en el espacio fase

$$S(x(t), p(t)) = \int_{t_0}^{t_1} p(t) dx(t) - \int_{t_0}^{t_1} H(x(t), p(t), t) dt,$$

y para puntos arbitrarios $p_0, \ldots, p_{\nu-1} \in \mathbb{R}^n$, ahora consideramos las trayectorias en el espacio de fases $(\gamma_{\Delta,x_i}(t), \gamma_{\Delta,p_i}(t))$, donde γ_{Δ,p_i} es una trayectoria constante por tramos tal que $\gamma_{\Delta,p_i}(t) = p_i$ para $t \in [t_i, t_{i+1}]$, y tomamos el operador integral oscilatorio:

$$C_\Delta f(x_\nu) = \frac{1}{(2\pi\hbar)^{n\nu}} \mathrm{Os} \int \exp\left(i\hbar^{-1} S(\gamma_{\Delta,x_i}, \gamma_{\Delta,p_i})\right) f(x_0) dx_0 dp_0 \ldots dx_{\nu-1} dp_{\nu-1},$$

y aplicamos los mismos pasos. En este trabajo investigaremos la construcción de la integral de camino en el espacio de fases utilizando el método de W. Ichinose para la ecuación pseudo-relativista de Schrödinger. Para ello, comenzaremos con el Hamiltoniano estándar para una partícula en un potencial dado por:

$$H(x,p) = \sqrt{m^2 + |p|^2} + V(x),$$

y definimos para este Hamiltoniano la versión en espacio de fases del operador fundamental, donde para trayectorias de la forma

$$\gamma(t_0, t_1, x_0, x_1, p)(\theta) = \left(x_1 + \frac{\theta - t_0}{t_1 - t_0}(x_0 - x_1), p \right),$$

tomamos

$$C(t_0, t_1) f(x_1) = \frac{1}{(2\pi\hbar)^n} \mathrm{Os} \int \exp\left(i\hbar^{-1} S(\gamma(t_0, t_1, x_0, x_1, p)) \right) f(x_0) dx_0 dp.$$

Nuestro resultado principal es demostrar que este operador está bien definido como una integral oscilatoria y si f es una función de Schwartz, entonces $C(t_0, t_1) f(x_1)$ es infinitamente diferenciable con respecto a x_1 y las derivadas son continuas con respecto a t_0, t_1 (ver 4.6). Luego discutimos las diferencias y similitudes con respecto a otros modelos y qué trabajo adicional se necesita. Sorprendentemente, para este modelo en particular, si se utiliza la integral de trayectoria en espacio fase, algunas de las demostraciones son más simples en comparación con sus contraparte no relativista. Esto demuestra que la formulación en espacio de fases tiene algunas ventajas y abre la posibilidad de completar la construcción, que sería, según nuestro conocimiento, la primera aplicación del método para un operador no local.

Introduction

Quantum mechanics is a theory from physics that describes the dynamics of microscopic particles. This theory has been highly influential in the development of modern physics and its rigorous mathematical treatment has been a source of great inspiration in the field of mathematical physics. The theory substantially differs from classical mechanics because the theory is non-deterministic. This means that one cannot in general predict the exact state of a physical system even if one knows the complete state of the system at some instant in time. One can only assign probabilities to set of possible states or measurements. To every quantum system, a Hilbert space H is assigned to it, where elements with norm one represent possible physical states. Temporal evolution is dictated by the Schrödinger equation:

$$i\hbar\frac{\partial\psi(t)}{\partial t} = H\psi(t),$$

where $\psi : [0,T] \to \mathsf{H}$ is a strongly differentiable map (with respect to the Hilbert space topology) and H is an unbounded self-adjoint operator. Using the functional calculus given by the spectral theorem applied to H one can solve this equation as

$$\psi(t) = \exp\left(-i\hbar^{-1}tH\right)\psi(0).$$

The family of operators $\exp\left(-i\hbar^{-1}tH\right)$ are known as the time evolution operator in physics texts and it forms a family of one parameter, strongly continuous, unitary operators that fully define the time evolution of the system [35, 8, 36]. Plenty of work has been done to study this formulation in a rigorous mathematical context and in this vein, many results from quantum mechanics have been rigorously justified and there is a big body of work on these topics [4, 32, 33, 30, 19].

In physics texts it is typically stated that an alternative formulation to quantum mechanics, namely the path integral formulation of quantum mechanics, exists and is in full agreement with the orthodox, operator-theoretic formulation. This formulation, chiefly developed by Richard Feynman, has been crucial in many theoretical developments of quantum theory. In the context of Quantum Electrodynamics (QED), it was a fundamental tool for the perturbative analysis of QED scattering amplitudes which ended up being the work for which R. Feynman received his Nobel prize in physics. This formulation is based on the construction of a functional integral over an infinite dimensional space of paths. The general prescription can be summarized with the following formal relation using bra-ket notation

$$\langle x_1, t_1 | x_0, t_0\rangle = \int_{C_{x_0,x_1}[t_0,t_1]} \exp(i\hbar^{-1}S(\gamma))D\gamma(t),$$

here, in the left hand side we have the complex transition of amplitude of a particle who is at x_0 at time t_0 and travels to x_1 at time t_1 and on the right hand side we have the path integral as an integral of the imaginary exponential of the classical action over the space $C_{x_0,x_1}[t_0,t_1]$ of all classical paths with fixed endpoints $\gamma(t_i) = x_i$ for $i = 1, 2$. The classical action is defined as:

$$S(\gamma) = \int_{t_0}^{t_1} \mathcal{L}\left(\gamma(t), \dot{\gamma}(t), t\right) dt,$$

with $\mathcal{L}$ the Lagrangian of the system. In other words, the prescription tells us that the Green's function for the Schrödinger equation can be computed as a path integral of the complex exponential of a corresponding classical action [10, 11, 13]. When he was developing this tool, Feynman gave an operational definition of this integral as a limit of integrals over finite dimensional subspaces of polygonal paths, but he considered that it might be possible to define the integral using the standard techniques of measure theory but this assertion was not further developed and till this day there is still no general agreed upon definition for the path integral (See [11] p. 34).

One approach to the problem By Gel'fand and Yaglom of giving a mathematically rigorous definition of this integral was to use the Wiener measure from the theory of stochastic processes, where an analogous construction had already been done for the heat equation. Gel'fand and Yaglom in [16] proposed defining the integral as a limit of Wiener integrals with complex variance. It was believed that this procedure could be used to define a proper measure on a space of paths, however in [6] Richard Cameron proved that this technique was fundamentally flawed and could not be used to define a measure. This negative result can be understood as an indication that a new mathematical definition is needed, and since, many proposed definitions have appeared. In [29], Nelson presented a definition that uses a generalization of the Lie product formula applied to self adjoint Schrödinger operators using a bottom up approach, but also gave a definition using Weiner integrals considering pointwise convergence over a complex parameter. In [26], Ito proposed defining the path integral using generalized complex measures over Hilbert spaces. Using this approach it was possible to consider all potentials that are the Fourier transform of a bounded complex measure and also second order polynomials. S. Albeverio, Krohn, and Mazzuchi extended this definition to the case of the anharmonic oscillator and the free bosonic scalar field; this approach has the advantage of being able to consider semiclassical expansions using the method of stationary phase [2]. In [20], Hida defined the Feynman integral using white noise calculus, which was later used in [1] by Albeverio, Hahn, and Sengupta to define the Feynman integral for topological Chern-Simons theories in three dimensions. Fujiwara Kumano-go, Tsuchida, and Ichinose developed the time-slicing approach, which will be used in this text. The time-slicing method is inspired by Feynman's prescription, in which the integral is considered as the limit of integrals over spaces of polygonal curves. In this direction, there are results for the time-dependent magnetic Schrödinger equation with growth bounds on the potentials, for the Dirac equation and even in the context of non-relativistic quantum electrodynamics [13, 14, 15, 22, 24].

The time-slicing approach contains two different formulation. One chiefly developed by Fujiwara, Kumano-go et. al. in which the propagator is constructed as a sequence of oscillatory integrals over spaces of picewise-classical paths [13, 14]. And another developed by Ichinose in which the time evolution operator is constructed as the strong limit of a sequence of of oscilatory integral operators defined over spaces of polygonal paths [22, 24]. Throughout this text we will focus on Ichinose's method in this text. For this method, for a fixed partition of a temporal interval $\Delta = \{t_0, t_1, \ldots, t_\nu\}$ and arbitrary points $x_0, x_1 \ldots, x_\nu = x \in \mathbb{R}^n$, if we consider γ_{Δ,x_i} the polygonal path that is defined such that in the interval $[t_i, t_{i+1}]$ it is a line segment joining x_i and x_{i+1}, then we define the oscillatory integral (see [27, 31]) operator

$$C_\Delta f(x_\nu) = K_\Delta \mathrm{Os}\int \exp(i\hbar^{-1}S(\gamma_{\Delta,x_i}))f(x_0)dx_0\dots dx_{\nu-1},$$

where f is a Schwartz function, and K_Δ is some normalization constant. The method concerns itself with establishing various properties of this family of operators in order to give a proof that it converges strongly to the time evolution operator of a corresponding quantum Hamiltonian. This is done generally in four steps: One starts by studying the regularity and smoothness of the fundamental operator, which is the path integral over lines segments, secondly one studies the L^2 and Sobolev extensions of the fundamental operator, then one gets stability results by applying the Schrödinger's equation to the fundamental operator, and finally the sequence of operators on polygonal paths is constructed as a composition of fundamental operators and convergence is proved.

The time slicing approach has also found success in constructing the phase-space version of the path integral using both methods [23, 28]. In particular, Ichinose has been successful in applying it for the Dirac Equation, even while considering non causal polygonal paths giving, essentially, a new approach to its construction [25]. In this approach, one starts with the classical Hamiltonian function $H(x,p,t)$ and for general paths $(x(t),p(t))$ in phase space, one starts with the phase space action functional

$$S(x(t),p(t)) = \int_{t_0}^{t_1} p(t)dx(t) - \int_{t_0}^{t_1} H(x(t),p(t),t)dt,$$

and for arbitrary points $p_0,\dots p_{\nu-1}\in\mathbb{R}^n$ we now consider the phase-space paths $(\gamma_{\Delta,x_i}(t),\gamma_{\Delta,p_i}(t))$ where γ_{Δ,p_i} is a piece-wise constant path such that $\gamma_{\Delta,p_i}(t)=p_i$ for $t\in[t_i,t_{i+1}]$ and we take the oscillatory integral operator

$$C_\Delta f(x_\nu) = \frac{1}{(2\pi\hbar)^{n\nu}}\mathrm{Os}\int \exp\left(i\hbar^{-1}S(\gamma_{\Delta,x_i},\gamma_{\Delta,p_i})\right)f(x_0)dx_0dp_0\dots dx_{\nu-1}dp_{\nu-1},$$

and apply the same steps. In this work we will investigate the construction of the phase-space path integral using W. Ichinose's method for the pseudo-relativistic Schrödinger equation. For this we will start with the standard Hamiltonian for a single particle in a potential given by

$$H(x,p) = \sqrt{m^2+|p|^2} + V(x),$$

and define for this Hamiltonian the phase-space version of the fundamental operator where for paths of the form

$$\gamma(t_0,t_1,x_0,x_1,p)(\theta) = \left(x_1 + \frac{\theta-t_0}{t_1-t_0}(x_0-x_1),p\right),$$

we take

$$C(t_0,t_1)f(x_1) = \frac{1}{(2\pi\hbar)^n}\mathrm{Os}\int \exp\left(i\hbar^{-1}S(\gamma(t_0,t_1,x_0,x_1,p))\right)f(x_0)dx_0dp.$$

Our main result is proving that this operator is well defined as an oscillatory integral and if f is a Schwartz function, $C(t_0, t_1)f(x_1)$ is infinitely differentiable with respect to x_1 and the derivatives are continuous with respect to t_0, t_1 (see 4.6). We discuss then the differences and similarities with respect other models and what further work is needed. Surprisingly, for this particular model, using the phase space integral, some of the proofs simplify compared with their non-relativistic counterpart. This shows that the phase-space formulation has some advantages and opens up the possibility for completing the construction which would be, to our knowledge, the first application of the method for a non-local operator.

Organization of the Text

To understand the main theorem of this text, logically one only needs to follow chapters 1,2, and 4 in that order. The general organization of the text is as follows:

On chapter 1 we introduce the preliminary tools needed to prove the main theorem. These include fixing the notation that will be used throughout the text and stating some basic theorems. Oscillatory integrals and the space of symbols is defined and expanded upon.

On chapter 2 we define the classical model. That is, we derive the classical Hamiltonian function defined on phase space $\mathbb{R}^{2n}$ and define the phase space action functional.

