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Binary Expansions of the Reciprocal of Prime Numbers and Polish Spaces for Bounded Variation Functions.

TESIS

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Contents

Chapter 1

Introduction

1.0.1 Introduction

This thesis includes two topics. The first topic is about binary expansions of prime reciprocals. We seek to obtain information about the prime numbers based on the binary expansion of their reciprocals. We obtained interesting results that are related to the Fermat numbers.

Prime numbers have been studied since the beginning of mathematics. Euclid in his work *Elements* circa 300 BC, showed that there are an infinite number of them. Many great mathematicians have worked with them, such as Euclid, Bertran, Legendre, Riemann see [12], Fermat, Leibnitz, Wiles see [9], Wilson, Lagrange see [31], Oppermann [29], Rosser [34], among others. And also, there are many conjectures about these numbers, such as the conjecture that *there are an infinite number of Mersenne primes*. A *Mersenne Prime* is a prime number of the form $M_n = 2^n - 1$ for some integer n. They are named after Marin Mersenne, a French Minim friar, who studied them in the early 17th century. There are many more open conjectures, such as Andrica's conjecture [1], Goldbach's conjecture, Brocard's conjecture see [32], [31], Artin's conjecture see [2] and [19], among others. See also [16], [18], [21], [25] and [33].

In Section 2.1 we provide some tools to prove Zsigmondy's Theorem, see [42]. This Theorem proves that for every positive integer $n \neq 6$, there exists at least one prime divisor of $2^n - 1$ that does not divide $2^s - 1$ for every $s < n$.

In Section 2.2 we explore binary expansion for reciprocal primes. In addition, we work with the sieve given by Zsigmondy's Theorem.

In Section 2.3 we work with the last digit of the new prime numbers that appear using the binary sieve. We find an interesting distribution in the last digit.

In Section 2.4 we worked with antisymmetric numbers. Let r be a positive integer, then r is called an antisymmetric number of size $m \in \mathbf{N}$ if $1/r$ has a binary expansion with period $2m$ and for each $i \in \{1, 2, \ldots, m\}$ the terms i and the $i+m$ of the binary expansion add up to 1. Furthermore, we relate these numbers to the Fermat numbers.

In Section 2.5 we include the tables that helped us in our analysis.
The new results are Theorem 2.3.1, Lemma 2.4.1, and Lemma 2.4.1.1, which we published in article *Binary expansions of prime reciprocals*, see [26].

The second topic is about the supremum metric and the total variation metric. We use a modification of the Skorohod metric to define a new metric. Our metric preserves the fractal properties of the functions and the concept of independence. In another section we study the different definitions of total variation for higher dimensions and we propose an alternative that we consider more appropriate to the unvaried definition.

In Section 3.1 we put the definitions and well-known results about the total variation.

In Section 3.2, we introduce the simplest metric generated by the total variation, $d_{\mathrm{TV}_a^b}$. We prove that $\left(BV([a,b]), d_{\mathrm{TV}_a^b}\right)$ is not a separable space. Additionally, we provide some relevant examples that motivate us to work with a metric involving total variation.

In Section 3.3, we introduce the concept of Absolutely Continuous functions and we add some important properties of these functions.

In Section 3.4, we propose a modification for the metric seen in Section 3.2. This modification is similar to the one suggested by Skorohod. Using these new metric we do not obtain a complete space.

Section 3.5 presents our final version of the metric.

In Section 3.6, we prove that the metric in 3.5 is a complete metric.

Section 3.7 we prove also separability to obtain a Polish Space.
Finally, in Section 3.8 we discuss the different definitions of total variation in larger dimensions. Furthermore, we propose a definition that seems to be an adequate extension of the definition in dimension 1.

1.0.2 Introducción

Esta tesis incluye dos temas. El primer tema es sobre las expansiones binarias de recíprocos de primos. Obtuvimos información sobre los números primos basandonos en la expansión binaria de sus reciprocos. Además, obtuvimos resultados muy interesantes relacionados con los números de Fermat.

Los números primos han sido estudiados desde el comienzo de las matemáticas. Euclides en su obra *Elementos* alrededor del 300 a.C. demostró que hay una cantidad infinita de ellos. Muchos grandes matemáticos han trabajado con ellos, como Euclides, Bertran, Legendre, Riemann ver [12], Fermat, Leibnitz, Wiles ver [9], Wilson, Lagrange ver [31], Oppermann [29], Rosser [34], entre otros. También hay muchas conjeturas acerca de estos números, como la conjetura de que *hay una cantidad infinita de primos de Mersenne*, un *primo de Mersenne* es un número primo de la forma $M_n = 2^n - 1$ para algún entero n. Reciben su nombre en honor a Marin Mersenne, un fraile francés, que los estudió a principios del siglo XVII. Hay muchas otras conjeturas abiertas, como la conjetura de Andrica [1], la conjetura de Goldbach, la conjetura de Brocard ver [32], [31], la conjetura de Artin ver [2] y [19], entre otras. Ver también [16], [18], [21], [25] y [33].

En la Sección 2.1 proporcionamos algunas herramientas para la demostración del Teorema de Zsigmondy, ver [42]. Este teorema demuestra que para cada entero positivo $n \neq 6$, existe al menos un divisor primo de $2^n - 1$ que no divide a $2^s - 1$ para cada $s < n$.

En la Sección 2.2 exploramos la expansión binaria para los recíprocos de primos. Además, trabajamos con la criba dada por el Teorema de Zsigmondy.

En la Sección 2.3 trabajamos con el último dígito de los nuevos números primos utilizando la criba binaria. Encontramos una distribución interesante en el último dígito.

En la Sección 2.4 trabajamos con números antisimétricos. Sea r un entero positivo, entonces r se llama un número antisimétrico de tamaño $m \in \mathbf{N}$ si $1/r$ tiene una expansión binaria con período $2m$ y para cada $i \in \{1, 2, \ldots, m\}$ los términos i y $i + m$ de la expansión binaria suman 1. Además, relacionamos estos números con los números de Fermat.

En la Sección 2.5 incluimos las tablas que nos ayudaron en nuestro análisis.

Los resultados nuevos son el Teorema 2.3.1, Lema 2.4.1, y Lema 2.4.1.1, los cuales publicamos en el articulo *Binary expansions of prime reciprocals*, ver [26].

El segundo tema trata sobre la métrica supremo y la métrica de variación total. Utilizamos una modificación de la métrica de Skorohod para definir una nueva métrica. Nuestra métrica conserva las propiedades fractales de las funciones y el concepto de independencia. En la última sección estudiamos las diferentes definiciones de variación total para dimensiones superiores y proponemos una alternativa que consideramos más apropiada a la definición univariada.

En la Sección 3.1 presentamos las definiciones y resultados conocidos sobre la variación total.

En la Sección 3.2, introducimos la métrica más simple generada por la variación total, $d_{\mathrm{TV}_a^b}$. Demostramos que $\left(BV([a,b]), d_{\mathrm{TV}_a^b}\right)$ no es un espacio separable. Además, proporcionamos algunos ejemplos relevantes que nos motivan a trabajar con una métrica que involucra variación total.

En la Sección 3.3, introducimos el concepto de funciones absolutamente continuas y agregamos algunas propiedades importantes de estas funciones.

En la Sección 3.4, proponemos una modificación para la métrica vista en la Sección 3.2. Esta modificación es similar a la sugerida por Skorohod. Usando esta nueva métrica no obtenemos un espacio completo.

La Sección 3.5 presenta nuestra versión final de la métrica.

En la Sección 3.6, demostramos que la métrica en 3.5 es una métrica completa.

La Sección 3.7 demuestra también la separabilidad obteniendo un Espacio Polaco.

Finalmente, en la Sección 3.8 discutimos las diferentes definiciones de variación total en dimensiones mayores. Además, proponemos una definición que parece ser una extensión adecuada de la definición en dimensión 1.

Chapter 2

Primes

2.1 Zsigmondy's Theorem

We want to prove that for every $n \in \mathbf{N} \setminus \{6\}$, where $\mathbf{N}$ is the set of that all positive integers, $2^n - 1$ has a prime divisor p such that p does not divide $2^k - 1$ with $1 \leq k < n$, except for $n = 6$. The first person to prove this result was A. S. Bang in 1886 [5], [6]. In 1892 Zsigmondy proved a more general result [42] which we present below. The proofs of this section are based on [8].

Definition 2.1.1 Let $a > b \geq 1$ be positive integers. And let us consider $\{x_n\}_{n\geq 1}$ the sequence of integers such that $x_i = a^i - b^i$ for every $i \in \mathbf{N}$. Let $n \in \mathbf{N}$, if p is a prime number such that p is a divisor of $a^n - b^n$ and p does not divide $a^k - b^k$ for every $1 \leq k < n$ then we call p a **primitive prime divisor** of $a^n - b^n$, see [40].

Remark 2.1.2 Recall that if a, b are integers, we say that a and b are **coprime integers** if and only if the greatest common divisor of a and b is 1. That is, if the only positive common divisor of a and b is 1. The greatest common divisor of a and b is denoted by $gcd(a, b)$.

Zsigmondy's Theorem 2.1.3 (1892) *Let $a > b \geq 1$ be coprime integers and let $n \geq 2$. Then there exists a primitive prime divisor of $a^n - b^n$, except when:*

- *$n = 2$ and $a + b$ is a power of 2; or*
- *$a = 2$, $b = 1$ and $n = 6$.*

Observe that the case $n = 2$ is easy to prove. In fact, if $a^2 - b^2 = (a+b)(a-b)$ has no primitive prime divisors then every prime p that divides $a+b$ must divide $a^2 - b^2$ and thus must divide $a-b$ (otherwise it would be a primitive prime divisor). Then if p is a prime that $p|a+b$ we have that $p|(a+b)-(a-b) = 2b$ and $p|(a+b)+(a-b) = 2a$.
Thus, $p = 2$ because a and b are coprime. This result tells us that the only prime number that divides $a+b$ is 2. Hence, $a+b$ is a power of 2.

Lifting the Exponent Lemma

For p a prime number, let $\nu_p\colon \mathbf{N} \to \mathbf{N} \cup \{0\}$ the function such that $\nu_p(n)$ denotes the exponent of p in the prime factorization of n. For example, $\nu_2(1) = 0$, $\nu_2(2) = 1$, $\nu_2(3) = 0$ and $\nu_2(4) = 2$.
Remark 2.1.4 Let us remember that for $r \in \mathbf{N}$ we consider the equivalence relation $\sim_r$ on $\mathbb{Z}$ given by: $x \sim_r y$ if and only if $r|x-y$. Then $\mathbb{Z}_r = \{[0]_r, [1]_r, \ldots, [r-1]_r\}$ are all different equivalence classes. Also $(\mathbb{Z}_r, +, \cdot)$ with

$$[x]_r + [y]_r := [x+y]_r \text{ and } [x]_r \cdot [y]_r := [x \cdot y]_r$$

is a field if and only if r is a prime number, see Proposition 3.19 [35]. When $x \sim_r y$ then we say that x **is congruent with** y **module** r and we denote it by $x \equiv y(\bmod r)$.

The following is the **Lifting The Exponent (LTE)** lemma, for the proof see [30].
Lemma 2.1.5 (LTE) *Let p be a prime number and let $x, y \in \mathbb{Z}$ and $m \in \boldsymbol{N}$ such that $x \equiv y(\bmod p)$ and $y, x \not\equiv 0(\bmod p)$.*

1. *If $p \geq 3$, then*

$$\nu_p(x^m - y^m) = \nu_p(x-y) + \nu_p(m). \tag{2.1}$$

2. *If $p = 2$, then*

$$\nu_2(x^m - y^m) = \begin{cases} \nu_2(x^2 - y^2) + \nu_2\left(\frac{m}{2}\right) & \text{if} \quad m \text{ is even,} \\ \nu_2(x-y) & \text{if} \quad m \text{ is odd.} \end{cases} \tag{2.2}$$

Remark 2.1.6 Let p be a prime number and let us consider the equivalence classes given in Remark 2.1.4. If $\mathbb{Z}_p^* = \{[1]_p, \ldots, [p-1]_p\}$ then $\left(\mathbb{Z}_p^*, \cdot\right)$ is the group of units of $\mathbb{Z}_p$ (with the product of Remark 2.1.4) and it has $p-1$ elements. For every $[s]_p \in \mathbb{Z}_p^*$, recall that if $m = order([s]_p)$ then m is the smallest natural number such that $[s]_p^m = [1]_p$ and m divides $\left|\mathbb{Z}_p^*\right| = p-1$. See Lagrange's Theorem 2.81 and Proposition 2.72 [35]. Also, $[s]_p^n = [1]_p$ if and only if $m|n$, see Lemma 2.53 [35].

2.2 Binary Expansions

We start this subsection with a Lemma which may be known, but we have not found its proof, so we have included it. First, we will introduce some notation. For every $n \in \mathbf{N}$, let D_n be the set

$$D_n = \{r \in \mathbf{N} \mid r|n \text{ and } r \neq 1, n\}.$$

We observe that D_n is the empty set if $n = 1$ or n is a prime number. For every $n, m \in \mathbf{N}$ such that $1 \leq m < n$ and n is not a prime number, let us consider

$$\gamma_m = \sum_{i=0}^{\lfloor \frac{n}{m} \rfloor - 1} 2^{im} \text{ and } \Gamma_n = \{l \in \mathbf{N} \mid \gamma_m \text{ does not divide } l \text{ for every } m \in D_n\}.$$

The next Lemma should be known but we could not find any reference for it.

Lemma 2.2.1 *Let q be a rational number in the unit closed interval $I = [0,1]$. Then the binary expansion of q takes the form*

$$q = 0.a_1a_2\cdots a_m\overline{b_1b_2\cdots b_n} \tag{2.3}$$

where $a_1, \ldots, a_m, b_1, \ldots, b_n \in \{0,1\}$, $m \in \boldsymbol{N} \cup \{0\}$ and $n \in \boldsymbol{N}$. The overlined terms is the periodic part of the number q, which includes zeros and ones if $n > 1$, and n is the size of the shortest period. Then

i) $m = 0$ and $n = 1$ if and only if $q = 0$ or $q = 1$.

ii) $m = 0$ and $n > 1$ if and only if $q = \frac{l}{2^n-1}$ for some integer $1 \leq l \leq 2^n - 2$ such that $l \in \Gamma_n$.

iii) $m \geq 1$ and $n = 1$ if and only if $q = \frac{l}{2^m}$ for some odd integer $1 \leq l \leq 2^m - 1$.

iv) If $m \geq 1$ *and* $n > 1$ *then* $q = \frac{l}{2^m(2^n-1)}$ *for some integer* $1 \leq l \leq 2^m(2^n-1)-1$.

Proof. Assume that $q \in I$ is a rational number with binary expansion given in equation (2.3).

i) If $m = 0$ and $n = 1$, then from (2.3) and using geometric series, we have that $q = 0.\overline{0} = 0$ or $q = 0.\overline{1} = \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1/2}{1-(1/2)} = 1$. The converse is clear.

ii) If $m = 0$ and $n > 1$ then from (2.3) we have that $q = 0.\overline{b_1 b_2 \cdots b_n}$, first we observe that $\overline{b_1 b_2 \cdots b_n} \neq \overline{11 \cdots 1}$ and $\overline{b_1 b_2 \cdots b_n} \neq \overline{00 \cdots 0}$, since we assumed the the periodic part is always the shortest possible. Therefore, the smallest periodic part would be $q = 0.\overline{00 \cdots 01}$, where the ones are located at the position $n, 2n, 3n, \ldots, kn, \ldots$ and the largest periodic part would be $q = \overline{11 \cdots 10}$, where the zeros are located at the positions $n, 2n, 3n, \ldots kn, \ldots$ In the first case

$$q = 0.\overline{00 \cdots 01} = \sum_{k=1}^{\infty} \frac{1}{2^{n \cdot k}} = \frac{\frac{1}{2^n}}{1 - \frac{1}{2^n}} = \frac{1}{2^n - 1}. \tag{2.4}$$

In the second case, using a finite number of geometric series we have that the largest value of q is

$$\begin{aligned} q &= 0.\overline{11 \cdots 10} \\ &= \sum_{k=0}^{\infty} \frac{1}{2^{n \cdot k+1}} + \sum_{k=0}^{\infty} \frac{1}{2^{n \cdot k+2}} + \cdots + \sum_{k=0}^{\infty} \frac{1}{2^{n \cdot k+(n-1)}} \\ &= \sum_{k=0}^{\infty} \frac{1}{2^{n \cdot k}} \left[\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} \right] \\ &= \frac{1}{1 - \frac{1}{2^n}} \left[\frac{\frac{1}{2} - \frac{1}{2^n}}{\frac{1}{2}} \right] \\ &= \frac{2^n}{2^n - 1} \left[\frac{2^{n-1} - 1}{2^{n-1}} \right] \\ &= \frac{2^n - 2}{2^n - 1}. \end{aligned} \tag{2.5}$$

In general, for $0.\overline{b_1 b_2 \cdots b_n}$ with $S \subseteq \{1, 2, \ldots n\}$ such that $b_i = 1$ for every $i \in S$ and $b_i = 0$ for every $i \in \{1, 2, \ldots, n\} \setminus S$. We can write $S = \{u_1, u_2, \ldots, u_t\}$ with $u_1 < u_2 < \cdots < u_t$ and $1 \leq t < n$. Then

$$\begin{aligned}
q &= 0.\overline{b_1 b_2 \cdots b_n} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{n\cdot k+u_1}} + \sum_{k=0}^{\infty} \frac{1}{2^{n\cdot k+u_2}} + \cdots + \sum_{k=0}^{\infty} \frac{1}{2^{n\cdot k+u_t}} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{n\cdot k}} \left[\frac{1}{2^{u_1}} + \frac{1}{2^{u_2}} + \ldots + \frac{1}{2^{u_t}} \right] \\
&= \frac{1}{1-\frac{1}{2^n}} \left[\frac{2^{n-u_1}}{2^n} + \frac{2^{n-u_2}}{2^n} + \cdots + \frac{2^{n-u_t}}{2^n} \right] \\
&= \frac{2^n}{2^n-1} \left[\frac{\sum_{i=1}^{t} 2^{n-u_i}}{2^n} \right] \\
&= \frac{\sum_{i=1}^{t} 2^{n-u_i}}{2^n-1}. \qquad (2.6)
\end{aligned}$$

Take $l = \sum_{i=1}^{t} 2^{n-u_i}$, note that

$$1 \leq \sum_{i=1}^{t} 2^{n-u_i} < \sum_{i=0}^{n-1} 2^i = \frac{1-2^n}{1-2} = 2^n - 1. \qquad (2.7)$$

Thus, $1 \leq l \leq 2^n - 2$ and by equation (2.27), $q = \frac{l}{2^n-1}$. If $l \notin \Gamma_n$ then there exists $g|n$, $g \neq n, 1$ such that

$$\sum_{i=0}^{\left\lfloor \frac{n}{g} \right\rfloor - 1} 2^{ig} = \gamma_g | l \qquad (2.8)$$

with $\left\lfloor \frac{n}{g} \right\rfloor = \frac{n}{g} = w$ for some $w \in \mathbf{N}$. We have that

$$\gamma_g = \sum_{i=0}^{\left\lfloor \frac{n}{g} \right\rfloor - 1} 2^{ig} = \sum_{i=0}^{w-1} 2^{ig} = \sum_{i=0}^{w-1} (2^g)^i = \frac{1-2^{gw}}{1-2^g} = \frac{2^n-1}{2^g-1}. \qquad (2.9)$$

And by equation (2.8), $l = \gamma_g z$ for some $z \in \mathbf{N}$, then

$$q = \frac{l}{2^n-1} = \frac{\gamma_g z}{\gamma_g(2^g-1)} = \frac{z}{2^g-1} < 1. \qquad (2.10)$$

Also, $z = \sum_{i\in J} 2^i$ such that $J \subsetneq \{0, 1, \dots, g-1\}$, because

$$\sum_{i=0}^{g-1} 2^i = \frac{1-2^g}{1-2} = 2^g - 1.$$

We can write $J = \{r_1, r_2, \dots, r_f\}$ such that $r_1 < r_2 < \cdots < r_f$. Take $c_1, \dots, c_g \in \{0, 1\}$ such that $c_i = 1$ if $i = g - j$ for $j \in J$ and $c_i's$ zero in other case,

$$\begin{aligned}
0.\overline{c_1 c_2 \cdots c_g} &= \sum_{k=0}^{\infty} \frac{1}{2^{g\cdot k+(g-r_f)}} + \sum_{k=0}^{\infty} \frac{1}{2^{g\cdot k+(g-r_{f-1})}} + \cdots + \sum_{k=0}^{\infty} \frac{1}{2^{g\cdot k+(g-r_1)}} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{g\cdot k}} \left[\frac{1}{2^{g-r_f}} + \frac{1}{2^{g-r_{f-1}}} + \ldots + \frac{1}{2^{g-r_1}} \right] \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{g\cdot k}} \left[\frac{2^{r_f}}{2^g} + \frac{2^{r_{f-1}}}{2^g} + \ldots + \frac{2^{r_1}}{2^g} \right] \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{g\cdot k}} \left[\frac{2^{r_f} + 2^{r_{f-1}} + \cdots + 2^{r_1}}{2^g} \right] = \sum_{k=0}^{\infty} \frac{1}{2^{g\cdot k}} \left[\frac{\sum_{i\in J} 2^i}{2^g} \right] \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{g\cdot k}} \left[\frac{z}{2^g} \right] = \frac{z}{2^g} \sum_{k=0}^{\infty} \frac{1}{2^{g\cdot k}} = \frac{z}{2^g} \left[\frac{1}{1-\frac{1}{2^g}} \right] = \frac{z}{2^g} \left[\frac{2^g}{2^g-1} \right] \\
&= \frac{z}{2^g-1}. \qquad (2.11)
\end{aligned}$$

Then n is not the smallest period of q, which is a contradiction. Therefore, $l \in \Gamma_n$.

Now, let $l \in \{1, 2, \dots, 2^n - 3, 2^n - 2\}$, then $l = \sum_{i\in H} 2^i$ for some $H \subsetneq \{0, 1, 2, \dots, n-1\}$. We can write the elements of H in the following form $H = \{r_1, r_2, \dots, r_w\}$ with $r_1 < r_2 < \cdots < r_w$. Let us define $b_i = 1$ for

$i = n - r_j$ for every $r_j \in H$ and for the rest $b'_j s$ zero.

$$\begin{aligned}
0.\overline{b_1 b_2 b_3 \cdots b_n} &= \sum_{k=0}^{\infty} \frac{1}{2^{n\cdot k+(n-r_w)}} + \sum_{k=0}^{\infty} \frac{1}{2^{n\cdot k+(n-r_{w-1})}} + \cdots + \sum_{k=0}^{\infty} \frac{1}{2^{n\cdot k+(n-r_1)}} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{n\cdot k}} \left[\frac{1}{2^{n-r_w}} + \frac{1}{2^{n-r_{w-1}}} + \ldots + \frac{1}{2^{n-r_1}} \right] \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{n\cdot k}} \left[\frac{2^{r_w}}{2^n} + \frac{2^{r_{w-1}}}{2^n} + \ldots + \frac{2^{r_1}}{2^n} \right] \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{n\cdot k}} \left[\frac{2^{r_w} + 2^{r_{w-1}} + \cdots + 2^{r_1}}{2^n} \right] = \sum_{k=0}^{\infty} \frac{1}{2^{n\cdot k}} \left[\frac{\sum_{i\in H} 2^i}{2^n} \right] \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{n\cdot k}} \left[\frac{l}{2^n} \right] = \frac{l}{2^n} \sum_{k=0}^{\infty} \frac{1}{2^{n\cdot k}} = \frac{l}{2^n} \left[\frac{1}{1 - \frac{1}{2^n}} \right] \\
&= \frac{l}{2^n} \left[\frac{2^n}{2^n - 1} \right] = \frac{l}{2^n - 1}. \qquad (2.12)
\end{aligned}$$

We need to see that n is the shortest period of q. We known that $q = 0.b_1 \cdots b_n b_1 \cdots b_n \cdots$ thus $m = 0$. If k is the shortest period of q with $k < n$, then

$$q = 0.\overline{c_1 \cdots c_k} = 0.b_1 \cdots b_n b_1 \cdots b_n b_1 \cdots = 0.\overline{b_1 \cdots b_n}. \qquad (2.13)$$

Therefore $c_1 \cdots c_k$ is part of $b_1 \cdots b_n$ and $k|n$. Also, $k \in D_n$. By the first section of the proof part *ii*) we have that there exists $1 \le r \le 2^k - 2$ such that $q = \frac{r}{2^k-1}$. Note that using that $k|n$ and $n = sk$ for some $s \in \mathbf{N}$,

$$\begin{aligned}
\gamma_k(2^k - 1) &= \left[\sum_{i=0}^{\lfloor \frac{n}{k} \rfloor - 1} 2^{ik} \right] (2^k - 1) = \left[\sum_{i=0}^{s-1} \left(2^k\right)^i \right] (2^k - 1) \\
&= \left[\frac{1 - 2^{sk}}{1 - 2^k} \right] (2^k - 1) = \frac{2^{sk} - 1}{2^k - 1} (2^k - 1) \\
&= 2^n - 1. \qquad (2.14)
\end{aligned}$$

Therefore

$$q = \frac{r}{2^k - 1} = \frac{l}{2^n - 1} = \frac{l}{\gamma_k(2^k - 1)} \qquad (2.15)$$

so, $l = r\gamma_k$ and $l \notin \Gamma_n$.
iii) If $m \geq 1$ and $n = 1$ then $q = 0.a_1a_2\cdots a_m\overline{0}$ with $a_m = 1$ or $q = 0.a_1a_2\cdots a_m\overline{1}$ with $a_m = 0$, that is, we have the two expansions of the same number. So, we can assume that $a_m = 1$ and then we have a tail of zeros. In this case we would have that

$$0.00\cdots 01 \leq q \leq 0.11\cdots 11$$

where the last one is in the position m. Then

$$\frac{1}{2^m} \leq q \leq \sum_{k=1}^{m} \frac{1}{2^k} = \frac{\frac{1}{2} - \frac{1}{2^{m+1}}}{\frac{1}{2}} = \frac{2^m - 1}{2^m}.$$

Also, $q = 0.a_1a_2\cdots a_m\overline{0}$ with $S \subseteq \{1, 2, \ldots, m\}$ such that $a_i = 1$ for every $i \in S$ and $a_i = 0$ for every $i \in \{1, 2, \ldots, m\} \setminus S$. Then $m \in S$, we can write $S = \{w_1, w_2, \ldots, w_r\}$ where $w_1 < w_2 < \cdots < w_r$ and $w_r = m$. We have that

$$q = a_1a_2\cdots a_m\overline{0} = \sum_{i=1}^{r} \frac{1}{2^{w_i}} = \sum_{i=1}^{r} \frac{2^{m-w_i}}{2^m} = \frac{1}{2^m}\sum_{i=1}^{r} 2^{m-w_i}. \quad (2.16)$$

If $l = \sum_{i=1}^{r} 2^{m-w_i}$ then $q = \frac{l}{2^m}$. l is an odd integer because $2^{m-w_r} = 1$ and the other addends are even, and

$$1 \leq l = \sum_{i=1}^{r} 2^{m-w_i} \leq \sum_{i=0}^{m-1} 2^i = \frac{1-2^m}{1-2} = 2^m - 1. \quad (2.17)$$

For the other implication, let l an odd integer such that $1 \leq l \leq 2^m - 1$. Then $l = \sum_{i \in H} 2^i$ with $0 \in H$ and $H \subseteq \{0, 1, \ldots, m-1\}$. Then

$$1 \leq l \leq \sum_{i=0}^{m-1} 2^i = \frac{1-2^m}{1-2} = 2^m - 1.$$

We can write $H = \{u_1, u_2, \ldots, u_t\}$ where $u_1 < u_2 < \cdots < u_t$ and $u_1 = 0$. Let $a_1, a_2, \ldots, a_m \in \{0, 1\}$ such that $a_i = 1$ if $i = m - j$ for some $j \in H$ and $a_i = 0$ in other case. So,

$$0.a_1a_2\cdots a_m\overline{0} = \sum_{i=1}^{t} \frac{1}{2^{m-u_i}} = \sum_{i=1}^{t} \frac{2^{u_i}}{2^m} = \frac{1}{2^m}\sum_{i=1}^{t} 2^{u_i} = \frac{l}{2^m} = q. \quad (2.18)$$

iv) If $m \geq 1$ and $n > 1$, then

$$\begin{aligned} q &= 0.a_1a_2\cdots a_m\overline{b_1b_2\cdots b_n} = 0.a_1a_2+\cdots a_m\overline{0} + 0.00\cdots 0\overline{b_1b_2\cdots b_n} \\ &= q_1+q_2 \end{aligned} \tag{2.19}$$

where q_2 has m zeros after the point and before the $b's$. Using *iii*) we know that

$$q_1 = \frac{k}{2^m}$$

for some k odd such that $1 \leq k \leq 2^m - 1$. For $b_1b_2\cdots b_n$ we take $S \subsetneq \{1,2,\ldots,n\}$, $S \neq \varnothing$ such that $b_i = 1$ for every $i \in S$ and $b_i = 0$ for every $i \in \{1,2,\ldots,n\} \setminus S$. We can write $S = \{u_1,u_2,\ldots,u_t\}$ such that $1 \leq u_1 < u_2 < \cdots < u_t \leq n$. Then

$$\begin{aligned} q_2 &= 0.0\cdots 0\overline{b_1\cdots b_n} = \sum_{j=0}^{\infty}\frac{1}{2^{m+nj+u_1}} + \cdots + \sum_{j=0}^{\infty}\frac{1}{2^{m+nj+u_t}} \\ &= \frac{1}{2^m}\sum_{j=1}^{\infty}\frac{1}{2^{nj}}\left[\frac{1}{2^{u_1}}+\frac{1}{2^{u_2}}+\cdots+\frac{1}{2^{u_t}}\right] \\ &= \frac{1}{2^m}\left[\frac{1}{1-\frac{1}{2^n}}\right]\left[\frac{2^{n-u_1}+2^{n-u_2}+\cdots+2^{n-u_t}}{2^n}\right] \\ &= \frac{1}{2^m}\left[\frac{2^n}{2^n-1}\right]\frac{\sum_{i=1}^{t}2^{n-u_t}}{2^n} = \frac{\sum_{i=1}^{t}2^{n-u_t}}{2^m(2^n-1)}. \end{aligned} \tag{2.20}$$

If $h = \sum_{i=1}^{t} 2^{n-u_t}$, then

$$1 \leq h = \sum_{i=1}^{t}2^{n-u_t} < \sum_{i=0}^{n-1}2^i = \frac{1-2^n}{1-2} = 2^n - 1.$$

So, we have that $q_2 = \frac{h}{2^m(2^n-1)}$ with $1 \leq h \leq 2^n - 2$. And

$$q_1+q_2 = \frac{k}{2^m} + \frac{h}{2^m(2^n-1)} = \frac{k(2^n-1)+h}{2^m(2^n-1)}. \tag{2.21}$$

Take $l = k(2^n-1)+h$, $1 \leq k \leq 2^m - 1$ and $1 \leq h \leq 2^n - 2$, hence, $1 \leq l \leq 2^m(2^n-1)-1$ and $q = q_1 + q_2 = \frac{l}{2^m(2^n-1)}$.

□

Note that the converse of Lemma 2.2.1 part *iv*) is not true. For example, for $n = 2$ and $m = 1$, $1 \leq l \leq 2^m(2^n - 1) - 1 = 5$ and $2^m(2^n - 1) = 6$. With $\frac{1}{2} = \frac{3}{6} = 0.1\overline{0} = 0.0\overline{1}$, so for $l = 3$, $\frac{l}{2^m(2^n-1)}$ does not have binary expansion with $m = 1$ and $n = 2$. Remember that n must be the shortest possible period.

Remark 2.2.2 *Note that in part ii) of Lemma 2.2.1, for every* $l \in \{1, 2, \ldots, 2^n - 3, 2^n - 2\}$ *equation* (10) *holds, that is, for every* $l \in \{1, 2, \ldots, 2^n - 3, 2^n - 2\}$,

$$\frac{l}{2^n - 1} = 0.\overline{b_1 \cdots b_n}$$

for some $b_1, \ldots, b_n \in \{0, 1\}$. *But n* ***may not be the shortest period of*** $\frac{l}{2^n-1}$.

Now we state a very well known result, whose proof follows directly from Fermat's little Theorem, we can find the proof of this theorem in [35]. We will use the notation $q|p$ to denote that q divides p.

Lemma 2.2.3 *Let p be a prime number greater than* 2*, then* $p|(2^{p-1} - 1)$.

Proof. Fermat's little Theorem states that:

If p is a prime number and a is any integer such that p does not divide a, then p divides $a^{p-1} - 1$.

Therefore, if $p > 2$ is a prime number then p does not divide $a = 2$. Hence p divides $2^{p-1} - 1$. □

Notice that, the converse of Lemma 2.2.3 does not hold. For $q = 341$, q is not prime: $341 = 11 \cdot 31$. And we have that $q|\left(2^{q-1} - 1\right)$

$$\begin{aligned}
2^{340} - 1 &= 3 \cdot 5^2 \cdot \underline{11 \cdot 31} \cdot 41 \cdot 137 \cdot 953 \cdot 1021 \cdot 442126317 \cdot 43691 \cdot 131071 \cdot \\
&\quad 550801 \cdot 23650061 \cdot 7226904352843746841 \cdot \\
&\quad 9520972806333758431 \cdot 26831423036065352611 \\
&= 3 \cdot 5^2 \cdot \underline{341} \cdot 41 \cdot 137 \cdot 953 \cdot 1021 \cdot 442126317 \cdot 43691 \cdot 131071 \cdot \\
&\quad 550801 \cdot 23650061 \cdot 7226904352843746841 \cdot \\
&\quad 9520972806333758431 \cdot 26831423036065352611.
\end{aligned}$$

Corollary 2.2.4 *Let p be a prime number greater than* 2*. Then* $0 < \frac{1}{p} \leq \frac{1}{3}$ *and*

$$\frac{1}{p} = \frac{r}{2^{p-1} - 1} \tag{2.22}$$

for some integer $1 \leq r \leq 2^{p-1} - 3$. *Besides,* $\frac{1}{p}$ *has a binary expansion which satisfies*

$$\frac{1}{p} = 0.\overline{b_1 b_2 \cdots b_{p-1}}, \tag{2.23}$$

with $b_1 = 0$ *and* $b_{p-1} = 1$. *Note also that the period in* (2.23) *may not be the shortest one.*

Proof. If p is a prime number greater than 2, it is obvious that $0 < \frac{1}{p} \leq \frac{1}{3}$, and by Lemma 2.2.3 p divides $2^{p-1} - 1$, so there exists r an integer such that $p \cdot r = 2^{p-1} - 1$. Therefore, (2.22) follows, and by Remark 2.2.2 we have that $\frac{1}{p}$ has the binary expansion given by equation (2.23). Since $\frac{1}{p} \leq \frac{1}{3}$ then $b_1 = 0$ and since p is an odd integer then $b_{p-1} = 1$. The last note can be observed, for example when $p = 7$, in the next paragraph. □

For example if $p = 3$ then $2^{p-1} - 1 = 3$ and in this case $\frac{1}{3} = 0.\overline{01}$, if $p = 5$ then $2^{p-1} - 1 = 15 = 3 \cdot 5$ and in this case $\frac{1}{5} = 0.\overline{0011}$, if $p = 7$ then $2^{p-1} - 1 = 63 = 3 \cdot 3 \cdot 7$ and in this case $\frac{1}{7} = 0.\overline{001001}$, here we observe that $2^3 - 1 = 7$, that is why $\frac{1}{7}$ has a shorter period of only three numbers, that is, $\frac{1}{7} = 0.\overline{001}$, this last example motivates the definition given below. If $p = 11$ then $2^{p-1} - 1 = 1023 = 3 \cdot 11 \cdot 31$ and in this case $\frac{1}{11} = 0.\overline{0001011101}$. Note that $p = 3 = 2^2 - 1$, $p = 7 = 2^3 - 1$, $p = 31 = 2^5 - 1$ and $p = 127 = 2^7 - 1$ are Mersenne's primes, but $p = 2047 = 2^{11} - 1 = 23 \cdot 89$ is not a prime, but a composite number.

A natural order to generate prime numbers p is to consider the size of the shortest period of the binary expansion of $\frac{1}{p}$.

Definition 2.2.5 *Let p be a prime number, then we will say that p is a* **long prime** *if and only if p does not divide* $2^{q-1} - 1$ *for any* $q < p$, *in any other case we will say that p is a* **short prime**.

From the example above $p = 3$, $p = 5$ and $p = 11$ are long primes, but $p = 7$ and $p = 31$ are short primes, since 7 divides $2^3 - 1$ and 31 divides $2^5 - 1$. Of course, 31 also divides $2^{30} - 1 = 1,073,741,823 = 3 \cdot 3 \cdot 7 \cdot 11 \cdot 31 \cdot 151 \cdot 331$. We will use later on the above Definition to generate prime numbers using numbers of the form powers of two minus one. Let us prove another useful result.

Lemma 2.2.6 *Let* $p, q \in \boldsymbol{N}$ *such that* $q|p$. *Then* $(2^q - 1)|(2^p - 1)$.

