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DYNAMIC WALKS IN EDGE-COLORED (DI)GRAPHS AVOIDING FORBIDDEN
TRANSITIONS

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To my parents Eduardo y Martha.
To my siblings Isabel y Eduardo.

"The beauty of mathematics only shows itself to more patient followers".
Maryam Mirzakhani

Preface

The existence of trails and cycles in directed and undirected graphs have been widely studied due to their wide variety of applications in different fields, for example, in molecular biology, physical sciences, social science, computer graphics, electronic circuit design, operations research, art among many others. In particular, they have been studied in edge-colored graphs, where several results have been obtained among them a well-known characterization of undirected multigraphs containing properly colored Eulerian trail. The main focus of this dissertation is to study the structure of edge-colored directed and undirected graphs with Eulerian trails or Hamiltonian cycles with restriction in the color transitions.

We intend this work to be self-contained, for this reason Chapter 1 is devoted to give definitions and notation that will be used throughout this work. Among the definitions included in Chapter 1 are graph, digraph and (directed) walk, which are our most basic objects that we will work with. In Chapter 2 the reader can find a brief historical introduction to the study of Eulerian trails and Hamiltonian cycles, from the beggining of graph theory with Euler's Theorem, through Icosain game created by Hamilton, who was one of the pioneers in the study of Hamiltonian cycles. We also brief summary of the results obtained on properly colored walks in edge-colored graphs, that are one of the most studied generalization of walks in directed and undirected graphs. We end this historical survey by introducing the reader to the study of H-walks in H-colored (di)graphs. These concepts were introduced by Linek and Sands, who proposed coloring the arcs of a digraph D with the vertices of H, a digraph with a loop at each vertex, and considered H-walks, which are walks satisfying that the colors of consecutive arcs form an arc in H.

In Chapter 3, we introduce the concept of dynamic H-trail, as well as, a notation that allows us to know the edges that belong to it. Subsequently, we study the existence of closed dynamic H-trails in H-colored multigraphs. In our investigation we found an auxiliary graph, that we called $L_{2,H}^{Dym}(G)$, and allows us to find a partition of the edges of the H-colored multigraph G into closed dynamic H-trails by finding two perfect matchings in $L_{2,H}^{Dym}(G)$. It is worth to mention that one of the perfect matching is obtained trivially and it is called the joint matching. This lead us to prove that G has closed Eulerian dynamic H-trail if and only if $L_{3,H}^{Dym}(G)$ is Hamiltonian, where $L_{3,H}^{Dym}(G)$ is obtained by subdividing certain edges of $L_{2,H}^{Dym}(G)$. All the original results in Chapter 3 are included in the paper "Euler dynamic H-trails in edge-colored graphs", which has been published in the international journal AKCE International Journal of Graph and Combinatorics (see [54]).

In Chapter 4, we extend the concepts studied in Chapter 3 to H-colored digraphs. We adapted the definition of the auxiliary graph $L_{n,H}^{Dym}(G)$ for H-colored digraphs. Although we obtained similar results, there are important differences between them. For example, we show that there exists a one-to-one function between the set of closed dynamic H-trails in D and the set of directed cycles in $L_{2,H}^{Dym}(D)$. As a consequence, we prove that D has closed Eulerian dynamic H-trails if and only if $L_{n,H}^{Dym}(D)$ is Hamiltonian, for each $n \geq 2$.

In the third part of this chapter, we show that all the results obtained for dynamic H-trail can be carry out for H-trails by considering an induced subdigraph of $L_{n,H}^{Dym}(D)$. In addition, we show conditions on an H-colored digraph that guarantee the existence of a closed Euler

H-trail. As a consequence, we obtain a characterization of a family of c-arc-colored digraphs, with $c \geq 2$, which contain a properly colored Eulerian trail. We end Chapter 4 by giving an infinite family of digraphs H fulfilling the hypotheses of the theorems stated throughout the chapter. The results presented in Chapter 4 were included in the article entitled "Characterizing arc-colored digraphs with an Eulerian trail with restrictions in the color transitions" submitted to an international journal.