On chapter 3 we quantize this classical Hamiltonian and discuss self adjointness. This chapter is not needed for the proof of our main theorem but it brings into context our proposed model and will be necessary for further work.

On chapter 4 we construct the fundamental operator, that is, the path integral over line segments using W. Ichinose's method. Then we prove the main theorem of this text; the smoothness and continuity of the fundamental operator.

Finally on chapter 5 we discuss the results and consider possible future work.

1 Preliminary Tools

In this section we will introduce all the basic tools that will be used throughout this text. We will define the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ as the set of natural, integer, rational, real and complex numbers respectively and for simplicity we will consider that $\mathbb{N}$ contains 0 throughout this text. For $n \in \mathbb{N}$ we consider $\mathbb{R}^n$ and $\mathbb{C}^n$ the vector spaces with the usual scalar and inner products $x \cdot y = \sum_{i=1}^{n} x_i y_i$ for $x, y \in \mathbb{R}^n$, $z \cdot w = \sum_{i=1}^{n} z_i^* \cdot w_i$. For $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$ we will denote the usual induced euclidean norm as

$$|x| = \sqrt{\sum_{i=1}^{n} x_i^2}, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

$$|z| = \sqrt{\sum_{i=1}^{n} |z_i|^2}, \qquad z \in \mathbb{C}^n, \tag{1.2}$$

if no confusion arises. Throughout this text multi-index notation will be liberally used (cf. [31] Ch. 1 [9] Ap. A.3)

Definition 1.1. *A multi-index of dimension n is any element of $\alpha \in \mathbb{N}^n$, that is, an n-dimensional vector with non-negative integer entries. We fix the following notation for $\alpha, \beta \in \mathbb{N}^n$*

1. *Addition*

$$\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$$

2. *Partial Order*

$$\alpha \geq \beta \;\leftrightarrow\; \alpha_i \geq \beta_i \;\forall i$$

3. *Absolute Value*

$$|\alpha| = \sum_{i=1}^{n} |\alpha_i| = \sum_{i=1}^{n} \alpha_i.$$

4. *Factorial*

$$\alpha! = \prod_{i=1}^{n} \alpha_i!$$

5. *Binomial Coefficient, if $\alpha \geq \beta$*

$$\binom{\alpha}{\beta} = \prod_{i=1}^{n} \binom{\alpha_i}{\beta_i} = \frac{\alpha!}{\beta!(\alpha - \beta)!}$$

6. *Multinomial Coefficient, for* $k \in \mathbb{N}$

$$\binom{k}{\alpha} = \frac{k!}{\alpha!}$$

7. *For* $z \in \mathbb{C}^n$*, Powers*

$$z^\alpha = \prod_{i=1}^{n} z_i^{\alpha_i}$$

8. *Partial Derivatives*

$$D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}} \qquad \alpha \neq 0$$
$$D_x^\alpha = 1 \qquad \alpha = 0$$

Remark. *Since we are considering in this text that* $0 \in \mathbb{N}$ *we remark that the multi-indices can have some or all entries equal to 0 which should be considering when reading formulas and proofs that use this notation.*

As a very useful application of multi-index notation, we obtain the following result (See [34] p. 159 and [31] Thm. 1.2).

Theorem 1.2. *[General Leibniz Rule] Let* $U \subset \mathbb{R}^n$ *be open and* $f, g \in C^\infty(U)$ *then, for all* $\alpha \in \mathbb{N}^n$ *multi-index, we have that*

$$D_x^\alpha(fg)(x) = \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} (D_x^{\alpha'} f(x))(D_x^{\alpha-\alpha'} g(x)) \qquad \forall x \in U. \tag{1.3}$$

Throughout this text we will use the following notation, for all $x \in \mathbb{R}^n$ we define

$$\langle x \rangle = \sqrt{1 + |x|^2}. \tag{1.4}$$

The following properties of $\langle x \rangle$ are basic but will be important in what follows

Proposition 1.3. *Let* $x, y \in \mathbb{R}^n$*, then*

$$\langle x + y \rangle \leq \sqrt{2} \langle x \rangle \langle y \rangle \tag{1.5}$$

$$\langle x + y \rangle^{-1} \leq \sqrt{2} \frac{\langle x \rangle}{\langle y \rangle}. \tag{1.6}$$

And for $s \in \mathbb{R}$ *we have that* $\langle x \rangle^s$ *as a function on* $\mathbb{R}^n$ *is integrable if and only if* $s < -n$.

Proof. First, we note that for all $a \in \mathbb{R}$ we have

$$-\frac{1}{2} \leq a(a-1), \tag{1.7}$$

by simply finding the vertex of the corresponding parabola. Now, let $x, y \in \mathbb{R}^n$ then it follows from the previous inequality that

$$\begin{aligned} &-\frac{1}{2} \leq |x||y|(|x||y|-1) \\ &\leftrightarrow 0 \leq 1+2|x||y|(|x||y|-1) = 1+2(|x|^2|y|^2-|x||y|), \end{aligned} \tag{1.8}$$

and by the Cauchy-Schwartz inequality we have that

$$\begin{aligned} 0 &\leq 1+2(|x|^2|y|^2-|x||y|) \\ &\leq 1+2|y|^2|x|^2-2x\cdot y \\ &\leq 1+|x|^2+|y|^2+2|y|^2|x|^2-2x\cdot y \\ &= 2+2(|x|^2+|y|^2+|y|^2|x|^2)-1-|x|^2-|y|^2-2x\cdot y, \end{aligned} \tag{1.9}$$

therefore we obtain

$$\begin{aligned} 1+|x+y|^2 = 1+|x|^2+|y|^2+2x\cdot y &\leq 2(1+|x|^2+|y|^2+|y|^2|x|^2) \\ &= 2(1+|x|^2)(1+|y|^2), \end{aligned} \tag{1.10}$$

and by taking square roots the first inequality follows. For the second inequality, we first note that since

$$\begin{aligned} 0 \leq (2|x|-|y|)^2 &= 4|x|^2-4|x||y|+|y|^2 \\ &\leq 4|x|^2+4x\cdot y+|y|^2 \\ &= 2(|x|^2+|x+y|^2)-|y|^2, \end{aligned} \tag{1.11}$$

again using Cauchy-Schwartz in the second inequality, it then follows that

$$\begin{aligned} 1+|y|^2 &\leq 1+2(|x|^2+|x+y|^2) \\ \leftrightarrow 1 &\leq \frac{1+2(|x|^2+|x+y|^2)}{1+|y|^2} \\ &\leq \frac{1+2(|x|^2+|x+y|^2)+1+2|x+y|^2|x|^2}{1+|y|^2} \\ &= \frac{2(1+|x+y|^2)(1+|x|^2)}{1+|y|^2} \\ \leftrightarrow \frac{1}{1+|x+y|^2} &\leq \frac{2(1+|x|^2)}{1+|y|^2}, \end{aligned} \tag{1.12}$$

so by taking square roots we obtain the second inequality. Now, we just need to see that $\langle x\rangle^s$ is integrable if and only if $s < -n$. If $s \geq 0$ it follows that it is not integrable from the fact that $\langle x\rangle \geq 1$, and so the integral is unbounded. Noting that

$$\prod_{j=1}^{n}\left(1+|x_i|^2\right) \leq \left(1+|x|^2\right)^n \tag{1.13}$$

which follows easily from the multinomial theorem, if we then take $-s < -n$ we note that

$$\langle x\rangle^{-s} = \frac{1}{\sqrt{1+|x|^2}^{s}} \leq \prod_{j=1}^{n} \frac{1}{\sqrt{1+|x_i|^2}^{s/n}}, \tag{1.14}$$

therefore, since

$$\frac{1}{\sqrt{1+|x_i|^2}^{s/n}} \leq \frac{1}{|x_i|^{s/n}} \tag{1.15}$$

for all $i = 1, \ldots, n$ and $x_i \in \mathbb{R} \setminus \{0\}$, we find that

$$\begin{aligned}\int_{|x_i|\geq 1} \frac{1}{\sqrt{1+|x_i|^2}^{s/n}} dx_i &\leq \int_{|x_i|\geq 1} \frac{1}{|x_i|^{s/n}} dx_i \\ &= 2\int_1^\infty x_i^{-s/n} dx_i \\ &= \frac{2x_i^{1-s/n}}{1-s/n}\Big|_1^\infty \\ &= \frac{2}{s/n-1}, \end{aligned} \tag{1.16}$$

where the last equality follows since $1 - s/n < 0$ so it goes to 0 at infinity. We then obtain

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{\sqrt{1+|x_i|^2}^{s/n}} dx_i &= \int_{|x_i|<1} \frac{1}{\sqrt{1+|x_i|^2}^{s/n}} dx_i + \int_{|x_i|\geq 1} \frac{1}{\sqrt{1+|x_i|^2}^{s/n}} dx_i \\ &\leq \int_{|x_i|<1} \frac{1}{\sqrt{1+|x_i|^2}^{s/n}} dx_i + \frac{2}{s/n-1} \\ &< \infty \end{aligned} \tag{1.17}$$

so using this, equation (1.13) and Tonelli's theorem we have that

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{1}{\sqrt{1+|x|^2}^{s}} dx &\leq \int_{\mathbb{R}^n} \prod_{i=1}^{n} \frac{1}{\sqrt{1+|x_i|^2}^{s/n}} dx \\ &= \prod_{i=1}^{n} \int_{\mathbb{R}} \frac{1}{\sqrt{1+|x_i|^2}^{s/n}} dx_i \\ &< \infty. \end{aligned} \tag{1.18}$$

Lastly, if $-n \leq -s < 0$ since we can verify that if $|x_i| \geq 1$ for all i then

$$1 + |x|^2 \leq (n+1) \prod_{j=1}^{n} |x_i|^2, \tag{1.19}$$

since each summand on the left is bounded by one product on the right, it follows that

$$\frac{1}{n+1}\prod_{i=1}^{n}\frac{1}{|x_i|^s} \le \frac{1}{\langle x\rangle^s} \qquad \text{if } |x_i| \ge 1. \tag{1.20}$$

But since $0 < s \le n$ we have that

$$\int_1^\infty \frac{1}{|x_i|^s} dx_i = \infty, \tag{1.21}$$

and so, again by Tonelli's theorem we obtain that in this case $\langle x\rangle^{-s}$ is not integrable. □

Remark. *We can generalize and combine inequalities (1.5),(1.6) into one by writing for* $s \in \mathbb{R}$

$$\langle x+y\rangle^s \le \sqrt{2}\,\langle x\rangle^{|s|}\langle y\rangle^s, \tag{1.22}$$

note that this inequality is symmetric for x and y.

Proposition 1.4. *For $s \in \mathbb{R}$ and $\alpha \in \mathbb{N}^n$ there exists $C_{s,\alpha} > 0$ such that for all $x \in \mathbb{R}^n$*

$$|D_x^\alpha \langle x\rangle^s| \le C_{s,\alpha}\langle x\rangle^{s-|\alpha|}. \tag{1.23}$$

Proof. The proof will be done by induction over the order of the multi-index $|\alpha|$. If $|\alpha| = 0$ the inequality follows trivially for $C_{s,0} = 1$. Now, let us assume that the inequality holds for all $s \in \mathbb{R}$ and for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| = K$ we will prove it holds for α with $|\alpha| = K+1$. For such α we can represent it as $\alpha = \hat{e}_i + \alpha'$ for some $i = 1,\dots,n$ and $|\alpha'| = K$ and so we have that

$$\begin{aligned} D_x^\alpha \langle x\rangle^s &= D_x^{\alpha'} D_x^{\hat{e}_i}\langle x\rangle^s \\ &= D_x^{\alpha'}\partial_{x_i}\langle x\rangle^s \\ &= D_x^{\alpha'} s x_i \langle x\rangle^{s-2} \\ &= s\sum_{\alpha''\le\alpha'}\binom{\alpha'}{\alpha''} D_x^{\alpha''} x_i D_x^{\alpha'-\alpha''}\langle x\rangle^{s-2}, \end{aligned} \tag{1.24}$$

where in the last line we used the general Leibniz rule 1.2. Now since for all multi-index $\beta \in \mathbb{N}^n$ we have that $\left|D^\beta x_i\right| \le 1+|x_i|^2 \le 1+|x|^2$, applying this and the induction hypothesis to the above identity we obtain

$$\begin{aligned} |D^\alpha \langle x\rangle^s| &\le s\sum_{\alpha''\le\alpha'}\binom{\alpha'}{\alpha''}\left|D_x^{\alpha''}x_i\right|\left|D_x^{\alpha'-\alpha''}\langle x\rangle^{s-2}\right| \\ &\le s\sum_{\alpha''\le\alpha'}\binom{\alpha'}{\alpha''}(1+|x|^2)C_{\alpha'-\alpha'',s-2}\langle x\rangle^{s-2-|\alpha'-\alpha''|} \\ &\le s\langle x\rangle^2\langle x\rangle^{s-2-|\alpha'|}\sum_{\alpha''\le\alpha'}\binom{\alpha'}{\alpha''}C_{\alpha'-\alpha'',s-2} \\ &= s\langle x\rangle^{s-|\alpha'|}\sum_{\alpha''\le\alpha'}\binom{\alpha'}{\alpha''}C_{\alpha'-\alpha'',s-2} \end{aligned} \tag{1.25}$$

so if we define

$$C_{\alpha,s} = s \sum_{\alpha'' \leq \alpha'} \binom{\alpha'}{\alpha''} C_{\alpha'-\alpha'',s-2}, \tag{1.26}$$

then we obtain that

$$|D_x^\alpha \langle x \rangle^s| \leq C_{\alpha,s} \langle x \rangle^{s-|\alpha'|}, \tag{1.27}$$

so the inequality is true also for $|\alpha| = K+1$ and the inductive step is complete. □

The next theorem obtained from [12] is a generalization of Leibniz integral rule from calculus using the language of measure theory.