Proof. Let us assume that $p, q \in \mathbf{N}$ such that $q|p$. Then there exists an integer $r \in \mathbf{N}$ such that $q \cdot r = p$. Define $k = \sum_{m=0}^{r-1}(2^q)^m$, then clearly k is an integer and using geometric sums

$$\begin{aligned}(2^q - 1)\left(\sum_{m=0}^{r-1}(2^q)^m\right) &= (2^q - 1)\frac{1-(2^q)^r}{1-2^q} \\ &= (2^q - 1)\frac{(2^q)^r - 1}{2^q - 1} \\ &= 2^p - 1. \end{aligned} \tag{2.24}$$

Therefore, $(2^q - 1)|(2^p - 1)$. □

Since $3|6$ then $7 = 2^3 - 1|2^6 - 1 = 63 = 3 \cdot 3 \cdot 7$, so 7 is a short prime, and since $2|6$ then $3 = 2^2 - 1|2^6 - 1 = 63 = 3 \cdot 3 \cdot 7$. We have seen that the case $2^6 - 1$ is an interesting exceptional case, when we consider all the numbers of the form $2^n - 1$, for any integer $n \geq 2$, see Zsigmondy's Theorem 2.1.3.

In Tables 2.3 we found the value of $2^n - 1$ for $2 \leq n \leq 75$ we give the prime decomposition of $2^n - 1$ **underlining** the new primes, which have not been found previously. And for $76 \leq n \leq 100$ we only provided of decompositions. The underlined primes will be of great importance in the interpretation of this Tables, and they will also help in finding the prime decomposition of the numbers $2^n - 1$ when n is not a prime number. We will also observe how to find the short primes when we evaluate the prime decompositions of the numbers $2^n - 1$ when n varies from 2 up to N for $N \leq 100$.

First, we note that from Lemma 2.2.3, if n is a prime greater than 2, then $n|(2^{n-1} - 1)$. So, if we find the prime decomposition of $2^m - 1$ for every $m \in \mathbf{N}$, then for every prime p greater that 2 we will find an $m \in \mathbf{N}$, such that $p|2^m - 1$, of course this holds for $m = p - 1$.

Let us assume that we are trying to find the prime decomposition of $2^n - 1$ when n is not a prime number. If n is not too large, it is possible to find its prime decomposition using for example the package Mathematica, which by the way has an amazing range to perform this task. Let us assume that $q_1 \leq q_2 \leq \cdots \leq q_{k-1} \leq q_k$ are the prime numbers such that

$$n = q_1 \cdot q_2 \cdot \cdots q_{k-1} \cdot q_k \quad \text{where} \quad k \in \mathbf{N}, \tag{2.25}$$

where (2.25) is of course the prime decomposition of n. Let

$$r_1 < r_2 < r_3 < \cdot < r_{m-1} < r_m \tag{2.26}$$

be all the different divisors of n obtained by multiplying one or more primes given in equation (2.26), of course $r_m = n$. So, for example if $n = 40$, its prime decomposition is given by $n = 2 \cdot 2 \cdot 2 \cdot 5$, that is, $k = 4$, and the different divisors of n are $2 < 4 < 5 < 8 < 10 < 20 < 40$, so, $m = 7$.

Now, we observed that the **only value of** n, for $2 \leq n \leq 100$, such that the decomposition of $2^n - 1$ does not include a new prime in its prime decomposition, is when $\mathbf{n = 6}$, see Tables 2 to 4. We prove that this holds for every $n > 100$. We also observe that as n increases the number of new primes may also increase, from Tables 2 to 4, we observe that $2^{11} - 1$ includes for the first time two new primes, then $2^{29} - 1$ for the first time includes three new primes, and finally $2^{92} - 1$ includes four new primes for the first time. The last observations leave us a new conjecture.

We proved that for every $n \in \mathbf{N}$, with $n \neq 6$, there is a prime number p such that $1/p$ has binary expansion of size n.

We are going to use the Zsigmondy's Theorem seen in section 2.1, Theorem 2.1.3.

Remark 2.2.7 *Observe that if p is a primitive prime divisor for* $2^n - 1$ *with* $n \in \boldsymbol{N} \setminus \{6\}$, *then* $p|(2^n - 1)$ *and* $p = \frac{l}{2^n - 1}$ *for some* $1 \leq l \leq 2^n - 2$. *By Remark 2.2.2,*

$$\frac{1}{p} = 0.\overline{b_1 \cdots b_n} \text{ for some } b_1, \ldots, b_n \in \{0, 1\}.$$

If n is not the size of the period of $\frac{1}{p}$, *then let* $k < n$ *be the size of the period of* $\frac{1}{p}$. *By Lemma 2.2.1, part ii)* $\frac{1}{p} = \frac{s}{2^k - 1}$ *for some* $1 \leq s \leq 2^k - 2$ *with* $s \in \Gamma_k$. *So,* $ps = 2^k - 1$ *and* $p|(2^k - 1)$ *with* $k < n$. *This is a contradiction since p is a primitive prime divisor of* $2^n - 1$. *Thus, n is the size of the period of* $\frac{1}{p}$.

Zsigmondy's theorem, Theorem 2.1.3, gives us the following theorem:

Theorem 2.2.8 *For every integer* $n \geq 2$ *with* $n \neq 6$ *the prime decomposition of the number* $2^n - 1$ *includes at least a new prime* q_n *such that* q_n *does not divide* $2^m - 1$ *for every* $2 \leq m < n$.

Part *ii*) of the Zsigmondy's Theorem proves that $n = 6$ is the only exception to the existence of primitive prime divisors for $2^n - 1$.

It is noteworthy that the last Theorem is related to the fact that between any natural number n and $2 \cdot n$ there exists a prime number p, but actually it is quite stronger, because it states that for every $n \geq 2$ with $n \neq 5$, if we consider the list of all prime numbers that have appeared in the prime decompositions of $2^k - 1$ for every $2 \leq k \leq n$, then we can find at least one new prime number in the prime decomposition of $2^{n+1} - 1 = 2 \cdot (2^n - 1) + 1$. Of course, in this case the new prime number found does not need to be between $2^n - 1$ and $2^{n+1} - 1$.

Let us assume that we want to find the prime decomposition of $2^{40} - 1$. As we observed above the divisors less than $p = 40$ are all the numbers $q \in \{2, 4, 5, 8, 10, 20\}$. Then using Lemma 2.2.6 we have that $2^2 - 1 | 2^{40} - 1$, $2^4 - 1 | 2^{40} - 1$, $2^5 - 1 | 2^{40} - 1$, $2^8 - 1 | 2^{40} - 1$, $2^{10} - 1 | 2^{40} - 1$ and $2^{20} - 1 | 2^{40} - 1$. Observing Table 2.3 we have that $3, 5, 31, 17, 11, 41$ all divide $2^{40} - 1$. Then $2^{40} - 1 = 1099511627775$, so $\frac{2^{40}-1}{3 \cdot 5 \cdot 11 \cdot 17 \cdot 31 \cdot 41} = 308405$. So, it is clear that the last number is divisible by 5 again and $\frac{308405}{5} = 61681$ and in a table of primes we find that $r = 61681$ is a prime number, which has not appeared in the new sieve of primes up to $n = 39$, see Table 2.3.

Even if it is not reported here, we have obtained the equivalence of Table 2.3 for $n = 1206$, see [26]. In Table 2.4, we report for $n = 50, 100, 150, 200, 250, \ldots$ and $n = 1000$, the number of different primes obtained from $2^m - 1$ when $2 \leq m \leq n$. These values are somehow related to the well known function $\pi(n)$ which counts the number of primes less than or equal n, which by the way has no close formula and it has been suggested that it may not exist, due to the capricious distribution of the primes. However, a nice approximation of this function is given by $\pi(n) \sim \frac{n}{\ln(n)}$ for large values of n. The first result for $\pi(n)$ was given by Carl Friedrich Gauss, in 1793, see [9] and [14].

In Table 2.1 for $1 \leq m \leq 10$ we have the first n such that $2^n - 1$ has m new primes in its decomposition of primes numbers. Also, we have how many of the values of the $1 \leq k \leq n$, $2^k - 1$ includes one new prime, two new primes, and so on up to m new primes, and we have a new conjecture.

We also observe in Table 2.1 that $2^{113} - 1$ includes five new primes, $2^{223} - 1$ includes six new primes, $2^{295} - 1$ includes seven new primes, $2^{333} - 1$ includes eight new primes, $2^{397} - 1$ includes nine new primes and 2^{1076} includes ten new primes. Hence, we may conjecture that for any $m \in \mathbf{N}$ there exists a value of n such that $2^n - 1$ includes m new primes for the first time.

We obtained Table 2.1, Table 2.2, Table 2.3, Table 2.4 and Table 2.5 with the help of Wolfram Mathematica and [28]. Figure 2.1 includes the values of n such that for the first time appear m new primes in the binary expansion of $\frac{1}{2^n-1}$ and it is standardize to be a probability density function, see [3].

Samuel Yates defined an **unique-prime** to be a prime p such that the decimal expansion of $\frac{1}{p}$ has a period that it shares with no other prime, see [41]. In general for decimal expansions Chris Caldwell and Harvey Dubner defined **bi-unique-primes** to be pairs of primes which have a period shared by no other primes. In a similar way, they defined **tri-unique-primes** and so on, see [10]. The analogous concept for *binary expansions* can be found in Table 2.1.

In Table 2.1 the first column for $m = 1$ we have the total of unique primes for the binary expansion of $1/p$ from $2^n - 1$ varying n in the set $\{2, 11, 29, 92, 113, 223, 295, 333, 397, 1076\}$, which corresponds to first time that we obtain $m = 1, m = 2, \cdots, m = 10$ new primes for $2^n - 1$.

Of course, the second column for $m = 2$ includes the total number of bi-unique primes, for $m = 3$ the column includes the total number of tri-unique primes, etc.

Figure 2.1 is a graphic representation of the results of the rows in Table 2.1 standardized by the sum of the rows. Now we state our conjecture based on the results of Table 2.1.

Conjecture 2.2.9 *For every $m \in \boldsymbol{N}$, there exists an $n \in \boldsymbol{N}$ such that the number of primitive prime divisors of $2^n - 1$ is m.*

n \ m	1	2	3	4	5	6	7	8	9	10
2^2-1	1	0	0	0	0	0	0	0	0	0
$2^{11}-1$	8	1	0	0	0	0	0	0	0	0
$2^{29}-1$	22	4	1	0	0	0	0	0	0	0
$2^{92}-1$	44	31	14	1	0	0	0	0	0	0
$2^{113}-1$	47	42	20	1	1	0	0	0	0	0
$2^{223}-1$	65	80	52	17	6	1	0	0	0	0
$2^{295}-1$	69	105	72	32	11	3	1	0	0	0
$2^{333}-1$	71	114	85	41	13	5	1	1	0	0
$2^{397}-1$	77	126	105	55	21	7	1	2	1	0
$2^{1076}-1$	107	240	260	208	134	79	23	19	4	1

Table 2.1: First n such that 2^n-1 has m primitive prime divisors

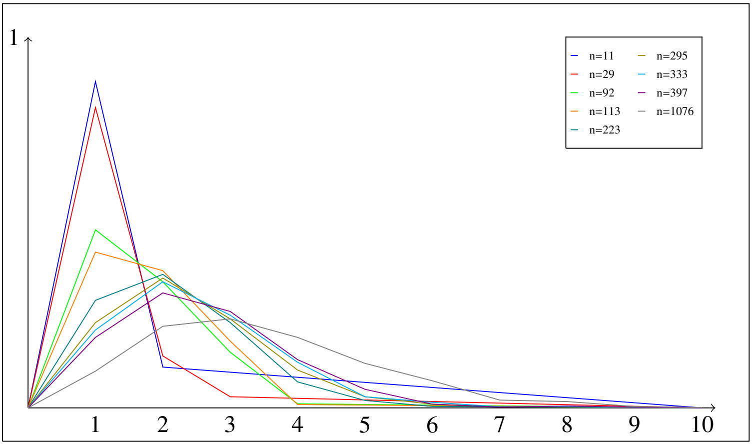

Figure 2.1: First n such that 2^n-1 has m primitive prime divisors

2.3 The Last digit

The Last digit of the new prime numbers using the binary sieve

In this subsection we present new results due to our research. Let p be a prime number and let us consider the equivalence class given in Remark 2.1.4. If $\mathbb{Z}_p^* = \{[1]_p, \ldots, [p-1]_p\}$ then $\left(\mathbb{Z}_p^*, \cdot\right)$ is the group of units of $\mathbb{Z}_p$ (with the product of Remark 2.1.4 and it has $p-1$ elements. For every $[s]_p \in \mathbb{Z}_p^*$, if $m = order([s]_p)$ then m is the smallest natural number such that $[s]_p^m = [1]_p$ and m divides $|\mathbb{Z}_p^*| = p-1$. See Lagrange's Theorem 2.81 and Proposition 2.72 in [35]. Also, $[s]_p^n = [1]_p$ if and only if $m|n$, see Lemma 2.53 in [35].

Note that if we want to see what is the last digit of an integer z, it is enough to see what is the remainder of dividing z by 10. That is, using Euclid's algorithm, we find $w \in \mathbb{Z}$ such that $z = 10w + r$ with $0 \leq r < 10$, this gives us that $z - r = 10w$ and $10|(z-r)$. So $z \equiv r(\mod 10)$ and r is the last digit in the decimal expansion of z.

Theorem 2.3.1 *If n is a multiple of 5, the last digit of the primitive prime divisors of $2^n - 1$ is always 1 in their decimal expansion.*

Proof. Let $n \in \mathbf{N}$ such that $n = 5k$ for some $k \in \mathbf{N}$. Let p be a primitive prime divisor of $2^n - 1$, so $p \neq 2$ and $p - 1$ is even. Thus $2|p-1$.

Also, $2^n \equiv 1(\mod p)$ and $n = order([2]_p)$. By Lemma 2.2.3, $p|(2^{p-1} - 1)$, that is, $2^{p-1} \equiv 1(\mod p)$. So, we have that $n|(p-1)$. Then, $2|(p-1)$ and $5k = n|(p-1)$. By the Fundamental Theorem of Arithmetic $10 = 5 \cdot 2|(p-1)$, that is, $p \equiv 1(\mod 10)$. The last digit of p is 1. □

In Figure 2.2 the graph shows the distribution of the last digit in the decimal expansions with the new order that we are considering taking up to $2^{1206} - 1$. Of course, the number 5 is the only prime whose last decimal digit is 5. Using [28] and R-studio we obtain that there are 1609 primes whose last decimal digit is 1, there are 879 primes whose last decimal digit is 3, there are 902 primes whose last decimal digit is 7 and finally, there are 884 primes whose last decimal digit is 9.

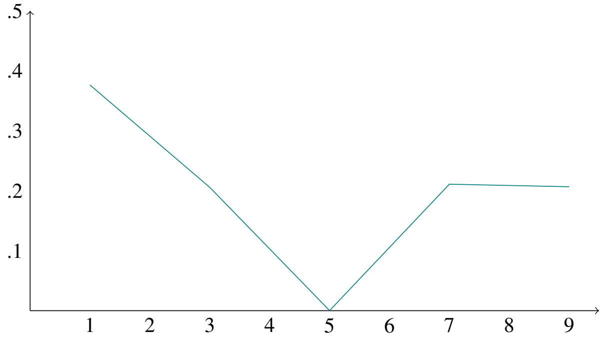

Figure 2.2: Graph of distribution in the last digit.

Open Question 2.3.2 *Why, in Figure 2.2 giving the last digit in the decimal expansions of primes in the new sieve, the number* 1 *appears almost twice more often than the digits* 3, 7 *and* 9 *?*

2.4 Antisymmetric numbers

Let r, m be positive integers, then r is called an **antisymmetric number of size** m if and only if $1/r$ has a **binary expansion with period of size 2m**, and the expansion is given by

$$\frac{1}{r} = 0.\overline{a_1 a_2 \cdots a_m \hat{a}_1 \hat{a}_2 \cdots \hat{a}_m}$$

for some $a_1, a_2, \ldots, a_m \in \{0, 1\}$ and $\hat{a}_i = 1 - a_i$ for every $i \in \{1, 2, \ldots, m\}$.

Observe that if r is an antisymmetric number of size m then the binary expansion of $1/r$ has a periodic part of even size, that is, $2m$.

The first idea of our antisymmetric numbers appeared first in [23], in a more restricted case. In the case of decimal expansions there is a similar result in the case of fractions with prime denominators first proved by E. Midy and generalized by A. Tripathi, see [24] and [39].
For every $m \geq 1$, let

$$S_m = \sum_{k=0}^{\infty} \frac{1}{(2^{2m})^k} = \frac{2^{2m}}{2^{2m} - 1}. \tag{2.27}$$

Let k be a positive integer such that for some integer $m \geq 1$,

$$\frac{1}{k} = 0.\overline{11\cdots 1\cdot 00\cdots 0} \tag{2.28}$$

where the last one is in the m^{th} position and it is followed by m consecutive zeros. Then k is an antisymmetric number of size m, in fact **the largest possible**, and

$$\frac{1}{k} = \frac{2^{2m-1} + 2^{2m-2} + \cdots + 2^{2m-m}}{2^{2m}} \sum_{k=0}^{\infty} \frac{1}{(2^{2m})^k} = \frac{2^m}{2^m+1}. \tag{2.29}$$

On the other hand, if k is the counterpart of equation (2.28), that is, an antysimmetric number of size m, such that

$$\frac{1}{k} = 0.\overline{00\cdots 0\cdot 11\cdots 1} \tag{2.30}$$

then $1/k$ is **the smallest possible** number with k antisymmetric of size m and

$$\frac{1}{k} = \frac{2^{m-1} + 2^{m-2} + \cdots 2^1 + 2^0}{2^{2m}} \sum_{k=0}^{\infty} \frac{1}{(2^{2m})^k} = \frac{1}{2^m+1}. \tag{2.31}$$

Lemma 2.4.1 *Let $k \geq 2$ be an integer. If k is antisymmetric of size m for some integer $m \geq 1$, then $\frac{1}{k} = \frac{l}{2^m+1}$ where l is an integer satisfying $1 \leq l \leq 2^m$. Furthermore, for every $l \in \{1, 2, \ldots, 2^m\}$, $\frac{l}{2^m+1}$ is antisymmetric of size less than or equal to m.*

Proof. Let $k \geq 2$ be an antisymmetric integer of size $m \in \mathbf{N}$, that is,

$$\frac{1}{k} = 0.\overline{a_1 \ldots a_m \cdot \hat{a}_1 \cdots \hat{a}_m}$$

where $\hat{a}_i = 1 - a_i$ for every $i \in \{1, 2, \ldots, m\} = M$. We define $J \subseteq M$ such that $a_i = 1$ for every $i \in J$ and $a_i = 0$ for every $i \in M \setminus J$. If $J = \emptyset$ then $M \setminus J = M$, which is the case given in equation (2.30), and by equation (2.31), $\frac{1}{k} = \frac{1}{2^m+1}$. If $J = M$ then $M \setminus J = \emptyset$, which is the case given in equation (2.28), and by equation (2.29), $\frac{1}{k} = \frac{2^m}{2^m+1}$.

So, assume that $\emptyset \subsetneq J \subsetneq M$ and let $J = \{u_1, \dots, u_s\}$ with $1 \le u_1 < \cdots < u_s \le m$ where $1 \le s < m$. And let $M \setminus J = \{v_1, \dots, v_r\}$ where $1 \le v_1 < \cdots < v_r \le m$ and $1 \le r < m$. Clearly $J \cap (M \setminus J) = \emptyset$, so $s + r = m$. Then

$$\begin{aligned} \frac{1}{k} &= 0.\overline{a_1 \dots a_m \cdot \hat{a}_1 \cdots \hat{a}_m} \\ &= \sum_{i=0}^{\infty} \frac{1}{2^{2mi+u_1}} + \cdots + \sum_{i=0}^{\infty} \frac{1}{2^{2mi+u_s}} + \sum_{i=0}^{\infty} \frac{1}{2^{2mi+m+v_1}} + \cdots + \sum_{i=0}^{\infty} \frac{1}{2^{2mi+m+v_r}} \\ &= \frac{1}{2^m + 1} \left[\sum_{j=1}^{s} 2^{m-u_j} + 1 \right]. \end{aligned}$$

If $l = \sum_{i=1}^{s} 2^{m-u_i} + 1$, then $1 \le l \le 2^m$.
For the converse, we have these observations:

i) For each $l \in \{1, \dots, 2^m - 1\}$, $l = \sum_{k \in \Omega} 2^k$ where $\Omega \subseteq \{0, \dots, m-1\}$ and $\Omega \neq \varnothing$.

ii) For each $l \in \{1, \dots, 2^m - 1\}$, $\frac{l+1}{2^m+1}$ has an antisymmetric binary expansion. In fact, let $l \in \{1, \dots, 2^m - 1\}$, then $l = \sum_{k \in J} 2^k$ with $J \subseteq \{0, \dots, m-1\} = N$. Then $J = \{i_1, \dots, i_r\}$ with $0 \le i_1 < \cdots < i_r \le m - 1$ for some $1 \le r \le m$. Observe that $S = m - J := \{m - i_r, \dots, m - i_1\} \subseteq \{1, \dots, m\} = M$.

   Let $a_i = 1$ for every $i \in S$, $a_i = 0$ for every $i \in M \setminus S$ and $\hat{a}_i = 1 - a_i$ for every $i \in M$.

   We note that $2^m - \sum_{j \in N \setminus J} 2^j = l + 1$ because $2^m - 1 = \sum_{k=0}^{m-1} 2^k$, so $2^m = \sum_{k=0}^{m-1} 2^k + 1$ and $2^m - \sum_{j \in N \setminus J} 2^j = \sum_{k=0}^{m-1} 2^k - \sum_{j \in N \setminus J} 2^j + 1 = \sum_{j \in J} 2^j + 1 = l + 1$. Then, using equation (2.27)

$$\begin{aligned} 0.\overline{a_1 \cdots a_m \cdot \hat{a}_1 \cdots \hat{a}_m} &= \frac{\sum_{k=0}^{2m-1} 2^k - \sum_{j \in J} 2^j - \sum_{j \in N \setminus J} 2^{m+j}}{2^{2m}} S_m \\ &= \frac{2^m \left(2^m - \sum_{j \in N \setminus J} 2^j\right) - (l+1)}{2^{2m} - 1} \\ &= \frac{(l+1)(2^m - 1)}{(2^m + 1)(2^m - 1)} = \frac{l+1}{2^m + 1}. \end{aligned}$$

Note that if $2m$ is not the shortest period of $\frac{l+1}{2^m+1}$, then any way it has an antisymmetric binary expansion.

iii) $\frac{1}{2^m+1}$ has an antisymmetric binary expansion of size m. See equation (2.30) and equation (2.31).

Let $m = 3$, $\frac{1}{k} = 0.\overline{a_1 a_2 a_3 \hat{a}_1 \hat{a}_2 \hat{a}_3} = 0.\overline{101010} = \frac{6}{2^3+1}$. But k is an antisymmetric number with size $m = 1$, since $\frac{1}{k} = 0.\overline{10} = \frac{2}{2^1+1}$. □

The last Lemma is new.

The following remark gives us a similar version of Midy's Theorem but with binary expansions, see [24].

Remark 2.4.2 *Let p be a prime number with period of size $2m$, that is, $\frac{1}{p} = 0.\overline{b_1 \cdots b_{2m}}$ with $m \geq 1$. Then p is an antisymmetric number of size m.*

Proof. Let p be a prime number such that $\frac{1}{p}$ has a binary expansion with period of size $2m$. Then $2m$ is the smallest number such that $p|(2^{2m} - 1)$. We have that $p|(2^{2m} - 1) = (2^m - 1)(2^m + 1)$. Using properties of prime numbers we have that $p|2^m + 1$ or $p|2^m - 1$. The case $p|2^m - 1$ is impossible. Then $p|2^m + 1$ and using Lemma 2.4.1 we have that p is an antisymmetric number of size m. □

Now let m be a positive integer and let $q_m := 2^m + 1$. Then q_m is an odd integer for every $m \geq 1$. Let $\{r_1, r_2, \ldots, r_{k(m)}\}$ be the prime decomposition of q_m, then $q_m = r_1 \cdot r_2 \cdots r_{k(m)}$ where we assume that $2 \leq r_1 \leq r_2 \leq \cdots \leq r_{k(m)}$, and $k(m)$ is a positive integer depending on m. In Table 2.2 we give the prime decomposition of $q_m = 2^m + 1$ for values of m between 1 and 10. In addition we give the binary expansion of the new primes of q_m

In Table 2.5 we included all binary expansions of the reciprocal primes $1/p$ up to $p = 521$. The last column indicates if the primes are short (S) or long (L), see Definition 2.5.

There exist different sieves based on the prime decomposition, for example of numbers of the form $10^n - 1$. This sieve does not include $p = 2$ and $p = 5$, since $10 = 2 \cdot 5$, see [37].

Using the order given by the size of the binary period of the reciprocals of prime numbers we have found new primes whose decimal expression have more of 200 digits, so it may be useful in order to generate security codes in cryptography. Also using the new sieve we can study properties

m	prime decomposition of $q_m = 2^m + 1$	expansion of $1/q$ for new q prime
1	3	$1/3 = 0.\overline{0 \cdot 1}$
2	5	$1/5 = 0.\overline{00 \cdot 11}$
3	$3 \cdot 3$	it does not exist
4	17	$1/17 = 0.\overline{0000 \cdot 1111}$
5	$3 \cdot 11$	$1/11 = 3/33 = 0.\overline{00010 \cdot 11101}$
6	$5 \cdot 13$	$1/13 = 5/65 = 0.\overline{000100 \cdot 111011}$
7	$3 \cdot 43$	$1/43 = 3/129 = 0.\overline{0000010 \cdot 1111101}$
8	257	$1/257 = 0.\overline{00000000 \cdot 11111111}$
9	$3 \cdot 3 \cdot 3 \cdot 19$	$1/19 = 0.\overline{000011010 \cdot 111100101}$
10	$5 \cdot 5 \cdot 41$	$1/41 = 25/1025 = 0.\overline{0000011000 \cdot 1111100111}$

Table 2.2: Binary expansion of the first ten antisymmetric numbers.

of the prime numbers using probabilistic and statistical methods, see for example Figure 2.1.

2.4.1 Fermat numbers

Recall that a **Fermat number** is a positive integer of the form $F_n = 2^{2^n} + 1$ for every $n \in \mathbf{N} \cup \{0\}$. It is well known that the first five Fermat Numbers are prime numbers, that is $F_0 = 2^{2^0} + 1 = 2^1 + 1 = 3$, $F_1 = 2^{2^1} + 1 = 2^2 + 1 = 5$, $F_2 = 2^{2^2} + 1 = 2^4 + 1 = 17$, $F_3 = 2^{2^3} + 1 = 2^8 + 1 = 257$, $F_4 = 2^{2^4} + 1 = 2^{16} + 1 = 65537$ are prime numbers. Fermat numbers are named after Pierre de Fermat (1607-1665) a French politician and mathematician who conjectured that for every $m \in \mathbf{N} \cup \{0\}$, F_m is a prime number. Later on Leonhard Euler (1707-1783) a Swiss mathematician found that Fermat's conjecture was not true, he proved that $F_5 = 2^{2^5} + 1 = 4294967297 = 641 \cdot 6700417$, so, F_5 is not a prime number. Furthermore, the equation of F_5 is its prime decomposition.

Even if Fermat's conjecture was wrong, the study of the Fermat's numbers continues up to present time, this is clear nowadays because Fermat's numbers increase very rapidly with n, since they behave like double exponential numbers. So far, nobody has found any more Fermat numbers F_m with $m \geq 5$ which are prime numbers, in fact, many mathematicians are still trying to find divisors of F_m for large values of m, a clear proof of this statement can be found on *www.prothsearch.com/fermat.html* that is a web page which is updated very often.

Some of the results that have been proved about the Fermat's numbers are:

1) For every $m \in \mathbf{N} \cup \{0\}$

$$\Pi_{i=0}^{m} F_i + 2 = F_0 \cdot F_1 \cdots F_m + 2 = F_{m+1}.$$

Two corollaries of (1) are

2) Any two different Fermat's numbers are coprimes. From this it follows that the prime numbers are infinite.

3) There is not a Fermat number which is the sum of two prime numbers.

4) Carl Friedrich Gauss proved a relation between the construction of regular poligonals with rule and compass and the prime numbers of Fermat.

5) For every $m \geq 2$ the last digit of F_m is 7.

6) A very important result is that every Fermat number $F_m = 2^{2^m} + 1$ which is a composite number, can be decomposed with primes of the form $k \cdot 2^{m+2} + 1$ where k is a positive integer.

To see the proofs of some of these results and some other results see *https://en.wikipedia.org/Fermat_number*

The last result has been used extensively, to find very large divisors of the Fermat numbers, in the generation of pseudo-random numbers.

We can relate Fermat's numbers with an infinite sequence of antisymmetric prime numbers, which corresponds to the prime decomposition of the Fermat's numbers. By Lemma 4.1, part ii) we have that $1/F_m = 1/(2^{2^m} + 1)$ has an antisymmetric expansion with period 2^{m+1} for every $m \in \mathbf{N} \cup \{0\}$. This period is of the form $0.00 \cdots 011 \cdots 1$ with 2^m zeros followed by 2^m ones.

We have found in the literature that for many $m \geq 5$ the divisors of F_m have a common feature. In fact, we state this property as a new Lemma.

Lemma 2.4.1.1 *For every $m \geq 5$ every prime divisor of F_m is an antisymmetric number whose whole period is of order 2^{m+1}. Furthermore, if p is a prime number such that $1/p$ has period 2^{m+1} then p divides to F_m.*

Proof: Let p be a prime divisor of F_m such that $p < F_m$. Since F_m is an odd integer then $p \neq 2$.

Also, $2^{2^m} + 1 = F_m$ divides to $F_m \left(2^{2^m} - 1\right) = 2^{2^{m+1}} - 1$. Thus $p | \left(2^{2^{m+1}} - 1\right)$. Using the field $\mathbb{Z}_p^*$ we have that $[2]_p^{2^{m+1}} = \left[2^{2^{m+1}}\right]_p = [1]_p$.

If $k = order\left([2]_p\right)$ then $k | 2^{m+1}$ and $k = 2^r$ with $0 \leq r \leq m+1$. Note that if $r = 0$, then $[2]_p^k = [2]_p = [1]_p$ and $p | (2-1) = 1$ which is a contradiction. So $k = 2^r$ with $1 \leq r \leq m+1$.

Using proof of Lemma 2.4.1, part *ii*), and the fact that $p|F_m$ we have that p is an antisymmetric number. We will show that $r = m + 1$ and $k = 2^{m+1}$.

If $2^r = k < 2^{m+1}$ then $1 \leq r < m+1$ by Lemma 2.4.1 $p|2^{2^{r-1}} + 1 = F_{r-1}$. But $r - 1 < m$ and F_m, F_{r-1} are coprimes which is a contradiction to result 2).

Therefore, $r = m + 1$ and $k = 2^{m+1}$ is the smallest integer such that $2^k \equiv 1 (\bmod p)$, that is, $k = 2^{m+1}$ is the smallest integer such that $p|(2^k - 1)$. Thus the period of $1/p$ is $k = 2^{m+1}$.

Besides, if p is a prime number such that $\frac{1}{p}$ has period $s = 2^{m+1}$, then $s = 2^{m+1}$ is the smallest integer such that $p|(2^s - 1) = (2^{2^{m+1}} - 1) = (2^{2m} - 1) \cdot (2^{2^m} + 1)$, by prime's properties $p|\left(2^{2^m} + 1\right)$ or $p|\left(2^{2^m} - 1\right)$. But $p|\left(2^{2^m} - 1\right)$ is not possible because $2^m < 2^{m+1} = s$ and $s = 2^{2^{m+1}}$ is the period of p by Theorem 2.2.8. Hence $p|\left(2^{2^m} + 1\right) = F_m$. $\square$

Two simple examples provide the prime factors of F^m for $m = 4$ and $m = 5$.

The first example, that is, $m = 4$ is the last known prime among the the Fermat's numbers. In this case $F_4 = 2^{2^4} + 1 = 65537$ is the last prime number found by Fermat. If we find the binary expansion of its reciprocal, that is, $1/F_4$ has an antisymmetric expansion, we have that

$$\frac{1}{65537} = 0.\overline{0000000000000000|1111111111111111}$$

which is a binary number with period p whose length is $32 = 2^5 = 2^{4+1}$,

In the case of $m = 5$ Euler proved that F_5 is not a primer number, since $F_5 = 4294967297 = 641 \cdot 6700417$ where both factors are prime, In this case we have that their reciprocals have binary expansions given by

$$\frac{1}{641} = 0.\overline{00000000011001100011110110000000|}\\ \overline{11111111100110011100001001111111}$$

and

$$\frac{1}{6700417} = 0.\overline{00000000000000000000001010000000|}\\ \overline{11111111111111111111110101111111}.$$

So in this case both factors have reciprocals with binary antisymmetric expansions with periods p of length $64 = 2^6 = 2^{5+1}$ as in the Lemma above.

Of course we also have that the antisymmetric expansion of the reciprocal of F_5 is given by

$$\frac{1}{F_5} = 0.\overline{00000000000000000000000000000000|}\\ \overline{11111111111111111111111111111111},$$

which has also period of size 64.