Gourvès et al. [30] proved that, deciding whether a c-arc-colored digraph contains a properly colored path from a vertex s to a vertex t is NP-complete, even for planar digraphs with no properly colored cycle. However, they also proved that the problem of finding a properly colored trail from a vertex s to a vertex t in a c-arc-colored digraph can be done within polynomial time. Since H-paths generalize properly colored paths, it follows that the problem of finding an H-path from a vertex v to a vertex u in an H-colored digraph is in NP. These results motivated us to study the computational complexity of finding H-trails between two given vertices. In Chapter 5, we prove that determining if there exists an H-trail starting with the arc e and ending at arc f can be done in polynomial time. As a consequence, we give a polynomial time algorithm to find (if any exists) the shortest H-trail from the vertex s to the vertex t. Moreover, we show that the problem of maximizing the number of arc disjoint $s - t$ H-trails in D can be solved in polynomial time. The results presented in Chapter 5 were included in the article entitled "Trails in arc-colored digraphs avoiding forbidden transitions", which has been published in the international journal Discrete Mathematics Letters (see [55]).

In Chapter 6, we study the existence and length of dynamic H-cycles and H-paths, in H-colored multigraphs, using a new concept of color degree, called dynamic degree. The dynamic degree allows us to extend some classical results, such as Ore's theorems for H-colored multigraphs. As a consequence, we partially solve a conjecture stated by Abouelaoualim et al. [2], which asserts that if every vertex of a c-edge-colored multigraph have at least $\lceil \frac{n}{2} \rceil$ incident edges of each of the c colors, with $c \geq 3$, then G has a properly colored Hamiltonian cycle. We end Chapter 6 by giving examples of H-colored multigraphs that fulfill the hypotheses of the theorems stated throughout Chapters 3 and 6. The results presented in Chapter 6 were included in the article entitled "Dynamic cycles in edge-colored multigraphs", submitted to an international journal.

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I could not have undertaken this journey without my parents, Eduardo and Martha, who provided unwavering support and encouragement, enabling me to devote my attention exclusively to my academic pursuits. *Esto es por ustedes y para ustedes.* I would like to express my deepest appreciation to my siblings, Isabel y Eduardo, who have consistently been a source of assistance and guidance. My profound gratitude is extended to my nephew Isaac for the immense joy he has brought into my life. Many thanks to my brother-in-law Oscar for his assistance and support. I would like to express my sincerest gratitude to the Saravia Alfaro family for the facilities provided during my studies in Querétaro. Special thanks to my aunt Elena for the *enchiladas.*

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Contents

Chapter 1

Basic concepts and notation

This chapter is devoted to provide most of the terminology, notation and classical results that will be used throughout this work. For basic concepts, terminology and notation not defined here, we refer the reader to [5] and [13], where the definitions were taken directly or influenced. In Section 1.1 we provide basic concepts of graph theory (such as, graph, multigraph, degree, walk, trail, cycle and connectivity) and some of the most well-known results on walks, connectivity and perfect matching. Section 1.2 is devoted to introduce the reader to the directed graph theory. A generalization of directed and undirected graphs, namely edge-colored graphs, are presented in Section 1.3. Finally, in Section 1.4 we give basic concepts and terminology of computational complexity.

1.1   Graphs

An **undirected graph** or just a **graph** G consists of a nonempty finite set $V(G)$ of elements called **vertices** and a finite set $E(G)$ of unordered pairs of distinct vertices called **edges**. We call $V(G)$ the **vertex set** and $E(G)$ the **edge set**. Each edge $\{u, v\}$ of $E(G)$ is denoted by uv or vu. If $uv \in E(G)$, we say that the vertices u and v are **adjacent** and that e **join** u and v. We also say that u and v are the **end-vertices** of e. Notice that the above definition of graph do not allow **loops** (i.e. pairs consisting of the same vertex) or **parallel edges** (i.e., multiple edges with the same end-vertices). When parallel edges and loops are admissible we speak of **pseudographs**. A pseudograph with no loops is a **multigraph**. For a pair u, v of vertices in a multigraph G, $\mu_G(uv)$ denotes the number of edges with end-vertices u and v.