Theorem 1.5. *[Leibniz Integral Rule (Measure Theory Version)] Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f : X \times [a,b] \to \mathbb{C}$ be such that $f(.,t) : X \to \mathbb{C}$ is integrable for each $t \in [a,b]$. Let $F(t) = \int_X f(x,t)d\mu(x)$.*

- *(i) Suppose that there exists $g \in L^1(\mu)$ such that $|f(x,t)| \leq g(x)$ for all x, t. If $\lim_{t\to t_0} f(x,t) = f(x,t_0)$ for every x, then $\lim_{t\to t_0} F(t) = F(t_0)$; in particular, if $f(x,\cdot)$ is continuous for each x, then F is continuous.*
- *(ii) Suppose that $\partial_t f$ exists and there is a $g \in L^1(\mu)$, such that $|\partial_t f(x,t)| \leq g(x)$ for all x, t. Then F is differentiable and $F'(t) = \int \partial_t f(x,t)d\mu(x)$.*

Next, we define a space of function typically called the Schwartz space which is important for the definition of oscillatory integrals and many operators that will be used.

Definition 1.6 (Schwartz Space). *We say a function belongs to the Schwartz space $f \in \mathcal{S}(\mathbb{R}^n)$ if $f \in C^\infty(\mathbb{R}^n)$ and for all $\alpha \in \mathbb{N}^n$ multi-index and $k \in \mathbb{N}$ non-negative integer, there is some $C_{\alpha,k,f} > 0$ such that, for all $x \in \mathbb{R}^n$*

$$|D_x^\alpha f(x)| \leq \frac{C_{\alpha,k,f}}{\sqrt{1+|x|^2}^k} = C_{\alpha,k} \langle x \rangle^{-k} \tag{1.28}$$

An important fact about Schwartz function is that for a large class of functions, one can "integrate by parts" as illustrated by the next theorem.

Theorem 1.7 (Integration by Parts Formula for Schwartz Functions). *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $g \in C^\infty(\mathbb{R}^n)$ be such that for all $\alpha \in \mathbb{N}^n$ multi-indices there is an $s \in \mathbb{R}$ and $C_{\alpha,g} > 0$ such that*

$$|D_x^\alpha g(x)| \leq C_{\alpha,g} \langle x \rangle^s, \tag{1.29}$$

then the following identity holds

$$\int_{\mathbb{R}^n} f(x) D_x^\alpha g(x) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} (D_x^\alpha f(x))\, g(x) dx. \tag{1.30}$$

Proof. To prove this we will employ induction over the order $|\alpha|$ of the multi-index. For $\alpha = 0$ the identity holds trivially. Now let us suppose that the identity holds for all for all multi-indices such that $|\alpha| = K$, we need to prove that it holds for $|\alpha| = K + 1 \leq k$. Let α be a multi-index of order $|\alpha| = K + 1$ then it can be written as $\alpha = \hat{e}_i + \alpha'$ with $|\alpha'| = K$, so by the induction hypothesis we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} f(x) D_x^{\alpha} g(x) dx &= \int_{\mathbb{R}^n} f(x) D_x^{\alpha'} D_x^{\hat{e}_i} g(x) dx \\ &= (-1)^{|\alpha'|} \int_{\mathbb{R}^n} \left(D_x^{\alpha'} f(x) \right) \partial_{x_i} g(x) dx. \qquad (1.31)\end{aligned}$$

Now if $\mathbb{1}_A$ is the characteristic function for given set A, and we take $A_k = B(0,k) = \{x \in \mathbb{R}^n \ : \ |x| < k\}$ for $k \in \mathbb{N}$. We note that $\lim_{k\to\infty} \mathbb{1}_{A_k}(x) = 1$ for all $x \in \mathbb{R}^n$. Now, since f is Schwartz and by the hypotheses on g, for $N \in \mathbb{N}$ such that $N - s > n$ there exists $C_{\alpha,N,f}, C_{\alpha,g} > 0$ such that equations (1.28,1.29) hold, and so

$$\begin{aligned}\left|\mathbb{1}_{A_k} \left(D_x^{\alpha} f(x)\right) \partial_{x_i} g(x)\right| &\leq \left|\left(D_x^{\alpha} f(x)\right) \partial_{x_i} g(x)\right| \\ &\leq C_{\alpha,N,f} C_{\alpha,g} \langle x \rangle^{s-N}. \qquad (1.32)\end{aligned}$$

And since $C_{\alpha,N,f} C_{\alpha,g} \langle x \rangle^{s-N}$ is Lebesgue integrable (since $s - N < -n$) by proposition 1.3, and this bound is for all $k \in \mathbb{N}$, by Lebesgue dominated convergence theorem we have that

$$\begin{aligned}\int_{\mathbb{R}^n} \left(D_x^{\alpha'} f(x) \right) \partial_{x_i} g(x) dx &= \int_{\mathbb{R}^n} \lim_{k\to\infty} \mathbb{1}_{A_k}(x) \left(D_x^{\alpha'} f(x) \right) \partial_{x_i} g(x) dx \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^n} \mathbb{1}_{A_k}(x) \left(D_x^{\alpha'} f(x) \right) \partial_{x_i} g(x) dx \\ &= \lim_{k\to\infty} \int_{B(0,k)} \left(D_x^{\alpha'} f(x) \right) \partial_{x_i} g(x) dx \qquad (1.33)\end{aligned}$$

Now, let $G : \mathbb{R}^n \to \mathbb{R}^n$ be given by $G(x) = g(x)\hat{e}_i$, then it follows that

$$\nabla_x \cdot G(x) = \partial_{x_i} g(x), \qquad (1.34)$$

and so by the divergence theorem (see [9] Ap. C.2 Thm 1) we obtain for all $k \in \mathbb{N}$

$$\begin{aligned}\int_{B(0,k)} \left(D_x^{\alpha'} f(x) \right) \partial_{x_i} g(x) dx &= \int_{B(0,k)} \left(D_x^{\alpha'} f(x) \right) \nabla_x \cdot G(x) dx \\ &= -\int_{B(0,k)} \left(\nabla_x D_x^{\alpha'} f(x) \right) \cdot G(x) dx + \int_{\partial B(0,k)} D^{\alpha'} f(x) G(x) \cdot \hat{n} dS_x \\ &= -\int_{B(0,k)} \left(D_x^{\alpha} f(x) \right) g(x) dx + \int_{\partial B(0,k)} D_x^{\alpha'} f(x) G(x) \cdot \hat{n} dS_x. \qquad (1.35)\end{aligned}$$

Now we note that using equation (1.32) we have

$$\begin{aligned}\left|\int_{\partial B(0,k)} D^{\alpha'} f(x)G(x)\cdot\hat{n}dS_x\right| &\leq \int_{\partial B(0,k)}\left|D^{\alpha'} f(x)G(x)\right||\hat{n}|dS_x\\ &\leq \int_{\partial B(0,k)}\left|D^{\alpha'} f(x)\partial_{x_i} g(x)\right|dS_x\\ &\leq \int_{\partial B(0,k)}\frac{C_{\alpha,N,f}C_{\alpha,g}}{\sqrt{1+|x|^2}^n}dS_x\\ &= \frac{2\pi^{n/2}C_{\alpha,N,f}C_{\alpha,g}k^{n-1}}{\Gamma(n/2)\sqrt{1+k^2}^n}. \end{aligned}\tag{1.36}$$

The last terms goes to 0 as $k\to\infty$ so in the limit this term goes to 0. And again using Lebesgue's dominated convergence theorem in the same way we got (1.33) we also have that

$$\lim_{k\to\infty}\int_{B(0,k)}\left(D_x^\alpha f(x)\right)g(x)dx = \int_{\mathbb{R}^n}\left(D_x^\alpha f(x)\right)g(x)dx. \tag{1.37}$$

So by using the above, (1.31), (1.33), (1.35) and (1.36) we get that

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)D_x^\alpha g(x)dx &= (-1)^{|\alpha'|}\int_{\mathbb{R}^n}\left(D_x^{\alpha'} f(x)\right)\partial_{x_i} g(x)dx\\ &= (-1)^{|\alpha'|}\lim_{k\to\infty}\int_{B(0,k)}\left(D_x^{\alpha'} f(x)\right)\partial_{x_i} g(x)dx\\ &= (-1)^{|\alpha'|}\lim_{k\to\infty} -\int_{B(0,k)}\left(D_x^{\alpha} f(x)\right)g(x)dx + \int_{\partial B(0,k)} D_x^{\alpha'} f(x)G(x)\cdot\hat{n}dS_x\\ &= (-1)^{|\alpha'|}(-1)\int_{\mathbb{R}^n}\left(D_x^{\alpha} f(x)\right)g(x)dx + 0\\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^n}\left(D_x^{\alpha} f(x)\right)g(x)dx, \end{aligned}\tag{1.38}$$

which ends the inductive step and the proof. □

In order to get careful growth estimates of certain functions, we will employ a generalization of the famous Faà di Bruno's formula for multi-variable function composition. For the original formulation that will be used throughout this work and a proof see [7]. First some preliminary definitions and notation.

Definition 1.8. *We define a total order $\prec$ in the space of multi-indices $\mathbb{N}^n$ as follows: For $\alpha,\beta\in\mathbb{N}^n$, then $\beta\prec\alpha$ if and only if one of the following holds*

(i) $|\beta|<|\alpha|$,

(ii) $|\beta|=|\alpha|$ *and* $\beta_1<\alpha_1$,

(iii) $|\beta|=|\alpha|$, $\beta_1=\alpha_1,\dots,\beta_k=\alpha_k$ *and* $\beta_{k+1}<\alpha_{k+1}$ *for some* $1\leq k<n$.

Definition 1.9. *For a given $s \in \mathbb{N} \setminus \{0\}$ and $\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m$ multi-indices, we define the set $\mathcal{P}_s(\alpha, \beta)$ as*

$$\begin{aligned}\mathcal{P}_s(\alpha,\beta) = \Bigg\{ &(\beta_1, \ldots, \beta_s, \alpha_1, \ldots, \alpha_s) \ : \ \alpha_i \in \mathbb{N}^n, \beta_i \in \mathbb{N}^m \ |\beta_i| > 0, \\ &0 \prec \alpha_1 \prec \cdots \prec \alpha_s, \ \sum_{i=1}^{s} \beta_i = \beta \text{ and } \sum_{i=1}^{s} |\beta_i| \alpha_i = \alpha \Bigg\} \end{aligned} \tag{1.39}$$

With this notation fixed, we can write the next compact formula

Theorem 1.10. *[General Faà di Bruno's Formula] Let $g : \mathbb{R}^n \to \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}$ be C^k functions then, for all $\alpha \in \mathbb{N}^n$ the next identity for the derivative of composition $h = f \circ g$ holds*

$$D^\alpha h(x) = \sum_{1 \le |\alpha'| \le |\alpha|} \left(D^{\alpha'} \right) (g(x)) \sum_{s=1}^{|\alpha|} \sum_{\mathcal{P}_s(\alpha,\alpha')} \alpha! \prod_{j=1}^{s} \frac{(D^{\alpha_j} g)^{\alpha'_j}(x)}{(\alpha'_j!)(\alpha_j!)^{|\alpha'_j|}}, \tag{1.40}$$

here $\alpha' \in \mathbb{N}^m$ and $D^{\alpha_j} g$ means the derivative acts component-wise on vector g.