2.5 Tables

$2^{2}-1$	=	$\underline{3}$
$2^{3}-1$	=	$\underline{7}$
$2^{4}-1$	=	$15 = 3 \cdot \underline{5}$
$2^{5}-1$	=	$\underline{31}$
$\mathbf{2^{6}-1}$	**=**	$\mathbf{63 = 3 \cdot 3 \cdot 7}$
$2^{7}-1$	=	$\underline{127}$
$2^{8}-1$	=	$255 = 3 \cdot 5 \cdot \underline{17}$
$2^{9}-1$	=	$511 = 7 \cdot \underline{73}$
$2^{10}-1$	=	$1023 = 3 \cdot 31 \cdot \underline{11}$
$2^{11}-1$	=	$2047 = \underline{23} \cdot \underline{89}$
$2^{12}-1$	=	$4095 = 3 \cdot 3 \cdot 5 \cdot 7 \cdot \underline{13}$
$2^{13}-1$	=	$\underline{8191}$
$2^{14}-1$	=	$16383 = 3 \cdot \underline{43} \cdot 127$
$2^{15}-1$	=	$32767 = 7 \cdot 31 \cdot \underline{151}$
$2^{16}-1$	=	$65535 = 3 \cdot 5 \cdot 17 \cdot \underline{257}$
$2^{17}-1$	=	$\underline{131071}$
$2^{18}-1$	=	$262143 = 3 \cdot 3 \cdot 3 \cdot 7 \cdot \underline{19} \cdot 73$
$2^{19}-1$	=	$\underline{524287}$
$2^{20}-1$	=	$1048575 = 3 \cdot 5 \cdot 5 \cdot 11 \cdot 31 \cdot \underline{41}$
$2^{21}-1$	=	$2097151 = 7 \cdot 7 \cdot 127 \cdot \underline{337}$
$2^{22}-1$	=	$4194303 = 3 \cdot 23 \cdot 89 \cdot \underline{683}$
$2^{23}-1$	=	$8388607 = \underline{47} \cdot \underline{178481}$
$2^{24}-1$	=	$16777215 = 3 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot \underline{241}$
$2^{25}-1$	=	$33554431 = 31 \cdot \underline{601} \cdot \underline{1801}$
$2^{26}-1$	=	$67108863 = 3 \cdot \underline{2731} \cdot 8191$
$2^{27}-1$	=	$134217727 = 7 \cdot 73 \cdot \underline{262657}$
$2^{28}-1$	=	$268435455 = 3 \cdot 5 \cdot \underline{29} \cdot 43 \cdot \underline{113} \cdot 127$
$2^{29}-1$	=	$536870911 = \underline{233} \cdot \underline{1103} \cdot \underline{2089}$
$2^{30}-1$	=	$1073741823 = 3 \cdot 3 \cdot 7 \cdot 11 \cdot 31 \cdot 151 \cdot \underline{331}$
$2^{31}-1$	=	$\underline{2147483647}$
$2^{32}-1$	=	$4294967295 = 3 \cdot 5 \cdot 17 \cdot 257 \cdot \underline{65537}$
$2^{33}-1$	=	$8589934591 = 7 \cdot 23 \cdot 89 \cdot \underline{599479}$
$2^{34}-1$	=	$17179869183 = 3 \cdot \underline{43691} \cdot 131071$
$2^{35}-1$	=	$34359738367 = 31 \cdot \underline{71} \cdot 127 \cdot \underline{122921}$
$2^{36}-1$	=	$68719476735 = 3 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot \underline{37} \cdot 73 \cdot \underline{109}$
$2^{37}-1$	=	$137438953471 = \underline{223} \cdot \underline{616318177}$
$2^{38}-1$	=	$274877906943 = 3 \cdot \underline{174763} \cdot 524287$
$2^{39}-1$	=	$549755813887 = 7 \cdot \underline{79} \cdot 8191 \cdot \underline{121369}$
$2^{40}-1$	=	$1099511627775 = 3 \cdot 5 \cdot 5 \cdot 11 \cdot 17 \cdot 31 \cdot 41 \cdot \underline{61681}$
$2^{41}-1$	=	$2199023255551 = \underline{13367} \cdot \underline{164511353}$
$2^{42}-1$	=	$4398046511103 = 3 \cdot 3 \cdot 7 \cdot 7 \cdot 43 \cdot 127 \cdot 337 \cdot \underline{5419}$

Table 2.3: Prime Numbers Found in Integer Powers of Two Minus One

$2^{43}-1$	=	$8796093022207 = \underline{431} \cdot \underline{9719} \cdot \underline{2099863}$
$2^{44}-1$	=	$17592186044415 = 3 \cdot 5 \cdot 23 \cdot 89 \cdot \underline{397} \cdot 683 \cdot \underline{2113}$
$2^{45}-1$	=	$35184372088831 = 7 \cdot 31 \cdot 73 \cdot 151 \cdot \underline{631} \cdot \underline{2331}$
$2^{46}-1$	=	$70368744177663 = 3 \cdot 47 \cdot 178481 \cdot \underline{2796203}$
$2^{47}-1$	=	$140737488355327 = \underline{2351} \cdot \underline{4513} \cdot \underline{13264529}$
$2^{48}-1$	=	$281474976710655 = 3 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot \underline{97} \cdot 241 \cdot 251 \cdot \underline{673}$
$2^{49}-1$	=	$562949953421311 = 127 \cdot \underline{4432676798593}$
$2^{50}-1$	=	$1125899906842623 = 3 \cdot 11 \cdot 31 \cdot \underline{251} \cdot 601 \cdot 1801 \cdot \underline{4051}$
$2^{51}-1$	=	$2251799813685247 = 7 \cdot \underline{103} \cdot \underline{2143} \cdot \underline{11119} \cdot 131071$
$2^{52}-1$	=	$4503599627370495 = 3 \cdot 5 \cdot \underline{53} \cdot \underline{157} \cdot \underline{1613} \cdot 2731 \cdot 8191$
$2^{53}-1$	=	$9007199254740991 = \underline{6361} \cdot \underline{69431} \cdot \underline{20394401}$
$2^{54}-1$	=	$18014398509481983 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 7 \cdot 19 \cdot 73 \cdot \underline{87211} \cdot 262657$
$2^{55}-1$	=	$36028797018963967 = 23 \cdot 31 \cdot 89 \cdot \underline{8811} \cdot \underline{3191} \cdot \underline{201961}$
$2^{56}-1$	=	$72057594037927935 = 3 \cdot 5 \cdot 17 \cdot 29 \cdot 43 \cdot 113 \cdot 127 \cdot \underline{20394401}$
$2^{57}-1$	=	$144115188075855871 = 7 \cdot \underline{32377} \cdot 524287 \cdot \underline{1212847}$
$2^{58}-1$	=	$288230376151711743 = 3 \cdot \underline{59} \cdot 233 \cdot 1103 \cdot 2089 \cdot \underline{3033169}$
$2^{59}-1$	=	$576460752303423487 = \underline{179951} \cdot \underline{3203431780337}$
$2^{60}-1$	=	$1152921504606846975 = 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 41 \cdot \underline{61} \cdot 151 \cdot 331 \cdot \underline{1321}$
$2^{61}-1$	=	$\underline{2305843009213693951}$
$2^{62}-1$	=	$4611686018427387903 = 3 \cdot \underline{715827883} \cdot 2147483647$
$2^{63}-1$	=	$9223372036854775807 = 7 \cdot 7 \cdot 73 \cdot 127 \cdot 337 \cdot \underline{92732} \cdot \underline{649657} \cdot$
$2^{64}-1$	=	$18446744073709551615 = 3 \cdot 5 \cdot 17 \cdot 257 \cdot \underline{641} \cdot 65537 \cdot \underline{6700417}$
$2^{65}-1$	=	$36893488147419103231 = 31 \cdot 8191 \cdot \underline{145295143558111}$
$2^{66}-1$	=	$73786976294838206463 = 3 \cdot 3 \cdot 7 \cdot 23\underline{67} \cdot 89 \cdot 683 \cdot \underline{20857} \cdot 599479$
$2^{67}-1$	=	$147573952589676412927 = \underline{193707721} \cdot \underline{761838257287}$
$2^{68}-1$	=	$295147905179352825855 = 3 \cdot 5 \cdot \underline{137} \cdot \underline{953} \cdot \underline{26317} \cdot 43691 \cdot 131071$
$2^{69}-1$	=	$590295810358705651711 = 7 \cdot 47 \cdot 178481 \cdot \underline{10052678938039}$
$2^{70}-1$	=	$1180591620717411303423 = 3 \cdot 11 \cdot 31 \cdot 43 \cdot 71 \cdot 127 \cdot \underline{281} \cdot \underline{86171} \cdot 122921$
$2^{71}-1$	=	$2361183241434822606847 = \underline{228479} \cdot \underline{48544121} \cdot \underline{212885883}$
$2^{72}-1$	=	$4722366482869645213695 = 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 37 \cdot 73 \cdot 109 \cdot 241 \cdot \underline{433} \cdot \underline{38737}$
$2^{73}-1$	=	$9444732965739290427391 = \underline{439} \cdot \underline{2298041} \cdot \underline{9361973132609}$
$2^{74}-1$	=	$18889465931478580854783 = 3 \cdot 223 \cdot \underline{1777} \cdot \underline{25781083} \cdot 616318177$
$2^{75}-1$	=	$37778931862957161709567 = 7 \cdot 31 \cdot 151 \cdot 601 \cdot 1801 \cdot \underline{100801} \cdot \underline{10567201}$
$2^{76}-1$	=	$3 \cdot 5 \cdot \underline{229} \cdot \underline{457} \cdot 174763 \cdot 524287 \cdot \underline{525313}$
$2^{77}-1$	=	$23 \cdot 89 \cdot 127 \cdot \underline{581283643249112959}$
$2^{78}-1$	=	$3 \cdot 3 \cdot 7 \cdot 79 \cdot 2731 \cdot 8191\underline{121369} \cdot \underline{22366891}$
$2^{79}-1$	=	$\underline{2687} \cdot \underline{202029703} \cdot \underline{1113491139767}$
$2^{80}-1$	=	$3 \cdot 5 \cdot 5 \cdot 11 \cdot 17 \cdot 31 \cdot 41 \cdot 257 \cdot 61681 \cdot \underline{4278255361}$
$2^{81}-1$	=	$7 \cdot 73 \cdot \underline{2593} \cdot \underline{71119} \cdot 262657 \cdot \underline{97685839}$
$2^{82}-1$	=	$3 \cdot \underline{81} \cdot 13367 \cdot 164511353 \cdot \underline{8831418697}$
$2^{83}-1$	=	$\underline{167} \cdot \underline{57912614113275649087721}$

Table 2.3: Prime Numbers Found in Integer Powers of Two Minus One

$2^{84}-1$	=	$3 \cdot 3 \cdot 5 \cdot 7 \cdot 7 \cdot 13 \cdot 29 \cdot 43 \cdot 113 \cdot 127 \cdot 337 \cdot \underline{1429} \cdot 5419 \cdot \underline{14449}$
$2^{85}-1$	=	$31 \cdot 131071 \cdot \underline{9520972806333758431}$
$2^{86}-1$	=	$3 \cdot 431 \cdot 9719 \cdot 2099863 \cdot \underline{2932031007403}$
$2^{87}-1$	=	$7 \cdot 233 \cdot 1103 \cdot 2089 \cdot \underline{4177} \cdot \underline{9857737155463}$
$2^{88}-1$	=	$3 \cdot 5 \cdot 17 \cdot 23 \cdot 89 \cdot \underline{353} \cdot 397 \cdot 683 \cdot 2113 \cdot \underline{2931542417}$
$2^{89}-1$	=	$\underline{618970019642690137449562111}$
$2^{90}-1$	=	$3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdot 31 \cdot 73 \cdot 151 \cdot 331 \cdot 631 \cdot 23311 \cdot \underline{18837001}$
$2^{91}-1$	=	$127 \cdot \underline{911} \cdot 8191 \cdot \underline{112901153} \cdot \underline{23140471537}$
$2^{92}-1$	=	$3 \cdot 5 \cdot 47 \cdot \underline{277} \cdot \underline{1013} \cdot \underline{1657} \cdot \underline{30269} \cdot 178481 \cdot 2796203$
$2^{93}-1$	=	$7 \cdot 2147483647 \cdot \underline{658812288653553079}$
$2^{94}-1$	=	$3 \cdot \underline{283} \cdot 2351 \cdot 4513 \cdot 13264529 \cdot \underline{165768537521}$
$2^{95}-1$	=	$31 \cdot \underline{191} \cdot 524287 \cdot \underline{420778751} \cdot \underline{30327152671}$
$2^{96}-1$	=	$3 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 97 \cdot \underline{193} \cdot 241 \cdot 257 \cdot 673 \cdot 65537\underline{22253377}$
$2^{97}-1$	=	$\underline{11447} \cdot \underline{13842607235828485645766393}$
$2^{98}-1$	=	$3 \cdot 43 \cdot 127 \cdot \underline{4363953127297} \cdot 4432676798593$
$2^{99}-1$	=	$7 \cdot 23 \cdot 73 \cdot 89 \cdot \underline{199} \cdot \underline{153649} \cdot 599479 \cdot \underline{33057806959}$
$2^{100}-1$	=	$3 \cdot 5 \cdot 5 \cdot 5 \cdot 11 \cdot 31 \cdot 41 \cdot \underline{101} \cdot 251 \cdot 601 \cdot 1801 \cdot 4051 \cdot \underline{8101} \cdot \underline{268501}$

Table 2.3: Prime Numbers Found in Integer Powers of Two Minus One

n	Number of new Primes until 2^n-1	Number of new Primes between 2^n-1 and $2^{n-50}-1$
50	67	67
100	168	101
150	277	109
200	420	143
250	571	151
300	721	150
350	880	159
400	1054	174
450	1222	168
500	1404	182
550	1583	179
600	1759	176
650	1953	194
700	2144	191
750	2343	199
800	2547	204
850	2752	205
900	2954	202
950	3178	224
1000	3384	206

Table 2.4: Number of new primes until 2^n-1

p	m	n	binary form of $1/p$	$p-1$	type
3	2	2	$0.\overline{0\cdot 1}$	2	L
5	4	3	$0.\overline{00\cdot 11}$	4	L
7	3	3	$0.\overline{001}$	6	S
11	10	4	$0.\overline{00010\cdot 11101}$	10	L
13	12	4	$0.\overline{000100\cdot 111011}$	12	L
17	8	5	$0.\overline{0000\cdot 1111}$	16	S
19	18	5	$0.\overline{000011010\cdot 111100101}$	18	L
23	11	5	$0.\overline{00001011001}$	22	S
29	28	5	$0.\overline{00001000110100\cdot 11110111001011}$	28	L
31	5	5	$0.\overline{00001}$	30	S
37	36	6	$0.\overline{000001101110101100\cdot 111110010001010011}$	36	L
41	20	6	$0.\overline{0000011000\cdot 1111100111}$	40	S
43	14	6	$0.\overline{0000010\cdot 1111101}$	42	S
47	23	6	$0.\overline{00000101011100100110001}$	46	S
53	52	6	$0.\overline{00000100110101001000011100\cdot}$ $\overline{11111011001010110111100011}$	52	L
59	58	6	$0.\overline{00000100010101101100011110010\cdot}$ $\overline{11111011101010010011100001101}$	58	L
61	60	6	$0.\overline{000001000011001001011100010100\cdot}$ $\overline{111110111100110110100011101011}$	60	L
67	66	7	$0.\overline{000000111101001000100110001101010\cdot}$ $\overline{111111000010110111011001110010101}$	66	L
71	35	7	$0.\overline{00000011100110110000101011010001001}$	70	S
73	9	7	$0.\overline{000000111}$	72	S
79	39	7	$0.\overline{000000110011110110010001110100101010001}$	78	S
83	82	7	$0.\overline{000000110001010110010}$ $\overline{11100100001111011010\cdot}$ $\overline{111111001110101001101}$ $\overline{000110111110000100101}$	82	L
89	11	7	$0.\overline{00000010111}$	88	S
97	48	7	$0.\overline{000000101010001110100000\cdot}$ $\overline{111111010101110001011111}$	96	S
101	100	7	$0.\overline{0000000101000100011011110}$ $\overline{0001100101011000101101100\cdot}$ $\overline{1111110101110111001000001}$ $\overline{1110011010100111010010011}$	100	L

Table 2.5: Let p be a prime number and let m be the smallest positive integer such that $p|(2^m-1)$. And let n be the smallest positive integer such that $2^{n-1} < p < 2^n$.

p	m	n	binary form of $1/p$	$p-1$	type
103	51	7	$0.\overline{000000100111111000100010110}$ $\overline{0101111001110010010101001}$	102	S
107	106	7	$0.\overline{000000100110010001111110001101001010001}$ $\overline{010110001000010 \cdot 1111110110011011100000 1}$ $\overline{11001011010111010100111011110 1}$	106	L
109	36	7	$0.\overline{000000100101100100 \cdot 111111011010011011}$	108	S
113	28	7	$0.\overline{00000010010000 \cdot 11111101101111}$	112	S
127	7	7	$0.\overline{0000001}$	126	S
131	130	8	$0.\overline{00000001111101000100011001011001 1}$ $\overline{11001001010010000100111000101010\cdot}$ $\overline{111111100000101110111001101001100}$ $\overline{00110110101101111011000111010101}$	130	L
137	68	8	$0.\overline{0000000111011110010111010110111000\cdot}$ $\overline{1111111000100001101000101001000111}$	136	S
139	138	8	$0.\overline{00000001110101110111110110110010101 0}$ $\overline{0101110000010110000110011100100010\cdot}$ $\overline{11111110001010001000010010011010101}$ $\overline{1010001111101001111001100011011101}$	138	L
149	148	8	$0.\overline{0000000110110111110101101100001111 01}$ $\overline{11011010001100110001011001010101111 00\cdot}$ $\overline{1111111001001000001010010011110000 10}$ $\overline{0010010111001100011101001101101000011}$	148	L
151	15	8	$0.\overline{000000011011001}$	150	S
157	52	8	$0.\overline{00000001101000010110110100\cdot}$ $\overline{11111110010111101001001011}$	156	S
163	162	8	$0.\overline{00000001100100100000111110110100100111010}$ $\overline{0001110001000101000110101011001100001010\cdot}$ $\overline{11111110011011011111000001001011011000101}$ $\overline{1110001110111010111001010100110011110101}$	162	L
167	83	8	$0.\overline{000000011000100001101110010}$ $\overline{11111000010101011101100001 0}$ $\overline{0100110010100101100011101001}$	166	S
173	172	8	$0.\overline{0000000101111010110100100010000010001110000}$ $\overline{0111011001100001101010100010110001100100100\cdot}$ $\overline{1111111010000101001011011101111110111000111 1}$ $\overline{1000100110011110010101011110100111001101101 1}$	172	L
179	178	8	$0.\overline{00000001011011100001111101110110101101 0000}$ $\overline{1100110111101100101100010101011100111101 0\cdot}$ $\overline{11111110100100011110000010001001010010111 1}$ $\overline{0011001000000111001001101001110101010001100001 01}$	178	L

Table 2.5: Let p be a prime number and let m be the smallest positive integer such that $p|(2^m-1)$. And let n be the smallest positive integer such that $2^{n-1} < p < 2^n$.

p	m	n	binary form of $1/p$	$p-1$	type
181	180	8	0.00000001011010100001001111001101000101010 0110	180	L
			111001010010000001000011111000111011011001 1100·		
			111111101001010111101100001100101110101011001		
			000110101101111110111100000011100010010011 00011		
191	95	8	0.0000000101010111000111101101 0011	190	S
			110001010000001101011001110011010		
			0010001011011001001000011000001		
193	96	8	0.00000001010100111001000010010 1001	192	S
			00011110100000 0 · 111111101010110		
			00110111101101010110111000010111111		
197	196	8	0.0000000101001100101010111000100001110010010110101	196	L
			111011011100111010011110100010011011111110000011 00·		
			111111101011001101010100011101111000110110100101 0		
			0001001000110001011000010111011001000000011111 0011		
199	99	8	0.1010010010101001110011110001110110010110100000	198	S
			110011011101010001000010111001001111110000 1001		
211	210	8	0.0000000100110110100110001101111100111101111000000 1110	210	L
			100011110010101001110110111001101000010101110101 1010·		
			111111101100100101100111001000000110000100001111110 001		
			0111000011010101100010010001100101111010100010100101		
223	37	8	0.1000110010100010100111	222	S
227	226	8	0.0000000100100000101101000111000011000110011111000000 11011	226	L
			00010000111010101001001010011011101000010100010011 001010		
			1111111011011111101001011100011110011100110000001111 1100100		
			1110111100010101011011010110010001011110101110110011 0101		
229	76	8	0.0000000100011110001011101111001110 1100	228	S
			11111110111000011101000100001100010011		
233	29	8	0.0000000100011001010001010 0111	232	S
239	119	8	0.0000000100010010001101011000111001110101	238	S
			1101001100000011001101101010000010101011		
			011000010111100100001001101000111110001		
241	24	8	0.000000010000 · 111111101111	240	S
251	50	8	0.0000000100000101000110010 · 1111111011111010111001101	250	S
257	16	9	0.00000000 · 11111111		
263	131	9	0000000011111001001011111101100100010000 10001	262	S
			100001010101101010000110010100111011011 00000		
			0101110101110001111000101100110001101001001		

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p	m	n	binary form of $1/p$	$p-1$	type
269	268	9	0.0000000011110011101000000110101010010110010111	268	L
			0101000011100100011001101100011111011001111101		
			110110111101011000100000010110110101110001OO·		
			111111110000110001011111001010101101001101000		
			101011110001101110011001001110000010011000010		
			00100100001010011101111110100100101000111011		
271	135	9	0.000000001111000111010100100010111100111011100	270	S
			000110100111001100111111010010101010000010010		
			111001001001101011101100001010011001000010001		
277	92	9	0.0000000011101100100101111001000100011000111100·	276	S
			1111111100010011011010000110111011100111000011		
281	70	9	0.00000000111010010011100101100101000·	280	S
			11111111000101101100011010011010111		
283	94	9	0.00000000111001111001001101110010111000100010010·	282	S
			11111111000110000110110010001101000111011101101		
293	292	9	0.0000000011011111101011000001111101110100001101000	292	L
			1101100010101110101111100111100010010010110010001		
			111010010100100010000100110011100011001010101100·		
			11111111001000000010100111110000010001011110010111		
			00100111010100010100000011000011101101101001101110		
			000101101011011101111011001100011100110101010011		
307		9	0.00000000110101010111100011101001011111000011111010·	306	S
			111111110010101010000111000101101000000111100000101		
311	155	9	0.000000001101001010111010000001000000111011010001000101	310	S
			00100101000010101011001101110010011010110000000100111		
			100000101110000110001011000111001100111101101111001		
313	156	9	0.000000001101000101100001010101000011111	312	S
			0001010001110010100000001001110100001000·		
			111111110010111010011110101010111100000		
			1110101110001101011111110110001011110111		
317	316	9	0.00000000110011101011110011111000101110110101101101000	316	L
			00101101001110010101011001101000111110111111011001001		
			1110010010001010111001101111011100011101111000010100·		
			11111111001100010100001100000111010001001010010010111		
			110100101100011010101001100101110000001000000100110110		
			0001101101110101000110010000100011100010000111101011		
331	30	9	0.000000001100010 · 111111110011101	330	S
337	21	9	0.000000001100001001111	336	S

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p	m	n	binary form of $1/p$	$p-1$	type
347	346	9	0.000000001011110011011101010100110101110110110001110011000 1 0110110111101100110100011000010110001000011110111111000100 111110101101010111110010101110000111000000100011011010010· 1111111101000011001000101010110010100010010011100011001110 1001001000010011001011100111101001110111100001000000111011 00000101001010100000110101000111100011111101110010010110 1	346	L
349	348	9	0.0000000010111011111001000010000001000110011010110001100000 1 1010011010000010100100010011110011100001111011001110110110 10100101110001101100100011111100010101010001011010111101 00· 1111111101000100001101111011111101110011001010011100111110 0101100101111110101101110110000110001111000010011000100100 1 0101101000111001001101110000000111010101011101001010000101 1	348	L
353	88	9	0.0000000010111001101001111000011000101010000 0· 11111111010001100101100001111001110101011111	352	S
359	179	9	0.000000001011011010001101001100010011010000001 11001000011000001111101100000001000100011101 00 11110010011100111000010101011001001000101111 0 0010000011001101010111101101011101011010100 1	358	S
367	183	9	0.000000001011001010010010011111000010100111011 0 10010101010001100111001111001010111111001000 00 11001000111001001100101110101111000101011001 11 111011110100001001000100010111000000100111000 1	366	S
373	372	9	0.000000001010111110110011001000011010000101001 00 101101111110111110000111001101001101100011100 00 100011101100000110001011010100110000101110101 01 0111001010011101110110101111000000110110111 00· 1111111101010000001001100110111100101111010110 11 0100100000001000000111100011001011001001110001111 0111000100111110011101001010110011110100010101 0 1000110101100010001001010000111111001001000 11	372	L
379	378	9	0.00000000101011001110101100001111100010010 0 01111001100101010100011011101100011010010 1 01111100111101011101111000111010000101101 1 11011100111000000010000001101100000010010111 0100110110101101100 10 · 1111111101010011000 10 10011110000011101101110000110011010101011 1 00100010011100101101010000001100001010001 00 00111000101111010010000100011000111111011 1 11100100111111011010001011001001010010011 01	378	L

Table 2.5: Let p be a prime number and let m be the smallest positive integer such that $p|(2^m - 1)$. And let n be the smallest positive integer such that $2^{n-1} < p < 2^n$.

p	m	n	binary form of $1/p$	$p-1$	type
383	191	9	0.1010101100011100101111011101001111100010100101	382	S
			1100001111010111111001010100001110000010010101		
			1011100100100110000110010110011001000100001011		
			010111001110100010011011000100100000110000001		
389	388	9	0.0000000010101000011110010001011100001000100011100	388	L
			01001100010101101101111001101111111000000011010010		
			1001011101011100110010101011000110101111110110110		
			0100101100000101111011000100000111001111101001100·		
			1111111101010111100001101110100011110111011100011		
			10110011101010010010000110010000000011111100101101		
			0110100010100011001101010100111001010000001001001		
			101101001111101000010011101111100011000010110011		
397	44	9	0.0000000010100101000100 · 1111111101011010111011	396	S
401	100	9	0.00000000101000110110111001110001101000101100101100	400	S
			00001100110001001010000001110000001011011111101110000·		
			11111111010111001001000110001110010111010011010011		
			111100110011101101011111000111110100100000010001111		
409	204	9	0.00000000101000000001111000000101101000100001110011001	408	S
			010110011000000110010000100101100011100001010101000·		
			1111111101011111111000011111010010111011110001100110		
			1010011001111111001101111011010011100011110101010111		
419	418	9	0.00000000100111000110100100010110100110110011000001000	418	L
			1000110110111111100111100011111001010001110111111000000		
			01110101010011101101000011110100011001000011001101010		
			010011110110110101011101011110101100111010000001010·		
			11111111011000111001011011101001011001001100111110111		
			0111001001000000011000011100000110101110001000000111111		
			10001010101100010010111100001011100110111100110010101		
			10110000100100101010001010000101001100010111110101		
421	420	9	0.00000000100110111010101011011110100011100100101000101	420	L
			111011011100000011111100111101011010100110100111001110		
			00100011010001001011011001101100001111001100111010111		
			11011101111110010100111110101000011011111111000101100·		
			11111111011001000101010100100001011100011011010111010		
			000100100011111000000011000010100101011001011000110001		
			11011100101110110100100110010011110000110011000101000		
			0010001000001101011000001010111100100000000111010011		
431	43	9	0.00000000100110000000011100100000101010110001	430	S
433	72	9	0.000000001001011101011010011101010000·	432	S
			11111111011010001010010110001010111		

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p	m	n	binary form of $1/p$	$p-1$	type
439	73	9	0.000000001001010101001000111001001001 0 1111001111000001000001010011111111001	438	S
443	442	9	0.0000000010010011111011111101000111000101000011100111001 0 0110101101111110000111110011010010010110001111111100100010 0001100001000101010110000110101001010100010111101100010 1 1010001001100010001111010100000101001100110110111001 0· 1111111101101100000010000000101110001110101111000110001101 1001010010000001111000001100101101101001110000000011011101 1110011110111010101001111001010110101011101000010011101 0 0101110110011101110000101011111010110011001001000110 1	442	L
449	224	9	0.0000000010010001111101011011110010111000101110110000001 0 110110011100110010101111100110111010011100001110010000 00· 1111111101101110000010100100001101000111010001001111110 1 0010011000110011010100000110010001011000111100011011111 1	448	S
457	76	9	0.000000001000111101100111101000011110 00· 11111111011100001001100001011110000111	456	S
461	460	9	0.000000001000111000101001000101111111000000111001 110000000101100011011001101011101110110010010 00 0011000011011110001000000011010101001111011010 0011110100010101101010000100001010100011010000 1100110001011011000100100101001101001100000100· 1111111101110001110101101110100000011111000110 0011111101001110010011001010001000100110110111 1100111100100001110111111100101010110000100101 1100001011101010010101111011110101011100101111 0011001110100100111011010101100101100111111011	460	L
463	231	9	0.0000000010001101100010111110001100111111110010 10 1110101110001010110010000001001111100111101010 1 1111101001111000100010010001111110000100001001 0110011001001010000101110011100011110011010010 0001101110100101010100100110001001101011010001	462	S

Table 2.5: Let p be a prime number and let m be the smallest positive integer such that $p|(2^m - 1)$. And let n be the smallest positive integer such that $2^{n-1} < p < 2^n$.

p	m	n	binary form of $1/p$	$p-1$	type
467	466	9	0.000000000100011000101010110000100000111001000000	466	L
			10101111011010101110010100100011101000011011011		
			01000101100111100110110010001010001001000010111		
			000001100000001111010110010101101001110011000111		
			100010011001011111011000100001111111001011010·		
			111111110111001110101010011110111111000110111111		
			01010000100101010001101011011100010111100100100		
			101110100110000110010011011101011101101111101000		
			111110011111100001010011010100101100011001111000		
			01110110011010000001001110111100000000110100101		
479	239	9	0.000000001000100011010001100000001100110100111010	478	S
			01000001001100111101011101100001110011011100000011		
			0001001010110100101001001001110000000111011110110		
			1110101000010110011100100101111100100001101010 11		
			10001110101100101000000101010110000001011110000 1		
487	243	9	0.000000001000011010010010001000101011000110101 1001	486	S
			1110001110011101001011001010111010000101100001001		
			1111110011011000100100110010111111010101111100100		
			1010101001010000111100111110100011011110111000100		
			000100101110110010001100111000001111110001010 1001		
491	490	9	0.000000001000010101111001011110111001000101110111 0	490	L
			1100101101010111000100100011111110011011111100100 1		
			1100011010100101110011001110011101111110101100100		
			1010000001001011000101000101010110000001110100110 0		
			1010010011000001111101001000011110001111011000010·		
			1111111101111010100001101000010001101110100010001		
			0011010010101000111011011100000001100100000011 0110		
			0011100101011010001100110001100010000000101001101 1		
			0101111101101001110101110101010011111100010110011		
			0101101100111110000010110111100001110000100111101		
499	166	9	0.000000001000001101010101101011001110001111	498	S
			0010001001011111011011000011111100111010·		
			111111110111110010101010010100110001110000		
			1101110110100000010010011110000000011000101		
503	251	9	0.000000001000001001001010010011100110000001011001001	502	
			001100010101111000100111101100101010001111100001011		
			1011010010101101000010101011000000001100001101101111		
			0111010110010001000011001011100101000000110100111011		
			1000101111110101110100100011000111100000000111001		

Table 2.5: Let p be a prime number and let m be the smallest positive integer such that $p|(2^m - 1)$. And let n be the smallest positive integer such that $2^{n-1} < p < 2^n$.

p	m	n	binary form of $1/p$	$p-1$	type
509	508	9	0.0000000010000000110000001001000011011001010001011110	508	L
			1000110111010100101111110001111010101110000000101000		
			001111000101101010000111110010111011000110001010010		
			011110111011100110010110011000011001001001011011100		
			01001010011011111010011101111011001110001101010100·		
			111111110111111100111110110111100100110101110100001		
			0111001000101011010000000111000010101000111111010111		
			110000111010010101111000001101000100111001110101101		
			100001000100011001101001100111100110110110100100011		
			1011010110010000001011000100000100110001110010101011		
521	260	10	0.00000000011111011100100111110011001110010111	520	S
			11010100110000101001010001100100001111001110		
			110111010001110011111101100010110000111000·		
			111111111000000100011011000001100110001101000		
			00101011001111010110101110011011110000110001		
			001000101110001100000010011101001111000111		

Table 2.5: Let p be a prime number and let m be the smallest positive integer such that $p|(2^m-1)$. And let n be the smallest positive integer such that $2^{n-1} < p < 2^n$.

This first part of the Thesis has been published, see [26].

Chapter 3

Bounded Variation

3.1 Preliminaries

Camille Jordan in 1881 introduced the concept of total variation for real functions defined on an interval $[a,b] \subset \mathbf{R}$. The concept of total variation for functions of one real variable can be defined using:

Definition 3.1.1 *Let* $-\infty < a < b < \infty$ *real numbers, consider the* **closed interval** $[a,b] = \{c \in \mathbb{R} \mid a \leq c \leq b\}$. *A* **partition** *of* $[a,b]$ *is a set of the form* $P = \{x_0, x_1, \ldots, x_{n_P-1}, x_{n_P}\}$, *where* $a = x_0 < x_1, \cdots < x_{n_P-1} < x_{n_P} = b$ *and* n_P *is a positive integer. The Family of all partitions of the interval* $[a,b]$ *will be denoted by* $\mathcal{P}([a,b])$.

Definition 3.1.2 *Let* $f : [a,b] \to \mathbb{R}$ *be a function where* $-\infty < a \leq b < \infty$. *the* **total variation** *of* f *is given by*

$$TV_a^b(f) = \sup_{P \in \mathcal{P}([a,b])} \sum_{i=1}^{n_P-1} |f(x_{i+1}) - f(x_i)|, \tag{3.1}$$

if the supremum in equation (3.1) is finite then we will say that f *is of* **finite total variation** *or* **bounded variation**, *and the family of functions* f *such that* $TV_a^b(f) < \infty$ *will be denoted by* $BV([a,b])$.

Definition 3.1.3 *Let* $-\infty < a < b < \infty$ *and define*

$$\mathbb{B}_{[a,b]} := \{f : [a,b] \to \mathbb{R} \mid f \text{ is a bounded function}\}, \tag{3.2}$$

where f is a **bounded function** *if and only if there exists $M \geq 0$ such that $|f(x)| \leq M$ for every $x \in [a, b]$.*

In order to see that not every real function defined on a closed interval $[a, b]$ belongs to by $BV([a, b])$. We have the following:

Example 3.1.4 *Let $a = 0$ and $b = 1$ and define the* **Dirichlet** *function on $[0, 1]$, given originally in 1829, by*

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{Q} \cap [0,1] \\ 1 & \text{if} \quad x \in \mathbb{I} \cap [0,1], \end{cases}$$

where $\mathbb{Q}$ is the set of rational numbers and $\mathbb{I}$ is the set of irrational numbers. Let $n \in \mathbb{N}$ be a positive integer and let $x_0 = 0, x_2 = 1/n, x_4 = 2/n, \ldots, x_{2k} = k/n, \ldots, x_{2n} = n/n = 1$. Since $\mathbb{I}$ is a dense in the set $\mathbb{R}$ of real numbers, then for every $k \in \{1, 2, \ldots, n-1\}$ we have that the non empty open intervals $(x_{2(k-1)/n}, x_{2k/n})$, are such that $\mathbb{I} \cap (x_{2(k-1)/n}, x_{2k/n}) \neq \emptyset$. So, define for every $k \in \{1, 2, \ldots, n\}$, $x_{2(k-1)+1} \in (\mathbb{I} \cap (x_{2(k-1)/n}, x_{2k/n}))$. Then $0 = x_0 < x_1 < x_2 = 1/n < x_3 < x_4 = 2/n < \cdots < x_{2(n-1)} = (n-1)/n < x_{2n-1} < x_{2n} = 1$ is a partition P^n of $[0, 1]$ of size $n_{P^n} = 2n$. We observe from equation (3.3) that

$$f(x_i) = \begin{cases} 0 & \text{if} \quad i \in \{0, 2, 4, \ldots, 2n\} \\ 1 & \text{if} \quad i \in \{1, 3, 5, \ldots, 2n-1\}. \end{cases}$$

Therefore, we obtain $\sum_{i=1}^{n_{P^n}-1} |f(x_{i+1}) - f(x_i)| = 2n$ for every $n \geq 1$. Hence, from (3.1), $TV_0^1(f) = \infty$.

Now we observe that a linear transformation always allows us to take the closed interval $[0, 1]$ instead of the general closed interval $[a, b]$.

Remark 3.1.5 *Let $-\infty < a < b < \infty$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be a function. Let $P = \{a = x_0, x_1, \ldots, x_{n_P-1}, x_{n_P} = b\} \in \mathcal{P}([a, b])$. Define*

$$t_i = \frac{x_i - a}{b - a} \text{ for every } i \in \{0, 1, \ldots, n_P\} \tag{3.3}$$

if $Q = \{0 = t_0, t_1, \ldots, t_{n_Q-1}, t_{n_Q} = 1\} \in \mathcal{P}([0, 1])$ then $n_Q = n_P$ and we have that

$$x_i = a + t_i(b - a) \quad \text{holds for every} \quad i \in \{0, 1, 2, \ldots, n-1, n\}. \tag{3.4}$$

Besides, if we define $g(t) = f(a+t(b-a))$ for every $t \in [0, 1]$, then $TV_a^b(f) = TV_0^1(g)$. Of course, the converse also holds.

The function $\phi : [0,1] \to [a,b]$ defined by $\phi(t) = a + t(b-a)$ is clearly an increasing bijection, whose inverse is given by $\phi^{-1} : [a,b] \to [0,1]$ where $\phi^{-1}(s) = (s-a)/(b-a)$, which provides the proof of Remark 3.1.5. As a consequence of Remark 3.1.5 we have

Remark 3.1.6 *Let $-\infty < a < b < \infty$ be real numbers and let $f : [a,b] \to \mathbb{R}$ be a function then the total variation of f can be written as*

$$TV_a^b(f) = \sup_{P\in\mathcal{P}([a,b])} \sum_{i=1}^{n_P-1} |f(x_{i+1}) - f(x_i)| = \sup_{Q\in\mathcal{P}([0,1])} \sum_{i=1}^{n_Q-1} |g(t_{i+1}) - g(t_i)|, \tag{3.5}$$

where $Q = \{0 = t_0, t_1, \ldots, t_{n_Q-1}, t_{n_Q} = 1\} \in \mathcal{P}([0,1])$, g is defined as in Remark 3.1.5 and $x_i = a + t_i(b-a)$ for every $i \in \{0,1,2\ldots,n_Q-1,n_Q\}$.

Remark 3.1.7 *Using Remarks 3.1.5 and 3.1.6 we will assume from now on that* $[a,b] = [0,1]$.

Now we have an important result about total variation of functions.

Lemma 3.1.8 *Let $f : [0,1] \to \mathbb{R}$ be a function, then $TV_0^1(f) = 0$ if and only if f is a constant function on* $[0,1]$. For the proof see (1.16) from [20].

Another important property is the following:

Lemma 3.1.9 *Let $f : [0,1] \to \mathbb{R}$ be a monotone function. Then $TV_0^1(f) =$* $|f(1) - f(0)|$. For the proof see (e), Proposition 1.3 from [20].

Lemma 3.1.10 *Let $f : [0,1] \to \mathbb{R}$ be a function and let* $0 < c < 1$. *Then*

$$TV_0^1(f) = TV_0^c(f) + TV_c^1(f). \tag{3.6}$$

For the proof see (g), Proposition 1.3 from [20].

Theorem 3.1.11 [Jordan] *A function $f\colon [0,1] \to \mathbb{R}$ is of bounded variation if and only if it may be represented in the form $f = g - h$, where both g and h are increasing functions.* See Theorem 1.5, [20].

Example 3.1.12 Let $f : [0,10] \to \mathbf{R}$ such that

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{1}{x} & \text{if} \quad 0 < x \leq 10. \end{cases}$$

We will compute $\text{TV}_0^{10}(f)$.

We first evaluate $\text{TV}_0^1(f)$. Let $M > 0$, define $x_0 = \frac{1}{M+1}$, then $0 < x_0 < 1$ and by Definition 3.1.2, $\text{TV}_0^{x_0}(f) \geq |f(0) - f(x_0)| = 1/(1/(M+1)) = M + 1 > M$. Therefore, using Lemma 3.1.9, for every $M > 0$, there exists

$0 < x_0 < 1$ such that $\mathrm{TV}_0^1(f) = \mathrm{TV}_0^{x_0}(f) + \mathrm{TV}_0^{x_0}(f) \geq \mathrm{TV}_0^{x_0}(f) > M$. Hence, $\mathrm{TV}_0^1(f) = \infty$, and again by Lemma 3.1.9, $\mathrm{TV}_0^{10}(f) = \mathrm{TV}_0^1(f) + \mathrm{TV}_1^{10}(f) \geq \mathrm{TV}_0^1(f) = \infty$. So, $\mathrm{TV}_0^{10}(f) = \infty$, that is, $f \notin BV([0, 10])$.

The last example, is useful to show when a function $f : [a, b] \to \mathbf{R}$ does not belong to $BV([a, b])$.

Example 3.1.13 Let $f : \mathbf{R} \to \mathbf{R}$ be a positive function with countable infinite support, then there exists $\{a_i \,|\, i \in \mathrm{N}\} \subseteq \mathbf{R}$ such that $supp(f) = \{a_i \,|\, i \in \mathrm{N}\} \subseteq \mathbf{R}$, that is, $f(a_i) > 0$ for every $i \in \mathrm{N}$, $f(x) = 0$ for every $x \in \mathbf{R} \setminus \{a_i \,|\, i \in \mathrm{N}\}$. Also assume that $\sum_{i=1}^{\infty} f(a_i) = 1$. Then

$$TV_{\mathbf{R}}(f) = \sum_{i=1}^{\infty} 2f(a_i) = 2. \tag{3.7}$$

In the case that f has a finite support, we also obtain $TV_{\mathbf{R}}(f) = 2$. That is, if f is a **discrete probability function**, then $TV_0^1(f) = 2$.