For a vertex v in a multigraph G, the set $N_G(v) = \{u \in V(G) : \mu(vu) > 0\}$ is called the **neighborhood** of v. The **degree** of v in a multigraph G, denoted by $d_G(v)$, is the number of edges incident with v, that is, $d_G(v) = \sum_{u \in V(G)} \mu(vu)$. The **minimum degree** of G is $\delta(G) = \min\{d(v) : v \in V(G)\}$ and the **maximum degree** of G is $\Delta(G) = \max\{d(v) : v \in V(G)\}$.

A graph H is said to be a **subgraph** of a graph G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and every edge of H has its end-vertices in $V(H)$. If $V(G) = V(H)$, then H is a **spanning** subgraph. For a nonempty subset S of $V(G)$, the subgraph $G[S]$ of G **induced by** S has S as its vertex set and two vertices u and v in S are adjacent in $G[S]$ if and only if u and v are adjacent in G. For a nonempty subset X of $E(G)$, the subgraph $G[X]$ of G **induced by** X has X as its edge set and a vertex v belongs to $G[X]$ if v is incident with at least one edge in X.

For a proper subset U of $V(G)$, the graph $G - U$ is the induced subgraph $G[V(G) \setminus U]$ of G. In particular, when $U = \{v\}$, $G - U$ will be denote by $G - v$. For a subset X of $E(G)$, the subgraph $G - X$ is the spanning subgraph of G with edge set $E(G) \setminus X$. When $X = \{e\}$, we will denote $G - X$ by $G - e$.

For a pair of graphs G and H, the **union** of G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

A graph G is **complete** if every pair of distinct vertices in G are adjacent. A graph H is **p-partite** if there exists a partition $V_1, V_2, \ldots, V_p$ of $V(H)$ into p partite set (i.e., $V(H) = V_1 \cup V_2 \cup \ldots \cup V_p$ and $V_i \cap V_j = \emptyset$, for every $i \neq j$) such that every edge of H has its end-vertices in different partite sets. The special case of a p-partite graph when $p = 2$ is called a **bipartite** graph. A p-partite graph H is **complete p-partite** or **complete multipartite** if, for every par $x \in V_i$ and $y \in V_j$, with $i \neq j$, an edge xy is in $E(H)$. We denote the complete p-partite graph with partite sets of cardinality $n_1, n_2, \ldots, n_p$ by $K_{n_1, n_2, \ldots, n_p}$.

Let G be a multigraph. A **walk** in G is a sequence of vertices and edges, $W = (v_0, e_0, v_1, e_1, v_2, \ldots, v_{k-1}, e_{k-1}, v_k)$, such that the end-vertices of e_j are v_j and v_{j+1}, for every $j \in \{0, \ldots, k-1\}$. The set of vertices $\{v_i : i \in \{1, \ldots, k\}\}$ is denoted by $V(W)$; and the set of edges $\{e_i : i \in \{1, \ldots, k-1\}\}$ is denoted by $A(W)$. We also say that W is a $v_0 - v_k$ walk and . The **length** of W, denoted by $l(W)$, is the number of edges (counting repetitions) encountered in W. A walk is **closed** if $v_0 = v_k$. When the arcs of W are defined from the context or simply unimportant, we will denote W by $(x_0, x_1, \ldots, x_k)$.