Remark. *While this theorem is quite useful to get sharp and explicit estimates for derivatives of the composition of functions, what is essentially needed from this result is that the derivative can be expressed as a sum of products of derivatives of f with a polynomial acting on derivatives of g of a certain order less than $|\alpha|$. It is much easier to prove that it takes this general form by using induction which is an alternative route if one wants to avoid using Faà di Bruno's Formula.*

The term "oscillatory integral" in this context will be used to extend integrals of the form

$$\int_{\mathbb{R}^n} e^{i\phi(x,y)} a(x,y) dy, \tag{1.41}$$

even when a is not Lebesgue integrable. Generally one can interpret this expression as a distribution that acts on a space of test functions as an iterated integral in the following way

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi(x,y)} a(x,y) u(y) dy dx. \tag{1.42}$$

This is well defined for a large class of functions ϕ, a. Nonetheless if a is not integrable, in general it is impossible to understand this integral as an integral in the product space of variables x, y. However, it is possible to extend the definition of oscillatory integral in order to be able to "exchange" the integration order while preserving many of the usual properties of regular integrals. For this we will use the definitions and general development presented in [27, 13], which is standard in the time slicing approach to the Feynman path integral.

Definition 1.11. *[Oscillatory Integral] Let $\phi, a \in C^\infty(\mathbb{R}^n)$ we say the oscillatory integral*

$$Os \int_{\mathbb{R}^n} e^{i\phi(x)} a(x) dx, \tag{1.43}$$

exists if there is some $L \in \mathbb{C}$ *such that for all Schwartz functions* $\chi \in \mathcal{S}(\mathbb{R}^n)$ *with* $\chi(0) = 1$ *we have that for all* $\epsilon > 0$ $e^{i\phi(x)}a(x)\chi(\epsilon x)$ *is Lebesgue integrable and*

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} e^{i\phi(x)}a(x)\chi(\epsilon x)dx = L. \tag{1.44}$$

In other words, the above limit of Lebesgue integrals exists and is independent of the selected χ*.*

In order to work more freely with the above definition, we will give the proof of a useful lemma as in [27] (lemma 6.3 p.47).

Lemma 1.12. *Let* $\chi \in \mathcal{S}(\mathbb{R}^n)$ *be a Schwartz function such that* $\chi(0) = 1$ *and let* $\chi_\epsilon(x) = \chi(\epsilon x)$

(i) $\chi_\epsilon(x) \to 1$ *uniformly on compact subsets of* $\mathbb{R}^n$*.*

(ii) $D^\alpha \chi_\epsilon(x) \to 0$ *uniformly for* $\alpha \in \mathbb{N}^n$ *multi-index such that* $\alpha \neq 0$*.*

(iii) For all multi-indices α *(including* $\alpha = 0$*) there is a constant* $C_{\alpha,\chi} > 0$ *such that for all* $0 < \epsilon < 1$ *and* $0 \leq \sigma \leq |\alpha|$ *we have*

$$|D_x^\alpha \chi_\epsilon(x)| \leq C_{\alpha,\chi}\epsilon^\sigma \langle x \rangle^{-(|\alpha|-\sigma)}. \tag{1.45}$$

Proof. For (i), let $K \subset \mathbb{R}^n$ be a compact set and $\epsilon' > 0$. Now since χ is continuous and $\chi(0) = 1$ there is a $\delta > 0$ such that if $|x| < \delta$ then $|\chi(x) - 1| < \epsilon'$. But since K is compact, it is bounded and there is some $\epsilon^* > 0$ such that $\epsilon^* K \subset B(0, \delta)$. Therefore, for all $\epsilon < \epsilon*$ and $x \in K$, we have that $\|\epsilon x\| < \delta$ so $|\chi_\epsilon(x) - 1| = |\chi(\epsilon x) - 1| < \epsilon'$. That is it satisfies the definition of uniform convergence on K.

For (iii), we first note that for all α multi-index, by the chain rule, we have that $D_x^\alpha \chi_\epsilon(x) = \epsilon^{|\alpha|}(D_x^\alpha \chi)(x\epsilon)$, and since χ is Schwartz, using eq (1.28) we get

$$\begin{aligned} |D^\alpha \chi_\epsilon(x)| \leq \epsilon^{|\alpha|}|D_x^\alpha \chi(x\epsilon)| &\leq \epsilon^{|\alpha|}\frac{C_{\alpha,|\alpha|,\chi}}{\langle \epsilon x \rangle^{|\alpha|}} \\ &= \epsilon^{|\alpha|}\frac{C_{\alpha,|\alpha|,\chi}}{\epsilon^{|\alpha|}\langle x \rangle^{|\alpha|}} \\ &\leq \frac{C_{\alpha,|\alpha|,\chi}}{\langle x \rangle^{|\alpha|-\sigma}}, \end{aligned} \tag{1.46}$$

Which is the inequality we wanted to prove.

For (ii), since the previous estimate is uniform in ϵ, and for $|\alpha| > 0$ the right-hand side goes to 0, it follows that the left hand side also does this uniformly. □

We now introduce a class of functions which have an oscillatory integral as it has been defined in 1.11. This will be a space of smooth functions whose derivatives (and the function itself) grow at most at a constant polynomial order.

Definition 1.13. *For $s \in \mathbb{R}$ we say a function $a \in \mathcal{A}^s(\mathbb{R}^n)$ if for every $\alpha \in \mathbb{N}^n$ multi-index, there is some $C_{\alpha,f} > 0$ such that for all $x \in \mathbb{R}^n$ we have that*

$$|D_x^\alpha a(x)| \leq C_{\alpha,f} \langle x \rangle^s . \tag{1.47}$$

We define for all $k \in \mathbb{N}$ a family of norms $\|\cdot\|_{k,\mathcal{A}^s} : \mathcal{A}^s(\mathbb{R}^n) \to \mathbb{R}$ as

$$\|a\|_{k,\mathcal{A}^s} = \max_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} \left| \frac{D_x^\alpha a(x)}{\langle x \rangle^s} \right|, \tag{1.48}$$

with which this space becomes a Fréchet space (see [27] P. 46)

For this family their oscillatory integral with respect to the Fourier phase $\phi(x, y) = -x \cdot y$ not only exists, but it can be represented as an absolutely convergent integral after a clever application of integration by parts. This mantra is crucial to construct more complicated oscillatory integrals.

Theorem 1.14. *Let $a \in \mathcal{A}^s(\mathbb{R}^{2n})$ then the oscillatory integral*

$$Os \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-y \cdot x} a(x, y) dx dy \tag{1.49}$$

exists as in definition 1.11. More precisely, there are $l_1, l_2 \in \mathbb{N}$ sufficiently large such that

$$\langle y \rangle^{-2l_2} (1 + \nabla_x^2)^{l_2} \langle x \rangle^{-2l_1} (1 + \nabla_y^2)^{l_1} a(x, y)$$

is Lebesgue integrable and

$$\begin{aligned} &Os \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-y \cdot x} a(x, y) dx dy \\ &= \\ &\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-iy \cdot x} \langle y \rangle^{-2l_2} (1 + \nabla_x^2)^{l_2} \langle x \rangle^{-2l_1} (1 + \nabla_y^2)^{l_1} a(x, y) dy dx. \end{aligned} \tag{1.50}$$

Furthermore, there is a constant $C > 0$ independent of a such that

$$\left| Os \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-y \cdot x} a(x, y) dx dy \right| \leq C \|a\|_{l_0, \mathcal{A}^s} \tag{1.51}$$

For a proof see [27] Thm. 6.4 p.47. This theorem encapsulates the usual techniques used to construct oscillatory integrals. The idea is that by integrating by parts enough times, one can represent the integral as a proper Lebesgue integral independent of the family χ_ϵ.

Remark. *In [27] they use a more general definition of the space of symbols $\mathcal{A}$ (See definition 6.1 at p. 46) however it contains our definition as a particular case.*

2 Classical Model

Our physical model starts with a relativistic Lagrangian for a single particle of positive mass in an external potential. We will assume that $c = 1$ and, for generality, that the configuration space of the particle is $\mathbb{R}^n$ for $n \in \mathbb{N} \setminus \{0\}$. So if the (rest) mass of the particle is given by $m > 0$ then the dynamics of the particle are generated by the relativistic Lagrangian

$$\mathcal{L}(x, v) = -m\sqrt{1 - |v|^2} - V(x), \tag{2.1}$$

for an external potential $V(x)$. Then, the classical action that acts on time-like paths $x : [0, T] \to \mathbb{R}^n$ is given by

$$S(x) = \int_0^T L(x(t), \dot{x}(t))dt = \int_0^T -m\sqrt{1 - |\dot{x}(t)|^2} - V(x(t))dt. \tag{2.2}$$

This system is not manifestly covariant, however it can be derived from a fully covariant action when one parametrizes the particle using the proper time (see [17] chap 7.9,7.10). Since this action becomes singular at the light cone (where $\dot{x}(t) = 1$ i.e. for particles going at the speed of light) this could be problematic if we naively apply the standard techniques when constructing the path integral But quite nicely, if we consider the Hamiltonian treatment through a Legendre transformation we get that

$$\mathcal{H}(x, p) = \sqrt{m^2 + |p|^2} + V(x), \tag{2.3}$$

and we define the phase space action for paths $(x, p) : [0, T] \to \mathbb{R}^n \times \mathbb{R}^n$ as

$$S((x, p)) = \int_0^T p(t) \cdot dx(t) - \int_0^T \mathcal{H}(q(t), p(t))dt, \tag{2.4}$$

where the first integral on the right can be understood as a Lebesgue-Stieltjes integral and here there are no singularities even for non causal paths (See [23] Eq 1.6).

Writing this more mathematically, for a given smooth potential $V : \mathbb{R}^n \to \mathbb{R}$, the Lagrangian of the system $\mathcal{L} : \mathbb{R}^n \times B(0, 1) \to \mathbb{R}$ is simply

$$\mathcal{L}(x, y) = -m\sqrt{1 - |y|^2} - V(x), \tag{2.5}$$

we need to restrict the second component of $\mathcal{L}$ to the unit ball since the possible magnitude of the velocities of relativistic particles must be less than $c = 1$. We define the canonical momentum $P : B(0, 1) \to \mathbb{R}^n$ as

$$P(y) = \nabla_y L(\cdot, y) = \frac{my}{\sqrt{1 - |y|^2}}, \tag{2.6}$$

we can find the inverse $P^{-1} : \mathbb{R}^n \to B(0, 1)$ explicitly as

$$P^{-1}(y) = \frac{y}{\sqrt{m^2 + |y|^2}}, \tag{2.7}$$

and through a Legendre transformation (see [17] Chp. 8 and [3] Chp. 14) , we define our classical Hamiltonian $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$\begin{aligned}\mathcal{H}(x,y) &= y \cdot P^{-1}(y) - L(x, P^{-1}(y)) \\ &= \frac{|y|^2}{\sqrt{m^2 + |y|^2}} + m\sqrt{1 - \frac{|y|^2}{m^2 + |y|^2}} + V(x) \\ &= \frac{|y|^2}{\sqrt{m^2 + |y|^2}} + m\sqrt{\frac{m^2 + |y|^2 - |y|^2}{m^2 + |y|^2}} + V(x) \\ &= \frac{|y|^2}{\sqrt{m^2 + |y|^2}} + \frac{m^2}{\sqrt{m^2 + |y|^2}} + V(x) \\ &= \sqrt{m^2 + |y|^2} + V(x). \end{aligned} \tag{2.8}$$

Now, for an arbitrary $T > 0$ we define

$$H^1([0,T], \mathbb{R}^n \times \mathbb{R}^n) := \left\{ (q,p) : [s,t] \to \mathbb{R}^n \times \mathbb{R}^n \ : \ x_i \in H^1[0,T], \ p_i \in L^2[0,T] \right\}, \tag{2.9}$$

where $H^1([0,T])$ is the classical Sobolev space of square integrable functions on the interval $[0,T]$ whose first weak derivative is also in $L^2[0,T]$ (See [9] Ch. 5.2.2). We can know define the phase space action as the map $S : H^1([0,T], \mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R}$ given by

$$S((q,p)) = \int_0^T p(\theta) \cdot dq(\theta) - \int_{\text{supp}((q,p)} \mathcal{H}(q(\theta), p(\theta)) d\theta \tag{2.10}$$

where

$$\int_0^T p(t) \cdot dx(t) := \sum_{i=1}^n \int_0^T p_i(t) dx_i(t), \tag{2.11}$$

where each of these is defined as Lebesgue-Stieltjes integral. These are well defined since x_i can be represented as an absolutely continuous function so in particular it is of bounded variation but also the induced measure is absolutely continuous with respect to the usual Lebesgue measure (See [12] Chap 3.5).