Remark 3.1.14 *We want to define the family of all the real functions such that they have finite total variation. Let $f : \mathbb{R} \to \mathbb{R}$ be a function then $f : (-\infty, \infty) \to \mathbb{R}$, in order to get finite total variation it is clear that f must belong to $BV([a, b])$ for every $-\infty < a < 0 < b < \infty$. Besides, it is clear that f must be bounded. So let us assume that f is such that $f(\infty) := \lim_{x \to \infty} f(x)$ exists and it is a real number, and similarly $f(-\infty) := \lim_{x \to -\infty} f(x)$ exists and it is also a real number. Then we can define*

$$\begin{aligned} BV([-\infty, \infty]) \quad := \quad & \{f : [-\infty, \infty] \to \mathbb{R} \;\; | \; f \in BV([a, b]) \textit{ for every} \\ & -\infty < a < 0 < b < \infty, \;\; f(\infty) := \lim_{x \to \infty} f(x) \textit{ and} \\ & f(-\infty) := \lim_{x \to -\infty} f(x) \textit{ are real numbers}\} \end{aligned} \tag{3.8}$$

Some examples to clarify the definition given in (3.8) are:

i) If $f(x) = x^3$ for every $x \in \mathbf{R}$ then clearly $f \in BV([a, b])$ for every $-\infty < a < 0 < b < \infty$, but $f(-\infty) = -\infty$ and $f(\infty) = \infty$. So, $f \notin BV([-\infty, \infty])$.

ii) If $f(x) = \sin(x)$ for every $x \in \mathbf{R}$ then clearly $f \in BV([a, b])$ for every $-\infty < a < 0 < b < \infty$, but $f(-\infty)$ and $f(\infty)$ are not well defined, since both limits do not exist. So, $f \notin BV([-\infty, \infty])$.

iii) If we define $f(x) = \sin(1/x)$ for every $x \in \mathbf{R} \setminus \{0\}$ and $f(0) = 0$, then $f(-\infty) = f(\infty) = 0$, but $f \notin BV([-1, 1])$, since f varies continuously between -1 and 1 infinitely often near $x = 0$. So, $f \notin BV([-\infty, \infty])$.

iv) If we define $f(x) = \sin(x)$ for $x \in [-2\pi, 2\pi]$ and $f(x) = \sin(1/x)$ for $|x| > 2\pi$ then it is not difficult to see $f \in BV([a, b])$ for every $-\infty < a < 0 < b < \infty$. Besides, $f(-\infty) = f(\infty) = 0$. Hence, $f \in BV([-\infty, \infty])$.

Remark 3.1.15 Let $-\infty < a < b < \infty$ be real numbers, let us consider $g\colon (a, b) \to \mathbb{R}$ a bounded function such that $g(b-)$ or $g(a+)$ does not exist. We assume without loosing generality that

$$g(a+) = \lim_{x \downarrow a} g(x)$$

does not exist. Since g is a bounded function, it is not possible that $g(a+) = \pm\infty$. Then there exist $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ two sequences in (a, b), such that $\{a_n\}_{n=1}^{\infty} \downarrow a$ and $\{b_n\}_{n=1}^{\infty} \downarrow a$ and $\lim_{n\to\infty} g(a_n) > \lim_{n\to\infty} g(b_n)$. Define $\{c_n\}_{n\geq 1}$ such that $c_1 = b_1$. $b_1 > a$ using that $\{a_n\}_{n=1}^{\infty} \downarrow a$ we can take $k \in \mathbf{N}$ such that $b_1 > a_k > a$ and $c_2 = a_k$. $a_k > a$ then we can take $s \in \mathbf{N}$ such that $a_k > b_s > a$ and we consider $c_3 = b_s$. In a similar way we obtain $c_1 > c_2 > c_3 > \cdots > c_n \downarrow a$ such that $\{c_{2i+1}\}_{i=0}^{\infty}$ is a subsequence of $\{b_n\}_{n=1}^{\infty}$ and $\{c_{21}\}_{i=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$. The sequence $\{c_n\}_{n\geq 1}$ has limit and it is a. In fact, let $\epsilon > 0$, then there exists $N_a \geq 1$ and $N_b \geq 1$ such that for every $n \geq N_a$ and for every $m \geq N_b$, $|a_n - a| < \epsilon$ and $|b_m - a| < \epsilon$. If $M > \max\{2N_a, 2(N_b - 1) + 1\}$, for $n \geq M$ even, $n = 2k$ for some $k \in \mathbf{N}$ then $n = 2k \geq M \geq 2N_a$ and $k \geq N_a$ therefore $|c_n - a| = |c_{2k} - a| = |a_k - a| < \epsilon$. If $n \geq M$ odd, $n = 2s + 1$ for some integer s, then $n = 2s + 1 \geq M \geq 2(N_b - 1) + 1$ and $s + 1 \geq N_b$ therefore $|c_n - a| = |c_{2s+1} - a| = |b_{s+1} - a| < \epsilon$. That is, for every $n \geq M$, $|c_n - a| < \epsilon$. Note that the sequence $\{g(c_n)\}_{n\geq 1}$ has no limit because if it has limit all sub sequence have a limit and each limit is the same but $\lim_{n\to\infty} g(a_n) \neq \lim_{n\to\infty} g(b_n)$.

Since $\mathbb{R}$ with the usual metric is a complete space, $\{g(c_n)\}_{n\geq 1}$ is not a Cauchy sequence. Then there exist $\epsilon_o > 0$ such that for every $N \geq 1$ exists $n, m \geq N$ with the property that $|f(c_n) - f(c_m)| \geq \epsilon_o$.

Let n be a positive integer. There exist $m_2 \geq m_1 \geq n$ such that $|g(c_{m_1}) - g(c_{m_2})| \geq \epsilon_o$. And for m_2 there exist $m_4 \geq m_3 \geq m_2 + 1$ such that $|g(c_{m_3}) -$

$g(c_{m_4})| \geq \epsilon_o$.Continuing in this way we can obtain $\{m_1, m_2, \ldots, m_{2n}\}$ such that $m_1 < m_2 < m_3 < \cdots < m_{2n}$ and

$$|g(c_{m_{2i}}) - g(c_{m_{2i-1}})| \geq \epsilon_o \tag{3.9}$$

for every $i \in \{1, 2, \ldots, n\}$.
If $\hat{g}\colon [a, b] \to \mathbb{R}$ is a function such that $\hat{g}(x) = g(x)$ for every $x \in (a, b)$ and $\hat{g}(a) = \hat{g}(b) = d$ for some real number d. Then $\{x_o, x_1, x_2, \ldots, x_{2n+2}\}$ such that $x_o = a$, $x_{2n+2} = b$ and $x_i = c_{m_i}$ for every $i \in \{1, 2, \ldots, n\}$ is a partition of $[a, b]$. And by using equation (3.9)

$$\begin{aligned}
\sum_{j=1}^{2n-1} |\hat{g}(x_{i+1}) - \hat{g}(x_i)| &\geq \sum_{j=1}^{n} |\hat{g}(x_{2i}) - \hat{g}(x_{2i-1})| \\
&= \sum_{j=1}^{n} |g(x_{2i}) - g(x_{2i-1})| \\
&\geq \sum_{j=1}^{n} \epsilon_o \\
&= n\epsilon_o
\end{aligned}$$

Then $\sum_{j=1}^{2n-1} |\hat{g}(x_{i+1}) - \hat{g}(x_i)| \geq n\epsilon_o$ for each $n \geq 1$. Hence, whatever that value of $\hat{g}$ is in a and b, from (3.1), $\mathrm{TV}_a^b(\hat{g}) = \infty$.

Definition 3.1.16 *Let $-\infty < a < b < \infty$ real numbers. And let $f\colon (a, b) \to \mathbb{R}$ be a function such that $f(b-)$ and $f(a+)$ exist and they are finite. Let us consider $\hat{f}\colon [a, b] \to \mathbb{R}$ such that $\hat{f}(x) = f(x)$ for every $x \in (a, b)$, $\hat{f}(a) = f(a+)$ and $\hat{f}(b) = f(b-)$. Then, we define the* ***total variation*** *of f as follows:*

$$TV_{(a,b)}(f) = TV_a^b(\hat{f}) \tag{3.10}$$

Notice that, if a function $h\colon (a, b) \to I\!R$ is not bounded, no matter how $h(a+)$ and $h(b-)$ are, we can define $TV_{(a,b)}(h) = \infty$.

Remark 3.1.17 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bounded function such that $f(+\infty) = \lim_{x\uparrow\infty} f(x)$ and $f(-\infty) = \lim_{x\downarrow-\infty} f(x)$ exist (and obviously they are finite). Let $h\colon (a, b) \to \mathbb{R}$ be a continuous biyection that is strictly increasing. By using Definition 3.1.16 and equation (3.10), we obtain

$$\mathrm{TV}_{\mathbb{R}}(f) = \mathrm{TV}_{(a,b)}(f \circ h) = \mathrm{TV}_a^b(\widehat{f \circ h}).$$

Now we recall the concept of metrics on non empty sets.

Example 3.1.18 We consider

$$\mathbf{B}_{[a,b]} := \{f : [a,b] \to \mathbf{R} \mid f \text{ is a bounded function}\}, \tag{3.11}$$

like in Definition 3.1.3. Let us define a function $d_{\sup}$ on $\mathbf{B}_{[a,b]} \times \mathbf{B}_{[a,b]}$ by

$$d_{\sup}(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|. \tag{3.12}$$

Recall that $d_{\sup}$ is a metric in $\mathbf{B}_{[a,b]}$.
Let us consider for every $f \in \mathbf{B}_{[a,b]}$ and $\overline{0}$ the constant function zero

$$\|f\|_{\sup} = d_{\sup}\left(f, \overline{0}\right) \tag{3.13}$$

Then, (3.13), is a norm on $\mathbf{B}_{[a,b]}$.

A very important subset of the family $\mathbf{B}_{[a,b]}$ is the family of continuous functions, that is, the family defined by

$$C([a,b]) = \{f : [a,b] \to \mathbf{R} \mid f \text{ is a continuous function}\} \tag{3.14}$$

It is well known that for every $f \in C([a,b])$, f is a bounded continuous function, in fact, there exist $c, d \in [a,b]$ such that $-\infty < f(c) = \inf\{f(x) \mid x \in [a,b]\} \leq \sup\{f(x) \mid x \in [a,b]\} = f(d) < \infty$, see [38].

Of course, $C([a,b])$ is a proper subset of $\mathbf{B}_{[a,b]}$, in fact, Dirichlet's function f given in equation (3.3) satisfies $f \in \mathbf{B}_{[0,1]} \backslash C_{[0,1]}$.

Definition 3.1.19 *Let (S,d) be a metric space and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in S, then $\{x_n\}_{n=1}^{\infty}$ is a* **Cauchy or Fundamental sequence** *if and only if for every $\epsilon > 0$ there exists an integer $N \in \boldsymbol{N}$, such that for every $n, m \geq N$, $d(x_n, x_m) < \epsilon$. The metric space (S,D) is a* **complete** *space if and only if for every $\{x_n\}_{n=1}^{\infty}$ Cauchy sequence there exists $x \in S$ such that $\{x_n\}_{n=1}^{\infty}$* **converges to** *x, that is, for every $\epsilon > 0$ there exists an integer $N \in \boldsymbol{N}$ such that for every $n \geq N$ we have $d(x_n, x) < \epsilon$.*
(S,d) is a **separable space** *if there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset S$, such that for any non empty open set A in S, there exists an integer $n \in \boldsymbol{N}$ such that $x_n \in A$, that is, any non empty open set intersects the sequence. In this case the sequence $\{x_n\}_{n=1}^{\infty}$ is called a* **countable dense** *subset of S. For a proof of this fact see for example [15].*
A metric space (S,d) which is complete and separable is called a **polish** *space.*

It is important to note that the total variation on a closed interval $[a, b]$ with $-\infty < a < b < \infty$ is not a metric on $BV([a, b])$. If we define for every $f, g \in BV([a, b])$

$$e(f, g) = \mathrm{TV}_a^b(f - g). \tag{3.15}$$

It is easy to observe that e is a symmetric function, such that $0 \leq e(f, g) < \infty$ for every $f, g \in BV([a, b])$, which also satisfies the triangle inequality. However, e does not satisfy that $e(f, g) = 0$ if and only if $f = g$. The statement $f = g$ implies $e(f, g) = 0$ holds, but the other implication does not hold. By Lemma 3.1.8 if $e(f, g) = 0$ it does not imply that $f = g$, it only implies that $f - g$ is a constant function, and this constant may be any real number different from zero. Therefore, we will ask for an extra condition to have a metric.

3.2 Bounded Variation Metric

Next, we provide a simple example to illustrate our goal of defining a metric using total variation. Let us consider $X \neq \emptyset$ and d the discrete metric, that is, $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. It is clear that this metric does not provide any information about the set X or its elements. If $X = C^o([0, 1])$ is the set of continuous functions with domain $[0, 1]$, considering $d_{\sup}$, the supremum metric, when calculating the distance between two functions, we only obtain only puntual information instead of global information.

For the proof of next Lemma see [20], Proposition 1.3.

Lemma 3.2.1 *Let $-\infty < a < b < \infty$ and let $f, g \in BV([a, b])$. If we define*

$$d_{TV_a^b}(f, g) = |f(a) - g(a)| + TV_a^b(f - g). \tag{3.16}$$

Then $\left(BV([a, b]), d_{TV_a^b}\right)$ is a metric space.

For the proof of next Lemma see [20], Proposition 1.10.

Lemma 3.2.2 *$\left(BV([a, b]), d_{TV_a^b}\right)$ is a complete space.*

Remark 3.2.3 *$\left(BV([a, b]), d_{TV_a^b}\right)$ is not a separable space.*

Let us consider $[a, b] = [0, 1]$. For $\alpha \in (0, 1)$, let us consider $1_{[\alpha,1]} : [0, 1] \to \mathbf{R}$ the characteristic function such that

$$1_{[\alpha,1]}(x) = \begin{cases} 1 & \text{if} \quad x \in [\alpha, 1] \\ 0 & \text{if} \quad x \in [0, \alpha) \end{cases} \tag{3.17}$$

Then, for $\alpha, \beta \in (0, 1)$ with $\alpha \neq \beta$, if $0 < \alpha < \beta < 1$

$$1_{[\alpha,1]}(x) - 1_{[\beta,1]}(x) = \begin{cases} 1 & \text{if} \quad x \in [\alpha, \beta) \\ 0 & \text{if} \quad x \in [0, \alpha) \cup [\beta, 1] \end{cases} \tag{3.18}$$

And $d_{TV_0^1}(1_{[\alpha,1]}, 1_{[\beta,1]}) = 2$. Now, let us consider the balls

$$B_{TV_0^1}(1_{[\alpha,1]}; 1) = \{f \in BV([0,1]) \mid d_{TV_0^1}(f, 1_{[\alpha,1]}) < 1\} \tag{3.19}$$

So, for $\alpha \neq \beta$, $B_{TV}(1_{[\alpha,1]}; 1) \cap B_{TV}(1_{[\beta,1]}; 1) = \varnothing$ because, if $\varphi \in B_{TV}(1_{[\alpha,1]}; 1) \cap B_{TV}(1_{[\beta,1]}; 1)$, then

$$d_{TV_0^1}(1_{[\alpha,1]}, 1_{[\beta,1]}) \leq d_{TV_0^1}(1_{[\alpha,1]}, \varphi) + d_{TV_0^1}(\varphi, 1_{[\beta,1]}) < 1 + 1 = 2 \tag{3.20}$$

but $d_{TV_0^1}(1_{[\alpha,1]}, 1_{[\beta,1]}) = 2$. Thus $B_{TV}(1_{[\alpha,1]}; 1) \cap B_{TV}(1_{[\beta,1]}; 1) = \varnothing$.

If $E \subsetneq BV([0, 1])$ is a dense set for $(BV([0, 1]), d_{TV})$, then for every $\alpha \in [0, 1]$ there exists $f_\alpha \in E \cap B_{TV}(1_{[\alpha,1]}; 1)$ and for $\alpha \neq \beta$, $f_\alpha \neq f_\beta$. Thus E has at least the cardinality of $(0, 1)$.

$(BV([0, 1]), d_{TV})$ is not a separable space. For this reason, we will make some modifications to d_{TV} so that we get a *Polish space*, that is, a separable and complete space.

Sometimes the supreme metric, $d_{\sup}$, does not detect the fractal-type behaviors of some functions. Therefore we want to propose a metric that does preserve them. To show this, we will look at some examples below.

Example 3.2.4 For n=1 Let us consider $F_1, G_1 : [0, 1] \to \mathbf{R}$ such that $F_1(x) = x^2$ and $G_1(x) = \sqrt{x}$

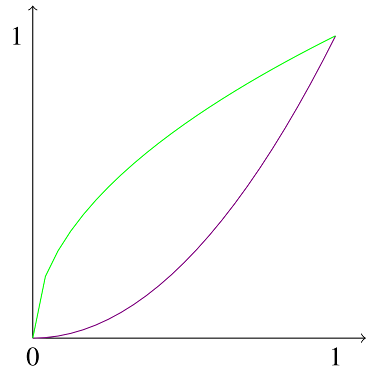

Figure 3.1 Graph of F_1 and G_1.

If $h_1 : [0,1] \to \mathbf{R}$ is such that $h_1(x) = G_1(x) - F_1(x)$, then $h_1'(x) = \frac{1}{2\sqrt{x}} - 2x$ on $(0,1)$. We have that $h_1'(x_0) = 0$ if and only if $x_0 = \left(\frac{1}{4}\right)^{2/3}$. Also, $h_1''(x) = -\frac{1}{4\sqrt{x^3}} - 2$ and $h_1''\left(\left(\frac{1}{4}\right)^{2/3}\right) = -1 - 2 = -3 < 0$. Thus x_0 is a maximum.

We have that $h_1(x_0) = \sqrt{x_0} - x_0^2 = \left(\frac{1}{4}\right)^{1/3} - \left(\frac{1}{4}\right)^{4/3} \approx 0.47247$ and $d_{\sup}(F_1, G_1) = h_1(x_0)$,

$$\begin{aligned} d_{TV_0^1}(F_1,G_1) &= 2h_1(x_0) = 2\left(\left(\frac{1}{4}\right)^{1/3} - 2\left(\frac{1}{4}\right)^{4/3}\right) \\ &= \left(\frac{1}{4}\right)^{1/3}\left(2 - 2\left(\frac{1}{4}\right)\right) = \left(\frac{1}{4}\right)^{1/3}\left(\frac{3}{2}\right) \approx 0.94494 \end{aligned}$$

For n=3 Let us consider

$$F_3(x) = \begin{cases} 3x^2 & if \quad 0 \le x \le 1/3 \\ \frac{1}{3} + \frac{1}{\sqrt{3}}\sqrt{x - 1/3} & if \quad 1/3 \le x \le 2/3 \\ \frac{2}{3} + 3(x - 2/3)^2 & if \quad 2/3 \le x \le 1 \end{cases} \tag{3.21}$$

and

$$G_3(x) = \begin{cases} \frac{1}{\sqrt{3}}\sqrt{x} & if \quad 0 \le x \le 1/3 \\ \frac{1}{3} + 3\,(x - 1/3)^2 & if \quad 1/3 \le x \le 2/3 \\ \frac{2}{3} + \frac{1}{\sqrt{3}}\sqrt{x - 2/3} & if \quad 2/3 \le x \le 1 \end{cases} \tag{3.22}$$

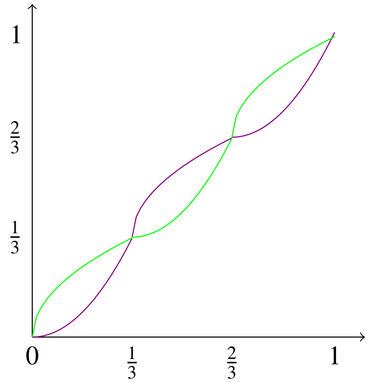

Figure 3.2 Graph of F_3 and G_3.

If $h_3 \colon \left[0, \frac{1}{3}\right] \to \mathbf{R}$ is such that $h_3(x) = G_3(x) - F_3(x)$, then $h_3'(x) = \frac{1}{2\sqrt{3}\sqrt{x}} - 6x$ on $\left(0, \frac{1}{3}\right)$. We have that $h_3'(x_0) = 0$ if and only if $x_0 = \left(\frac{1}{12\sqrt{3}}\right)^{2/3}$. Also, $h_3''(x) = -\frac{1}{4\sqrt{3}\sqrt{x^3}} - 6$ and $h_3''\left(\left(\frac{1}{12\sqrt{3}}\right)^{2/3}\right) = -3 - 6 = -9 < 0$. Thus x_0 is a maximum.

We have that $h_3(x_0) = \frac{1}{\sqrt{3}}\sqrt{x_0} - 3x_0^2 = \frac{1}{\sqrt{3}}\left(\frac{1}{12\sqrt{3}}\right)^{1/3} - 3\left(\frac{1}{12\sqrt{3}}\right)^{4/3} \approx 0.157490$ and $d_{\sup}(F_3, G_3) = h_3(x_0)$, $d_{TV_0^1}(F_3, G_3) = 6h_3(x_0) \approx 0.94494$

For n=5 Let us consider

$$F_5(x) = \begin{cases} 5x^2 & if \quad 0 \le x \le 1/5 \\ \frac{1}{5} + \frac{1}{\sqrt{5}}\sqrt{x - 1/5} & if \quad 1/5 \le x \le 2/5 \\ \frac{2}{5} + 5(x - 2/5)^2 & if \quad 2/5 \le x \le 3/5 \\ \frac{3}{5} + \frac{1}{\sqrt{5}}\sqrt{x - 3/5} & if \quad 3/5 \le x \le 4/5 \\ \frac{4}{5} + 5(x - 4/5)^2 & if \quad 4/5 \le x \le 1 \end{cases} \tag{3.23}$$

and

$$G_5(x) = \begin{cases} \frac{1}{\sqrt{5}}\sqrt{x} & if \quad 0 \le x \le 1/5 \\ \frac{1}{5} + 5\,(x - 1/5)^2 & if \quad 1/5 \le x \le 2/5 \\ \frac{2}{5} + \frac{1}{\sqrt{5}}\sqrt{x - 2/5} & if \quad 2/5 \le x \le 3/5 \\ \frac{3}{5} + 5\,(x - 3/5)^2 & if \quad 3/5 \le x \le 4/5 \\ \frac{4}{5} + \frac{1}{\sqrt{5}}\sqrt{x - 4/5} & if \quad 4/5 \le x \le 1 \end{cases} \tag{3.24}$$

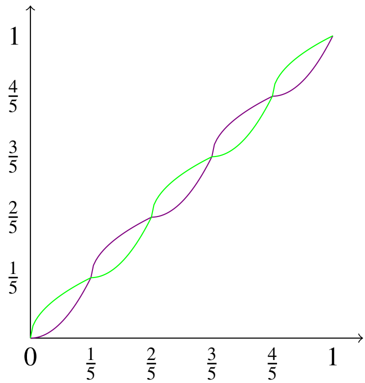

Figure 3.3 Graph of F_5 and G_5.

If $h_5 \colon \left[0, \frac{1}{5}\right] \to \mathbf{R}$ is such that $h_5(x) = G_5(x) - F_5(x)$, then $h_5'(x) = \frac{1}{2\sqrt{5}\sqrt{x}} - 10x$ on $\left(0, \frac{1}{5}\right)$. We have that $h_5'(x_0) = 0$ if and only if $x_0 = \left(\frac{1}{20\sqrt{5}}\right)^{2/3}$. Also, $h_5''(x) = -\frac{1}{4\sqrt{5}\sqrt{x^3}} - 10$ and $h_5''\left(\left(\frac{1}{20\sqrt{5}}\right)^{2/3}\right) < 0$. Thus x_0 is a maximum.

We have that $h_5(x_0) = \frac{1}{\sqrt{5}}\sqrt{x_0} - 5x_0^2 = \frac{1}{\sqrt{5}}\left(\frac{1}{20\sqrt{5}}\right)^{1/3} - 5\left(\frac{1}{20\sqrt{5}}\right)^{4/3} \approx 0,094494$ and $d_{\sup}(F_5, G_5) = h_5(x_0)$, $d_{TV_0^1}(F_5, G_5) = 10h_5(x_0) \approx 0.94494$

In general, for $n \geq 5$, if $h_n \colon \left[0, \frac{1}{n}\right] \to \mathbf{R}$ is such that $h_n(x) = G_n(x) - F_n(x) = \frac{1}{\sqrt{n}}\sqrt{x} - nx^2$ we have that $x_0 = \left(\frac{1}{4n\sqrt{n}}\right)^{2/3}$ is a maximum.

Also

$$
\begin{aligned}
d_{TV_0^1}(F_n, G_n) = 2nh_n(x_0) &= 2n\left(\frac{1}{\sqrt{n}}\left(\frac{1}{4n\sqrt{n}}\right)^{1/3} - n\left(\frac{1}{4n\sqrt{n}}\right)^{4/3}\right) \\
&= \left(\frac{1}{4n\sqrt{n}}\right)^{1/3}\left(\frac{2n}{\sqrt{n}} - 2n^2\left(\frac{1}{4n\sqrt{n}}\right)\right) \\
&= \left(\frac{1}{4n\sqrt{n}}\right)^{1/3}\sqrt{n}\left(2 - \frac{1}{2}\right) \\
&= \left(\frac{\sqrt{n}^3}{4n\sqrt{n}}\right)^{1/3}\frac{3}{2} = \left(\frac{1}{4}\right)^{1/3}\frac{3}{2} := K \approx 0.94494
\end{aligned}
$$

And $d_{\sup}(F_n, G_n) = h_n(x_0) = \frac{K}{2n} \to_{n\to\infty} 0$

Example 3.2.5 For every $m \in \mathbf{N}$, let us consider $f_{m,1}, g_{m,1} : [0,1] \to \mathbf{R}$ such that $f_{m,1}(x) = x^m$ and $g_{m,1}(x) = \sqrt[m]{x}$. And for every $n \geq 2$, let us consider $f_{m,n}, g_{m,n} : [0,1] \to \mathbf{R}$

$$f_{m,n}(x) = \frac{i}{n} + n^{m-1}\left(x - \frac{i}{n}\right)^m \text{ if } x \in \left[\frac{i}{n}, \frac{i+1}{n}\right] \text{ for every } i \in \{0, \ldots, n-1\}.$$

and

$$g_{m,n}(x) = \frac{i}{n} + \frac{1}{\sqrt[m]{n^{m-1}}}\sqrt[m]{x - \frac{i}{n}} \text{ if } x \in \left[\frac{i}{n}, \frac{i+1}{n}\right] \text{ for every } i \in \{0, \ldots, n-1\}.$$

In the following graph we can to see $f_{3,3}$, $g_{3,3}$, $f_{4,3}$, $g_{4,3}$, $f_{5,3}$, $g_{5,3}$, $f_{6,3}$, $g_{6,3}$, $f_{7,3}$, $g_{7,3}$, $f_{8,3}$, $g_{8,3}$

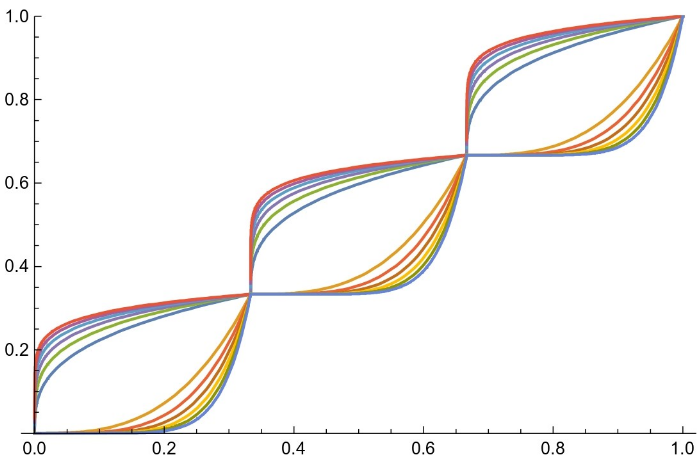

Figure 3.4 Graphs of $f_{m,n}$ and $g_{m,n}$ for $n = 3$.

For every $m, n \in \mathbf{N}$, let us consider $h_{m,n} : [0,1] \to \mathbf{R}$ such that $h_{m,n} =$

$g_{m,n} - f_{m,n} = |g_{m,n} - g_{m,n}|$. Note that, for every $x \in \left(0, \frac{1}{n}\right)$,

$$\begin{aligned} h'_{m,n}(x) &= g'_{m,n}(x) - f'_{m,n}(x) = \frac{1}{m}\frac{1}{\sqrt[m]{n^{m-1}}}x^{\frac{1}{m}-1} - n^{m-1}mx^{m-1} \\ &= \frac{1}{m}\frac{1}{\sqrt[m]{n^{m-1}}}x^{\frac{1-m}{m}} - n^{m-1}mx^{m-1} = \frac{1}{m}\frac{1}{\sqrt[m]{n^{m-1}}}x^{\frac{1-m}{m}} - n^{m-1}mx^{-\frac{(1-m)m}{m}} \\ &= x^{\frac{1-m}{m}}\left(\frac{1}{m}\frac{1}{\sqrt[m]{n^{m-1}}} - n^{m-1}mx^{\frac{-(1-m)(m+1)}{m}}\right) \end{aligned} \tag{3.25}$$

Then, $h'_{m,n}(x) = 0$ when

$$x^{\frac{(1-m)(m+1)}{m}} = n^{m-1}m^2\sqrt[m]{n^{m-1}}$$

that is,

$$x_0 = \left(n^{m-1}m^2\sqrt[m]{n^{m-1}}\right)^{\frac{m}{(1-m)(1+m)}}$$

$$\begin{aligned} h_{m,n}(x_0) &= \frac{1}{\sqrt[m]{n^{m-1}}}\left(n^{m-1}m^2\sqrt[m]{n^{m-1}}\right)^{\frac{1}{(1-m)(1+m)}} - n^{m-1}\left(n^{m-1}m^2\sqrt[m]{n^{m-1}}\right)^{\frac{m^2}{(1-m)(1+m)}} \\ &= \left(n^{m-1}m^2\sqrt[m]{n^{m-1}}\right)^{\frac{1}{(1-m)(1+m)}}\left(\frac{1}{\sqrt[m]{n^{m-1}}} - n^{m-1}\left(n^{m-1}m^2\sqrt[m]{n^{m-1}}\right)^{\frac{m^2-1}{(1-m)(1+m)}}\right) \\ &= \left(n^{m-1}m^2\sqrt[m]{n^{m-1}}\right)^{\frac{1}{(1-m)(1+m)}}\left(\frac{1}{\sqrt[m]{n^{m-1}}} - n^{m-1}\left(n^{m-1}m^2\sqrt[m]{n^{m-1}}\right)^{-1}\right) \\ &= \left(n^{m-1}m^2\sqrt[m]{n^{m-1}}\right)^{\frac{1}{(1-m)(1+m)}}\frac{1}{\sqrt[m]{n^{m-1}}}\left(1 - \frac{1}{m^2}\right) \\ &= n^{\frac{-1}{m}}\left(m^2\right)^{\frac{1}{(1-m)(1+m)}}\frac{1}{\sqrt[m]{n^{m-1}}}\left(1 - \frac{1}{m^2}\right) \\ &= \left(m^2\right)^{\frac{1}{(1-m)(1+m)}}\frac{1}{n}\left(1 - \frac{1}{m^2}\right) \end{aligned} \tag{3.26}$$

We have that

$$
\begin{aligned}
TV_0^1(h_{m,n}) &= \sum_{i=0}^{n-1} TV_{i/n}^{(i+1)/n}(h_{m,n}) = \sum_{i=0}^{n-1} 2h_{m,n}\left(x_0 + \frac{i}{n}\right) \\
&= 2n\left(m^2\right)^{\frac{1}{(1-m)(1+m)}} \frac{1}{n}\left(1 - \frac{1}{m^2}\right) \\
&= 2\left(m^2\right)^{\frac{1}{(1-m)(1+m)}}\left(1 - \frac{1}{m^2}\right) \\
&= 2\frac{1}{m^{\frac{2}{m^2-1}}}\left(\frac{m^2-1}{m^2}\right)
\end{aligned}
\tag{3.27}
$$

Using L'H$\hat{o}$pital we obtain

$$
\begin{aligned}
\lim_{m\to\infty} TV_0^1(h_{m,n}) &= \lim_{m\to\infty} 2\frac{1}{m^{\frac{2}{m^2-1}}}\left(\frac{m^2-1}{m^2}\right) \\
&= 2\lim_{m\to\infty}\frac{1}{m^{\frac{2}{m^2-1}}}\lim_{m\to\infty}\left(\frac{m^2-1}{m^2}\right) \\
&= 2\frac{1}{\lim_{m\to\infty} m^{\frac{2}{m^2-1}}}\lim_{m\to\infty}\left(\frac{m^2-1}{m^2}\right) \\
&= 2\frac{1}{\lim_{m\to\infty} m^{\frac{2}{m^2-1}}} \\
&= 2\frac{1}{e^{2\lim_{m\to\infty}\frac{\log(m)}{m^2-1}}} = 2\frac{1}{e^{2\lim_{m\to\infty}\frac{1/m}{2m}}} \\
&= 2\frac{1}{e^{2\lim_{m\to\infty}\frac{1}{2m^2}}} = 2\frac{1}{e^0} = 2
\end{aligned}
$$

Since, using equation (3.27), we have

$$
\lim_{m\to\infty} TV_0^1(h_{m,n}) = \lim_{m\to\infty} TV_0^1(g_{m,n} - f_{m,n}) = 2
$$

which is the maximum of the total variation between two distribution functions.

Example 3.2.6 Let us consider $[0, 2\pi]$ the closed interval. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions defined on $[0, 2\pi]$ such that f_0 is the constant function 0 and for $n \geq 1$, $f_n(x) = \frac{1}{n}\sin(n^2 x)$ for all $x \in [0, 2\pi]$.

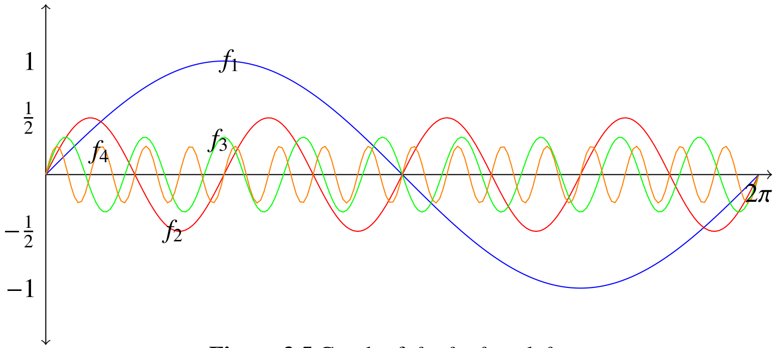

Figure 3.5. Graph of f_1, f_2, f_3 and f_4.

Now, for $f_1(x) = \sin(x)$, we have

$$\begin{aligned} \mathrm{TV}_0^{2\pi}(f_1) &= \mathrm{TV}_0^{\pi}(f_1) + \mathrm{TV}_{\pi}^{2\pi}(f_1) \\ &= 2 + 2 = 4 \end{aligned}$$

also, $f_2(x) = \frac{1}{2}\sin(4x)$, then

$$\mathrm{TV}_0^{2\pi}(f_2) = \sum_{i=0}^{7} \mathrm{TV}_{i\pi/4}^{(i+1)\pi/4}(f_2) = 8 \cdot 2\left(\frac{1}{2}\right) = 8 = 4 \cdot 2.$$

In general, for $n \geq 3$, $f_n(x) = n \cdot \frac{1}{n^2}\sin(n^2 x)$, so

$$\mathrm{TV}_0^{2\pi}(f_n) = \sum_{i=0}^{2n^2-1} \mathrm{TV}_{i\pi/(n^2)}^{(i+1)\pi/(n^2)}(f_n) = 2n^2 \cdot 2\left(\frac{1}{n}\right) = 4n.$$

so,

$$\begin{aligned} d_{\mathrm{TV}_0^{2\pi}}(f_n, f_0) &= |f_n(0) - f_0(0)| + \mathrm{TV}_0^{2\pi}(f_n - f_0) \\ &= 0 + 4n = 4n \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} d_{\mathrm{TV}_0^{2\pi}}(f_n, f_0) = \lim_{n\to\infty} 4n = \infty. \tag{3.28}$$

On the other hand

$$d_{\sup}(f_n, f_0) = \sup_{x\in[0,2\pi]} |f_n(x) - f_0(x)| = \sup_{x\in[0,2\pi]} |f_n(x) - 0| = \sup_{x\in[0,2\pi]} |f_n(x)| = \frac{1}{n}. \tag{3.29}$$

So,

$$0 \leq \lim_{n\to\infty} d_{\sup}(f_n, f_0) = \lim_{n\to\infty} \frac{1}{n} = 0. \tag{3.30}$$

And,

$$\infty = \lim_{n\to\infty} d_{\mathrm{TV}_0^{2\pi}}(f_n, f_0) \neq \lim_{n\to\infty} d_{\sup}(f_n, f_0) = 0. \tag{3.31}$$

Example 3.2.7 Recall that a function $C : \mathbf{I}^2 \to \mathbf{R}$ is a **copula** if and only if C satisfies:
i) For every $u, v \in \mathbf{I}$

$$C(u, 0) = 0 = C(0, v), \quad C(u, 1) = u \quad \text{and} \quad C(1, v) = v. \tag{3.32}$$

ii) For every $0 \leq u_1 \leq u_2 \leq 1$ and $0 \leq v_1 \leq v_2 \leq 1$ we have that

$$V_C([u_1, u_2] \times [v_1, v_2]) = C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0. \tag{3.33}$$

In particular,

$$C(u, v) = V_C([0, u] \times [0, v]). \tag{3.34}$$

Then C is a **bivariate distribution function** with uniform marginals $(0, 1)$ that is jointly uniformly continuous. See [27].
It can be shown that any copula is twice differentiable almost everywhere in $(0, 1)^2$ with respect to λ^2 the Lebesgue measure. Furthermore, if we define

$$c(u, v) = \frac{\partial^2}{\partial u \partial v} C(u, v), \tag{3.35}$$

then $c(u, v)$ is the **bivariate density** of C, and it exists almost everywhere with respect to λ^2 the Lebesgue measure. It is important to note that a copula C can be a singular continuous function. In fact, we known that

$$W(u, v) = \max\{0, u + v - 1\} \leq C(u, v) \leq \min\{u, v\} = M(u, v)$$

for all $(u, v) \in I^2$, where W and M are copulas that are singular continuous. W has its support on the secondary diagonal $D_2 = \{(u, v) \in I^2 \,|\, u + v = 1\}$, and M has its support on the main diagonal $D_1 = \{(u, v) \in I^2 \,|\, u = v\}$. Both D_1 and D_2 have Lebesgue measure λ^2 equal to zero. W and M are known as the **lower and upper Fréchet-Hoeffding bounds**. Note that if $(u, v) \in I^2$ and $u < v$, then $M(u, v) = u$, meaning $M(u, v) = u$ for every point in I^2 on the main diagonal. Similarly, $M(u, v) = v$ for every point in I^2 below the main diagonal. Therefore, using (3.35) the density satisfies that $c(u, v) = 0$ for all $(u, v) \in I^2 \backslash D_1$.