To verify our classical model is correct, that is, that the Hamiltonian does generate the correct equations of motion we can verify directly by applying Hamilton's equation of motion to (2.9) (see [17] p. 337). We are looking for paths in phase space $(x(t), p(t)) : [0,T] \to \mathbb{R}^{2n}$ that are at least C^1 which satisfy

$$\frac{dx(t)}{dt} = \frac{\partial \mathcal{H}(x(t), p(t))}{\partial y} = \frac{p(t)}{\sqrt{m^2 + p^2(t)}} \tag{2.12}$$

$$\frac{dp(t)}{dt} = -\frac{\partial \mathcal{H}(x(t), p(t))}{\partial x} = -V'(x(t)) \tag{2.13}$$

and we see that the first equation is just the relativistic, coordinate velocity with the appropriate Lorentz factor and the second one is then just Newton's second law. We again emphasize that these equations are in a particular reference frame, but this Classical system is fully relativistic.

3 Quantum Model

In this section we will define the corresponding quantum mechanical Hamiltonian. This is only to bring into context the proposed model, but since the main result does not mention this operator, we will just briefly discuss it.

Definition 3.1. *For a given $V \in \mathcal{A}^m(\mathbb{R}^n)$ as in definition 1.13, we define the Hamiltonian operator on Schwartz space $H : \mathcal{S}(\mathbb{R}) \to C^\infty(\mathbb{R}^n)$ as*

$$\begin{aligned}(Hf)(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot y} \mathcal{H}(x,y) \hat{f}(y) dy \\ &= \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot y} \left(\sqrt{m^2 + |y|^2} + V(x) \right) \hat{f}(y) dy \end{aligned} \tag{3.1}$$

where $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$ is the Fourier transform of f given by

$$\hat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} f(y) dy. \tag{3.2}$$

The Hamiltonian as defined above is a pseudo-differential operator (cf. [27] Chp. 2). However, by separating the potential term which doesn't depend on y, using Fourier inversion theorem ([32] Thm. IX.1), and defining

$$(H_0 f)(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot y} \sqrt{m^2 + |y|^2} \hat{f}(y) dy \tag{3.3}$$

we can re-write it as

$$\begin{aligned}(Hf)(x) &= \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot y} \sqrt{m^2 + |y|^2} \hat{f}(y) dy + V(x) f(x) \\ &= (H_0 + V(x)) f(x). \end{aligned} \tag{3.4}$$

So, if we consider the domain

$$D(H_0) = \left\{ f \in L^2(\mathbb{R}^n) \ : \ \sqrt{m + |x|^2} \mathcal{F}(f) \in L^2(\mathbb{R}^n) \right\}, \tag{3.5}$$

where $\mathcal{F}$ is the L^2 unitary extension of the Fourier transform that exists thanks to Plancherel's theorem (See [32] Thm IX.6), then we can extend the previous definition as

$$\overline{H}_0 f(x) = \mathcal{F}^{-1} \left(\sqrt{m^2 + |\cdot|^2} \mathcal{F}(f) \right). \tag{3.6}$$

In this form we can see that, if we take $-\nabla^2$ the self-adjoint extension of the negative Laplacian, then $\overline{H}_0$ actually is just the operator

$$\sqrt{m^2 - \nabla^2}, \tag{3.7}$$

defined by the functional calculus (which for the negative Laplacian it is essentially constructed using the Fourier transform). In this form we see can more clearly that the question about self-adjointness of $H_0 + V(x)$ can be seen using classical perturbation theory. In [21] general self-adjointness of these types of operators is studied, and as a special case, the next theorem is proved (In the mentioned article see Thm. 2.1 and the discussion in page 243).

Theorem 3.2. *Let $V \in L^2(\mathbb{R}^n)_{loc}$, which means that for all compact sets $K \subset \mathbb{R}^n$, we have that $V\restriction_K \in L^2(K)$ and assume that there are some $C, m \geq 0$ such that*

$$-C\langle x\rangle^m \leq V(x), \qquad \forall x \in \mathbb{R}^n, \tag{3.8}$$

the operator $H_0 + V : \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined in 3.1 is essentially self-adjoint.

This shows that for a large class of potentials we have a proper quantum models and the assumptions are actually quite weaker to the ones assumed in the typical literature for the time slicing approach to the Feynman path integral where the potential also has to be smooth. This asymmetry between the formulations has not to our knowledge been fully addressed as even basic Coulomb-type potentials can not be treated using these techniques. This caveat is also well known from the theoretical physics side and many techniques have been established to extend path integrals to worse behaved potentials (See [18]) but to our knowledge this has not been done rigorously.

4 Fundamental Operator

In this section we will construct the fundamental operator for the Feynman path integral, that is, the path integral over the space of line segments. Here we will present the main result and prove it. We will define the Feynman path integral over line segments. Since we will be considering phase-space path integrals, these paths will be line segments over the configuration space variables and will be constant over the momentum variables. To wit, let us define first the phase space action over line segments.

Definition 4.1. *Let $T > 0$ be fixed, and let $\triangle_T = \{(s,t) \in [0,T]^2 \mid s \leq t\}$ we consider, using (2.9), the map $\tilde{\Gamma} : \triangle_T \times \mathbb{R}^{3n} \to H^1([s,t], \mathbb{R}^{2n})$ given by*

$$\tilde{\Gamma}(s,t,x,y,z)(\theta) = \left(y + \frac{\theta - s}{t-s}(x-y), z \right), \tag{4.1}$$

whose image represents the line segment joining x and y parameterized in the interval $[s,t]$ on the first coordinate and a constant path on the second coordinate. For s,t,x,y,z fixed we define $\tilde{\gamma}^{x,y,z}_{s,t} \in H^1([s,t], \mathbb{R}^{2n})$ as

$$\tilde{\gamma}^{x,y,z}_{s,t} := \tilde{\Gamma}(s,t,x,y,z). \tag{4.2}$$

And for a given smooth potential $V \in C^\infty(\mathbb{R}^n)$, we define using (2.11) $\mathcal{S} : \triangle_T \times \mathbb{R}^{3n} \to \mathbb{R}$ as

$$\begin{aligned} \mathcal{S}(s,t,x,y,z) = S(\tilde{\gamma}^{x,y,z}_{s,t}) &= \int_s^t \frac{z \cdot (x-y)}{t-s} d\theta - \int_s^t \sqrt{m^2 + |z|^2} + V\left(y + \frac{\theta - s}{t-s}(x-y) \right) d\theta \\ &= z \cdot (x-y) - (t-s)\sqrt{m^2 + |z|^2} - (t-s) \int_0^1 V\left(y + \theta(x-y) \right) d\theta \end{aligned} \tag{4.3}$$

Remark. *An important thing to note is that, even if we started with a relativistic classical model the paths given by equation (4.8) are not by any means relativistic since they have unbounded momenta. In the physics literature, this is stated as the reason why the quantum model is not fully relativistic since the probability amplitudes have contributions from non-causal paths which allow some tunnelling. Also, since the momentum variable z is not fixed these paths do not even satisfy the classical definition of momentum. While this initially can seems to conflict with the heuristic prescription of integrating over all "classical" paths, however, since quantum mechanics is considered to be the fundamental theory and not classical mechanics, it is a basic feature of quantum theory that these type of paths exist due to the uncertainty principle and they are perfectly valid in such a framework. With this in mind one can also interpret the path integral as being defined over all "quantum" paths with the integrand giving probabilities to these and the $\hbar$ dependency in the exponential explains why at some scales certain paths seem to be selected (cf. [5])*

Remark. *It is also significant that here the kinetic term is decoupled and since it is $\sqrt{m^2 + |z|^2}$, by 1.4, all of its derivatives are bounded. This will simplify some arguments that parallel the relativistic case in which the term was quadratic. Also, note that the temporal terms factor out of the kinetic and potential terms which is also not the case for the non-relativistic case.*

Definition 4.2. *Let* $V \in C^{\infty}(\mathbb{R}^n)$*, and* $p(x,y,z) \in C^{\infty}(\mathbb{R}^{3n})$ *be a smooth functions for which there is some* $m \in \mathbb{R}$ *such that for all* α, β, γ *multi-indices, there is a constant* $C_{\alpha,\beta,\gamma} \geq 0$ *such that*

$$\left|D_x^{\alpha} D_y^{\beta} D_z^{\gamma} p(x,y,z)\right| \leq C_{\alpha,\beta,\gamma} \left(\sqrt{1 + |x|^2 + |y|^2 + |z|^2}\right)^m \qquad \forall (x,y,z) \in \mathbb{R}^{3n}, \tag{4.4}$$

that is $p \in \mathcal{A}^m(\mathbb{R}^{3n})$ *as in definition 1.13. Then we define the family of formal operator* $\hat{p}(s,t)$*, with* $(s,t) \in \Delta_T$*, acting on the Schwartz space* $S(\mathbb{R}^n)$ *as*

$$(\hat{p}(s,t)f)(x) = \frac{1}{(2\pi\hbar)^n} Os \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(i\hbar^{-1}\mathcal{S}(s,t,x,y,z)\right) p(x,y,z) f(y) dy dz, \tag{4.5}$$

where the oscillatory integral is defined as in 1.11.

A priori, the oscillatory integral (4.5) doe not necessarily exist. In what follows we will focus on proving that this operator is well defined and study its regularity which is a basic step towards constructing the full path integral (cf. [23]). To begin, we will first prove some lemmas.

Lemma 4.3. *Let* $V \in \mathcal{A}^{m_V}(\mathbb{R}^n)$*,* $p \in \mathcal{A}^{m_p}(\mathbb{R}^{3n})$ *and* $\Omega \subset \mathbb{C}$ *a compact subset of the complex plane, and define*

$$\delta(x,y,z) = \sqrt{m^2 + |z|^2} + \int_0^1 V\left(y + x - \theta y\right) d\theta, \tag{4.6}$$

then for all $\alpha \in \mathbb{N}^{2n}$*,*$\beta \in \mathbb{N}^n$ *multi-indices, there is some* $C_{\alpha,\beta,\Omega} > 0$ *such that for all* $c \in \Omega$

$$\left|D_z^{\beta} D_{x,y}^{\alpha} e^{c\delta(x,y,z)}\right| \leq \left|e^{c\delta(x,y,z)}\right| \langle x \rangle^{m_V|\alpha|} \langle y \rangle^{|m_V\alpha|} C_{\alpha,\beta,\Omega}, \tag{4.7}$$

that is $C_{\alpha,\beta,\Omega}$ *depends only on the set* Ω*.*

Proof. Let $\alpha \in \mathbb{N}^{2n}$ be a multi-index, then using Faa di Bruno's formula 1.10 for the case $m = 1$, and setting $\alpha' = a \in \mathbb{N}$, we get that for all $c \in \mathbb{C}$:

$$\begin{aligned} D_{x,y}^{\alpha} \exp\left(c\delta(x,y,z)\right) &= \sum_{1 \leq a \leq |\alpha|} c^a \left(D^a \exp\right)\left(c\delta(x,y,z)\right) \sum_{\substack{\mathcal{P}_\tau(\alpha,a) \\ \tau=1,\dots,|\alpha|}} \alpha! \prod_{j=1}^{\tau} \frac{\left[D_{x,y}^{\alpha_j}\delta(x,y,z)\right]^{a_j}}{(a_j)(\alpha_j!)^{a_j}} \\ &= e^{c\delta(x,y,z)} \sum_{1 \leq a \leq |\alpha|} c^a \sum_{\substack{\mathcal{P}_\tau(\alpha,a) \\ \tau=1,\dots,|\alpha|}} \alpha! \prod_{j=1}^{\tau} \frac{\left[D_{x,y}^{\alpha_j}\delta(x,y,z)\right]^{a_j}}{(a_j)(\alpha_j!)^{k_j}}, \end{aligned} \tag{4.8}$$

and similarly for $\beta \in \mathbb{N}^n$

$$D_z^{\beta} \exp\left(c\delta(x,y,z)\right) = e^{c\delta(x,y,z)} \sum_{1 \leq b \leq |\beta|} c^b \sum_{\substack{\mathcal{P}_\tau(\beta,b) \\ \tau=1,\dots,|\beta|}} \beta! \prod_{j=1}^{\tau} \frac{\left[D_z^{\beta_j}\delta(x,y,z)\right]^{b_j}}{(b_j)(\beta_j!)^{b_j}}. \tag{4.9}$$