If we define $\Pi(u, v) = uv$ for all $(u, v) \in I^2$, then it is straightforward to see that Π is a copula called the **product copula**. It is the unique copula representing two independent uniformly distributed random variables on the interval I. Using (3.35) its density satisfies that $\pi(u, v) = 1$ for all $(u, v) \in \mathbf{I}^2$.

Now, we will define what we will call **Shuffles of** M.

"The mass of a shuffle of M is obtained by using the support D_1 of M in I^2, making vertical cuts in I^2 in a finite number of vertical strips, and randomly shuffling the strips and allowing some or all of them to be flipped. The result of these steps will be a new singular copula, called a shuffle of M, with support equal to the union of those strips."

The formal definition is

Definition *Since the support of the copula M is the main diagonal $D_1 = \{(x, x) \in I^2 \mid x \in I\}$. Let $a_0, a_1, a_2, \ldots a_{n-1}, a_n$ be a* **partition of** *I, i.e., $0 = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = 1$. Let $J_1 = [a_0, a_1) = [0, a_1)$, $J_2 = [a_1, a_2)$, $\ldots$, $J_i = [a_{i-1}, a_i) \ldots J_n = [a_{n-1}, a_n) = [a_{n-1}, 1)$ and $K_n = \{1\}$, for each partition of I and for each $n \geq 1$. Then $\{J_i\}_{i=1}^n$ is a disjoint family of non-empty intervals closed on the left and open on the right, and for each $a_0, a_1, a_2, \ldots a_{n-1}, a_n$ partition of I with $n \geq 1$ $\cup_{i=1}^n J_i \cup K_n = I$, which is a disjoint union. Given $a_0, a_1, a_2, \ldots a_{n-1}, a_n$ a partition of I with $n \geq 1$, and the intervals $\{J_i\}_{i=1}^n$, defined as above, we define the* **strips** *$\{J_i \times [0, 1]\}_{i=1}^n \subset I^2$ and $K_n \times [0, 1] \subset I^2$. Then it is obvious that $I^2 = (\cup_{i=1}^n J_i \times I) \cup (K_n \times I)$, which is a disjoint union. Let $T_n = \{1, 2, \ldots, n\}$ for each $n \in \mathbb{N}$, the set of the first n natural numbers, and remember that a* **permutation of** *n is simply a bijective map from T_n onto T_n, which we will write as $\sigma : T_n \to T_n$, and denote the range of σ by $Ran(\sigma) = (\sigma) = \{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$. For a fixed permutation σ of T_n, we permute the strips $\{J_i \times I\}_{i=1}^n$ by taking $\{J_{\sigma(i)} \times I\}_{i=1}^n$ and leaving $K_n \times I = \{1\} \times [0, 1]$ fixed. Therefore, the original strips contain a non-empty piece of D_1, and by permuting the strips, the new strips have a linear segment of D_1, but in a new order. We will denote by $\omega : T_n \to \{-1, 1\}$ a function that indicates whether we flip the strips*

180 degrees, when $\omega(i) = -1$, or do not flip them when $\omega(i) = 1$, the linear segment of D_1. The support of the shuffle of M can be written as a function of $n \in \mathbb{N}$, a partition of I that induces the intervals $\{J_i\}_{i=1}^n$, a permutation σ of T_n and the function ω. The resulting copula is denoted by $M(n, \{J_i\}_{i=1}^n, (\sigma), \omega)$.

Observation *Any shuffle $M(n, J_{i\,i=1}^{\,n}, (\sigma), \omega)$ is always a singular copula.*

As a simple example, let $n = 6$ and $0 = a_0 < a_1 = \frac{1}{10} < a_2 = \frac{2}{10} < a_3 = \frac{4}{10} < a_4 = \frac{5}{10} < a_5 = \frac{9}{10} < a_6 = \frac{10}{10} = 1$, and let $i = 3$ and $j = 5$. After applying the permutation $(\sigma) = (1, 2, 5, 4, 3, 6)$, we obtain $0 = b_0 = a_0$, $\frac{1}{10} = b_1 = a_1$, $\frac{2}{10} = b_2 = a_2$, $\frac{6}{10} = \frac{2}{10} + (\frac{9}{10} - \frac{5}{10}) = b_3 = b_i = a_{i-1} + (a_j - a_{j-1})$, $\frac{7}{10} = \frac{6}{10} + (\frac{5}{10} - \frac{4}{10}) = b_i + (a_{i+1} - a_i) = b_4 = b_{i+1}$, $\frac{9}{10} = \frac{7}{10} + (\frac{4}{10} - \frac{2}{10}) = b_4 + (a_3 - a_2) = b_{j-1} + (a_i - a_{i-1}) = a_j$, and $1 = \frac{10}{10} = a_{j+1} = a_6 = b_6$.

In Figure 1, we plot the original strips associated with the partition $0 = a_0 < a_1 = \frac{1}{10} < a_2 = \frac{2}{10} < a_3 = \frac{4}{10} < a_4 = \frac{5}{10} < a_5 = \frac{9}{10}, a_6 = \frac{10}{10} = 1$, then apply the permutation $(\sigma) = (1, 2, 5, 4, 3, 6)$, and in Figure 2, we plot the permuted strips along with the corresponding strips of the support of M. In this case, the new partition of $\mathbf{I}$ is given by $0 = b_0 < \frac{1}{10} = b_1 < \frac{2}{10} = b_2 < \frac{6}{10} = b_3 < \frac{7}{10} = b_4 < \frac{9}{10} = b_5 < \frac{10}{10} = 1 = b_6$. Note that the partition of $\mathbf{I}$ on the v-axis does not change in both figures. It can be observed that we are only moving the support of M to a new bijection from $\mathbf{I}$ onto $\mathbf{I}$, which is the support of $M(n = 6, J_i i = 1^6, \sigma, \omega)$, where $\omega(i) = 1$ for each $i \in T_n$. Of course, if we take a new function ω, such that $\omega(i) = -1$, for some $i \in T_n$, we are only flipping the i-th strip to obtain a line segment with slope -1 instead of 1, which is part of the permuted support of the copula W. Therefore, any shuffle $M(n, J_i i = 1^n, (\sigma), \omega)$ is always a singular copula.

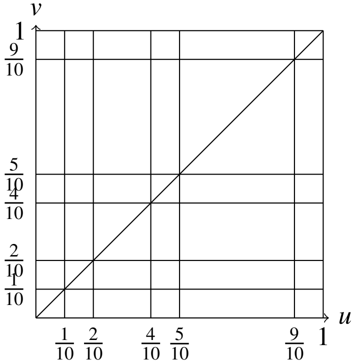

Figure 3.6. Graph of the original strips with the partition $0 < \frac{1}{10} < \frac{2}{10} < \frac{4}{10} < \frac{5}{10} < \frac{9}{10} < \frac{10}{10}$

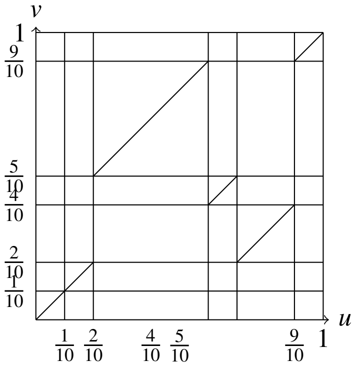

Figure 3.7. Graph with the permuted strips with the partition $0 < \frac{1}{10} < \frac{2}{10} < \frac{6}{10} < \frac{7}{10} < \frac{9}{10} < \frac{10}{10}$

The theory of copulas is strongly related to the study of the concept of independence. It is known that two continuous random variables, X and Y, are independent if and only if their copula $C_{X,Y} = \Pi$ is the product copula Π. It is crucial to define degrees of dependence, and indeed, it is relevant to determine when two variables are "extremely" dependent, that is, what constitutes a proximity to independence. Many authors have proposed defining "extreme dependence." The definition we consider most appropriate is the concept defined by Lancaster in 1963 for mutually completely dependent random variables X and Y, see [22]. In other words,

Definition: *Two random variables X and Y defined on the same probability space* $(\Omega, \mathcal{F}, P)$ *are* **mutually completely dependent** *if and only if there exists a bijection* φ *such that*

$$P(\{\omega \in \Omega \,|\, Y(\omega) = \varphi(X(\omega)) \text{ and } X(\omega) = \varphi^{-1}(Y(\omega))\}) = 1,$$

meaning X and Y are almost surely invertible functions of each other.

Thus, there exists a set $B \in \mathcal{F}$ with $P(B) = 1$, such that if we know the value of $X(\omega)$ for $\omega \in B$, then we also know the value of $Y(\omega)$, and vice versa.

Let X and Y be two random variables with copula $C_{X,Y}$ that is a shuffle of M, i.e., $C_{X,Y} = M(n, \{J_i\}_{i=1}^n, (\sigma), \omega)$. If U and V are the variables determined by the shuffle, then the support of the copula is a bijection φ from $\mathbf{I}$ to $\mathbf{I}$, and φ^{-1} is well-defined. Therefore, X and Y are mutually completely dependent.

We will show that the copula Π is the uniform limit of shuffles that are mutually completely dependent. This turns out to be very surprising because independence can be uniformly approximated by random variables that are mutually completely dependent.

Theorem: *For every* $\epsilon > 0$*, there exists a shuffle of M denoted by* C_ϵ *such that*

$$\sup_{(u,v)\in\mathbf{I}^2} |C_\epsilon(u, v) - \Pi(u, v)| < \epsilon.$$

Proof: Let $\epsilon > 0$ and let $m \in \mathbb{N}$ be such that $\frac{1}{m} \le \frac{\epsilon}{4}$.

Let $n = m^2$, let $0 = a_0 < a_1 = \frac{1}{n} < a_2 = \frac{2}{n} < \ldots < a_{n-1} = \frac{n-1}{n} < a_n = \frac{n}{n}$ be the **uniform partition** of $\mathcal{I}$ of size n, and define $J_i = [a_{i-1}, a_i) = \left[\frac{i-1}{n}, \frac{i}{n}\right)$ for each $i \in T_n$. Let σ be the permutation of T_n such that

$$\sigma(m(j-1) + k) = m(k-1) + j \quad \text{for every} \quad k, j \in \{1, 2, \ldots, m\} = T_m,$$

and let $\omega : T_n \to \{-1, 1\}$ be an arbitrary function. Define

$$C_\epsilon = M(n = m^2, \{J_i\}_{i=1}^n, (\sigma), \omega).$$

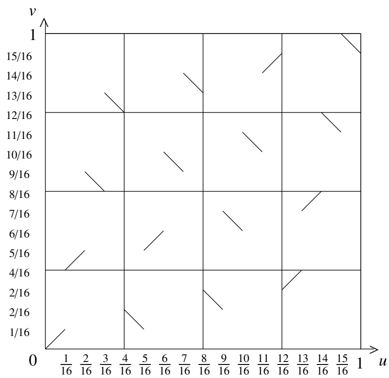

Figure 3.8 Graph of one of the possible supports of the shuffle of M when $m = 4$

Then it can be seen that

$$|C_\epsilon(u, v) - \Pi(u, v)| \quad \leq \quad \epsilon. \tag{3.36}$$

It is not difficult to see that if $\epsilon > 0$ and C is a copula, then there exists a shuffle of M, denoted by C_ϵ, such that

$$\sup_{(u,v)\in \mathbf{I}^2} |C(u, v) - C_\epsilon(u, v)| < \epsilon.$$

In summary, the family $\mathcal{G}$ of all shuffles of M, with the **supremum distance**, is **dense** in the family of all copulas $\mathcal{C}$.

If we use the Total Variation using any shuffle of M the total variation is

$$d_{TV}(c_\epsilon, \pi) = \frac{1}{2}\int\int |c_\epsilon(u, v) - \pi(u, v)| dudv = \frac{1}{2}\int\int |0 - 1| dudv = \frac{1}{2}$$

3.3 Continuous and Absolutely Continuous functions.

Remark 3.3.1 *There exist continuous functions that have no finite bounded variation, that is,* $C[0, 1] \nsubseteq BV([0, 1])$*. Also, there are density functions that have no finite bounded variation.*

For example, let us consider the odd integer numbers, this numbers form a countable set, then we can write all odd numbers as $\{a_i \mid i \in \mathbf{N}\}$ with $a_1 < a_2 < \cdots < a_k < \cdots$ and let us consider $f\colon [0,1] \to \mathbf{R}$ such that

$$f(x) = \begin{cases} 2\left(x - \frac{1}{2}\right) & if \quad x \in \left(\frac{1}{2}, 1\right] \\ -3\left(x - \frac{1}{3}\right) + \frac{1}{2} & if \quad x \in \left(\frac{1}{3}, \frac{1}{2}\right] \\ 6(x - \frac{1}{4}) & if \quad x \in \left(\frac{1}{4}, \frac{1}{3}\right] \\ \vdots & \vdots \quad \vdots \\ \frac{-a_k(a_k-1)}{k}\left(x - \frac{1}{a_k}\right) + \frac{1}{k} & if \quad x \in \left(\frac{1}{a_k}, \frac{1}{a_k-1}\right] \\ \frac{a_k(a_k+1)}{k}\left(x - \frac{1}{a_k+1}\right) & if \quad x \in \left(\frac{1}{a_k+1}, \frac{1}{a_k}\right] \\ \vdots & \vdots \quad \vdots \\ 0 & if \quad x = 0 \end{cases} \tag{3.37}$$

f is a continuous function, see figure 1, it is clear that f is continuous in each $\left(\frac{1}{k+1}, \frac{1}{k}\right]$ for every $k \in \mathbf{N}$. And, for $\epsilon > 0$, there exists $k \in \mathbf{N}$ such that k is odd $\frac{1}{k} < \epsilon$ and $a_k > k$, then, if $0 \leq y < \frac{1}{a_k - 1}$ we have $|f(0) - f(y)| = |0 - f(y)| = |f(y)| \leq \frac{1}{k} < \epsilon$ and f is continuous in 0.

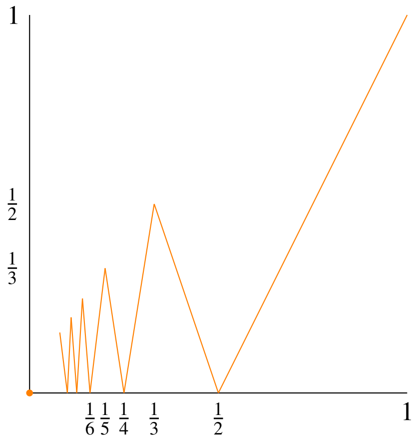

Figure 3.9 Graph of f.

Also, if $\overline{0}$ is the constant function zero,

$$\begin{aligned}
d_{TV_0^1}\left(f,\overline{0}\right) = TV_0^1(f) &= TV_{[\frac{1}{2},1]}(f) + \sum_{k=1}^{\infty} TV_{[\frac{1}{2k+2},\frac{1}{2k}]}(f) \\
&= TV_{[\frac{1}{2},1]}(f) + \sum_{k=2}^{\infty} \frac{2}{k} \\
&= 1 + 2\sum_{k=2}^{\infty} \frac{1}{k} \\
&= 2 + 2\sum_{k=2}^{\infty} \frac{1}{k} - 1 \\
&= 2\left(\sum_{k=1}^{\infty} \frac{1}{k}\right) - 1 = \infty. \qquad (3.38)
\end{aligned}$$

Also, note that the area under the curve on $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ for $n \in \mathbf{N}$ is

$$\begin{aligned}
&\begin{cases} \frac{\left(\frac{1}{n}-\frac{1}{n+1}\right)\frac{2}{n+1}}{2} & if \quad n+1 \text{ is even} \\ \frac{\left(\frac{1}{n}-\frac{1}{n+1}\right)\frac{2}{n+2}}{2} & if \quad n+1 \text{ is odd} \end{cases} \\
= &\begin{cases} \frac{1}{n(n+1)^2} & if \quad n+1 \text{ is even} \\ \frac{1}{n(n+1)(n+2)} & if \quad n+1 \text{ is odd} \end{cases} \qquad (3.39)
\end{aligned}$$

If $K = \int_0^1 f(t)dt$, then

$$\begin{aligned}
K &= \sum_{\{n\geq 1 \,|\, n+1 \text{ is even }\}} \frac{1}{n(n+1)^2} + \sum_{\{n\geq 1 \,|\, n+1 \text{ is odd }\}} \frac{1}{n(n+1)(n+2)} \\
&< \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} < \infty \qquad (3.40)
\end{aligned}$$

So, if $g\colon [0,1] \to \mathbf{R}$ is such that $g(x) = \frac{f(x)}{K}$ for every $x \in [0,1]$, then $TV_0^1(g) = \frac{1}{K} TV_0^1(f) = \infty$ and g is a density function. Also, $d_{\sup}\left(f,\overline{0}\right) = 1$. □

In order to work with a continuous function with bounded variation we will introduce the absolutely continuous functions, also will show some properties of absolutely continuous functions.

For $-\infty < a < b < \infty$ we denote $\sum([a,b])$ the family of all finite collections $S = \{[a_1,b_1],\ldots,[a_n,b_n]\}$ of pairwise nonoverlapping subintervals of $[a,b]$, that is, S is a finite collection of subintervals of $[a_i,b_i]$ with $a \le a_i < b_i \le b$ and for every $I,J \in S$ with $I \neq J$, $I \cap J = \emptyset$ or $I \cap J$ is a singleton. That is, for every $I,J \in S$ with $I \neq J$ we have that $int(I) \cap int(J) = \emptyset$. For the following definition see [20].

Definition 3.3.2 *Let $f\colon [a,b] \to \mathbb{R}$ be a function. f is called an* **absolutely continuous** *function if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all collections $S = \{[a_1,b_1],\ldots,[a_n,b_n]\} \in \sum([a,b])$, the condition*

$$\sum_{k=1}^{n}(b_k - a_k) \le \delta \tag{3.41}$$

implies that

$$\sum_{k=1}^{n}|f(b_k) - f(a_k)| \le \epsilon. \tag{3.42}$$

Denote $AC([a,b])$ the space of the absolute continuous functions $f\colon [a,b] \to \mathbb{R}$.

It is easy to see the following result:

Lemma 3.3.3 *If $f \in AC([a,b])$ then f is uniformly continuous.*

Note that $AC([a,b]) \subseteq C([a,b])$.

We will also use the following equivalences of the definition of absolutely continuous function.

Theorem 3.3.4 *Let $f\colon [a,b] \to \mathbb{R}$ be a function. The following statements are equivalent:*

a) f is absolutely continuous

b) f has a derivative f' almost everywhere, the derivative is Lebesgue integrable, and

$$f(x) = f(a) + \int_a^x f'(t)dt \text{ for every } x \in [a,b] \tag{3.43}$$

c) There exists $g\colon [a,b] \to \mathbb{R}$ a function Lebesgue integrable such that

$$f(x) = f(a) + \int_a^x g(t)dt \text{ for every } x \in [a,b] \tag{3.44}$$

and $f' = g$ a.e.$[\lambda]$.

For the proof see [4] theorems 4.4.1 and 4.4.2.

Theorem 3.3.5 *Let* $f, g \in AC([a,b])$ *and* $\alpha, \beta \in \mathbb{R}$. *Then* $\alpha f + \beta g \in AC([a,b])$ *and* $f^+, f^-, |f| \in AC([a,b])$. *Also,* $fg \in AC([a,b])$ *and if* $g(x) \neq 0$ *for every* $x \in [a,b]$ *then* $f/g \in AC([a,b])$. For the proof see [20], Proposition 3.2.

Proposition 3.3.6 *Every function* $f \in AC([0,1])$ *belongs to* $BV([0,1])$. For the proof see [20], Proposition 1.22.

Definition 3.3.7 *Let us consider* $-\infty < a < b < \infty$. *A continuous function* $f\colon [a,b] \to \mathbb{R}$ *is called* **piecewise linear** *if there exists a partition* $P = \{t_0.t_1, \ldots, t_m\} \in \mathcal{P}([a,b])$ *such that for every* $1 \leq i \leq m$, f *is a straight line on* $[t_{i-1}, t_i]$. *And denote*

$$PL([a,b]) = \{f\colon [a,b] \to \mathbb{R} \mid f \textit{ is piecewise linear}\}. \tag{3.45}$$

Remark 3.3.8 *Note that* $PL([a,b]) \subseteq BV([a,b])$.
In fact, for $f \in PL([a,b])$ there exists $P = \{t_0.t_1, \ldots, t_m\} \in \mathcal{P}([a,b])$ such that for every $1 \leq i \leq m$, f is a straight line on $[t_{i-1}, t_i]$. Then $TV_{t_{i-1}}^{t_i}(f) = |f(t_i) - f(t_{i-1})|$. Then,

$$TV_a^b(f) = \sum_{i=1}^{m} TV_{t_{i-1}}^{t_i}(f) = \sum_{i=1}^{m} |f(t_i) - f(t_{i-1})| < \infty. \tag{3.46}$$

Also, note that $PL([a,b]) \subseteq AC([a,b])$. In fact, let $f \in PL([a,b])$ there exists $P = \{t_0.t_1, \ldots, t_m\} \in \mathcal{P}([a,b])$ such that for every $1 \leq i \leq m$, f is a straight line on $[t_{i-1}, t_i]$. Then $f'(x) = \frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}}$ on (t_{i-1}, t_i). Thus f has a derivative f' almost everywhere. And for $x \in [a,b]$ there exists $1 \leq i_0 \leq m$ such that $x \in [t_{i_0-1}, t_{i_0}]$, then if $x \in (t_{i_0-1}, t_{i_0})$,

$$\begin{aligned}
\int_a^x f'(s)ds &= \sum_{k=1}^{i_0-1} \int_{t_{k-1}}^{t_k} f'(s)ds + \int_{t_{i_0-1}}^{x} f'(s)ds \\
&= \sum_{k=1}^{i_0-1} (f(t_k) - f(t_{k-1})) + \frac{f(t_{i_0}) - f(t_{i_0-1})}{t_{i_0} - t_{i_0-1}}(x - t_{i_0-1}) \\
&= f(t_{i_0-1}) - f(t_0) + \frac{f(t_{i_0}) - f(t_{i_0-1})}{t_{i_0} - t_{i_0-1}}(x - t_{i_0-1}) \\
&= f(t_{i_0-1}) + \frac{f(t_{i_0}) - f(t_{i_0-1})}{t_{i_0} - t_{i_0-1}}(x - t_{i_0-1}) - f(a) \\
&= f(x) - f(a).
\end{aligned} \tag{3.47}$$

Note that

$$\int_a^a f'(s)ds = 0 = f(a) - f(a)$$

and in the case $x = t_i$ for some $1 \leq i \leq m$

$$\begin{aligned}\int_a^x f'(s)ds &= \int_a^{t_i} f'(s)ds = \sum_{k=1}^{i} \int_{t_{k-1}}^{t_k} f'(s)ds \\ &= \sum_{k=1}^{i} (f(t_k) - f(t_{k-1})) \\ &= f(t_i) - f(t_0) = f(x) - f(a). \qquad (3.48)\end{aligned}$$

Thus $f \in AC([a, b])$.
Also, for every $f, g \in PL(a, b)$ and $\alpha, \beta \in \mathbf{R}$, $\alpha f + \beta g \in PL([a, b])$. It is clear that $\alpha f \in PL([a, b])$ for every $f \in PL([a, b])$ and $\alpha \in \mathbf{R}$. Then it is enough to show that for every $f, g \in PL([a, b])$, $f + g \in PL([a, b])$.
Let $f, g \in PL(a, b)$, $f + g$ is a continuous function because f and g are continuous functions. There exists $P_f = \{t_0, t_1, \ldots, t_n\} \in \mathcal{P}([a, b])$ such that f is a straight line on $[t_{i-1}, t_i]$ for every $1 \leq i \leq n$. And there exists $P_g = \{s_0, s_1, \ldots, s_m\} \in \mathcal{P}([a, b])$ such that g is a straight line on $[s_{i-1}, s_i]$ for every $1 \leq i \leq m$. Let us consider $P = \{r_0, r_1, \ldots, r_k\} \in \mathcal{P}([a, b])$ such that $P_f \cup P_g = P$. Then for every $1 \leq i \leq k$, $[r_{i-1}, r_i] = [t_{j_0-1}, t_{j_0}] \cap [s_{j_1-1}, s_{j_1}]$ for some $1 \leq j_0 \leq n$ and $1 \leq j_1 \leq m$. f is linear on $[t_{j_0-1}, t_{j_0}]$ then $f(x) = a_1 x + b_1$ for every $x \in [t_{j_0-1}, t_{j_0}]$ and for some $a_1, b_1 \in \mathbf{R}$. In the same way g is linear on $[s_{j_1-1}, s_{j_1}]$ then $g(x) = a_2 x + b_2$ for every $x \in [s_{j_1-1}, s_{j_1}]$ and for some $a_2, b_2 \in \mathbf{R}$. Thus for every $x \in [r_{i-1}, r_i]$, $f(x) + g(x) = (a_1 x + b_1) + (a_2 x + b_2) = (a_1 + a_2)x + (b_1 + b_2)$. Therefore $f + g$ is a straight line on $[r_{i-1}, r_i]$ and $f + g \in PL([a, b])$.
Definition 3.3.8 *Let* $-\infty < a < b < \infty$. $f \colon [a, b] \to I\!R$ *is called a* **step function** *if there exists* $P = \{t_0.t_1, \ldots, t_m\} \in \mathcal{P}([a, b])$ *such that for every* $1 \leq i \leq m$, f *is constant on* (t_i, t_{i-1}). *We will write* $S([a, b])$ *for the space of all step functions on* $[a, b]$.

Recall that for a set X, $B(X) = \{f \colon X \to \mathbf{R} \mid f \text{ is a bounded function}\}$.
Definition 3.3.9 *We defined the operator* J, $J \colon L^1([a, b]) \to B([a, b])$ *such that*

$$J(f)(x) = \int_a^x f(y)dy. \qquad (3.49)$$

Lemma 3.3.10 *Let $-\infty < a < b < \infty$. $J(S([a,b])) \subseteq PL([a,b])$ and for every $f \in PL([a,b])$ we have that $f - f(a) \in J(S([a,b]))$.*

Proof: Let $f \in PL([a,b])$. Then f is a continuous function and there exists a partition $P = \{t_0.t_1, \ldots, t_m\} \in \mathcal{P}([a,b])$ such that for every $1 \le i \le m$, f is a straight line on $[t_{i-1}, t_i]$. Let $h\colon [a,b] \to \mathbf{R}$ such that $h(x) = \frac{f(t_i)-f(t_{i-1})}{t_i-t_{i-1}}$ on (t_{i-1}, t_i) and $h(t_i) = 0$ for every $0 \le i \le m$. Note that $f' = h$ on $[a,b] \setminus \{t_0.t_1, \ldots, t_m\}$. Then, by equation (3.47), $J(h)(x) = \int_a^x h(t)dt = f(x) - f(a)$. Thus $f - f(a) \in J(S([a,b]))$.

Let $f \in J(S([a,b]))$. So, there exists $h \in S([a,b])$ such that for every $x \in [a,b]$, $f(x) = \int_a^x h(t)dt$ and there exist $P = \{x_0, x_1, \ldots, x_k\} \in \mathcal{P}([a,b])$ and $\{c_1, c_2, \ldots, c_k\} \subseteq \mathbf{R}$ such that for every $1 \le i \le k$, $h(x) = c_i$ on (x_{i-1}, x_i). Let $1 \le j \le k$ and $x_{j-1} \le x \le x_j$ then

$$\begin{aligned}
f(x) &= \int_a^x h(t)dt = \sum_{i=1}^{j-1} \int_{x_{i-1}}^{x_i} h(t)dt + \int_{x_{j-1}}^{x} h(t)dt \\
&= \sum_{i=1}^{j-1} \int_{x_{i-1}}^{x_i} c_i dt + \int_{x_{j-1}}^{x} c_j dt \\
&= \sum_{i=1}^{j-1} c_i \int_{x_{i-1}}^{x_i} dt + c_j \int_{x_{j-1}}^{x} dt \\
&= \sum_{i=1}^{j-1} c_i(x_i - x_{i-1}) + c_j(x - x_{j-1}) \\
&= \alpha_{j-1} + c_j(x - x_{j-1}) \qquad (3.50)
\end{aligned}$$

Where $\alpha_{j-1} = \sum_{i=1}^{j-1} c_i(x_i - x_{i-1}) = \int_a^{x_{j-1}} h(t)dt$. Note that $f(x) = c_j(x - x_{j-1}) + \alpha_{j-1}$ is a straight line on $[x_{j-1}, x_j]$. We have that $f(x_0) = f(x_0+)$ and $f(x_k) = f(x_k-)$. Also, if $2 \le j \le k$, $f(x_{j-1}) = \alpha_{j-1}$, for $0 < \epsilon < \min\{x_j - x_{j-1}, x_{j-1} - x_{j-2}\}$

$$\begin{aligned}
f(x_{j-1} + \epsilon) &= \int_a^{x_{j-1}+\epsilon} h(t)dt = \int_a^{x_{j-1}} h(t)dt + \int_{x_{j-1}}^{x_{j-1}+\epsilon} h(t)dt \\
&= \alpha_{j-1} + c_j(x_{j-1} + \epsilon - x_{j-1}) = \alpha_{j-1} + c_j\epsilon \qquad (3.51)
\end{aligned}$$

Thus, $\lim_{\epsilon\downarrow 0} f(x_{j-1} + \epsilon) = \lim_{\epsilon\downarrow 0}\left(\alpha_{j-1} + c_j\epsilon\right) = \alpha_{j-1} = f(x_{j-1})$. And

$$\begin{aligned} f(x_{j-1}-\epsilon) &= \int_a^{x_{j-1}-\epsilon} h(t)dt = \int_a^{x_{j-1}} h(t)dt - \int_{x_{j-2}}^{x_{j-1}} h(t)dt \\ &+ \int_{x_{j-2}}^{x_{j-1}-\epsilon} h(t)dt = \alpha_{j-1} - c_{j-1}(x_{j-1}-x_{j-2}) \\ &+ c_{j-1}(x_{j-1}-\epsilon-x_{j-2}) = \alpha_{j-1} - c_{j-1}\epsilon \end{aligned}$$

So, $\lim_{\epsilon\downarrow 0} f(x_{j-1}-\epsilon) = \lim_{\epsilon\downarrow 0}\left(\alpha_{j-1}-c_{j-1}\epsilon\right) = \alpha_{j-1} = f(x_{j-1})$ and f is continuous on $[a,b]$. □

Remark 3.3.11 Let us consider $C = \{(a,b] \mid -\infty < a \leq b < \infty\}$. Thus C is a semiring of subsets of $\mathbf{R}$. (See [15] Proposition 3.2.2). And

$$\mathcal{R} = \left\{\cup_{i=1}^n A_i \mid \{A_i\}_{i=1}^n \subseteq C \text{ is a disjoint family, for some } n \geq 1\right\}$$

is the smallest ring that includes C, see [15] Proposition 3.2.3.
We known that the Borel σ−algebra in $\mathbf{R}$ is such that $\mathcal{B}(\mathbf{R}) = \sigma(C) = \sigma(\mathcal{R})$, see [15] Proposition 3.2.10.
By Remark 3.1.33 [15], for every $E \in \mathcal{B}(\mathbf{R})$ such that $\lambda(E) < \infty$ and for every $\epsilon > 0$ there exists $\{A_n\}_{n=1}^\infty \subseteq \mathcal{R}$ such that $E \subseteq \cup_{n=1}^\infty A_n$ and

$$\lambda(E) \leq \lambda\left(\cup_{n=1}^\infty A_n\right) \leq \lim_{k\to\infty}\sum_{n=1}^k \lambda(A_n) = \sum_{n=1}^\infty \lambda(A_n) \leq \lambda(E) + \epsilon \tag{3.52}$$

So, there exists $N \in \mathbf{N}$ such that for every $m \geq N$, $\left|\sum_{i=1}^m \lambda(A_i) - \lambda(E)\right| < \epsilon$
Let $-\infty < a < b < \infty$ and let $f\colon [a,b] \to \mathbf{R}$ be a simple function nonnegative. Then, there exists $\{B_i\}_{i=1}^n \subseteq \mathcal{B}([a,b])$ a partition of $[a,b]$ and $\{b_i\}_1^n \subseteq (0,\infty)$ such that $f(x) = \sum_{i=1}^n b_i 1_{B_i}(x)$ for every $x \in [a,b]$. Note That $\lambda(B_i) \leq \lambda([a,b]) < \infty$ for every $1 \leq i \leq n$.
Then for every $\epsilon > 0$, there exist $\{C_j\}_{j=1}^l \subseteq C$ a disjoint family and $\{c_j\}_{j=1}^l \subseteq (0,\infty)$ such that $\int_a^b |f - \sum_{j=1}^l c_j 1_{C_j}| d\lambda < \epsilon$. To see this, first we consider $n = 1$
For $\epsilon > 0$ and B_1 there exist $\{A_i\}_{i=1}^\infty \subseteq \mathcal{R}$ such that $B_1 \subseteq \cup_{i=1}^\infty A_i := H$

$$0 \leq \lambda(H) - \lambda(B_1) < \frac{\epsilon}{2b_1} \tag{3.53}$$

$H \in \mathcal{B}(\mathbf{R})$ and

$$\begin{aligned}
\int_a^b \left|b_1 1_{B_1} - b_1 1_H\right| d\lambda &= b_1 \int_a^b |1_{H\setminus B_1}| d\lambda \\
&= b_1 \int_a^b 1_{H\setminus B_1} d\lambda \\
&= b_1 \lambda(H \setminus B_1) = b_1 \left(\lambda(H \setminus B_1) + \lambda(H \cap B_1)\right. \\
&\quad \left. -\lambda(H \cap B_1)\right) \\
&= b_1 \left(\lambda(H) - \lambda(B_1)\right) < \frac{\epsilon}{2} \qquad (3.54)
\end{aligned}$$

And there exists $m \in \mathbf{N}$ such that $0 \leq \sum_{i=1}^{\infty} \lambda(A_i) - \sum_{i=1}^{m} \lambda(A_i) < \frac{\epsilon}{2b_1}$. Using that $\mathcal{R}$ is a ring, we have that $\cup_{i=1}^{m} A_i \in \mathcal{R}$ and we can write it as a the disjoint union of elements of C, so we can consider the family $\{A_i\}_{i=1}^{m}$ as a disjoint family of elements of C.

$$\begin{aligned}
\int_a^b |b_1 1_H - b_1 1_{\cup_{i=1}^m A_i}| d\lambda &= b_1 \int_a^b |1_{H\setminus \cup_{i=1}^m A_i}| = b_1 \int_a^b 1_{H\setminus \cup_{i=1}^m A_i} \\
&= b_1 \lambda\left(H \setminus \cup_{i=1}^m A_i\right) = b_1 \left(\lambda\left(H \setminus \cup_{i=1}^m A_i\right)\right. \\
&\quad \left. +\lambda\left(H \cap \cup_{i=1}^m A_i\right) - \lambda\left(H \cap \cup_{i=1}^m A_i\right)\right) \\
&= b_1 \left(\lambda(H) - \sum_{i=1}^{m} \lambda(A_i)\right) \\
&\leq b_1 \left(\sum_{i=1}^{\infty} \lambda(A_i) - \sum_{i=1}^{m} \lambda(A_i)\right) < \frac{\epsilon}{2} \qquad (3.55)
\end{aligned}$$

Then,

$$\begin{aligned}
\int_a^b |f - 1_{\cup_{i=1}^m A_i}| d\lambda &= \int_a^b |f - 1_H + 1_H - 1_{\cup_{i=1}^m A_i}| d\lambda \\
&\leq \int_a^b |f - 1_H| + \int_a^b |1_H - 1_{\cup_{i=1}^m A_i}| d\lambda \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad (3.56)
\end{aligned}$$

If $n = 2$, let $\epsilon > 0$. For the case $n = 1$, there exist $\{A_i\}_{i=1}^{m} \in C$ disjoint family and $\{C_i\}_{i=1}^{k} \subseteq C$ disjoint family such that $\int_a^b |b_1 1_{B_1} - b_1 1_{\cup_{i=1}^m A_i}| d\lambda < \frac{\epsilon}{2}$ and

$\int_a^b |b_2 1_{B_2} - b_2 1_{\cup_{i=1}^k C_i}| d\lambda < \frac{\epsilon}{2}$. Then,

$$\begin{aligned}&\int_a^b |b_1 1_{B_1} + b_2 1_{B_2} - b_1 1_{\cup_{i=1}^m A_i} - b_2 1_{\cup_{i=1}^k C_i}| d\lambda \\ &\leq \int_a^b |b_1 1_{B_1} - b_1 1_{\cup_{i=1}^m A_i}| d\lambda + \int_a^b |b_2 1_{B_2} - b_2 1_{\cup_{i=1}^k C_i}| d\lambda \\ &< \epsilon \end{aligned} \tag{3.57}$$

Note that $\left(\cup_{i=1}^m A_i\right) \cup \left(\cup_{i=1}^k C_i\right) \in \mathcal{R}$ and we can write this as $\cup_{i=1}^s D_i$ a disjoint family of elements of $\mathcal{C}$. And using the proof of Proposition 4.1.2 [15] for $n = 2$ the result holds. The proof is similar for $n \geq 3$.

Theorem 3.3.12 *Let $-\infty < a < b < \infty$. If $f\colon [a,b] \to \mathbb{R}$ is an increasing function, then the derivative exists a.e. on $[a,b]$, and it belongs to $L^1([a,b])$ and satisfies*

$$\int_a^b f'(t)dt \leq f(b) - f(a). \tag{3.58}$$

Proof: By Lebesgue's Theorem for the Differentiability of Monotone Functions, see [36], we have that f' exists a.e. on $[a,b]$. We can extend f by $f\colon [a-1, b+1] \to \mathbf{R}$ such that

$$f(x) = \begin{cases} f(a) & \text{if} \quad a-1 \leq x < a \\ f(x) & \text{if} \quad a \leq x \leq b \\ f(b) & \text{if} \quad b < x \leq b+1. \end{cases}$$

And for every $n \geq 1$, we defined $f_n, g_n\colon [a-1, b+1] \to \mathbf{R}$ and $g\colon [a,b] \to \mathbf{R}$ such that

$$f_n(t) = f\left(t + \frac{1}{n}\right), \; g_n(t) = n\left[f_n(t) - f(t)\right] \text{ and } g(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \tag{3.59}$$

Note that

$$g(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = \lim_{n\to\infty} \frac{f_n(t) - f(t)}{1/n} = \lim_{n\to\infty} g_n(t) \tag{3.60}$$

Thus, g is defined a.e. on $[a,b]$. f and f_n are continuous a.e. on $[a,b]$ because they are differentiable a.e. on $[a,b]$ thus f and f_n are measurable functions, so $\{g_n\}_{n=1}^{\infty}$ is a sequence of measurable functions and by Theorem 4.2.3 [15], g is a measurable function.

Using that f is an increasing function we have that $g_n(t) \geq 0$ for every $t \in [a,b]$ and $n \in \mathbb{N}$. Also, $\int_a^b 0dt = 0$. By Fatou's Lemma (see Lemma 4.3.11[15]),

$$\begin{aligned}
\int_a^b g(t)dt &= \int_a^b \left(\liminf_{n\to\infty} g_n(t)\right) dt \\
&\leq \liminf_{n\to\infty} \int_a^b g_n(t)dt = \liminf_{n\to\infty} n \int_a^b \left[f\left(t+\frac{1}{n}\right) - f(t)\right] dt
\end{aligned}$$

And

$$\begin{aligned}
\int_a^b \left[f\left(t+\frac{1}{n}\right) - f(t)\right] dt &= \int_a^b f\left(t+\frac{1}{n}\right) dt - \int_a^b f(t)dt \\
&= \int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(t)dt - \int_a^b f(t)dt \\
&= \int_b^{b+\frac{1}{n}} f(t)dt - \int_a^{a+\frac{1}{n}} f(t)dt \\
&= \frac{f(b)}{n} - \int_a^{a+\frac{1}{n}} f(t)dt
\end{aligned} \tag{3.61}$$

We have that $f(t) \geq f(a)$ for every $t \in [a, a+1/n]$, so

$$\int_a^{a+\frac{1}{n}} f(t)dt \geq \int_a^{a+\frac{1}{n}} f(a)dt = \frac{f(a)}{n} \tag{3.62}$$

Thus,

$$\int_a^b g(t)dt \leq \liminf_{n\to\infty} n\left(\frac{f(b)}{n} - \frac{f(a)}{n}\right) = f(b) - f(a). \tag{3.63}$$

□

Corollary 3.3.13 *Let $-\infty < a < b < \infty$. If $f \in BV([a,b])$ then the derivative exists a.e. on $[a,b]$ and it belongs to $L^1([a,b])$.*

Proof: Using that f can be write as $g - h$ where g and h are increasing function and the Theorem 1.2.12 the result follows. □

For the Theorem 3.3.15 we need the following result. For the proof of Theorem 3.3.14 see Theorem 12.34 [17]
Theorem 3.3.14 *(Absolute continuity of the integral) Let $M \in \mathcal{B}(\mathbb{R})$ and $f: M \to \mathbb{R}$ be a integrable function. Then for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any measurable $N \subseteq M$ with $\lambda(N) \leq \delta$, we have*

$$\int_N |f(t)|dt \leq \epsilon. \tag{3.64}$$

Theorem 3.3.15 *Let $f \in BV([0,1])$. Then*

$$\int_0^1 |f'(t)|dt \leq TV_0^1(f) \tag{3.65}$$

In case $f \in AC([0,1])$, we have the equality

$$\int_0^1 |f'(t)|dt = TV_0^1(f) \tag{3.66}$$

Proof: By Corollary 3.3.13, f' exists a.e. on $[a,b]$ and $f' \in L^1([a,b])$, so f' is a measurable function. By Theorem 3.3.14, for $\epsilon > 0$ there exists $\delta > 0$ such that if $N \in \mathcal{B}([a,b])$ and $\lambda(N) \leq \delta$, then

$$\int_N |f'(t)|dt \leq \frac{\epsilon}{2}. \tag{3.67}$$

f' is a measurable function, then

$$D_+ = \{t \in (a,b) \,|\, f'(t) \geq 0\} = (f')^{[-1]}([0,\infty)) \in \mathcal{B}([a,b])$$

and

$$D_- = \{t \in (a,b) \,|\, f'(t) < 0\} = (f')^{[-1]}((-\infty,0)) \in \mathcal{B}([a,b])$$

By Remark 3.3.11, there exists $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{R}$ such that $D_+ \subseteq \cup_{n=1}^{\infty} A_n$ and

$$\lambda(D_+) \leq \lambda\left(\cup_{n=1}^{\infty} A_n\right) \leq \lim_{k\to\infty} \sum_{n=1}^{k} \lambda(A_n) = \sum_{n=1}^{\infty} \lambda(A_n) \leq \lambda(D_+) + \frac{\delta}{2} \tag{3.68}$$

And there exists $k \in \mathbf{N}$ such that

$$0 \leq \sum_{n=1}^{\infty} \lambda(A_n) - \sum_{n=1}^{k} \lambda(A_n) < \frac{\delta}{2} \tag{3.69}$$

$\mathcal{R}$ in Remark 3.3.11 is a ring, then $\cup_{n=1}^{k} A_n \in \mathcal{R}$ and we can write this sum as a disjoint sum of elements of C. Then we can assume that for every $1 \le i \le k$, $A_i = (a_i, b_i]$ with $a \le a_1 < b_1 < a_2 < b_2 < a_3 < \cdots < b_i < a_{i+1} < \cdots < b_{k-1} < a_k < b_k \le b$. Thus, if $B = \cup_{n=1}^{k} A_n$ and $C = \cup_{n=1}^{\infty} A_n$,

$$\begin{aligned} \lambda(D_+ \triangle B) &= \lambda(D_+ \setminus B) + \lambda(B \setminus D_+) \\ &\le \lambda(C \setminus B) + \lambda(C \setminus D_+) \\ &= [\lambda(C) - \lambda(B)] + [\lambda(C) - \lambda(D_+)] \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta \end{aligned} \tag{3.70}$$

We can sum $a = b_0$ and $b = a_{k+1}$, if necessary, to obtain

$$\{b_0, a_1, b_1, a_2, b_2, a_3, \ldots, b_i, a_{i+1}, \ldots, b_{k-1}, a_k, b_k, a_{k+1}\}$$

a partition of $[a, b]$.
Note that

$$\begin{aligned} D_- \triangle [(a,b) \setminus B] &= (D_- \cap [(a,b) \cap B^c]^c) \cup ((a,b) \cap B^c \cap D_-^c) \\ &= (D_- \cap [(a,b)^c \cup B]) \cup ((a,b) \cap B^c \cap D_+) \\ &= (D_- \cap B) \cup (B^c \cap D_+) = (B \setminus D_+) \cup (D_+ \setminus B) \\ &= B \triangle D_+ \end{aligned}$$

And using the Fundamental Theorem of Calculus (FTC), the inequality $a - b \le |a| + |b|$

$$\begin{aligned} TV_a^b(f) &\ge \sum_{i=1}^{k} |f(b_i) - f(a_i)| + \sum_{j=1}^{k+1} |f(a_i) - f(b_{i-1})| \\ &= \sum_{i=1}^{k} \left| \int_{a_i}^{b_i} f'(t)dt \right| + \sum_{j=1}^{k+1} \left| \int_{a_i}^{b_{i-1}} f'(t)dt \right| \\ &\ge \left| \sum_{i=1}^{k} \int_{a_i}^{b_i} f'(t)dt \right| + \left| \sum_{j=1}^{k+1} \int_{a_i}^{b_{i-1}} f'(t)dt \right| \\ &= \left| \int_B f'(t)dt \right| + \left| \int_{(a,b)\setminus B} f'(t)dt \right| \end{aligned}$$

$$
\begin{aligned}
&\geq \int_B f'(t)dt - \int_{B^c\cap(a,b)} f'(t)dt \\
&= \int_{D_+\cap B} f'(t)dt - \int_{D_-\cap B^c} f'(t)dt + \int_{B\cap D_-} f'(t)dt - \int_{D_+\cap B^c} f'(t)dt \\
&= \int_{D_+\cap B} f'(t)dt - \int_{D_-\cap B^c} f'(t)dt + \int_{B\setminus D_+} f'(t)dt - \int_{D_+\setminus B} f'(t)dt \\
&= \int_{D_+\cap B} f'(t)dt + \int_{D_+\setminus B} f'(t)dt - \int_{D_-\cap B^c} f'(t)dt - \int_{D_-\setminus B^c} f'(t)dt \\
&\quad - \int_{D_+\setminus B} f'(t)dt + \int_{B\setminus D_+} f'(t)dt + \int_{B\setminus D_+} f'(t)dt - \int_{D_+\setminus B} f'(t)dt \\
&= \int_{D_+} f'(t)dt - \int_{D_-} f'(t)dt - \int_{D_+\setminus B} f'(t)dt + \int_{B\setminus D_+} f'(t)dt \\
&\quad + \int_{B\setminus D_+} f'(t)dt - \int_{D_+\setminus B} f'(t)dt \\
&= \int_{D_+} f'(t)dt - \int_{D_-} f'(t)dt - \int_{B\triangle D_+} |f'(t)|dt - \int_{D_-\triangle[(a,b)\setminus B]} |f'(t)|dt \\
&= \int_a^b |f'(t)|dt - \int_{B\triangle D_+} |f'(t)|dt - \int_{D_-\triangle[(a,b)\setminus B]} |f'(t)|dt \\
&\geq \int_a^b |f'(t)|dt - \epsilon
\end{aligned}
$$

Since $\epsilon > 0$ was arbitrary, we conclude that $TV_a^b(f) \geq \int_a^b |f'(t)|dt$.
In the case $f \in AC([0,1])$ take $\{t_0, t_1, \ldots, t_k\} \in \mathcal{P}([0,1])$ and using Theorem 3.3.4.

$$
\begin{aligned}
\sum_{i=1}^k |f(t_i) - f(t_{i-1})| &= \sum_{i=1}^k \left| f(0) + \int_0^{t_i} f'(t)dt - \left(f(0) + \int_0^{t_{i-1}} f'(t)dt \right) \right| \\
&= \sum_{i=1}^k \left| \int_{t_{i-1}}^{t_i} f'(t)dt \right| \\
&\leq \sum_{i=1}^k \int_{t_{i-1}}^{t_i} |f'(t)|dt = \int_0^1 |f'(t)|dt
\end{aligned}
$$

Taking supreme over $\mathcal{P}([0,1])$ we have that $TV_0^1(f) \leq \int_0^1 |f'(t)|dt$. □

Corollary 3.3.16 *If $f : [0,1] \to I\!R$ is a function and we have a partition $P \in \mathcal{P}([0,1])$, where $P = \{0 = t_0, t_1, \ldots, t_{n_P-1}, t_{n_P} = 1\}$, such that f is an absolutely continuous function on all the intervals $[0 = t_0, t_1]$, $(t_1, t_2], \ldots, (t_{n_P-1}, t_{n_P} = 1]$ then*

$$TV_0^1(f) = TV_0^{t_1}(f) + \sum_{j=1}^{n_P} TV_{(t_{j-1},t_j]}(f) + \sum_{j=1}^{n_P-1} |f(t_j+) - f(t_j-)|. \tag{3.71}$$

Proof. Let $f : [0,1] \to \mathbf{R}$ be a function and $P = \{0 = t_0, t_1, \ldots, t_{n_P-1}, t_{n_P} = 1\} \in \mathcal{P}([0,1])$ be a partition of $[0,1]$ such that f is a differentiable and Riemann integrable function on all the intervals $[0 = t_0, t_1]$, $(t_1, t_2], \ldots,$ $(t_{n_P-1}, t_{n_P} = 1]$. By the continuity of f on each interval we can assume that $f(t_j+) = k_j \in \mathbf{R}$ for every $j \in \{1, 2, \ldots, n_P - 1\}$. Let us define $g_1(x) = f(x)$ if $x \in [t_0 = 0, t_1]$ and for every $j \in \{2, 3, \ldots, n_P\}$ let

$$g_j(x) = \begin{cases} k_j & \text{if} \quad x = t_{j-1} \\ f(x) & \text{if} \quad x \in (t_{j-1}, t_j]. \end{cases} \tag{3.72}$$

Then, the family of functions $\{g_j\}_{j\in\{1,2,\ldots,n_P\}}$ is differentiable and defined on closed intervals whose intersections are at most single points. Using limits, we can define

$$\mathrm{TV}_{(t_{j-1},t_j]}(f) = \mathrm{TV}_{t_{j-1}}^{t_j}(g_j) \text{ for every } j \in \{2, \ldots, n_P\}. \tag{3.73}$$

Besides, obviously $\mathrm{TV}_0^{t_1}(f) = \mathrm{TV}_0^{t_1}(g_1)$. Since f is not necessarily continuous on $t_1, t_2, \ldots, t_{n_P-1}$, but it has left hand limits $f((t_j)-)$ and right hand limits $f((t_j)+)$ for every $j \in \{1, 2, \ldots, t_{n_P} - 1\}$. Then, using (3.72), (3.73), the last note and Theorem 3.1.10 we conclude that

$$\begin{aligned} \mathrm{TV}_0^1(f) &= \mathrm{TV}_0^{t_1}(f) + \sum_{j=2}^{n_P} \mathrm{TV}_{(t_{j-1},t_j]}(f) + \sum_{j=1}^{n_P-1} |f((t_j)+) - f((t_j)-)| \\ &= \sum_{j=1}^{n_P} \mathrm{TV}_{t_j-1}^{t_j}(g_j) + \sum_{j=1}^{n_P-1} |f((t_j)+) - f((t_j)-)|, \end{aligned} \tag{3.74}$$

and the result follows. □

Definition 3.3.17 *A nonconstant function $f \in BV([a,b]) \cap C([a,b])$. f is called* **singular function** *if f is differentiable a.e. on $[a,b]$ with $f'(x) = 0$.*

Theorem 3.3.18 *Let $f \in BV([a,b]) \cap C([a,b])$. Then f may be represented as*

$$f(x) = f_{ac}(x) + f_{sg}(x) \text{ for every } x \in [a,b] \tag{3.75}$$

where f_{ac} is absolutely continuous and f_{sg} is singular or $f_{sg} \equiv 0$. This functions are uniquely determined up to additive constants, and so the representation (3.75) may be made unique by requiring that $f(a) = f_{ac}(a)$. See Proposition 3.21 [20].

Theorem 3.3.19 $(AC([0,1]), d_{TV})$ *is a complete space.* See [20].

Remark 3.3.20 Recall that $f\colon [a,b] \to \mathbf{R}$ is called **Lipschitz-continuous**, $f \in Lip([a,b])$, if and only if there exists $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for every $x, y \in [a,b]$.

Any such L is referred to as a Lipschitz constant for the function f and f may also be referred to as L–Lipschitz. The smallest constant is sometimes called **the (best) Lipschitz constant of f** or **the dilation of f**. If f is not a constant function, we called $Lip(f) = \sup_{s<t} \left|\frac{f(t)-f(s)}{t-s}\right| > 0$ the best constant of f.

Note that $Lip([a,b]) \subseteq AC([a,b])$. In fact, for $f \in Lip([a,b])$ and $\epsilon > 0$ if $\delta = \frac{\epsilon}{Lip(f)}$, then for $S = \{[a_1,b_1], \ldots, [a_n,b_n]\} \in \sum([a,b])$, with

$$\sum_{k=1}^{n} (b_k - a_k) \leq \delta \tag{3.76}$$

implies that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq \sum_{k=1}^{n} Lip(f)(b_k - a_k) = Lip(f) \sum_{k=1}^{n} (b_k - a_k) < \epsilon. \tag{3.77}$$

Also, if $f, g \in Lip([a,b])$ and g is biyective with $g\colon [a,b] \to [a,b]$ then $f \circ g \in Lip([a,b])$.

3.4 Another Bounded Variation Metric

We will propose a Skorohod-type metric. Let us remember the Skorohod metric.

Definition 3.4.1 *Let $D = D[0,1]$ be the space of real functions f on $[0,1]$ that are right-continuous and have left-hand limits:*

i) For $t \in [0,1)$, $f(t+) = \lim_{s\downarrow t} f(s)$ exists and $f(t+) = f(t)$

ii) For $t \in (0,1]$, $f(t-) = \lim_{s\uparrow t} f(s)$ exists.

Functions having these two properties are called ***cadlag****, continue à droite limite à gauche.*
Let $\Lambda'([0,1])$ denote the class of strictly increasing, continuous mappings of $[0,1]$ onto itself. If $\lambda \in \Lambda'([0,1])$, then $\lambda(0) = 0$ and $\lambda(1) = 1$. For g and f in D, define $d(f,g)$ to be the infimum of those positive ϵ for which there exists in $\Lambda'([0,1])$ a λ satisfying

$$\sup_{t\in[0,1]} |\lambda(t) - t| = \sup_{t\in[0,1]} |t - \lambda^{-1}(t)| < \epsilon \tag{3.78}$$

and

$$\sup_{t\in[0,1]} |f(t) - g(\lambda(t))| = \sup_{t\in[0,1]} |f(\lambda^{-1}(t)) - g(t)| < \epsilon \tag{3.79}$$

To express this in more compact form, let Id be the identity map on $[0,1]$ and use the notation d_{sup} for the supremum metric. Then the definition becomes

$$\begin{aligned} d(f,g) \quad = \quad & \inf\{\epsilon > 0 \mid \exists\lambda \in \Lambda'([0,1]) \text{ such that } d_{sup}(\lambda, Id) < \epsilon \\ & \text{and } d_{sup}(f, g\circ\lambda) < \epsilon\} \end{aligned} \tag{3.80}$$

(D,d) is not a complete space, see [7]. So, Skorohod provided a new metric d^o such that (D,d^o) is a Polish space. For $\lambda \in \Lambda'([0,1])$ such that

$$\|\lambda\|^o := \sup_{s<t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| < \infty. \tag{3.81}$$

He defined

$$\begin{aligned} d^o(f,g) \quad = \quad & \inf\{\epsilon > 0 \mid \exists\lambda \in \Lambda'([0,1]) \text{ such that } \|\lambda\|^o < \epsilon \\ & \text{and } d_{sup}(f, g\circ\lambda) < \epsilon\}. \end{aligned} \tag{3.82}$$

In order to present our metric we propose the following modification to $\Lambda'([0,1])$

$$\Lambda([0,1]) = \{\lambda \in Lip([0,1]) \mid \lambda(0) = 0,\ \lambda(1) = 1,\ \lambda \text{ is strictly increasing}\} \tag{3.83}$$

Definition 3.4.2 *For g and f in $BV([0,1])$, define $d(f,g)$ to be the infimum of those positive ϵ for which there exists in $\lambda \in \Lambda([0,1])$ satisfying*

$$d_{TV}(\lambda, Id) = |\lambda(0) - Id(0)| + TV_0^1(\lambda - Id) = TV_0^1(\lambda - Id) < \epsilon \qquad (3.84)$$

and

$$\begin{aligned} d_{TV}(f, g\circ\lambda) &= |f(0) - g(\lambda(0))| + TV_0^1(f - g\circ\lambda) \\ &= |f(0) - g(0)| + TV_0^1(f - g\circ\lambda) < \epsilon \qquad (3.85) \end{aligned}$$

where $Id\colon [0,1] \to [0,1]$ is the identity map. Then the definition becomes

$$\begin{aligned} d(f,g) &= \inf\{\epsilon > 0 \mid \exists\lambda \in \Lambda([0,1]) \ \textit{such that} \ d_{TV}(\lambda, Id) < \epsilon \\ &\qquad \textit{and}\ d_{TV}(f, g\circ\lambda) < \epsilon\}. \qquad (3.86) \end{aligned}$$

Remark 3.4.3 Let $\lambda \in \Lambda([0,1])$. For every $P = \{x_0, x_1, \ldots, x_P\} \in \mathcal{P}([0,1])$, then $\lambda(P) := \{\lambda(x_0), \lambda(x_1), \ldots, \lambda(x_P)\} \in \mathcal{P}([0,1])$, because λ is strictly increasing and $\lambda(0) = 0$, $\lambda(1) = 1$, thus $0 = \lambda(0) = \lambda(x_0) < \lambda(x_1) < \cdots < \lambda(x_P) = \lambda(1) = 1$. Also, λ is a bijective strictly increasing function, hence, there exists $y_0 < y_1 < \cdots < y_P \in [0,1]$ such that $\lambda(y_i) = x_i$ for every $i \in \{0, 1, \ldots, P\}$. $\lambda(y_0) = x_0 = 0$ and $\lambda(y_P) = x_P = 1$, thus $y_0 = 0$ and $y_P = 1$. So, $\{y_0, y_1, \ldots, y_P\} \in \mathcal{P}([0,1])$, then we have

$$\{P \mid P \in \mathcal{P}([0,1])\} = \{\lambda(P) \mid P \in \mathcal{P}([0,1])\} \qquad (3.87)$$

By equation (3.87),

$$\begin{aligned} TV_0^1(f) &= \sup_{\{P\in\mathcal{P}([0,1])\}} \sum_{i=1}^{n_P} |f(x_i) - f(x_{i-1})| \\ &= \sup_{\{\lambda(P)\in\mathcal{P}([0,1])\}} \sum_{i=1}^{n_P} |f(\lambda(x_i)) - f(\lambda(x_{i-1}))| \\ &= \sup_{\{P\in\mathcal{P}([0,1])\}} \sum_{i=1}^{n_P} |(f\circ\lambda)(x_i) - (f\circ\lambda)(x_{i-1})| \\ &= TV_0^1(f\circ\lambda) \qquad (3.88) \end{aligned}$$

Therefore the total variation of a function is invariant under compositions of elements of Λ, that is, for every $f \in BV([0,1])$ and for every $\lambda \in \Lambda$, $TV_0^1(f \circ \lambda) = TV_0^1(f) < \infty$.
We will see that d is a metric. Using $\lambda = Id$, $d_{TV}(\lambda, Id) = 0 < \infty$ and $d_{TV}(f, g \circ \lambda) = d_{TV}(f,g) < \infty$. Thus, $d(f,g) < \infty$. Of course, $d(f,g) \geq 0$. $d(f,g) = 0$ implies that, for equation (3.84), $\lambda = Id$, because d_{TV} is a metric. If $\lambda = Id$, by equation (3.85), $f = g$.
So, by Remark 1.3.2,

$$TV_0^1(\lambda_2 \circ \lambda_1 - \lambda_1) = TV_0^1((\lambda_2 - Id) \circ \lambda_1) = TV_0^1(\lambda_2 - Id) \qquad (3.89)$$

and if $\lambda_1 \in \Lambda([0,1])$ so does λ_1^{-1}, using again Remark 1.3.2,

$$\begin{aligned} d_{TV}(\lambda_1, Id) = TV_0^1(\lambda_1 - Id) &= TV_0^1((\lambda_1 - Id) \circ \lambda_1^{-1}) = TV_0^1(Id - \lambda_1^{-1}) \\ &= d_{TV}(Id, \lambda_1^{-1}). \qquad (3.90) \end{aligned}$$

and

$$\begin{aligned} d_{TV}(f, g \circ \lambda_1) &= |f(0) - g(0)| + TV_0^1(f - g \circ \lambda_1) \\ &= |f(0) - g(0)| + TV_0^1((f - g \circ \lambda_1) \circ \lambda_1^{-1}) \\ &= |f(0) - g(0)| + TV_0^1(f \circ \lambda_1^{-1} - g) = d_{TV}(g, f \circ \lambda_1^{-1}) \\ & \qquad (3.91) \end{aligned}$$

$d(f,g) = d(g,f)$ follows from equations (3.90) and (3.91). Using the definition of d for $\epsilon > 0$ there exist $\lambda_1, \lambda_2 \in \Lambda([0,1])$ such that

$$d_{TV}(\lambda_1, Id) \leq d(f,g) + \frac{\epsilon}{2} \text{ and } d_{TV}(f, g \circ \lambda_1) \leq d(f,g) + \frac{\epsilon}{2} \qquad (3.92)$$

and

$$d_{TV}(\lambda_2, Id) \leq d(g,h) + \frac{\epsilon}{2} \text{ and } d_{TV}(g, h \circ \lambda_2) \leq d(g,h) + \frac{\epsilon}{2} \qquad (3.93)$$

By equations (3.89), (3.92), (3.93) and the inequality triangle of d_{TV},

$$\begin{aligned} d_{TV}(\lambda_2 \circ \lambda_1, Id) &\leq d_{TV}(\lambda_2 \circ \lambda_1, \lambda_1) + d_{TV}(\lambda_1, Id) \\ &= d_{TV}(\lambda_2, Id) + d_{TV}(\lambda_1, Id) \\ &\leq d(g,h) + \frac{\epsilon}{2} + d(f,g) + \frac{\epsilon}{2} \\ &= d(g,h) + d(f,g) + \epsilon. \end{aligned} \tag{3.94}$$

and

$$\begin{aligned} & d_{TV}(f, h \circ \lambda_2 \circ \lambda_1) \leq d_{TV}(f, g \circ \lambda_1) + d_{TV}(g \circ \lambda_1, h \circ \lambda_2 \circ \lambda_1) \\ &= d_{TV}(f, g \circ \lambda_1) + |g(0) - h(0)| + TV_0^1(g \circ \lambda_1 - h \circ \lambda_2 \circ \lambda_1) \\ &= d_{TV}(f, g \circ \lambda_1) + |g(0) - h(0)| + TV_0^1((g - h \circ \lambda_2) \circ \lambda_1) \\ &= d_{TV}(f, g \circ \lambda_1) + |g(0) - h(0)| + TV_0^1(g - h \circ \lambda_2) \\ &= d_{TV}(f, g \circ \lambda_1) + d_{TV}(g, h \circ \lambda_2) \\ &\leq d(f,g) + \frac{\epsilon}{2} + d(g,h) + \frac{\epsilon}{2} = d(f,g) + d(g,h) + \epsilon \end{aligned} \tag{3.95}$$

equations (3.94) and (3.95) are valid for every $\epsilon > 0$, thus $d(f,h) \leq d(f,g) + d(g,h)$. Thus d is a metric.

Definition 3.4.4 *Let $\{f_n\}_{n=1}^{\infty}$ a sequence in $BV([0,1])$. And let $f \in BV([0,1])$. $\{f_n\}_{n=1}^{\infty}$* ***converges to*** *f with the metric d if and only if there exists $\{\lambda_n\}_{n=1}^{\infty} \in \Lambda([0,1])$ such that $\lim_{n\to\infty} d_{TV}(f_n \circ \lambda_n, f) = 0$ and $\lim_{n\to\infty} d_{TV}(\lambda_n, Id) = 0$. Also, we say that a sequence $\{g_n\}_{n=1}^{\infty} \in BV([0,1])$ is a* ***Cauchy sequence*** *with the metric d if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbf{N}$ such that if $n,m \geq N_\epsilon$ then $d(g_n, g_m) < \epsilon$.*

Remark 3.4.5 If there exist $\{\lambda_n\}_{n=1}^{\infty} \in \Lambda([0,1])$ such that $\lim_{n\to\infty} d_{TV}(g_{n+1} \circ \lambda_n, g_n) = 0$ and $\lim_{n\to\infty} d_{TV}(\lambda_n, Id) = 0$, then $\{g_n\}_{n=1}^{\infty}$ is a Cauchy sequence. In fact, for every $n \in \mathbf{N}$

$$d(g_n, g_{n+1}) \leq \max\{d_{TV_0^1}(\lambda_n, Id), d_{TV_0^1}(g_n, g_{n+1} \circ \lambda_n)\} \tag{3.96}$$

and

$$\lim_{n\to\infty} d(g_n, g_{n+1}) \leq \lim_{n\to\infty} \max\{d_{TV_0^1}(\lambda_n, Id), d_{TV_0^1}(g_n, g_{n+1} \circ \lambda_n)\} = 0.$$

Then, $\{g_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Remark 3.4.6 Let $\{f_n\}_{n=1}^{\infty}, f \in BV([0,1])$. If $\lim_{n\to\infty} d_{TV}(f_n, f) = 0$, then we take $\lambda_n = Id$ for every $n \geq 1$ and we have that $\{f_n\}_{n=1}^{\infty}$ converges to f with the metric d.

Example 3.4.7 ($BV([0,1])$ is not complete under the metric d). For every $n \geq 1$, let us consider $f_n : [0,1] \to \mathbf{R}$ such that $f_n(x) = 1_{[0,1/2^n)}(x)$ for every $x \in [0,1]$. Observe that if $n \neq m$, then $d_{TV}(f_n, f_m) = 2$.

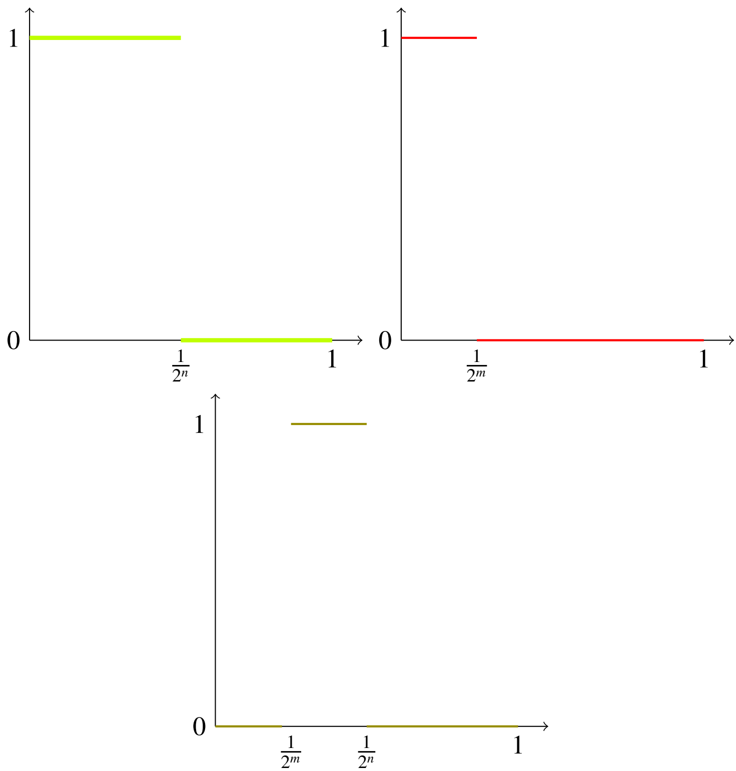

Figure 3.10 Graphs of f_n, f_m and $f_n - f_m$ for $n < m$.

For every $n \geq 1$, let $\lambda_n : [0,1] \to [0,1]$ such that

$$\lambda_n(x) = \begin{cases} \frac{1}{2}x & \text{if} \quad x \in \left[0, \frac{1}{2^n}\right] \\ \left(x - \frac{1}{2^n}\right)\frac{2^{n+1}-1}{(2^n-1)2} + \frac{1}{2^{n+1}} & \text{if} \quad x \in \left[\frac{1}{2^n}, 1\right] \end{cases} \tag{3.97}$$

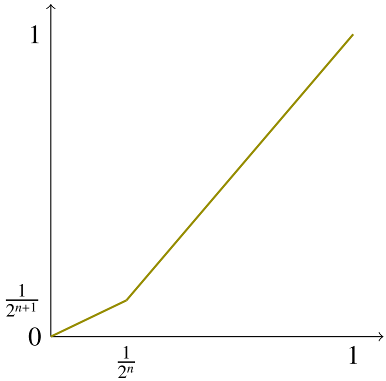

Figure 3.11 Graph of λ_n.

λ_n is linear on $\left[0, \frac{1}{2^n}\right]$ and on $\left[\frac{1}{2^n}, 1\right]$ and $\lambda_n\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}}$. Then $f_{n+1} \circ \lambda_n = f_n$ and for every $n \geq 1$, $d_{TV}(f_{n+1} \circ \lambda_n, f_n) = 0$. And

$$\begin{aligned} d_{TV}(\lambda_n, Id) &= 2\left(Id\left(\frac{1}{2^n}\right) - \lambda_n\left(\frac{1}{2^n}\right)\right) = 2\left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right) \\ &= 2\left(\frac{2}{2^{n+1}} - \frac{1}{2^{n+1}}\right) = \frac{2}{2^{n+1}} = \frac{1}{2^n} \end{aligned} \tag{3.98}$$

By equation (3.98), $\lim_{n\to\infty} d_{TV}(\lambda_n, Id) = 0$. Thus, $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence with the metric d. Let $f = 1_{\{0\}}$. For every $\lambda \in \Lambda$, $\lambda(0) = 0$, thus $f_n \circ \lambda(0) = f_n(0) = 1$ and $f(0) - f_n \circ \lambda(0) = 0$. λ is a bijection, thus there exists $v \in (0, 1)$ such that $f_n(\lambda(v)) = 1$ and $f_n(\lambda(1)) = 0$. Therefore, $d_{TV}(f_n, f) = 2$ for every $n \geq 1$. $\{f_n\}_{n=1}^{\infty}$ is not convergent to f with the metric d.