Now, if we denote without loss of generality $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_i \in \mathbb{N}^n$ by the Leibniz, integral rule and the chain rule we have that

$$\begin{aligned}
\left| D^{\alpha}_{x,y} \int_0^1 V\left(y + \theta(x-y)\right) d\theta \right| &\leq \int_0^1 \left| D^{\alpha}_{x,y} V\left(y + \theta(x-y)\right)\right| d\theta \\
&\leq \int_0^1 \left| (\theta)^{|\alpha|} \left(D^{\alpha_1+\alpha_2}_{x} V\right)\left(y + \theta(x-y)\right)\right| d\theta \\
&\leq \int_0^1 (\theta)^{|\alpha|} C^V_{\alpha_1+\alpha_2} \left\langle y + \theta(x-y)\right\rangle^{m_V} d\theta \\
&\leq \int_0^1 1^{|\alpha|} C^V_{\alpha_1+\alpha_2} \sqrt{2} \left\langle y(1-\theta)\right\rangle^{|m_V|} \left\langle \theta x\right\rangle^{m_V} d\theta \\
&\leq \sqrt{2} C^V_{\alpha_1+\alpha_2} \left\langle y\right\rangle^{|m_V|} \left\langle x\right\rangle^{m_V}, \qquad (4.10)
\end{aligned}$$

where in the second to last inequality we used equation (1.22). Now, since it follows from proposition 1.4 that for $s \in \mathbb{R}$ fixed and for all $\alpha' \in \mathbb{N}^n$ multi index, there is some $C^{\text{ek}}_{\alpha',s} \geq 0$ such that

$$\left| D^{\alpha'}_z \sqrt{m^2 + |z|^2}^{\,s} \right| \leq C^{\text{ek}}_{\alpha',s} \left(m^2 + |z|^2\right)^{(s-|\alpha'|)/2}. \qquad (4.11)$$

Now, recalling definition 1.9 and equations (4.10,4.9,4.30), if α, β are arbitrary multi-indices we obtain

$$\begin{aligned}
\left| D^{\beta}_z D^{\alpha}_{x,y} e^{c\delta(x,y,z)} \right| &= \left| D^{\beta}_z e^{c\delta(x,y,z)} \sum_{\substack{1\leq a\leq|\alpha| \\ \mathcal{P}_{\tau_1}(\alpha,a) \\ \tau_1=1,\ldots,|\alpha|}} c^a \alpha! \prod_{j=1}^{\tau_1} \frac{\left[D^{\alpha_j}_{x,y}\delta(x,y,z)\right]^{a_j}}{(a_j)(\alpha_j!)^{a_j}} \right| \\
&\leq \left| e^{c\delta(x,y,z)} \right| \sum_{\substack{1\leq a\leq|\alpha| \\ 1\leq b\leq|\beta| \\ \mathcal{P}_{\tau_1}(\alpha,a) \\ \mathcal{P}_{\tau_2}(\beta,b) \\ \tau_1=1,\ldots,|\alpha| \\ \tau_2=1,\ldots,|\beta|}} |c|^{a+b} \beta! \prod_{k=1}^{\tau_2} \frac{\left|D^{\beta_k}_z \delta(x,y,z)\right|^{b_k}}{(b_k)(\beta_k!)^{b_k}} \alpha! \prod_{j=1}^{\tau_1} \frac{\left|D^{\alpha_j}_{x,y}\delta(x,y,z)\right|^{a_j}}{(a_j)(\alpha_j!)^{a_j}} \\
&\leq \left| e^{c\delta(x,y,z)} \right| \\
&\times \sum_{\substack{1\leq a\leq|\alpha| \\ 1\leq b\leq|\beta| \\ \mathcal{P}_{\tau_1}(\alpha,a) \\ \mathcal{P}_{\tau_2}(\beta,b) \\ \tau_1=1,\ldots,|\alpha| \\ \tau_2=1,\ldots,|\beta|}} |c|^{a+b} \beta! \prod_{k=1}^{\tau_2} \frac{C^{\text{ek}}_{\beta_k}}{(b_k)(\beta_k!)^{b_k} \left\langle z\right\rangle^{|\beta_k|b_k-1}} \alpha! \prod_{j=1}^{\tau_1} \frac{\left(C^V_{\alpha_{j,1}+\alpha_{j,2}}\sqrt{2}\right)^{a_j} \left\langle x\right\rangle^{m_V a_j} \left\langle y\right\rangle^{|m_V|a_j}}{(a_j)(\alpha_j!)^{a_j}}.
\end{aligned}$$

$$(4.12)$$

Now recalling the definition of $\mathcal{P}$ by equation (1.39) we have that $\sum_{j=1}^{\tau_1} a_j = a \leq |\alpha|$ and $|\beta| = \sum_{k=1}^{\tau_2} |\beta_k| b_k$, so if we set $K_\Omega = \max_{c\in\Omega}(1+|c|)^{|\alpha|+|\beta|}$ we can factor the $\left\langle z\right\rangle, \left\langle x\right\rangle$ terms in the above expression and get

$$\begin{aligned}\left|D_z^{\beta} D_{x,y}^{\alpha} e^{c\delta(x,y,z)}\right| &\leq \left|e^{c\delta(x,y,z)}\right| \frac{\langle x\rangle^{m_V|\alpha|} \langle y\rangle^{|m_V||\alpha|} K_{\Omega}\sqrt{2}^{|\alpha|}}{\langle z\rangle^{0}} \\ &\times \sum_{\substack{1\leq a\leq|\alpha| \\ 1\leq b\leq|\beta| \\ \mathcal{P}_{\tau_1}(\alpha,a) \\ \mathcal{P}_{\tau_2}(\beta,b) \\ \tau_1=1,\ldots,|\alpha| \\ \tau_2=1,\ldots,|\beta|}} \beta! \prod_{k=1}^{\tau_2} \frac{(C_{\beta_k}^{\mathrm{ek}})^{b_j}}{(b_k)(\beta_k!)^{b_k}} \alpha! \prod_{j=1}^{\tau_1} \frac{(C_{\alpha_{j,1}+\alpha_{j,2}}^{V})^{1_j}}{(a_j)(\alpha_j!)^{a_j}} \\ &= \left|e^{c\delta(x,y,z)}\right| \langle x\rangle^{m_V|\alpha|} \langle y\rangle^{|m_V||\alpha|} K_{\Omega}\sqrt{2}^{|\alpha|} \\ &\times \sum_{\substack{1\leq a\leq|\alpha| \\ 1\leq b\leq|\beta| \\ \mathcal{P}_{\tau_1}(\alpha,a) \\ \mathcal{P}_{\tau_2}(\beta,b) \\ \tau_1=1,\ldots,|\alpha| \\ \tau_2=1,\ldots,|\beta|}} \beta! \prod_{k=1}^{\tau_2} \frac{(C_{\beta_k}^{\mathrm{ek}})^{b_j}}{(b_k)(\beta_k!)^{b_k}} \alpha! \prod_{j=1}^{\tau_1} \frac{(C_{\alpha_{j,1}+\alpha_{j,2}}^{V})^{a_j}}{(a_j)(\alpha_j!)^{a_j}}, \qquad (4.13)\end{aligned}$$

so by defining

$$C_{\alpha,\beta,\Omega} = K_{\Omega}\sqrt{2}^{|\alpha|} \sum_{\substack{1\leq a\leq|\alpha| \\ 1\leq b\leq|\beta| \\ \mathcal{P}_{\tau_1}(\alpha,a) \\ \mathcal{P}_{\tau_2}(\beta,b) \\ \tau_1=1,\ldots,|\alpha| \\ \tau_2=1,\ldots,|\beta|}} \beta! \prod_{k=1}^{\tau_2} \frac{(C_{\beta_k}^{\mathrm{ek}})^{b_j}}{(b_k)(\beta_k!)^{b_k}} \alpha! \prod_{j=1}^{\tau_1} \frac{(C_{\alpha_{j,1}+\alpha_{j,2}}^{V})^{a_j}}{(a_j)(\alpha_j!)^{a_j}}, \qquad (4.14)$$

we obtain the desired constant since

$$\left|D_z^{\beta} D_{x,y}^{\alpha} e^{c\delta(x,y,z)}\right| \leq C_{\alpha,\beta,\Omega}\left|e^{c\delta(x,y,z)}\right| \langle x\rangle^{m_V|\alpha|} \langle y\rangle^{|m_V\alpha|}. \qquad (4.15)$$

□

Lemma 4.4. *Let* $p \in \mathcal{A}^{m_p}(\mathbb{R}^{3n})$*,* $\chi \in \mathcal{S}(\mathbb{R}^{2n})$ *such that* $\chi(0,0) = 1$ *and* $f \in \mathcal{S}(\mathbb{R}^n)$*, then for all* $\alpha \in \mathbb{N}^{3n}$ *multi-index and* $k \in \mathbb{N}$*, there is a* $C_{\alpha,k,f,p,\chi} > 0$ *such that for all* $1 > \epsilon > 0$ *we have that*

$$\left|D_{x,y,z}^{\alpha} \chi(\epsilon(y+x), \epsilon z) p(x, y+x, z) f(x+y)\right| \leq C_{\alpha,k,f,p,\chi} \frac{\langle x\rangle^{|m_p|} \langle z\rangle^{m_p}}{\langle y+x\rangle^{k}} \qquad (4.16)$$

Proof. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^{3n}$, $\chi_{\epsilon}(y,z) = \chi(\epsilon y, \epsilon z)$, and let

$$h_{\epsilon}(x,y,z) = \chi_{\epsilon}(y+x, z) p(x, y+x, z) f(x+y) \qquad (4.17)$$

then by the general Leibniz Rule 1.2 we obtain

$$\begin{aligned}
&D^{\alpha}_{x,y,z} h_\epsilon(x,y,z) \\
&= \\
&\sum_{\substack{\alpha' \leq \alpha \\ \alpha'' \leq \alpha'}} \binom{\alpha}{\alpha'} \binom{\alpha'}{\alpha''} \left(D^{(\alpha''_2, \alpha''_3)}_{y,z} \chi_\epsilon(y+x,z) \right) \left(D^{\alpha'-\alpha''}_{x,y,z} p(x, y+x, z \right) \left(D^{(\alpha_1-\alpha'_1, \alpha_2-\alpha'_2)}_{x,y} f(x+y) \right).
\end{aligned} \tag{4.18}$$

So using lemma 1.12, equation 1.22 and the hypothesis on p and f, for all $1 > \epsilon > 0$ and $k' \in \mathbb{N}$ we obtain the following estimate

$$\begin{aligned}
&\left| D^{\alpha}_{x,y,z} h_\epsilon(x,y,z) \right| \\
&\leq \\
&\sum_{\substack{\alpha' \leq \alpha \\ \alpha'' \leq \alpha'}} \binom{\alpha}{\alpha'} \binom{\alpha'}{\alpha''} \frac{C_{(\alpha''_2,\alpha''_3),\chi} C_{\alpha'-\alpha'',p} C_{(\alpha_1-\alpha'_1,\alpha_2-\alpha'_2),k',f} \left\langle (x, y+x, z) \right\rangle^{m_p}}{\left\langle (y+x, z) \right\rangle^{|(\alpha''_2,\alpha''_3)|} \left\langle x+y \right\rangle^{k'}} \\
&\leq \\
&\sum_{\substack{\alpha' \leq \alpha \\ \alpha'' \leq \alpha'}} \binom{\alpha}{\alpha'} \binom{\alpha'}{\alpha''} \frac{2C_{(\alpha''_2,\alpha''_3),\chi} C_{\alpha'-\alpha'',p} C_{(\alpha_1-\alpha'_1,\alpha_2-\alpha'_2),k',f} \left\langle x \right\rangle^{|m_p|} \left\langle z \right\rangle^{m_p}}{\left\langle y+x \right\rangle^{k'-m_p}} \\
&\leq \\
&\frac{\left\langle x \right\rangle^{|m_p|} \left\langle z \right\rangle^{m_p}}{\left\langle y+x \right\rangle^{k'-m_p}} \sum_{\substack{\alpha' \leq \alpha \\ \alpha'' \leq \alpha'}} 2 \binom{\alpha}{\alpha'} \binom{\alpha'}{\alpha''} C_{(\alpha''_2,\alpha''_3),\chi} C_{\alpha'-\alpha'',p} C_{(\alpha_1-\alpha'_1,\alpha_2-\alpha'_2,k',f)},
\end{aligned} \tag{4.19}$$

so for $k \in \mathbb{N}$, taking $k' = k + m_p$ and defining

$$C_{\alpha,k,f,p,\chi} = 2 \sum_{\substack{\alpha' \leq \alpha \\ \alpha'' \leq \alpha'}} \binom{\alpha}{\alpha'} \binom{\alpha'}{\alpha''} C_{(\alpha''_2,\alpha''_3),\chi} C_{\alpha'-\alpha'',p} C_{(\alpha_1-\alpha'_1,\alpha_2-\alpha'_2,k',f)}, \tag{4.20}$$

we obtain

$$\left| D^{\alpha}_{x,y,z} h_\epsilon(x,y,z) \right| \leq C_{\alpha,k,f,p,\chi} \frac{\left\langle x \right\rangle^{|m_p|} \left\langle z \right\rangle^{m_p}}{\left\langle y+x \right\rangle^{k}}, \tag{4.21}$$

which is what we wanted to prove. □

Lemma 4.5. *For all $f \in C^\infty(\mathbb{R}^n)$ and $l \in \mathbb{N}$ we have that*

$$(1+\nabla^2)^l f(x) = \left(\sum_{|\alpha|=l} \binom{l}{\alpha} D^{2\alpha/\hat{e_0}} \right) f(x) \qquad \forall x \in \mathbb{R}^n \tag{4.22}$$

with $\alpha \in \mathbb{N}^{n+1}$, $\alpha/\hat{e_0}$ being the multi-index in $\mathbb{N}^n$ obtained from removing the first entry of α.