3.5 A Metric of Skorohod type

Definition 3.5.1 *Let us define $\Lambda^o([a, b])$ as the set of functions $\lambda\colon [a, b] \to [a, b]$ such that λ is a strictly increasing function with $\lambda(a) = a$, $\lambda(b) = b$ and there exists $P_\lambda = \{x_0, x_1, \ldots, x_{k_\lambda}\} \in \mathcal{P}([a, b])$ such that λ is linear on $[x_{i-1}, x_i]$ for every $i \in \{1, 2, \ldots, k_\lambda\}$ with*

$$m_i = \frac{\lambda(x_i) - \lambda(x_{i-1})}{x_i - x_{i-1}} \neq \frac{\lambda(x_{i+1}) - \lambda(x_i)}{x_{i+1} - x_i} = m_{i+1} \text{ for every } i \in \{1, 2, \ldots, k_\lambda - 1\}$$

Definition 3.5.2 *Let* $\lambda \in \Lambda^o([0,1])$ *such that there exists* $P_\lambda = \{x_0, \dots, x_{k_\lambda}\} \in \mathcal{P}([0,1])$ *such that* λ *is linear on* $[x_{i-1}, x_i]$ *for every* $i \in \{1, 2, \dots, k_\lambda\}$ *and* $m_i = \frac{\lambda(x_i)-\lambda(x_{i-1})}{x_i - x_{i-1}} \neq m_{i+1} = \frac{\lambda(x_{i+1})-\lambda(x_i)}{x_{i+1}-x_i}$ *for every* $i \in \{1, 2, \dots, k_\lambda\}$*, thus we define* $\|\lambda\|^o$ *by*

$$\|\lambda\|^o = \max_{1\le i\le k_\lambda} |\ln(m_i)| = \max_{1\le i\le k_\lambda} \left|\ln \frac{\lambda(x_i) - \lambda(x_{i-1})}{x_i - x_{i-1}}\right| = \max_{1\le i\le k_\lambda} \left|\ln \frac{TV_{x_{i-1}}^{x_i}(\lambda)}{TV_{x_{i-1}}^{x_i}(Id)}\right| \tag{3.99}$$

Note that $\Lambda^o([0,1]) \subseteq \Lambda([0,1])$. *In general, for* $\lambda \in \Lambda([0,1])$ *given in* (77),

$$\|\lambda\|^o = \sup_{s<t} \left|\ln\left(\frac{\lambda(t) - \lambda(s)}{t - s}\right)\right| \tag{3.100}$$

Remark 3.5.3 See Theorem 3.7.11 and by Remark 3.3.20 $\Lambda^o([a,b])$ is dense in $\Lambda([a,b])$ with the metric d_{TV}. Observe that if λ satisfies Definition 3.5.1, then $\lambda - Id$ is a monotonic function on $[x_i, x_{i-1}]$ for every $i \in \{1, 2, \dots, k_\lambda\}$ and

$$\begin{aligned} TV_0^1(\lambda - Id) &= \sum_{i=1}^{k_\lambda} TV_{x_{i-1}}^{x_i}(\lambda - Id) \\ &= \sum_{i=1}^{k_\lambda} |(\lambda - Id)(x_i) - (\lambda - Id)(x_{i-1})| \end{aligned} \tag{3.101}$$

Definition 3.5.4 *For* g *and* f *in* $BV([0,1])$*, define* $d^o(f,g)$ *to be the infimum of those positive* ϵ *for which there exists in* $\lambda \in \Lambda([0,1])$ *satisfying*

$$\|\lambda\|^o < \epsilon \tag{3.102}$$

and

$$d_{TV}(f, g\circ\lambda) = |f(0) - g(\lambda(0))| + TV_0^1(f - g\circ\lambda) = |f(0) - g(0)| + TV_0^1(f - g\circ\lambda) < \epsilon. \tag{3.103}$$

The definition becomes

$$d^o(f,g) = \inf\{\epsilon > 0 \mid \exists\lambda \in \Lambda([0,1]) \text{ such that } \|\lambda\|^o < \epsilon \text{ and } d_{TV}(f, g\circ\lambda) < \epsilon\}. \tag{3.104}$$

Note that $|u - 1| \le e^{|\log(u)|} - 1$ for $u > 0$, because if $u \ge 1$, $|u - 1| = u - 1 = e^{\log(u)} - 1 = e^{|\log(u)|} - 1$. And if $0 < u < 1$, then $0 \le (u - 1)^2$,

$0 \leq u^2 - 2u + 1$, then $2u \leq u^2 + 1$, so $2 \leq u + 1/u$, therefore $1 - u \leq 1/u - 1$ and $|u - 1| = 1 - u \leq 1/u - 1 = e^{\log(1/u)} - 1 = e^{-\log(u)} - 1 = e^{|\log(u)|} - 1$. For $s \in (0, 1)$

$$\begin{aligned} |\lambda(s) - Id(s)| &= |\lambda(s) - \lambda(0) - (s - 0)| \\ &= |s| \left| \frac{\lambda(s) - \lambda(0)}{s - 0} - 1 \right| \\ &\leq s \left(\exp \left| \ln \left(\frac{\lambda(s) - \lambda(0)}{s - 0} \right) \right| - 1 \right) \\ &\leq \exp |\ln (\|\lambda\|^o)| - 1 \end{aligned} \tag{3.105}$$

So, $d_{sup}(\lambda, Id) \leq \exp |\ln (\|\lambda\|^o)| - 1$

And since $v \leq e^v - 1$ for all v. It is because if $f : \mathbb{R} \to \mathbb{R}$ is such that $f(v) = e^v - 1 - v$, then $f'(v) = e^v - 1$ and $f'(v) = 0$ if and only if $v = 0$. Moreover, $f''(v) = e^v$ and $f''(0) = 1 > 0$. Then $v = 0$ is the minimum of the function. That is, for each $v \in \mathbb{R}$, $e^v - 1 - v = f(v) \geq f(0) = 0$.
Then, using the inequality $v \leq e^v - 1$ for all v we have that

$$d_{TV}(f, g \circ \lambda) \leq \exp (d_{TV}(f, g \circ \lambda)) - 1 \tag{3.106}$$

For $\lambda \in \Lambda$ and $0 \leq s < t \leq 1$

$$\begin{aligned} \|\lambda\|^o = \sup_{s<t} \left| \ln \frac{\lambda(t) - \lambda(s)}{t - s} \right| &= \sup_{\lambda^{-1}(s)<\lambda^{-1}(t)} \left| \ln \frac{\lambda(\lambda^{-1}(t)) - \lambda(\lambda^{-1}(s))}{\lambda^{-1}(t) - \lambda^{-1}(s)} \right| \\ &= \sup_{\lambda^{-1}(s)<\lambda^{-1}(t)} \left| \ln \frac{t - s}{\lambda^{-1}(t) - \lambda^{-1}(s)} \right| \\ &= \sup_{\lambda^{-1}(s)<\lambda^{-1}(t)} \left| - \ln \frac{\lambda^{-1}(t) - \lambda^{-1}(s)}{t - s} \right| = \|\lambda^{-1}\|^o \end{aligned}$$

Therefore, $\|\lambda\|^o = \|\lambda^{-1}\|^o$ and symmetry for d^o follows. The triangle inequality for d^o follows from the inequality

$$\|\lambda_1 \circ \lambda_2\|^o \leq \|\lambda_1\|^o + \|\lambda_2\|^o \tag{3.107}$$

equation (3.107) holds because

$$
\begin{aligned}
\|\lambda_1 \circ \lambda_2\|^o &= \sup_{t<s} \left| \ln \frac{\lambda_1(\lambda_2(t)) - \lambda_1(\lambda_2(s))}{t-s} \right| \\
&= \sup_{s<t} \left| \ln \left(\frac{\lambda_1(\lambda_2(t) - \lambda_1(\lambda_2(s))}{t-s} \cdot \frac{\lambda_2(t) - \lambda_2(s)}{\lambda_2(t) - \lambda_2(s)} \right) \right| \\
&= \sup_{s<t} \left| \ln \left(\frac{\lambda_1(\lambda_2(t)) - \lambda_1(\lambda_2(s))}{\lambda_2(t) - \lambda_2(s)} \cdot \frac{\lambda_2(t) - \lambda_2(s)}{t-s} \right) \right| \\
&= \sup_{s<t} \left| \ln \frac{\lambda_1(\lambda_2(t)) - \lambda_1(\lambda_2(s))}{\lambda_2(t) - \lambda_2(s)} + \ln \frac{\lambda_2(t) - \lambda_2(s)}{t-s} \right| \\
&\leq \sup_{s<t} \left(\left| \ln \frac{\lambda_1(\lambda_2(t)) - \lambda_1(\lambda_2(s))}{\lambda_2(t) - \lambda_2(s)} \right| + \left| \ln \frac{\lambda_2(t) - \lambda_2(s)}{t-s} \right| \right) \\
&\leq \sup_{s<t} \left| \ln \frac{\lambda_1(\lambda_2(t)) - \lambda_1(\lambda_2(s))}{\lambda_2(t) - \lambda_2(s)} \right| + \sup_{s<t} \left| \ln \frac{\lambda_2(t) - \lambda_2(s)}{t-s} \right| \\
&\leq \sup_{s<t} \left| \ln \frac{\lambda_1(\lambda_2(t)) - \lambda_1(\lambda_2(s))}{\lambda_2(t) - \lambda_2(s)} \right| + \sup_{s<t} \left| \ln \frac{\lambda_2(t) - \lambda_2(s)}{t-s} \right| \\
&= \sup_{w<r} \left| \ln \frac{\lambda_1(r) - \lambda_1(w)}{r-w} \right| + \sup_{s<t} \left| \ln \frac{\lambda_2(t) - \lambda_2(s)}{t-s} \right| \\
&= \|\lambda_1\|^o + \|\lambda_2\|^o. \qquad (3.108)
\end{aligned}
$$

That $d^o(f, g) = 0$ implies $\|\lambda\|^o = 0$ thus $\lambda = Id$ and $f = g$ follows using the corresponding property for d_{sup} and d_{TV}. Therefore, d^o is a metric.

3.6 Completeness

Let $\{f_n\}_{n=1}^{\infty} \subseteq BV([0,1])$ a **Cauchy sequence**, that is, for every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that if $n, m \geq N$ then $d^o(f_n, f_m) < \epsilon$.

Remark 3.6.1 *If $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence then there exists a sequence $\{\lambda_i\}_{i=1}^{\infty} \subseteq \Lambda([0,1])$ such that for every $i \in \mathbf{N}$, $\|\lambda_i\|^o < \frac{1}{2^i}$. And there exists $\{f_{N_i}\}_{i=1}^{\infty}$ a subsequence of $\{f_n\}_{n=1}^{\infty}$ such that for every $i \in \mathbf{N}$, $d_{TV}(f_{N_i}, f_{N_i+1} \circ \lambda_i) < \frac{1}{2^i}$.*

Proof: Let $\epsilon_1 = \frac{1}{2}$ then there exists $N_1 > 0$ and such that for every $j \geq 1$, $d^o(f_{N_1}, f_{N_1+j}) < \epsilon_1$. For $\epsilon_2 = \frac{1}{2^2}$ there exists $N_2 > 0$ such that $N_2 > N_1$ such that for every $j \geq 1$, $d^o(f_{N_2}, f_{N_2+j}) < \epsilon_2$.

Since $N_2 > N_1$, then there exists $\lambda_1 \in \Lambda([0,1])$ such that

$$d^o(f_{N_1}, f_{N_2}) \leq \max\{\|\lambda_1\|^o, d_{TV_0^1}(f_{N_1}, f_{N_2} \circ \lambda_1)\} < \frac{1}{2} = \epsilon_1. \qquad (3.109)$$

Inductively for $\epsilon_k = \frac{1}{2^k}$ there exist $N_k > N_{k-1}$ and $\lambda_k \in \Lambda([0,1])$ such that

$$d^o(f_{N_{k-1}}, f_{N_k}) \leq \max\{\|\lambda_k\|^o, d_{TV_0^1}(f_{N_{k-1}}, f_{N_k} \circ \lambda_{k-1})\} < \frac{1}{2^k} = \epsilon_k. \quad (3.110)$$

Lemma 3.6.2 *Let* (M, d) *be a metric space. If* $\{x_n\}_{n=1}^{\infty} \subseteq M$ *is a Cauchy sequence such that there exists* $\{x_{n_i}\}_{i=1}^{\infty}$ *a subsequence of* $\{x_n\}_{n=1}^{\infty}$ *and* $x \in M$ *such that* $\lim_{i\to\infty} d(x_{n_i}, x) = 0$ *then* $\lim_{n\to\infty} d(x_n, x) = 0$. See [15].

Theorem 3.6.3 $(BV([0,1]), d^o)$ *is a complete metric space.*

Proof: Let $\{f_n\}_{n=1}^{\infty} \subseteq BV([0,1])$ a Cauchy sequence. Then there exists $\{f_{N_i}\}_{i=1}^{\infty}$ a subsequence of $\{f_n\}_{n=1}^{\infty}$ and $\{\lambda_i\}_{i=1}^{\infty} \subseteq \Lambda([0,1])$ such that

$$\|\lambda_i\|^o < \frac{1}{2^i}, \text{ and } d_{TV}(f_{N_i}.f_{N_i+1} \circ \lambda_i) < \frac{1}{2^i} \qquad (3.111)$$

Let us see that $e^u - 1 \leq 2u$ for $0 \leq u \leq \frac{1}{2}$. If $f\colon \left[0, \frac{1}{2}\right] \to \mathbb{R}$ is such that $f(u) = 2u - e^u + 1$. $f'(u) = 2 - e^u$, $f'(0) = 1$ and $f'\left(\frac{1}{2}\right) \approx 0.3512$ then $f'(u) = 2 - e^u > 0$. Thus f is increasing and $0 = f(0) \leq f(u)$, that is, $0 \leq 2u - e^u + 1$. Also note that for $m, n \in \mathbf{N}$, $\lambda_{n+m} \circ \cdots \circ \lambda_{n+1} \circ \lambda_n \in Lip([0,1])$, and

$$\begin{aligned} d_{sup}(\lambda_{n+m+1} \circ \cdots \circ \lambda_{n+1} \circ \lambda_n, \lambda_{n+m} \circ \cdots \circ \lambda_{n+1} \circ \lambda_n) &= d_{sup}(\lambda_{n+m+1}, Id) \\ &\leq 2\|\lambda_{n+m+1}\|^o < \frac{1}{2^{n+m}} \end{aligned}$$

Thus $\{\lambda_{n+m} \circ \cdots \circ \lambda_{n+1} \circ \lambda_n\}_{m=0}^{\infty}$ is a Cauchy sequence with the metric d_{sup}. So, the sequence converges to γ_n

$$\gamma_n(t) = \lim_{m\to\infty} \lambda_{n+m} \circ \cdots \circ \lambda_{n+1} \circ \lambda_n(t) \text{ for every } t \in [a,b] \qquad (3.112)$$

γ_n is continuous

$$\begin{aligned}\left|\log\frac{\lambda_{n+m}\circ\cdots\circ\lambda_n(t)-\lambda_{n+m}\circ\cdots\circ\lambda_n(s)}{t-s}\right| &\leq \|\lambda_{n+m}\circ\cdots\circ\lambda_{n+1}\circ\lambda_n\|^o\\ &\leq \|\lambda_{n+m}\|^o+\cdots+\|\lambda_n\|^o\\ &\leq \frac{1}{2^{n-1}}\end{aligned}\tag{3.113}$$

Taking limit when $m\to\infty$ we have that $\|\gamma_n\|^o \leq \frac{1}{2^{n-1}} < \infty$. So γ_n is strictly increasing (in other case $\|\gamma_n\|^o = \infty$). Also, $Lip(\gamma) < \infty$ in the other case $\|\gamma_n\|^o = \infty$, thus $\gamma_n \in \Lambda$.
We have that $\gamma_n = \gamma_{n+1}\circ\lambda_n$, so $\gamma_{n+1}^{-1} = \lambda_n\circ\gamma_n^{-1}$ and

$$\begin{aligned}d_{TV}(f_{N_i}\circ\gamma_i^{-1}, f_{N_{i+1}}\circ\gamma_{i+1}^{-1}) &= d_{TV}(f_{N_i}\circ\gamma_i^{-1}, f_{N_{i+1}}\circ\lambda_i\circ\gamma_i^{-1})\\ &= d_{TV}(f_{N_i}, f_{N_{i+1}}\circ\lambda_i) < \frac{1}{2^i}\end{aligned}\tag{3.114}$$

Thus $\{f_{N_i}\circ\gamma_i^{-1}\}_{i=1}^{\infty} \subseteq BV([0,1])$ is a Cauchy sequence with the metric d_{TV}. So, there exists $f\in BV([0,1])$ such that $0 = \lim_{n\to\infty} d_{TV}(f_{N_i}\circ\gamma_i^{-1}, f)$ and $\lim_{i\to\infty} d^o(f_{N_i}, f) = 0$. By Lemma 3.6.2 we have the result. □

Theorem 3.6.4 ($AC([0,1]), d^o$) *is a complete space*

Proof: Let $\{f_n\}_{n=1}^{\infty} \subseteq AC([0,1])$ a Couchy sequence with respect to the metric d^o.
Using that $AC([0,1]) \subseteq BV([0,1])$ we have that $\{f_n\}_{n=1}^{\infty} \subseteq BV([0,1])$ and is a Cauchy sequence on $BV([0,1])$. $(BV([0,1]), d^o)$ is a complete space then there exists $f\in BV([0,1])$ such that $\lim_{n\to\infty} d^o(f_n, f) = 0$.
Then there exists a strictly increasing sequence $\{N_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ and $\{\lambda_i\}_{i=1}^{\infty} \subseteq \Lambda([0,1])$ such that

$$\|\lambda_i\|^o < \frac{1}{2^i}, \text{ and } d_{TV}(f.f_{N_i}\circ\lambda_i) < \frac{1}{2^i}\tag{3.115}$$

Also, $f_{N_i}\circ\lambda_i \in AC([0,1])$ because for $\epsilon > 0$ there exists $\delta > 0$ such that for $S = \{[a_1,b_1],\ldots,[a_n,b_n]\} \in \sum([0,1])$, with

$$\sum_{k=1}^{n}(b_k - a_k) \leq \delta\tag{3.116}$$

implies that

$$\sum_{k=1}^{n} |f_{N_i}(b_k) - f_{N_i}(a_k)| < \epsilon. \tag{3.117}$$

And for $\delta > 0$ there exists $\eta > 0$ such that if

$$\sum_{k=1}^{n} (b_k - a_k) \leq \eta \tag{3.118}$$

implies that

$$\sum_{k=1}^{n} |\lambda_i(b_k) - \lambda_i(a_k)| < \delta. \tag{3.119}$$

Note that $S = \{[\lambda_i(a_1), \lambda_i(b_1)], \ldots, [\lambda_i(a_n), \lambda_i(b_n)]\} \in \sum([0,1])$ because λ_i is strictly increasing and $\lambda_i(0) = 0$ and $\lambda_i(1) = 1$, so

$$\sum_{k=1}^{n} |f_{N_i} \circ \lambda_i(b_k) - f_{N_i} \circ \lambda_i(a_k)| < \epsilon. \tag{3.120}$$

Then $f_{N_i} \circ \lambda_i \in AC([0,1])$ and $\lim_{i\to\infty} d_{TV}(f_{N_i} \circ \lambda_i, f)$. By Theorem 3.3.19, $f \in AC([0,1])$. □

3.7 Separability

Definition 3.7.1 *Let us denote $\mathcal{P}_{\mathbb{Q}}([0,1])$ the set of partitions $P \in \mathcal{P}([0,1])$ such that all elements of P are rational numbers. Let us define the following set*

$$\begin{aligned} \mathcal{E}([0,1]) \quad = \quad & \left\{ \sum_{i=1}^{s} c_i(x) 1_{(q_{i-1},q_i)}(x) + \sum_{i=0}^{s} k_i 1_{\{q_i\}}(x) \mid k_0, k_1 \ldots, k_s \in \mathbb{Q}, \right. \\ & \textit{and for every } i \in \{1, \ldots, s\}\ c_i(x) = m_i x + b_i \\ & \left. \textit{for some } m_i, b_i \in \mathbb{Q} \textit{ and } \{q_0, q_1, \ldots, q_s\} \in \mathcal{P}_{\mathbb{Q}}([0,1]) \right\} \end{aligned}$$

Remark 3.7.2 *Note that if $f \in \mathcal{E}([0,1])$, then $-f \in \mathcal{E}([0,1])$. Moreover, $\mathcal{E}([0,1])$ is a countable set. Also if $m_i = 0$, for some i, then c_i is a constant function.*

Remark 3.7.3 *Note that we can define* $(BV([a,b]), d^o)$ *in a similar way as in Definition* 3.5.4 *and we have the same properties.*

Similar to equations (3.126) and (3.127),

$$\Lambda([a,b]) = \{\lambda \in Lip([a,b]) \,|\, \lambda(a) = a,\ \lambda(b) = b,\ \lambda \text{ is strictly increasing}\} \tag{3.121}$$

For $\lambda \in \Lambda([a,b])$

$$||\lambda||^o = \sup_{a\leq s<t\leq b} \left| \ln\left(\frac{\lambda(t)-\lambda(s)}{t-s}\right)\right| \tag{3.122}$$

The definition becomes

$$d^o(f,g) = \inf\{\epsilon > 0 \,|\, \exists\lambda \in \Lambda([a,b]) \ \textit{such that}\ ||\lambda||^o < \epsilon \ \textit{and}\ d_{TV}(f, g\circ\lambda) < \epsilon\}. \tag{3.123}$$

Lemma 3.7.4 *Let* $f \in BV([0,1])$ *be a linear function or a constant function. Then, for every* $\epsilon > 0$ *there exists* $f_\epsilon \in \mathcal{E}([0,1])$ *such that* $d_{TV}(f, f_\epsilon) < \epsilon$.

Proof: If f is an increasing linear function, that is, $f(x) = ax + b$ for every $x \in [0,1]$ and for some $a, b \in \mathbf{R}$. For $\epsilon > 0$, let $a_0 \in \mathbb{Q} \cap \left(f(0) - \frac{\epsilon}{3}, f(0)\right)$ and $a_1 \in \mathbb{Q}\cap(f(1) - (f(0) - a_0), f(1))$. Note that $0 < f(1)-a_1 < f(0)-a_0$. And using that f is an increasing function, $0 \leq f(1) - f(0) < a_1 - a_0$ and $a_0 < a_1$.

And let us consider $f_\epsilon\colon [0,1] \to \mathbf{R}$ the straight line such that $f_\epsilon(0) = a_0$ and $f_\epsilon(1) = a_1$, that is, $f_\epsilon(x) = (a_1 - a_0)x + a_0$. Note that $f_\epsilon \in \mathcal{E}([0,1])$ with partition $\{0,1\} \in \mathcal{P}_{\mathbb{Q}}([0,1])$. f_ϵ is an increasing function because $a_0 < a_1$. Also $f - f_\epsilon$ is a straight line that passes through the points $(0, f(0) - a_0)$ and $(1, f(1) - a_1)$.

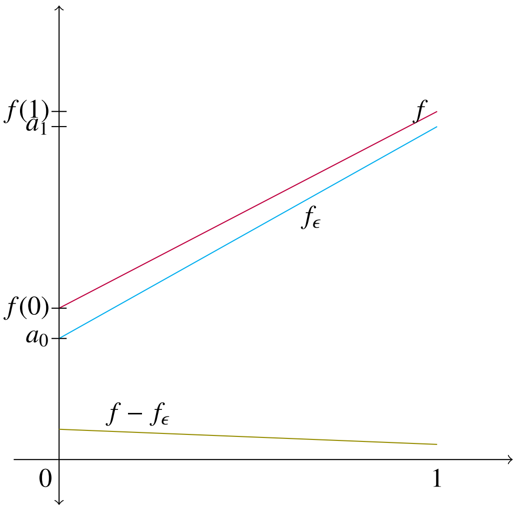

Figure 3.12 Graphs of f, f_ϵ and $f - f_\epsilon$.

So,

$$\begin{aligned} d_{TV}(f.f_\epsilon) &= |f(0) - f_\epsilon(0)| + TV_0^1(f - f_\epsilon) \\ &= |f(0) - a_0| + |(f - f_\epsilon)(1) - (f - f_\epsilon)(0)| \\ &= |f(0) - a_0| + |(f(1) - a_1) - (f(0) - a_0)| \\ &\leq |f(0) - a_0| + |f(1) - a_1| + |f(0) - a_0| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned} \tag{3.124}$$

If f is a decreasing linear function, then $-f$ is an increasing linear function and there exists $h_\epsilon \in \mathcal{E}([0,1])$ such that $d_{TV}(-f, h_\epsilon)$. Also

$$\begin{aligned} d_{TV}(-f, h_\epsilon) &= |-f(0) - h_\epsilon(0)| + TV_0^1(-f - h_\epsilon) \\ &= |f(0) - (-h_\epsilon(0))| + TV_0^1(f - (-h_\epsilon)) \\ &= d_{TV}(f, -h_\epsilon) \end{aligned} \tag{3.125}$$

And by Remark 1.6.2, $-h_\epsilon \in \mathcal{E}([0,1])$.
In the case f a constant function the proof is similar. □

Proposition 3.7.5 *Let $f\colon [0,1] \to \mathbb{R}$ be a function such that*

$$f(x) = b_1 1_{[a_0,a_1]}(x) + \sum_{i=2}^{n} b_i 1_{(a_{i-1},a_i]}(x)$$

for some $\{a_0,\ldots,a_n\} \in \mathcal{P}([0,1])$ and $\{b_1,\ldots,b_n\} \in \mathbb{R}$. Then there exists $f_\epsilon \in \mathcal{E}([0,1])$ such that $d^o(f,f_\epsilon) < \epsilon$

Proof: Suppose that $n = 1$, in this case f is a constant function and by the Lemma 3.7.4 for $\epsilon > 0$ there exists f_ϵ such that $d^o(f,f_\epsilon) < \epsilon$. If $n = 2$, then there exists $\{a_0,a_1,a_2\} \in \mathcal{P}([0,1])$ and $b_1, b_2 \in \mathbf{R}$ such that $f(x) = b_1 1_{[a_0,a_1]}(x) + b_2 1_{(a_1,a_2]}(x)$. Let $\epsilon > 0$ then using that $\ln\colon (0,\infty) \to \mathbf{R}$ is a continuous function we have that there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $x \in (0,\infty)$ and $|x - a_1| < \delta_1$ then $|\ln(x) - \ln(a_1)| < \epsilon$ and if $x \in (0,\infty)$ and $|x - (1 - a_1)| < \delta_2$ then $|\ln(x) - \ln(1 - a_1)| < \epsilon$.

Let $\delta = \min\{\delta_1, \delta_2\}$, $q_1 \in (b_1, b_1 + \frac{\epsilon}{3}) \cap \mathbb{Q}$, $q_2 \in (b_2, b_2 + \frac{\epsilon}{3}) \cap \mathbb{Q}$ and $r_1 \in (a_1, a_2) \cap (a_1, a_1 + \delta) \cap \mathbb{Q}$. Then $|1 - r_1 - (1 - a_1)| = |r_1 - a_1| < \delta$ and $|\ln(r_1) - \ln(a_1)| < \epsilon$ and $|\ln(1 - r_1) - \ln(1 - a_1)| < \epsilon$. Let us consider $\lambda\colon [0,1] \to [0,1]$ such that

$$\lambda(x) = \begin{cases} \left(\frac{r_1}{a_1}\right) x & \text{if} \quad x \in [0,a_1] \\ \left(\frac{1-r_1}{1-a_1}\right)(x - a_1) + r_1 & \text{if} \quad x \in [a_1,1] \end{cases} \tag{3.126}$$

Then $\lambda(0) = 0$, $\lambda(a_1) = r_1$, $\lambda(1) = 1$ and

$$\begin{aligned} \|\lambda\|^o &= \max\left\{\left|\ln\left(\frac{r_1}{a_1}\right)\right|, \left|\ln\left(\frac{1-r_1}{1-a_1}\right)\right|\right\} \\ &= \max\{|\ln(r_1) - \ln(a_1)|, |\ln(1 - r_1) - \ln(1 - a_1)|\} < \epsilon \end{aligned} \tag{3.127}$$

Also, let us consider $f_\epsilon\colon [0,1] \to \mathbf{R}$ such that

$$f_\epsilon(x) = \begin{cases} q_1 & \text{if} \quad x \in [0,r_1] \\ q_2 & \text{if} \quad x \in (r_1,1] \end{cases} \tag{3.128}$$

Note that $f_\epsilon \in \mathcal{E}([0,1])$ and

$$f_\epsilon \circ \lambda(x) = \begin{cases} q_1 & \text{if} \quad x \in [a_0,a_1] \\ q_2 & \text{if} \quad x \in (a_1,1] \end{cases} \tag{3.129}$$

Note that

$$(f - f_\epsilon \circ \lambda)(x) = \begin{cases} b_1 - q_1 & \text{if} \quad x \in [a_0, a_1] \\ b_2 - q_2 & \text{if} \quad x \in (a_1, 1] \end{cases} \tag{3.130}$$

Then, $f - f_\epsilon \circ \lambda$ is a monotonic function. Therefore

$$\begin{aligned} d_{TV}(f, f_\epsilon \circ \lambda) &= |f(0) - f_\epsilon \circ \lambda(0)| + TV_0^1(f - f_\epsilon \circ \lambda) \\ &= |b_1 - q_1| + |(f - f_\epsilon \circ \lambda)(1) - (f - f_\epsilon \circ \lambda)(0)| \\ &= |b_1 - q_1| + |b_2 - q_2 - (b_1 - q_1)| \\ &\leq |b_1 - q_1| + |b_2 - q_2| + |b_1 - q_1| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned} \tag{3.131}$$

Using equations (3.127) and (3.131) we have that

$$d^o(f, f_\epsilon) \leq \max\{d_{TV}(f, f_\epsilon \circ \lambda), \|\lambda\|^o\} < \epsilon. \tag{3.132}$$

Now, suppose that the results is holds for a $n \geq 2$. Let $f\colon [0,1] \to \mathbf{R}$ such that

$$f(x) = b_1 1_{[a_0,a_1]}(x) + \sum_{i=2}^{n+1} b_i 1_{(a_{i-1},a_i]}(x)$$

for $\{a_0, a_1, \ldots, a_n, a_{n+1}\} \in \mathcal{P}([0,1])$ and $b_1, b_2, \ldots, b_n, b_{n+1} \in \mathbf{R}$. For $\epsilon > 0$ let $q_0 \in (a_{n-1}, a_n) \cap (a_{n-1}, a_{n-1} + \frac{\epsilon}{3}) \cap \mathbb{Q}$

Let $g\colon [0, q_0] \to \mathbf{R}$ such that $g = f|_{[0,q_0]}$, that is,

$$g(x) = b_1 1_{[a_0,a_1]}(x) + \sum_{i=2}^{n-1} b_i 1_{(a_{i-1},a_i]}(x) + b_n 1_{(a_{n-1},q_0]}(x)$$

Thus by Induction hypothesis there exists $g_1 \in \mathcal{E}([0, q_0])$ such that $d^o(g, g_1) < \frac{\epsilon}{3}$.
Let $h\colon [q_0, 1] \to \mathbf{R}$ such that $h = f|_{[q_0,1]}$, that is,

$$h(x) = b_n 1_{[q_0,a_n]}(x) + b_{n+1} 1_{(a_n,a_{n+1}]}(x) \tag{3.133}$$

By induction's base there exists $h_1 \in \mathcal{E}([q_0, 1])$, $h_1(x) = q_1 1_{[q_0,r_1]}(x) + q_2 1_{(r_1,1]}$ for some $q_1, q_2, r_1 \in \mathbb{Q}$ and such that $d^o(h, h_1) < \frac{\epsilon}{3}$. Thus there exist $\lambda_1 \colon [0, q_0] \to [0, q_0]$ and $\lambda_2 \colon [q_0, 1] \to [q_0, 1]$ such that

$$\begin{aligned} d^o(g, g_1) &\leq \max\{\|\lambda_1\|^o, d_{TV}(g, g_1 \circ \lambda_1)\} < \frac{\epsilon}{3} \text{ and} \\ d^o(h, h_1) &\leq \max\{\|\lambda_2\|^o, d_{TV}(h, h_1 \circ \lambda_2)\} < \frac{\epsilon}{3} \end{aligned} \tag{3.134}$$

Note that

$$\begin{aligned} |g(q_0) - g_1(q_0)| &= |g(q_0) - g_1 \circ \lambda_1(q_0)| \\ &\leq |g(q_0) - g_1 \circ \lambda_1(q_0) - (g(0) - g_1 \circ \lambda_1(0))| \\ &\quad + |g(0) - g_1 \circ \lambda_1(0)| \\ &\leq TV_0^1(g - g_1 \circ \lambda_1) + |g(0) - g_1 \circ \lambda_1(0)| \\ &= d_{TV}(g, g_1 \circ \lambda_1) < \frac{\epsilon}{3} \end{aligned} \tag{3.135}$$

Let $\lambda \colon [0, 1] \to [0, 1]$ such that

$$\lambda(x) = \begin{cases} \lambda_1(x) & \text{if} \quad x \in [0, q_0] \\ \lambda_2(x) & \text{if} \quad x \in [q_0, 1] \end{cases} \tag{3.136}$$

Then $\lambda(0) = 0$, $\lambda(q_0) = \lambda_1(q_0) = \lambda_2(q_0) = q_0$ and $\lambda(1) = 1$. Note that

$$\|\lambda\|^o = \max\{\|\lambda_1\|^o, \|\lambda_2\|^o\} < \frac{\epsilon}{3} \tag{3.137}$$

And let

$$f_\epsilon(x) = \begin{cases} g_1(x) & \text{if} \quad x \in [0, q_0] \\ h_1(x) & \text{if} \quad x \in (q_0, 1] \end{cases} \tag{3.138}$$

Then, using equations (3.135), (3.134) we have that

$$\begin{aligned}
d_{TV}(f, f_\epsilon \circ \lambda) &= |f(0) - f_\epsilon \circ \lambda(0)| + TV_0^{q_0}(f - f_\epsilon \circ \lambda) + TV_{q_0}^1(f - f_\epsilon \circ \lambda) \\
&= |g(0) - g_1 \circ \lambda_1(0)| + TV_0^{q_0}(g - g_1 \circ \lambda_1) + TV_{q_0}^1(f - f_\epsilon \circ \lambda) \\
&= d_{TV}(g, g_1 \circ \lambda_1) + |(f - f_\epsilon \circ \lambda)(q_0) - (f - f_\epsilon \circ \lambda)(q_0+)| \\
&\quad + TV_{(q_0,1]}(f - f_\epsilon \circ \lambda) \\
&= d_{TV}(g, g_1 \circ \lambda_1) + |(g(q_0) - g_1(q_0) - (h(q_0) - h_1(q_0))| \\
&\quad + TV_{(q_0,1]}(h - h_1 \circ \lambda_2) \\
&= d_{TV}(g, g_1 \circ \lambda_1) + |(g(q_0) - g_1(q_0) - (h(q_0) - h_1(q_0))| \\
&\quad + TV_{q_0}^1(h - h_1 \circ \lambda_2) \\
&\leq d_{TV}(g, g_1 \circ \lambda_1) + |(g(q_0) - g_1(q_0)| + |h(q_0) - h_1(\lambda_2(q_0))| \\
&\quad + TV_{q_0}^1(h - h_1 \circ \lambda_2) \\
&= d_{TV}(g, g_1 \circ \lambda_1) + |(g(q_0) - g_1(q_0)| + d_{TV}(h, h_1 \circ \lambda_2) < \epsilon
\end{aligned} \tag{3.139}$$

Using equations (3.139) and (3.137) we have that

$$d^o(f, f_\epsilon) \leq \max\{\|\lambda\|^o, d_{TV}(f, f_\epsilon \circ \lambda)\} < \epsilon. \tag{3.140}$$

□

Remark 3.7.6 Let $f \in \mathcal{E}([a, b])$ where a, b may not be rational numbers. Then there exists $\{r_0, r_1, \ldots, r_n\} \in \mathcal{P}([a, b])$ a partition as such that for every $i \in \{1, \ldots, n-1\}$ $r_i \in \mathbb{Q}$. Also there exist $k_0, k_1, \ldots, k_n \in \mathbb{Q}$ and $c_1(x), \ldots, c_n(x)$ such that for every $i \in \{2, \ldots, n-1\}$, $c_i\colon (r_{i-1}, r_i) \to \mathbf{R}$ is such that $c_i(x) = z_i$ a constant function with $z_i \in \mathbb{Q}$ or $c_i(x) = m_i x + b_i$ for some $m_i, b_i \in \mathbb{Q}$. In this case $c_1\colon (r_0, r_1) \to \mathbf{R}$ is a rational constant function or it is a straight line with $k_0 = f(r_0) = f(a), k_1 = f(r_1) \in \mathbb{Q}$, in the same way $c_n\colon (r_{n-1}, r_n) \to \mathbf{R}$ is a rational constant function or it is a straight line with $k_{n-1} = f(r_{n-1}), k_n = f(r_n) \in \mathbb{Q}$. That is, in the case that c_1 and c_n are not constant functions then c_1 and c_n may not have rational slopes, but they are straight lines with extremes rationals.

$$f(x) = \sum_{i=1}^{n} c_i(x) 1_{(r_{i-1}, r_i)}(x) + \sum_{i=0}^{n} k_i 1_{\{r_i\}}(x).$$

Let q_0, q_n be rational numbers such that $q_0 < a < r_{n-1} < q_n < b$. And let

$\alpha\colon [q_0, q_n] \to [a, b]$ such that

$$\alpha(x) = \begin{cases} \left(\frac{r_1-a}{r_1-q_0}\right)(x-q_0)+a & \text{if} \quad x \in [q_0, r_1] \\ \left(\frac{b-r_{n-1}}{q_n-r_{n-1}}\right)(x-r_{n-1})+r_{n-1} & \text{if} \quad x \in [r_{n-1}, q_n] \\ x & \text{if} \quad x \in [r_1, r_{n-1}] \end{cases} \tag{3.141}$$

that is, $\alpha(q_0) = a$, $\alpha(q_n) = b$ and α is an increasing function. Note that $\alpha(q_0) = a$, $\alpha(q_n) = b$ and for every $i \in \{1, 2, \ldots, n-1\}$ $\alpha(r_i) = r_i$

We have that $f \circ \alpha(x) = f(x)$ for every $x \in [r_1, r_{n-1}]$. Also, $f \circ \alpha(q_0) = f(a) = f(r_0) = k_0 \in \mathbb{Q}$, $f \circ \alpha(q_n) = f(b) = f(r_n) = k_n \in \mathbb{Q}$. And $f \circ \alpha|_{(q_0, r_1)} = c_1 \circ \alpha\colon (q_0, r_1) \to \mathbf{R}$ is a straight line with rational extremes, $f \circ \alpha|_{(r_{n-1}, q_n)} = c_n \circ \alpha\colon (r_{n-1}, q_n) \to \mathbf{R}$ also is a straight line with rational extremes. Then $f \circ \alpha \in \mathcal{E}([q_0, q_n])$.

Proposition 3.7.7 *Let $f \in PL([0,1])$ then there exists $f_\epsilon \in \mathcal{E}([0,1])$ such that $d^o(f, f_\epsilon) < \epsilon$*

Proof: Let $f \in PL([0,1])$ then there exists $P = \{t_0, t_1, \ldots, t_k\} \in \mathcal{P}([0,1])$ such that f is a straight line on $[t_{i-1}, t_i]$ for every $1 \le i \le k$. Let $\epsilon > 0$ and for every $0 \le i \le k$ let $a_i \in \mathbb{Q} \cap \left(f(t_i) - \frac{\epsilon}{2k}, f(t_i)\right)$.

We consider $\delta = \min\{t_i - t_{i-1} \mid 1 \le i \le k\} > 0$. $\ln\colon \left[\frac{\delta}{10}, 1\right] \to \mathbf{R}$, thus ln is uniformly continuous because $\left[\frac{\delta}{10}, 1\right]$ is a compact set. So there exists $\eta > 0$ such that for every $x, y \in \left[\frac{\delta}{10}, 1\right]$ such that $|x - y| < \eta$ then $|\ln(x) - \ln(y)| < \epsilon$.