Proof. This proof will be done by induction over l. If $l = 1$, then it follows directly since

$$\begin{aligned}\sum_{|\alpha|=1} \binom{1}{\alpha} D^{2(\alpha/\hat{e_0})} f(x) &= \sum_{i=0}^{n} 1 D^{2(\hat{e}_i/\hat{e_0})} f(x) \\ &= 1 f(x) + \sum_{i=1}^{n} D^{2(\hat{e}_i/\hat{e_0})} f(x) \\ &= \left(1 + \sum_{i=1}^{n} \partial_{x_i}^2\right) f(x) \\ &= (1+\nabla^2) f(x). \end{aligned} \tag{4.23}$$

Now let us assume the identity holds true for some $l = K$ we will prove it for $l = K+1$, so using the induction hypothesis we get

$$\begin{aligned}(1+\nabla^2)^{K+1} f(x) &= (1+\nabla^2)(1+\nabla^2)^K f(x) \\ &= \left(\sum_{|\alpha_1|=1} D^{2\alpha_1/\hat{e_0}}\right)\left(\sum_{|\alpha_2|=K} \binom{K}{\alpha_2} D^{2\alpha_2/\hat{e_0}}\right) f(x) \\ &= \left(\sum_{i=0}^{n} \sum_{|\alpha_2|=K} \binom{K}{\alpha_2} D^{2(\alpha_2+\hat{e_i})/\hat{e_0}}\right) f(x). \end{aligned} \tag{4.24}$$

Now, for a fixed $\alpha \in \mathbb{N}^{n+1}$ such that $|\alpha| = K+1$ we can represent it uniquely as $\alpha = \alpha^i + \hat{e}_i$ for $i = 0, \ldots, n$, so we can rewrite the above sum as

$$\begin{aligned}(1+\nabla^2)^{K+1} f(x) &= \sum_{|\alpha|=K+1} \sum_{i=0}^{n} \binom{K}{\alpha^i} D^{2(\alpha^i+\hat{e}_i)/\hat{e_0}} f(x) \\ &= \sum_{|\alpha|=K+1} \left(\sum_{i=0}^{n} \binom{K}{\alpha^i}\right) D^{2\alpha/\hat{e_0}} f(x) \end{aligned} \tag{4.25}$$

Now since $\alpha^i = \alpha - \hat{e}_i$ we have that

$$\binom{K}{\alpha^i} = \frac{K!}{\alpha_1! \ldots (\alpha_i - 1)! \ldots \alpha_{n+1}!}, \tag{4.26}$$

therefore it is easily verified that

$$\sum_{i=0}^{n} \binom{K}{\alpha^i} = \binom{K+1}{\alpha}, \tag{4.27}$$

so finally we get

$$(1+\nabla^2)^{K+1} f(x) = \sum_{|\alpha|=K+1} \binom{K+1}{\alpha} D^{2\alpha/\hat{e_0}} f(x), \tag{4.28}$$

so the inductive step is complete. □

Theorem 4.6. *Let $T > 0$, $V \in \mathcal{A}^{m_V}(\mathbb{R}^n)$ and $p \in \mathcal{A}^{m_p}(\mathbb{R}^{3n})$, and $f \in \mathcal{S}(\mathbb{R}^n)$, then the oscillatory integral operator $\hat{p}(s,t)f$, as in 4.5, is well defined with respect to definition 1.11 and for all α multi-index $D_x^\alpha(\hat{p}(s,t)f)(x)$ exists and is continuous in $\Delta_T \times \mathbb{R}^n$.*

Proof. To see if $\hat{p}(s,t)$ is well defined as an oscillatory integral in the sense of definition 1.11, for a fixed Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$ we must show that for all Schwartz functions $\chi \in \mathcal{S}(\mathbb{R}^n)$ with $\chi(0) = 1$ that if we define for all $\epsilon > 0$ the family of integrals (the integrals exists because of 4.3,4.4 and the definition of Schwartz functions 1.6)

$$\begin{aligned}(\hat{p}_\epsilon(s,t)f)(x) = \lim_{\epsilon\to 0}\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big(e^{-i\hbar^{-1}\mathcal{S}(s,t,x,y,z)}\\ \times\chi_\epsilon(y,z)p(x,y,z)f(y)\Big)\,dydz,\end{aligned} \tag{4.29}$$

then the limit as $\epsilon \to 0$ exist and is independent of χ. Let us define

$$\delta(x,y,z) = \sqrt{m^2+|z|^2} + \int_0^1 V\left(y+x-\theta y\right)d\theta, \tag{4.30}$$

then using (4.5) and (4.3), we have that

$$\begin{aligned}(\hat{p}_\epsilon(s,t)f)(x) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big(e^{i\hbar^{-1}z\cdot(x-y)}e^{-(t-s)i\hbar^{-1}\left(\sqrt{m^2+|z|^2}+\int_0^1 V(y+\theta(x-y))d\theta\right)}\\ &\times\chi_\epsilon(y,z)p(x,y,z)f(y)\Big)\,dydz\\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big(e^{-i\hbar^{-1}z\cdot y}e^{-(t-s)i\hbar^{-1}\delta(x,y,z)}\\ &\times\chi_\epsilon(y+x,z)p(x,y+x,z)f(y+x)\Big)\,dydz\end{aligned} \tag{4.31}$$

for all $\epsilon > 0$ by a simple change of variable. We first note that by direct computation it follows that

$$(1+\nabla_y^2)e^{-i\hbar^{-1}z\cdot y} = (1+\left|\hbar^{-1}z\right|^2)e^{-i\hbar^{-1}z\cdot y}, \tag{4.32}$$

$$(1+\nabla_z^2)e^{-i\hbar^{-1}z\cdot y} = (1+\left|\hbar^{-1}y\right|^2)e^{-i\hbar^{-1}z\cdot y}, \tag{4.33}$$

so by using integration by parts 1.7, for arbitrary $l_1, l_2 \in \mathbb{N}$ we can rewrite the integral in (4.31) as

$$\begin{aligned}\hat{p}_\epsilon(s,t)f(x) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\left(\frac{(1+\nabla_y^2)^{l_1}}{(1+|\hbar^{-1}z|^2)^{l_1}}e^{-i\hbar^{-1}z\cdot y}\right)e^{-(t-s)i\hbar^{-1}\delta(x,y,z)}\right.\\ &\left.\times\chi_\epsilon(y+x,z)p(x,y+x,z)f(y+x)\right)dydz\\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(e^{-i\hbar^{-1}z\cdot y}\left(\frac{(1+\nabla_y^2)^{l_1}}{(1+|\hbar^{-1}z|^2)^{l_1}}e^{-(t-s)i\hbar^{-1}\delta(x,y,z)}\right.\right.\\ &\left.\left.\times\chi_\epsilon(y+x,z)p(x,y+x,z)f(y+x)\right)\right)dydz\\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(e^{-i\hbar^{-1}z\cdot y}\left(\frac{(1+\nabla_z^2)^{l_2}}{(1+|\hbar^{-1}y|^2)^{l_2}}\frac{(1+\nabla_y^2)^{l_1}}{(1+|\hbar^{-1}z|^2)^{l_1}}e^{-(t-s)i\hbar^{-1}\delta(x,y,z)}\right.\right.\\ &\left.\left.\times\chi_\epsilon(y+x,z)p(x,y+x,z)f(y+x)\right)\right)dydz.\end{aligned} \tag{4.34}$$

Now, by lemma 4.5, we obtain formally

$$\left(1+\nabla_y^2\right)^{l_1} = \sum_{|\alpha_1|=l_1} \binom{l_1}{\alpha_1} D_y^{2\alpha_1/\hat{e}_0}, \tag{4.35}$$

$$\left(1+\nabla_z^2\right)^{l_2} = \sum_{|\alpha_2|=l_1} \binom{l_2}{\alpha_2} D_z^{2\alpha_2/\hat{e}_0}, \tag{4.36}$$

with $\alpha_j \in \mathbb{N}^{n+1}$ and $\alpha_i/\hat{e_0}$ being the multi-index in $\mathbb{N}^n$ obtained from removing the first entry of α_i. To simplify notation we fix

$$\alpha_{i,0} = 2\alpha_i/\hat{e_0} \qquad i=1,2, \tag{4.37}$$

and

$$h_\epsilon(x,y,z) = \chi_\epsilon(y+x,z)p(x,y+x,z)f(y+x). \tag{4.38}$$

So using (4.36) and the general Leibniz rule, for $\alpha \in \mathbb{N}^n$ multi-index we have that

$$\begin{aligned}
&D_x^\alpha \frac{(1+\nabla_z^2)^{l_2}}{(1+|\hbar^{-1}y|^2)^{l_2}} \frac{(1+\nabla_y^2)^{l_1}}{(1+|\hbar^{-1}z|^2)^{l_1}} e^{-(t-s)i\hbar^{-1}\delta(x,y,z)} \chi_\epsilon(y+x,z)p(x,y+x,z)f(y+x) \\
&= \\
&\frac{(1+\nabla_z^2)^{l_2}}{(1+|\hbar^{-1}y|^2)^{l_2}} \sum_{\substack{|\alpha_1|=l_1 \\ (\alpha',\beta_1)\leq(\alpha,\alpha_{1,0})}} \binom{l_1}{\alpha_1} \binom{(\alpha,\alpha_{1,0})}{(\alpha'\beta_1)} \frac{D_{x,y}^{(\alpha',\beta_1)} e^{-(t-s)i\hbar^{-1}\delta(x,y,z)} D_{x,y}^{(\alpha-\alpha',\alpha_{1,0}-\beta_1)} h_\epsilon(x,y,z)}{(1+|\hbar^{-1}z|^2)^{l_1}} \\
&= \\
&\frac{1}{(1+|\hbar^{-1}y|^2)^{l_2}} \sum_{\substack{|\alpha_1|=l_1 \\ (\alpha',\beta_1)\leq(\alpha,\alpha_{1,0}) \\ |\alpha_2|=l_2 \\ \beta_2\leq\alpha_{2,0}}} \binom{l_1}{\alpha_1} \binom{\alpha_{1,0}}{\beta_1} \binom{l_2}{\alpha_2} \binom{\alpha_{2,0}}{\beta_2} D_z^{\alpha_{2,0}-\beta_2} \frac{1}{(1+|\hbar^{-1}z|^2)^{l_1}} \\
&\times D_z^{\beta_2} \left(D_{x,y}^{(\alpha',\beta_1)} e^{-(t-s)i\hbar^{-1}\delta(x,y,z)} D_{x,y}^{(\alpha-\alpha',\alpha_{1,0}-\beta_1)} h_\epsilon(x,y,z) \right).
\end{aligned} \tag{4.39}$$

Let us analyze the terms separately, again by the general Leibniz rule we have that

$$\begin{aligned}
&D_z^{\beta_2} \left(D_{x,y}^{(\alpha',\beta_1)} e^{-(t-s)i\hbar^{-1}\delta(x,y,z)} D_{x,y}^{(\alpha-\alpha',\alpha_{1,0}-\beta_1)} h_\epsilon(x,y,z) \right) \\
&= \\
&\sum_{\beta_2'\leq\beta_2} \binom{\beta_2}{\beta_2'} D_z^{\beta_2'} D_{x,y}^{(\alpha',\beta_1)} e^{-(t-s)i\hbar^{-1}\delta(x,y,z)} D_z^{\beta_2-\beta_2'} D_{x,y}^{(\alpha-\alpha',\alpha_{1,0}-\beta_1)} h_\epsilon(x,y,z)
\end{aligned} \tag{4.40}$$

and, using lemma 4.3 with $\Omega = \{-(t-s)\hbar^{-1}i \ : \ 0 \leq s \leq t \leq T\} \subset \mathbb{C}$ and lemma 4.4 we have for all $1 > \epsilon > 0$ and $k \in \mathbb{N}$

$$\begin{aligned}
&\left|\sum_{\beta_2'\leq\beta_2}\binom{\beta_2}{\beta_2'} D_z^{\beta_2'} D_{x,y}^{(\alpha',\beta_1)} e^{-(t-s)i\hbar^{-1}\delta(x,y,z)} D_z^{\beta_2-\beta_2'} D_{x,y}^{(\alpha-\alpha'\alpha_{1,0}-\beta_1)} h_\epsilon(x,y,z)\right| \\
&\leq \\
&\sum_{\beta_2'\leq\beta_2}\binom{\beta_2}{\beta_2'} \frac{C_{3.4;(\alpha',\beta_1),\beta_2',\Omega} C_{(\alpha-\alpha',\alpha_{1,0}-\beta_1,\beta-\beta_2'),k,f,p,\chi} \langle x\rangle^{m_V(|\alpha'|+|\beta_1|)+|m_p|} \langle y\rangle^{|m_V|(|\alpha'|+|\beta_1|)} \langle z\rangle^{m_p}}{\langle x+y\rangle^k} \\
&\leq \\
&\frac{\sqrt{2}\langle x\rangle^{m_V(|\alpha|+|\alpha_{1,0}|)+|m_p|+k} \langle y\rangle^{|m_V|(|\alpha|+|\alpha_{1,0}|)} \langle z\rangle^{m_p}}{\langle y\rangle^k} \sum_{\beta_2'\leq\beta_2}\binom{\beta_2}{\beta_2'} C_{3.4;(\alpha',\beta_1),\beta_2',\Omega} C_{(\alpha-\alpha',\alpha_{1,0}-\beta_1,\beta-\beta_2'),k,f,p,\chi} \\
&\leq \\
&\sqrt{2}\langle x\rangle^{m_V(|\alpha|+2l_1)+|m_p|+k} \langle y\rangle^{|m_V|(|\alpha|+2l_1)-k} \langle z\rangle^{m_p} \sum_{\beta_2'\leq\beta_2}\binom{\beta_2}{\beta_2'} C_{3.4;(\alpha',\beta_1),\beta_2',\Omega} C_{(\alpha-\alpha',\alpha_{1,0}-\beta_1,\beta-\beta_2'),k,f,p,\chi}
\end{aligned} \tag{4.41}$$

Now using proposition 1.4 for $s = -2l_1$ and the chain rule we get that

$$\begin{aligned}
\left|D_z^{\alpha_{2,0}-\beta_2}\frac{1}{(1+|\hbar^{-1}z|^2)^{l_1}}\right| &\leq \hbar^{-|\alpha_{2,0}-\beta_2|} C^{\text{ek}}_{(\alpha_{2,0}-\beta_2),-2l_1} \langle \hbar^{-1}z\rangle^{-2l_1-|\alpha_{2,0}-\beta_2|} \\
&\leq C^{\text{ek}}_{(\alpha_{2,0}-\beta_2),-2l_1} \langle \hbar^{-1}z\rangle^{-2l_1} \\
&\leq C^{\text{ek}}_{(\alpha_{2,0}-\beta_2),-2l_1} \langle z\rangle^{-2l_1}.
\end{aligned} \tag{4.42}$$

so applying the above estimate and 4.41 to equation (4.39) we obtain

$$\begin{aligned}
&\left|D_x^\alpha \frac{(1+\nabla_z^2)^{l_2}}{(1+|\hbar^{-1}y|^2)^{l_2}} \frac{(1+\nabla_y^2)^{l_1}}{(1+|\hbar^{-1}z|^2)^{l_1}} e^{-(t-s)i\hbar^{-1}\delta(x,y,z)} h_\epsilon(x,y,z)\right| \\
&\leq \\
&\frac{\sqrt{2}\langle x\rangle^{m_V(|\alpha|+2l_1)+|m_p|+k} \langle y\rangle^{|m_V|(|\alpha|+2l_1)-k} \langle z\rangle^{m_p-2l_1}}{(1+|y|^2)^{l_2}} \\
&\times \sum_{\substack{|\alpha_1|=l_1 \\ (\alpha',\beta_1)\leq(\alpha,\alpha_{1,0}) \\ |\alpha_2|=l_2 \\ \beta_2\leq\alpha_{2,0} \\ \beta_2'\leq\beta_2}} \binom{l_1}{\alpha_1}\binom{\alpha_{1,0}}{\beta_1}\binom{l_2}{\alpha_2}\binom{\alpha_{2,0}}{\beta_2}\binom{\beta_2}{\beta_2'} C_{3.4;(\alpha',\beta_1),\beta_2',\Omega} C_{(\alpha-\alpha',\alpha_{1,0}-\beta_1,\beta-\beta_2'),k,f,p,\chi} C^{\text{ek}}_{(\alpha_{2,0}-\beta_2),-2l_1},
\end{aligned} \tag{4.43}$$

so if we take

$$\begin{aligned}&C_{l_1,l_2,\alpha,k,f,p,\chi,\Omega}\\ &:=\\ &\sqrt{2}\sum_{\substack{|\alpha_1|=l_1\\(\alpha',\beta_1)\leq(\alpha,\alpha_{1,0})\\|\alpha_2|=l_2\\\beta_2\leq\alpha_{2,0}\\\beta_2'\leq\beta_2}}\binom{l_1}{\alpha_1}\binom{\alpha_{1,0}}{\beta_1}\binom{l_2}{\alpha_2}\binom{\alpha_{2,0}}{\beta_2}\binom{\beta_2}{\beta_2'}C_{3.4;(\alpha',\beta_1),\beta_2',\Omega}C_{(\alpha-\alpha',\alpha_{1,0}-\beta_1,\beta-\beta_2'),k,f,p,\chi}C^{\text{ek}}_{(\alpha_{2,0}-\beta_2),-2l_1},\end{aligned}\tag{4.44}$$

we get the estimate

$$\begin{aligned}&\left|D_x^\alpha\frac{(1+\nabla_z^2)^{l_2}}{(1+|\hbar^{-1}y|^2)^{l_2}}\frac{(1+\nabla_y^2)^{l_1}}{(1+|\hbar^{-1}z|^2)^{l_1}}e^{-(t-s)i\hbar^{-1}\delta(x,y,z)}h_\epsilon(x,y,z)\right|\\ &\leq\\ &\langle x\rangle^{m_V(|\alpha|+2l_1)+|m_p|+k}\langle y\rangle^{|m_V|(|\alpha|+2l_1)-k-2l_2}\langle z\rangle^{m_p-2l_1}C_{l_1,l_2,\alpha,k,f,p,\chi,\Omega}.\end{aligned}\tag{4.45}$$

Remembering that l_1, l_2 where taken arbitrary, for α, k fixed if we then fix l_1 such that $m_p - 2l_1 < -n$ and then fix l_2 such that $|m_V|(|\alpha| + 2l_1) - k - 2l_2 < -n$ then, by Fubini's theorem and proposition 1.3, for each $x \in \mathbb{R}^n$ fixed, the function

$$\hbar^{-2n}\langle x\rangle^{m_V(|\alpha|+l_1+2n+1)+k}\langle y\rangle^{|m_V|(|\alpha|+l_1+2n)-k-2l_2}\langle z\rangle^{m_p-(2l_1+2n)}C_{l_1,l_2,\alpha,k,f,p,\chi,\Omega}\tag{4.46}$$

is Lebesgue integrable in $\mathbb{R}^{2n}$ with respect to (y,z). So first fixing l_1, l_2 for the case $\alpha = 0$, and since the estimate is independent of $1 > \epsilon > 0$, then we can use Lebesgue dominated convergence theorem) to interchange the limit with the integral so we obtain

$$\begin{aligned}&(\hat{p}(s,t)f)(x)\\ &=\\ &\frac{1}{(2\pi\hbar)^n}\mathrm{Os}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(i\hbar^{-1}\mathcal{S}(s,t,x,y,z))\right)p(x,y,z)f(y)dydz\\ &=\\ &\lim_{\epsilon\to0}(\hat{p}_\epsilon(s,t)f)(x)\\ &=\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Bigg(e^{-i\hbar^{-1}z\cdot y}\Bigg(\frac{(1+\nabla_z^2)^{l_2}}{(1+|\hbar^{-1}y|^2)^{l_2}}\frac{(1+\nabla_y^2)^{l_1}}{(1+|\hbar^{-1}z|^2)^{l_1}}e^{-(t-s)i\hbar^{-1}\delta(x,y,z)}\\ &\times p(x,y+x,z)f(y+x)\Bigg)\Bigg)dydz,\end{aligned}\tag{4.47}$$

remembering from lemma 1.12 that $\chi_\epsilon \to 1$ pointwise and their derivatives converge to 0 uniformly. So equation (4.47) shows us that we can represent our oscillatory integral as a proper Lebesgue integral. But since the estimate is independent of (s,t) it follows from theorem 1.5 that this function is continuous in (s,t). Now, to see that this function is smooth, we consider the estimate 4.45 for $\alpha \neq 0$ fixed. We again define l_1', l_2' sufficiently big so that 4.46 is Lebesgue integrable for our α and we take $l_1^* = \max\{l_1, l_1'\}, l_2^* = \max\{l_2, l_2'\}$ so we first have that.

$$
\begin{aligned}
&(\hat{p}(s,t)f)(x) \\
&\qquad = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(e^{-i\hbar^{-1} z \cdot y} \left(\frac{(1+\nabla_z^2)^{l_2^*}}{(1+|\hbar^{-1}y|^2)^{l_2^*}} \frac{(1+\nabla_y^2)^{l_1^*}}{(1+|\hbar^{-1}z|^2)^{l_1^*}} e^{-(t-s)i\hbar^{-1}\delta(x,y,z)} \right.\right. \\
&\qquad \left.\left. \times p(x, y+x, z) f(y+x) \right)\right) dydz. \qquad (4.48)
\end{aligned}
$$

But since the derivatives of the integrand are also bounded by 4.46 by the dominated convergence theorem, it then follows again from theorem 1.5 that the derivatives of the function exist and it is given by

$$
\begin{aligned}
&D_x^\alpha (\hat{p}(s,t)f)(x) \qquad (4.49) \\
&\qquad = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} D_x^\alpha \left(e^{-i\hbar^{-1} z \cdot y} \left(\frac{(1+\nabla_z^2)^{l_2^*}}{(1+|\hbar^{-1}y|^2)^{l_2^*}} \frac{(1+\nabla_y^2)^{l_1^*}}{(1+|\hbar^{-1}z|^2)^{l_1^*}} e^{-(t-s)i\hbar^{-1}\delta(x,y,z)} \right.\right. \\
&\qquad \left.\left. \times p(x, y+x, z) f(y+x) \right)\right) dydz. \qquad (4.50)
\end{aligned}
$$

and for a fixed α, since again, the bounds are independent of s, t then it also follows the above expression is continuous. This finishes the proof. □

Remark. *In this proof, we not only gave an argument for the smoothness of the fundamental operator, we also gave an explicit representation of the oscillatory integral as a Lebesgue integrable function by repeatedly integrating by parts. We also have an explicit representation of the derivatives using this technique but a higher order representation is needed for each multi-index* $\alpha \in \mathbb{N}$

Remark. *We emphasise that this proof is much simpler that the usual proof when considering non-relativistic particle over configuration space. This can be seen from the simplicity of the kinetic term which is also continuous with respect to* t, s *which allows us to apply operators (4.32) to the Fourier phase and integrate by parts. So it appears that for this model the phase space path integral is better behaved in this regard.*

5 Discussion and Further work

In this work we proposed a definition for the fundamental operator for the phase space Feynman path integral for the pseudo-relativistic Schrödinger equation. Since the path integral formulation has been used to explain many peculiarities of the model, a mathematically rigorous formulation of this development is of theoretical importance. Theorem 4.6, which is the main result of this thesis is the first step on giving a full construction of the Feynman path integral through W. Ichinose's time slicing approach and surprisingly, by formulating everything in phase space, the proof of the theorem simplifies significantly due to the particular form of the kinetic energy term.

With this result in place, it is now easier to move into the next steps on the construction of the Feynman path integral. The next step would be to investigate boundedness of the fundamental operator in L^2 and some Sobolev-space norms. However we mention that due to the derivatives of the Kinetic term being bounded, then for potentials in $\mathcal{A}^0$ we immediately get L^2 boundedness from the Calderon-Vaillancourt theorem ([27] p.79 thm 4.1).

After that the next step would be to apply the quantized Hamiltonian to the fundamental operator to get some stability results. Since in this case the Hamiltonian is a non-local pseudo-differential operator, further investigation is warranted since the typical techniques used for this step may not apply directly. But after this we could potentially have all the elements to prove convergence of the sequence of operators to the time evolution of the corresponding quantum system.

It is also of great interest to investigate this model with Fujiwara's approach and get an explicit construction of the semi-classical approximation which could give a direct argument as to why this system is not fully relativistic.

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