Take $y_i \in \mathbb{Q} \cap \left(t_i - \frac{\delta}{10}, t_i\right) \cap (t_i - \frac{\eta}{2}, t_i)$ for every $1 \le i \le k-1$, $y_0 = 0$ and $y_k = 1$. Thus,

$$\begin{aligned} |t_i - t_{i-1} - (y_i - y_{i-1})| &= |t_i - y_i - (t_{i-1} - y_{i-1})| \le |t_i - y_i| + |t_{i-1} - y_{i-1}| \\ &< \frac{\eta}{2} + \frac{\eta}{2} = \eta \end{aligned} \tag{3.142}$$

And $t_i - t_{-1} \ge \delta > \frac{\delta}{10}$, $y_i - y_{i-1} > y_i - t_{i-1} > \frac{9\delta}{10} > \frac{\delta}{10}$, so $t_i - t_{i-1}, y_i - y_{i-1} \in \left[\frac{\delta}{10}, 1\right]$,

$$|\ln(t_i - t_{i-1}) - \ln(y_i - y_{i-1})| < \epsilon \tag{3.143}$$

Let us define $h\colon [0,1] \to \mathbf{R}$ such that for every $1 \le i \le k$,

$$h(x) = \frac{a_i - a_{i-1}}{y_i - y_{i-1}}(x - y_{i-1}) + a_{i-1} \text{ for } x \in [y_{i-1}, y_i] \tag{3.144}$$

Thus, $h \in \mathcal{E}([0,1])$ Also, we consider $\lambda\colon [0,1] \to [0,1]$, such that for every $1 \leq i \leq k$,

$$\lambda(x) = \frac{y_i - y_{i-1}}{t_i - t_{i-1}}(x - t_{i-1}) + y_{i-1} \text{ for } x \in [t_{i-1}, t_i] \tag{3.145}$$

Then,

$$||\lambda||^o = \max_{1 \leq i \leq k} \left| \ln \frac{y_i - y_{i-1}}{t_i - t_{i-1}} \right| = \max_{1 \leq i \leq k} |\ln(y_i - y_{i-1}) - \ln(t_i - t_{i-1})| < \epsilon \tag{3.146}$$

And $h \circ \lambda(x)$ on $[t_{i-1}, t_i]$ is the straight line such that $h \circ \lambda(t_i) = a_i$ and $h \circ \lambda(t_{i-1}) = a_{i-1}$, then

$$\begin{aligned}
d_{TV}(f, h \circ \lambda) &= TV_0^1(f - h \circ \lambda) = \sum_{i=1}^{k} TV_{t_{i-1}}^{t_i}(f - h \circ \lambda) \\
&= \sum_{i=1}^{k} TV_{t_{i-1}}^{t_i} |f(t_i) - a_i - (f(t_{i-1}) - a_{i-1}))| \\
&\leq \sum_{i=1}^{k} |f(t_i) - a_i| + |f(t_{i-1}) - a_{i-1})| \\
&< \sum_{i=1}^{k} \frac{\epsilon}{k} = \epsilon
\end{aligned}$$

Finally $d^o(f, h) \leq \max\{||\lambda||^o, d_{TV}(f, h \circ \lambda)\} < \epsilon$ $\square$.

Example 3.7.8 Let us consider the set $B = \left\{\frac{1}{i} \mid i \in \mathbf{N}\right\}$, and let $f\colon [0,1] \to \mathbf{R}$ such that

$$f(x) = \begin{cases} \frac{1}{2^i} & if \quad x = \frac{1}{i} \text{ for every } i \in \mathbf{N} \\ 0 & if \quad x \in [0,1] \setminus B \end{cases} \tag{3.147}$$

Then

$$\begin{aligned} TV_0^1(f) &= \frac{1}{2} + 2\sum_{i=2}^{\infty}\frac{1}{2^i} = \frac{1}{2} + \sum_{i=2}^{\infty}\frac{1}{2^{i-1}} \\ &= \frac{1}{2} + \sum_{j=1}^{\infty}\frac{1}{2^j} + 1 - 1 = \sum_{i=0}^{\infty}\frac{1}{2^i} - \frac{1}{2} \\ &= \frac{1}{1-\frac{1}{2}} - \frac{1}{2} = 2 - \frac{1}{2} = \frac{3}{2} \end{aligned}$$

Also $\lim_{n\to\infty}\sum_{i=2}^{n}\frac{1}{2^i} = \sum_{i=2}^{\infty}\frac{1}{2^i} < \infty$, then for $\epsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that

$$\left|\sum_{i=2}^{\infty}\frac{1}{2^i} - \sum_{i=2}^{n_0}\frac{1}{2^i}\right| < \frac{\epsilon}{2}$$

Let $g\colon [0,1] \to \mathbf{R}$ such that

$$g(x) = \begin{cases} \frac{1}{2^i} & if \quad x = \frac{1}{i} \text{ for every } i \in \{1,2,\dots,n_0\} \\ 0 & \text{in other case} \end{cases} \tag{3.148}$$

Then $f - g = f1_{[0,1/(n_0+1)]}$ and

$$TV_0^1(f-g) = 2\sum_{i=n_0+1}^{\infty}\frac{1}{2^i} < \epsilon \tag{3.149}$$

Therefore $d_{TV}(f,g) = |f(0) - g(0)| + TV_0^1(f-g) = TV_0^1(f-g) < \epsilon$ and $g \in \mathcal{E}([0,1])$. □

Proposition 3.7.9 *Let $f \in BV([0,1])$ be a function such that $f(x) = \sum_{i=1}^{\infty} b_i 1_{\{a_i\}}(x)$ for some $\{b_i\}_{i=0}^{\infty} \subseteq \mathbb{R}^+$ and $\{a_i\}_{i=1}^{\infty} \subseteq (0,1)$ with $a_i \neq a_j$ for every $i \neq j$, $i,j \in \mathbb{N}$. Then there exists $f_\epsilon \in \mathcal{E}([0,1])$ such that $d^o(f, f_\epsilon) < \epsilon$*

Proof: Note that

$$\infty > TV_0^1(f) = 2\sum_{i=1}^{\infty} b_i \geq 0 \tag{3.150}$$

So, $\lim_{n\to\infty}\sum_{i=1}^{n} b_i = \sum_{i=1}^{\infty} b_i < \infty$. Then for $\epsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that

$$\left|\sum_{i=1}^{\infty} b_i - \sum_{i=1}^{n_0} b_i\right| < \frac{\epsilon}{4} \tag{3.151}$$

Let $g\colon [0,1] \to \mathbf{R}$ such that $g(x) = \sum_{i=1}^{n_0} b_i 1_{\{a_i\}}(x)$ for every $x \in \mathbf{R}$. Then $(f-g)(x) = \sum_{i=n_0+1}^{\infty} b_i 1_{\{a_i\}}(x)$ and $d_{TV}(f,g) = |f(0)-g(0)| + TV_0^1(f-g) \leq 2\sum_{i=n_0+1}^{\infty} |b_i| < \frac{\epsilon}{2}$.

We can rearrange $a_1, a_2, \ldots, a_{n_0}$ such that $a_1 < a_2 < \cdots < a_{n_0}$. We consider $\delta = \min\{a_{i+1} - a_i \,|\, 1 \leq i \leq n_0 - 1\} > 0$. $\ln\colon \left[\frac{\delta}{10}, 1\right] \to \mathbf{R}$ is an uniformly continuous function, the there exists $\eta > 0$ such that for every $x, y \in \left[\frac{\delta}{10}, 1\right]$ such that $|x-y| < \eta$, we obtain $|\ln(x) - \ln(y)| < \frac{\epsilon}{2}$.
Let $q_i \in \mathbb{Q} \cap (a_i - \frac{\delta}{10}, a_i) \cap (a_i - \frac{\eta}{2}, a_i)$ for every $1 \leq i \leq n_0$. Note that

$$\begin{aligned} |a_{i+1} - a_i - (q_{i+1} - q_i)| &= |a_{i+1} - q_{i+1} - (a_i - q_i)| \leq |a_{i+1} - q_{i+1}| + |a_i - q_i| \\ &< \frac{\eta}{2} + \frac{\eta}{2} = \eta. \end{aligned} \tag{3.152}$$

Also, $a_{i+1} - a_i \geq \delta > \frac{\delta}{10}$, $q_{i+1} - q_i > q_{i+1} - a_i > \frac{9\delta}{10} > \frac{\delta}{10}$, so $a_{i+1} - a_i, q_{i+1} - q_i \in \left[\frac{\delta}{10}, 1\right]$ and $|\ln(q_{i+1} - q_i) - \ln(a_{i+1} - a_i)| < \frac{\epsilon}{2}$.
And let $\lambda \in \Lambda^o([0,1])$ such that $\lambda(a_i) = q_i$ for every $1 \leq i \leq n_o$, that is,

$$\lambda(x) = \left(\frac{q_{i+1} - q_i}{a_{i+1} - a_i}\right)(x - a_i) + q_i \text{ for every } x \in [a_i, a_{i+1}] \text{ and for every } 1 \leq i \leq n_0 - 1.$$

We have that

$$\|\lambda\|^o = \max_{1 \leq i \leq n_0 - 1} \left|\ln \frac{q_{i+1} - q_i}{a_{i+1} - a_i}\right| = \max_{1 \leq i \leq n_0 - 1} |\ln(q_{i+1} - q_i) - \ln(a_{i+1} - a_i)| < \frac{\epsilon}{2}.$$

For every $i \in \{1, 2, \ldots, n_0\}$, take $r_i \in \mathbb{Q} \cap (b_i - \frac{\epsilon}{4n_0}, b_i)$ and we consider $h\colon [0,1] \to \mathbf{R}$ such that $h(x) = \sum_{i=1}^{n_0} r_i 1_{\{q_i\}}(x)$. Note that $h \in \mathcal{E}([0,1])$ and $h \circ \lambda(x) = \sum_{i=1}^{n_0} r_i 1_{\{a_i\}}(x)$ for every $x \in [0,1]$, then

$$\begin{aligned} d_{TV_0^1}(g, h\circ\lambda) &= |g(0) - h\circ\lambda(0)| + TV_0^1(g - h\circ\lambda) \\ &= = TV_0^1\left(\sum_{i=1}^{n_0}(b_i - r_i)1_{\{a_i\}}(x)\right) \\ &= 2\sum_{i=1}^{n_0} |b_i - r_i| < \frac{\epsilon}{2}. \end{aligned} \tag{3.153}$$

Then,

$$d^o(g,h) \le \max\{\|\lambda\|^o, d_{TV_0^1}(g, h\circ\lambda)\} < \frac{\epsilon}{2}.$$

Therefore

$$\begin{aligned} d^o(f,h) &\le d^o(f,g) + d^o(g,h) \le d_{TV_0^1}(f,g) + d^o(g,h) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned} \tag{3.154}$$

with $h \in \mathcal{E}([0,1])$. □

In a similar way to the Proposition 3.7.9 we can obtain the following proposition

Proposition 3.7.10 *Let $f\colon [0,1] \to \mathbb{R}$ be a function such that*

$$f(x) = b_1 1_{[a_0,a_1]}(x) + \sum_{i=2}^{\infty} b_i 1_{(a_{i-1},a_i]}(x)$$

for some $\{a_i\}_{i=0}^{\infty} \subseteq [0,1]$, $a_0 < a_1 < \cdots < a_n < \cdots$ and $\{b_i\}_{i=1}^{\infty} \in \mathbb{R}$. Then there exists $f_\epsilon \in \mathcal{E}([0,1])$ such that $d^o(f, f_\epsilon) < \epsilon$

Theorem 3.7.11 *$PL([a,b])$ is dense in $(AC([a,b]), d^o)$.*

Proof: Using Section 3.3 let $f \in AC([a,b])$ a positive function then there exists $g \in L^1([a,b])$ such that

$$f(x) = f(a) + \int_a^x g(t)dt \tag{3.155}$$

And $f' = g$ a.e. By Integration's Theorem there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of nonnegative simple functions such that $0 \le g_n(x) \uparrow g(x)$ for every $x \in [a,b]$ and for any of this sequences

$$\int_a^b g_n(t)dt \uparrow \int_a^b g(t)dt \tag{3.156}$$

For $\epsilon > 0$ there exists $M \in \mathbf{N}$ such that for every $n \ge M$

$$\int_a^b |g(t) - g_n(t)|dt < \frac{\epsilon}{4} \tag{3.157}$$

By Remark 3.3.11, there exists $h_n \in S(a,b)$ such that

$$\int_a^b |g_n(t) - h_n(t)|dt < \frac{\epsilon}{4} \tag{3.158}$$

so,

$$\int_a^b |g(t) - h_n(t)|dt < \frac{\epsilon}{2} \tag{3.159}$$

Let $n \geq M$ and $\hat{h}_n \colon [a,b] \to \mathbf{R}$ such that $\hat{h}_n(x) = \int_a^x h_n(t)dt + f(a) = J(h_n)(x) + f(a)$. Note that $\hat{h}_n \in PL([a,b])$ because $J(h_n) \in PL([a,b])$. h_n is a step function then there exists $P = \{t_0, t_1, \ldots, t_s\} \in \mathcal{P}([a,b])$ such that h_n is a constant function on (t_{i-1}, t_i) for every $1 \leq i \leq a$. Thus h_n is a continuous function on $[a,b] \setminus \{t_0, t_1, \ldots, t_s\} = [a,b] \setminus P$. By the Fundamental Theorem of Calculus (FTC) [38] for every $x \in [a,b] \setminus P$, $\hat{h}'_n(x) = h_n(x)$.

$$\begin{aligned} d_{TV}(f, \hat{h}_n) &= |f(a) - \hat{h}_n(a)| + TV_a^b\left(f - \hat{h}_n\right) \\ &= |f(a) - f(a)| + \sum_{i=1}^{s} TV_{t_{i-1}}^{t_i}(f - \hat{h}_n) \\ &= \sum_{i=1}^{s} \int_{t_{i-1}}^{t_i} |f' - h_n| = \int_a^b |f' - h_n| = \int_a^b |g - h_n| < \epsilon \end{aligned} \tag{3.160}$$

Also,

$$\begin{aligned} d^o(f, \hat{h}_n) &\leq \max\{\|Id\|^o, d_{TV}(f, \hat{h}_n \circ Id)\} \\ &= \max\{0, d_{TV}(f, \hat{h}_n)\} = d_{TV}(f, \hat{h}_n) < \epsilon \end{aligned}$$

Finally, for $f \colon [a,b] \to \mathbf{R}$ absolutely continuous function taking the positive and negative parts of f, $f(x) = f^+(x) - f^-(x)$ for every $x \in [a,b]$. And for $\epsilon > 0$ there exist $h, g \in PL([a,b])$ such that $d_{TV}(f^+, h) < \frac{\epsilon}{2}$ and $d_{TV}(f^-, g) < \frac{\epsilon}{2}$. Note that $h - g \in PL([a,b])$

$$\begin{aligned} d_{TV}(f, h - g) &= |f(a) - h(a) + g(a)| + TV_a^b(f - h + g) \\ &= |f^+(a) - f^-(a) - h(a) + g(a)| + TV_a^b(f^+ - f^- - h + g) \\ &\leq |f^+(a) - h(a)| + |f^-(a) - g(a)| + TV_a^b(f^+ - h) \\ &\quad + TV_a^b(f^- - g) \\ &= d_{TV}(f^+, h) + d_{TV}(f^-, g) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{3.161}$$

□

Theorem 3.7.12 *For every* $f \in AC([0,1])$ *there exists* $f_\epsilon \in \mathcal{E}([0,1]) \cap AC([0,1])$ *such that* $d^o(f, f_\epsilon) < \epsilon$.

Proof: By Theorem 3.7.11, $PL([0,1])$ is dense in $(AC([0,1]), d^o)$. Then, for $f \in AC([0,1])$ and $\epsilon > 0$ there exists $g \in PL([0,1])$ such that $d^o(f, g) < \frac{\epsilon}{2}$. By Proposition 3.7.7 there exists $h \in \mathcal{E}([0,1])$ such that $d^o(g, h) < \frac{\epsilon}{2}$. Thus $d^o(f, h) \leq d^o(f, g) + d^o(g, h) \leq d^o(f, g) + d^o(g, h) < \epsilon$. And by the proof of 3.7.7 and the fact that $g \in AC([0,1])$ we have that $h \in PL([0,1])$ and h is absolutely continuous. □

3.8 Bounded Variation for higher dimensions

We recall some well-know definitions of bounded variation for functions of several variables. We start with two variables, see [11].

Given $f\colon [a,b] \times [c,d] \to \mathbf{R}$ and two partitions $P = \{s_0, s_1, \ldots, s_m\} \in \mathcal{P}([a,b])$ and $Q = \{t_0, t_1, \ldots, t_n\} \in \mathcal{P}([c,d])$, consider the expressions

$$\begin{aligned} & V_2(f, P \times Q, [a,b] \times [c,d]) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} |f(s_i, t_j) - f(s_{i-1}, t_j) - f(s_i, t_{j-1}) + f(s_{i-1}, t_{j-1})| \end{aligned} \tag{3.162}$$

and

$$\begin{aligned} V_2(f, [a,b] \times [c,d]) \quad = \quad & \sup\{V_2(f, P \times Q; [a,b] \times [c,d]) \mid \\ & p \in \mathcal{P}([a,b]),\ Q \in \mathcal{P}([c,d])\} \end{aligned} \tag{3.163}$$

Observe that the equation (3.162) provides the volume of the box $R = [a,b] \times [c,d]$, when f is a distribution function in a Probability sense. Therefore, we can define this bounded variation for every $k \geq 2$ for any distribution functions f in $\mathbf{R}^k$. Of course this definition may be use even for general signed measures.

Definition 3.8.1(**Hardy**) *Let* $f\colon [a,b] \times [c,d] \to \mathbb{R}$, *define*

$$V_H(f; [a,b] \times [c,d]) = TV_a^b(f(\cdot, c)) + TV_c^d(f(a, \cdot)) + V_2(f, [a,b] \times [c,d])$$

the **Hardy total variation** *of* f *on* $[a,b] \times [c,d]$. *In case* $V_H(f; [a,b] \times [c,d]) < \infty$, *we say that* f *has bounded variation on* $[a,b] \times [c,d]$ *in the sense of Hardy and* $f \in BVH([a,b] \times [c,d])$.

For the following Proposition see Proposition 1.43 and Proposition 1.44 in [20].

Proposition 3.8.2 *The Hardy total variation has the following properties*

i) *For* $f, g\colon [a,b]\times[c,d] \to \mathbb{R}$ *,*

$$V_H(f+g;[a,b]\times[c,d]) \le V_H(f;[a,b]\times[c,d]) + V_H(g;[a,b]\times[c,d]).$$

ii) *For* $f\colon [a,b]\times[c,d] \to \mathbb{R}$ *and* $\mu \in \mathbb{R}$ *,*

$$V_H(\mu f;[a,b]\times[c,d]) = |\mu| V_H(f;[a,b]\times[c,d])$$

iii) *For* $a \le \xi < x \le b$ *and* $c \le \eta < y \le d$*, we have that*

$$|f(x,y) - f(\xi,\eta)| \le V_H(f;[\xi,x]\times[\eta,y]).$$

iv) *For every* $f \in BVH([a,b]\times[c,d])$*,*

$$\begin{aligned} \sup\{|f(x,y)| \mid a \le x \le b,\ c \le y \le d\} &= \|f\|_{\sup} \\ &\le |f(a,c)| + V_H(f;[a,b]\times[c,d]). \end{aligned}$$

v) $BVH([a,b]\times[c,d])$ *is a complete space and for every* $f, g \in BVH([a,b]\times[c,d])$*,*

$$\|fg\|_{BVH} \le 4\|f\|_{BVH}\|g\|_{BVH}.$$

where

$$\|f\|_{BVH} = |f(a,c)| + V_H(f;[a,b]\times[c,d]).$$

Definition 3.8.3 (**Vitali**) *A function* $f\colon [a,b]\times[c,d] \to \mathbb{R}$ *has bounded variation in the sense of Vitali if* $V_V(f;[a,b]\times[c,d]) := V_2(f;[a,b]\times[c,d]) < \infty$*. In this case,* $f \in BVV([a,b]\times[c,d])$*.*

Definition 3.8.4 (**Fréchet**) *Let* $f\colon [a,b]\times[c,d] \to \mathbb{R}$ *,* $P = \{s_0, s_1, \ldots, s_m\} \in \mathcal{P}([a,b])$ *and* $Q = \{t_0, t_1, \ldots, t_n\} \in \mathcal{P}([c,d])$*, consider the expression*

$$\begin{aligned} &V_2^{\pm}(f, P\times Q;[a,b]\times[c,d]) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} \epsilon_i\epsilon_j |f(s_i,t_j) - f(s_{i-1},t_j) - f(s_i,t_{j-1}) + f(s_{i-1},t_{j-1})| \end{aligned} \tag{3.164}$$

where $\epsilon_i, \epsilon_j \in \{-1, 1\}$. *If*

$$V_F^{\pm}(f; [a,b] \times [c,d])$$
$$= \sup\{V_2^{\pm}(f; P \times Q : [a,b] \times [c,d]) \mid P \in \mathcal{P}([a,b]),\ Q \in \mathcal{P}([c,d])\} < \infty \tag{3.165}$$

we say that $f \in BVF([a,b] \times [c,d])$ *and that* f *has bounded variation in the sense of Fréchet.*

Definition 3.8.5 (**Arzelá**) *Let* $f\colon [a,b] \times [c,d] \to I\!R$ *,* $P = \{s_0, s_1, \ldots, s_m\} \in \mathcal{P}([a,b])$ *and* $Q = \{t_0, t_1, \ldots, t_m\} \in \mathcal{P}([c,d])$ *(P and Q have the same size), let us consider the expression*

$$V_A(f, P \times Q; [a,b] \times [c,d])$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{m} |f(s_i, t_j) - f(s_{i-1}, t_{j-1})|. \tag{3.166}$$

When

$$V_A(f; [a,b] \times [c,d])$$
$$= \sup\{V_A(f, P \times Q; [a,b] \times [c,d]) \mid P \in \mathcal{P}([a,b]),\ Q \in \mathcal{P}([c,d])\} < \infty$$

we say that $f \in BVA([a,b] \times [c,d])$ *and that* f *has bounded variation in the sense of Arzelá.*

Definition 2.8.6 (**Tonelli**) *Let* $f\colon [a,b] \times [c,d] \to I\!R$ *. If*

i) $TV_a^b(f(\cdot, y)) < \infty$ *for almost all* $y \in [c,d]$ *and* $TV_c^d(f(x, \cdot)) < \infty$ *for almost all* $x \in [a,b]$

ii) $\int_a^b TV_c^d(f(x,\cdot))dx < \infty$ *and* $\int_c^d TV_a^b(f(\cdot, y))dy < \infty$

we say that $f \in BVT([a,b] \times [c,d])$ *an that* f *has bounded variation in the sense of Tonelli.*

Example 3.8.7 Let $f\colon [a,b] \times [c,d] \to \mathbf{R}$ be a function such that $f(x,y) = h(x)$. Then

$$V_2(f, P \times Q, [a,b] \times [c,d])$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} |f(s_i, t_j) - f(s_{i-1}, t_j) - f(s_i, t_{j-1}) + f(s_{i-1}, t_{j-1})|$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} |h(s_i) - h(s_{i-1}) - h(s_i) + h(s_{i-1})| = 0$$

In the same way, if $f\colon [a,b]\times[c,d] \to \mathbf{R}$ is a function such that $f(x,y) = g(y)$, so

$$\begin{aligned} &V_2(f, P\times Q, [a,b]\times[c,d]) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} |f(s_i,t_j) - f(s_{i-1},t_j) - f(s_i,t_{j-1}) + f(s_{i-1},t_{j-1})| \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} |g(t_i) - g(t_{i-1}) - g(t_i) + g(t_{i-1})| = 0 \end{aligned}$$

Also, if f is a strictly increasing function and $c < d$, then

$$\begin{aligned} V_A(f, P\times Q; [a,b]\times[c,d]) &= \sum_{i=1}^{m}\sum_{j=1}^{m} |f(s_i,t_j) - f(s_{i-1},t_{j-1})| \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} |g(t_i) - g(t_{i-1})| \\ &= \sum_{i=1}^{m} g(t_m) - g(t_0) = m(d-c) \end{aligned}$$

and

$$V_A(f; [a,b]\times[c,d]) = \lim_{m\to\infty} m(g(d) - g(c)) = \infty$$

Remark 3.8.8 *From the previous example, we can see that the definitions of Arzelà, Vitali, and Fréchet are not a extensions of the geometric concept of the total variation definition in one variable. However, Hardy´s definition 3.8.1 is an extension on the univariate case.*

Example 3.8.9 Let us consider the Dirichlet extension to dimension two

$$f(x,y) = \begin{cases} 0 & if \quad x,y \in \mathbb{Q} \\ 1 & if \quad x \in \mathbb{I} \text{ or } y \in \mathbb{I} \end{cases} \tag{3.167}$$

We take $m \in \mathbf{N}$ and $n = 2m$. We can take $\{x_0, x_1, x_2, \ldots, x_{2m-1}, x_{2m}\} \in \mathcal{P}([0,1])$ such that $x_k \in \mathbb{Q}$ for every $k \in \{0,2,4,\ldots,2m\}$ and $x_k \in \mathbb{I}$ for every $k \in \{1,3,5,\ldots,2m-1\}$. Also, we take $y_i = x_i$ for every $i \in \{0,1,\ldots,2m\}$. Then, for odd $k \geq 1$

$$|f(x_k,x_k) - f(x_k,x_{k-1}) - f(x_{k-1},x_k) + f(x_{k-1},x_{k-1})| = |1-1-1+0| = 1 \tag{3.168}$$

and for even $k \geq 1$

$$|f(x_k, x_k) - f(x_k, x_{k-1}) - f(x_{k-1}, x_k) + f(x_{k-1}, x_{k-1})| = |0 - 1 - 1 + 1| = 1 \tag{3.169}$$

So,

$$\sum_{i=1}^{2m} \sum_{i=1}^{2m} |(f(x_i, x_i) - f(x_{i-1}, x_i) - f(x_i, x_{i-1}) + f(x_{i-1}, x_{i-1})| = 4m^2 \rightarrow_{m \to \infty} \infty \tag{3.170}$$

Observe that we obtain the total variation of the function using the main diagonal.

Remark 3.8.10 *From the previous example, we can see that the definition of Arzelà is a good extension when the variation occurs along the main diagonal of the domain box.*

The definition that provides us with a better extension is Hardy's. However, we want to propose a new definition. The following examples show how we would like our definition to work.

Example 3.8.11 Let us consider $m \geq 2$ and $\underline{x_1}, \underline{x_2}, \ldots, \underline{x_n} \in A$, where $A \subset \mathbf{R}^m$, such that for every $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ we have $\underline{x}_i \neq \underline{x}_j$. Let be $f \colon A \to \mathbf{R}$ a function such that $f(\underline{x}) = 0$ for every $\underline{x} \in A \setminus \{\underline{x_1}, \underline{x_2}, \ldots, \underline{x_n}\}$. And $f(\underline{x}_i) = k_i$ for every $i \in \{1, 2, \ldots, n\}$ with $k_i \neq k_j$ when $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$.

Let us consider $\alpha \in C^0(I, \mathbf{R}^m)$ such that $\underline{x}_i \in Im(\alpha)$, $\alpha(0) \neq \underline{x}_i$ and $\alpha(1) \neq \underline{x}_i$ for every $i \in \{1, 2, \ldots, n\}$.

Then, we have $f \circ \alpha \colon I \to \mathbf{R}$ and we can obtain $TV_0^1(f \circ \alpha)$ as the usual form.

There exist $a_1, a_2, \ldots, a_n \in I$ such that $\alpha(a_i) = \underline{x}_i$ for every $i \in \{1, 2, \ldots, n\}$. Then $(f \circ \alpha)(a_i) = k_i$ for every $i \in \{1, 2, \ldots, n\}$ and $(f \circ \alpha)(x) = 0$ for every $x \in I \setminus \{a_1, a_2, \ldots, a_n\}$.

Therefore

$$TV_0^1(f \circ \alpha) = 2 \sum_{i=1}^{n} |k_i|.$$

This example generalizes the total variation in dimension one to the total variation of a function defined on $\mathbf{R}^m$.

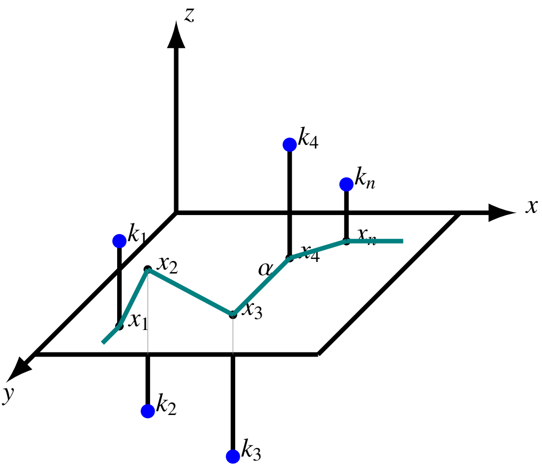

Figure 3.13 Graphs of f and α.

Example 3.8.12 Let $f\colon [-1,1]^2 \to \mathbf{R}$ such that $f(x,y) = \sin(x^2+y^2)$. If we consider the bounded variation of f through straight lines we have that the biggest bounded variation of f is in the straight line $y = x$ and in the straight line $x = -y$.

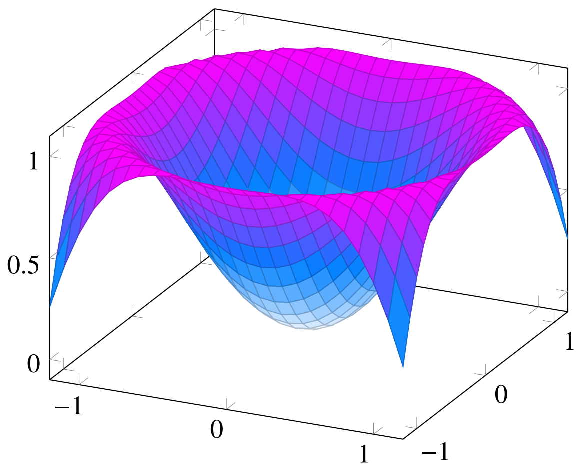

Figure 3.14 Graph of $f(x,y) = \sin(x^2+y^2)$.

Then,

$$\begin{aligned}
TV_{-1}^{1}(f(x,x)) &= TV_{-1}^{1}(\sin(2x^2)) = \int_{-1}^{1} |\sin(2x^2)'| = \int_{-1}^{1} |\cos(2x^2)4x| \\
&= \int_{-\sqrt{\pi}/2}^{\sqrt{\pi}/2} |\cos(2x^2)4x| + \int_{\sqrt{\pi}/2}^{1} |\cos(2x^2)4x| \\
&\quad + \int_{-1}^{-\sqrt{\pi}/2} |\cos(2x^2)4x| \\
&= 2\sin(\pi/2) - 2\sin(0) + 2\sin(\pi/2) - 2\sin(2) \\
&= 4\sin(\pi/2) - 2\sin(2) = 4 - 2\sin(2) \qquad (3.171)
\end{aligned}$$

And

$$TV_{-1}^{1}(f(x,-x)) = TV_{-1}^{1}(\sin(2x^2)) = 4 - 2\sin(2) \qquad (3.172)$$

Example 3.8.13 Let us consider $A = \{(x,y) \mid x^2 + y^2 \leq 1^2\}$. And for every $n \geq 1$, let $f_n : A \to \mathbf{R}$ such that $f_n(a\cos(\theta), a\sin(\theta)) = a\cos(n\theta) + a\sin(n\theta)$ for all $\theta \in [0, 2\pi]$ and $a \in [0,1]$. For $n = 1$, if we consider the bounded variation of f through straight lines we have that the biggest bounded variation of f is in the straight line when $a = 1$. Then,

$$\begin{aligned}
TV_{-1}^{1}(f(\cos(\theta), \sin(\theta))) &= TV_{0}^{2\pi}(\cos(\theta) + \sin(\theta)) \\
&= \int_{0}^{2\pi} |\cos(\theta) - \sin(\theta)| d\theta = 4\sqrt{2}.
\end{aligned}$$

In general, when $a = 1$ and $n \geq 1$, using the frontier of A, $\delta(A)$

$$\begin{aligned}
TV_{-1}^{1}(f_n(\cos(\theta), \sin(\theta))) &= TV_{0}^{2\pi}(\cos(n\theta) + \sin(n\theta)) \\
&= \int_{0}^{2\pi} |n\cos(n\theta) - n\sin(n\theta)| d\theta \\
&= n\int_{0}^{2\pi} |\cos(n\theta) - \sin(n\theta)| d\theta \\
&= \int_{0}^{2\pi n} |\cos(u) - \sin(u)| du \\
&= n\int_{0}^{2\pi} |\cos(u) - \sin(u)| du = n4\sqrt{2}.
\end{aligned}$$

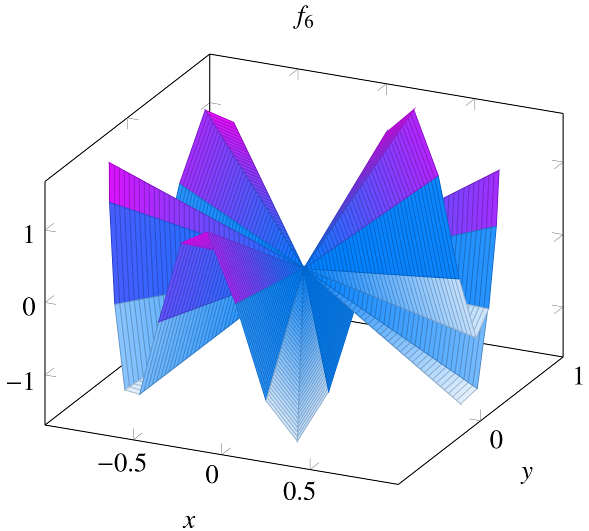

Figure 3.15 Graph of f_6.

Example 3.8.14 Let us consider $f\colon [0,1]^2 \to \mathbf{R}$ such that

$$f(x,y) = \begin{cases} 0 & if \quad x \neq y \\ 1 & if \quad x = y \end{cases} \tag{3.173}$$

And for $n \in \mathbf{N}$, let us consider

$$g_n(x) = \begin{cases} 2nx & if \quad 0 \le x < \frac{1}{2n} \\ -2n\left(x - \frac{1}{2n}\right) + 1 & if \quad \frac{1}{2n} \le x < \frac{2}{2n} \\ 2n\left(x - \frac{2}{2n}\right) & if \quad \frac{2}{2n} \le x < \frac{3}{2n} \\ \cdots & \cdots \quad \cdots \\ 2n\left(x - \frac{2n-2}{2n}\right) & if \quad \frac{2}{2n-2} \le x < \frac{2n-1}{2n} \\ -2n\left(x - \frac{2n-1}{2n}\right) + 1 & if \quad \frac{2n-1}{2n} \le x < \frac{2n}{2n} = 1. \end{cases} \tag{3.174}$$

Note that $g_n(x) = x$

1) $x = 0$ on $\left[0, \frac{1}{2n}\right)$

2) $x = \frac{2}{2n+1}$ on $\left[\frac{1}{2n}, \frac{2}{2n}\right)$

3) $x = \frac{2}{2n-1}$ on $\left[\frac{2}{2n}, \frac{3}{2n}\right)$

4) $x = \frac{4}{2n+1}$ on $\left[\frac{3}{2n}, \frac{4}{2n}\right)$

3) $x = \frac{4}{2n-1}$ on $\left[\frac{4}{2n}, \frac{5}{2n}\right)$

... ...

$2n-1$) $x = \frac{2n-2}{2n-1}$ on $\left[\frac{2n-2}{2n}, \frac{2n-1}{2n}\right)$

$2n$) $x = \frac{2n}{2n+1} = 1$ on $\left[\frac{2n-1}{2n}, \frac{2n}{2n}\right]$

$$\begin{aligned} TV_0^1(f \circ (Id, g_n)) &= TV_0^1(f(x, g_n)) = 1 + 2 + 2 + \cdots + 2 = 1 + 2(2n-1) \\ &= 4n - 1 \to_{n\to\infty} \infty \end{aligned}$$

Now, we provide a proposal of a definition for the multivariate case which may extend the univariate total variation in the sense that we can use continues curves with values on the domain $A \subseteq \mathbf{R}^k$ of a function $f: A \to \mathbf{R}$. In some cases we will ask some properties on the domain of f, and the curve will be define in terms of a function $\alpha: [0,1] \to A$.

Recall that $C^0([0,1], A) = \{f: [0,1] \to A \,|\, f \text{ is a continuous function}\}$. In this case, $A \subseteq \mathbf{R}^k$ for $k \geq 2$.

Definition 3.8.15 Let $f: A \to \mathbf{R}$ such that $A \subseteq \mathbf{R}^k$ for some $k \geq 2$. And we consider $\alpha \in \mathcal{G} \subset C^0([0,1], A)$. Then, define $TV_A(f)$ as following

$$TV_A(f) = \sup_{\alpha \in \mathcal{G}} TV_0^1(f \circ \alpha), \tag{3.175}$$

where, $\mathcal{G}$ must satisfy certain conditions which need to be studied.

In the Example 3.8.9, $A = [0,1]^2$ and in this case we define $\alpha: [0,1] \to A$ such that $\alpha(x) = (x, x)$. We find the total variation using $f(\alpha(x))$.

In the Example 3.8.12, $A = [-1,1]^2$ and let $l: [0,1] \to [-1,1]$ such that $l(x) = 2x - 1$ for every $x \in [0,1]$. And let $\alpha: [0,1] \to A$ such that $\alpha(x) = (l(x), l(x))$, and we find the total variation of f using the values of $f(\alpha(x))$.

In the Example 3.8.13, $A = \{(x,y) \,|\, x^2 + y^2 \leq 1\}$ and we define $h_n: [0,1] \to [0, 2\pi n]$ such that $h_n(t) = 2\pi n t$. And we define $\alpha_n: [0,1] \to A$ such that $\alpha_n(t) = (\cos(h_n(t)), \sin(h_n(t)))$ that corresponds to points on the frontier of

A, $\delta(A) = \{(x, y) \in A \mid x^2 + y^2 = 1\}$. We find the total variation of f_n using α_n.

In the Example 3.8.14, $A = [0, 1]^2$ and $\alpha_n \colon [0, 1] \to A$ is such that $\alpha_n(x) = (x, g_n(x))$. And we find the total variation of f using α_n and taking the limit.

3.8.1 Conclusions about Section 3.8

Of course Definition 3.8.15 is a preliminary version of a Definition of Total Variation on $\mathbf{R}^k$ with $k \geq 2$, and it needs to be studied more profoundly. This study is an open problem which needs to be improved in the future. Using the curve α allows to use the univariate definition of total variation.

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