A **trail** is a walk such that no edge occurs more than once. A trail in which no vertex is repeated is called **path**. If the vertices $v_0, v_1, \ldots, v_{k-1}$ are distinct, $k \geq 3$ and $v_0 = v_k$, W is a **cycle**. A trail W in G is an **Euler** or **Eulerian** trail if and only if $E(W) = E(G)$. A multigraph that contains a closed Eulerian trail is called **Eulerian**. A multigraph G is **superulerian** if it has a spanning Eulerian subgraph. We also say that a cycle (path) C in G is a **Hamilton** or **Hamiltonian** cycle (path) whenever $V(C) = V(G)$. A graph G is **Hamiltonian** if and only if G contains a Hamiltonian cycle.

Two vertices u and v in a multigraph G are **connected** if G contains a $u - v$ walk. The graph G itself is **connected** if every two vertices of G are connected.

A **matching** M in a graph G is a set of edges with no common end-vertices. If M is a matching, then we say that the edges in M are **independent**. A matching M in G is **maximum** if M contains the maximum possible number of edges. A matching in G is **perfect** if every vertex of G is incident with some edge in M.

1.2   Digraphs

A **directed graph** (or just **digraph**) D consists of a nonempty finite set $V(D)$ and a finite multiset $A(D)$ of ordered pairs of vertices. We called $V(D)$ the **vertex set** and its elements will be called **vertices** of D. The multiset $A(D)$ is called the **arc set** of D and its elements are called **arcs**. The **order** (**size**) of D is the number of vertices (arcs) in D. Let (u, v) be an arc of D the vertex u is its **tail**, the vertex v is its **head** and u and v are its **end-vertices**. We also say that the vertex u is **adjacent to** v. We say that a vertex u is **incident** to an arc a if u is the head or the tail of a. Note that the definition of arc set allows to have more than one arc with the same tail and head. We will say that two different arcs are **parallel** whenever they have the same head and tail. The number of arcs in D with tail x and head y will be denoted by $\mu_D(x, y)$, in particular, $\mu_D(x, y) = 0$ means that there is no arc from x to y in D. We say that an arc is a **loop** if and only if its tail and head coincide. A digraph with parallel arcs and without loops will be called **multidigraphs**. A **simple digraph** is a digraph with neither parallel arcs nor loops.

For a vertex v in D, the sets $N_D^+(v) = \{u \in V(D) : (v, u) \in A(D)\}$, $N_D^-(v) = \{u \in V(D) : (u, v) \in A(D)\}$ and $N_D(v) = N_D^+(v) \cup N_D^-(v)$ are called the **out-neighborhood**,**in-neighborhood** and **neighborhood** of v, respectively. The **out-degree** of v is the number of arcs with tail v, the **in-degree** of v is the number of arcs with head v, and will be denoted by $d_D^+(v)$ and $d_D^-(v)$, respectively.

A digraph H is a **subdigraph** of a digraph D if $V(H) \subseteq V(D)$, $A(H) \subseteq A(D)$ and every arc in $A(H)$ has both end-vertices in $V(H)$. If $V(H) = V(D)$, we say that H is a **spanning subdigraph** of D. If every arc of $A(D)$ with both end-vertices in $V(H)$ is in $A(H)$, we say

that H is **induced** by $V(H)$ and call H an **induced subdigraph** of D.

Let D be a digraph. For a set $B \subseteq A(D)$, the digraph $D - B$ is the spanning subdigraph of D with arc set $A(D) \setminus B$. If $X \subseteq V(D)$, the digraph $D - X$ is the subdigraph induced by $V(D) \setminus X$. Let $e = (x, y)$ be an arc of D, the **contraction** of the arc e is a digraph D/e with vertex set $V(D/e) = \{v\} \cup (V(D) \setminus \{x, y\})$, where $v \notin V(D)$, and $\mu_{D/e}(u, w) = \mu_D(u, w)$, for every u and w in $V(D) \setminus \{x, y\}$, $\mu_{D/e}(u, v) = \mu_D(u, x) + \mu_D(u, y)$, $\mu_{D/e}(v, u) = \mu_d(x, u) + \mu_D(y, u)$ and $\mu_{D/e}(v, v) = 0$. The **subdivision** of an arc $e = (x, y)$ of D consists of replacing e by two arcs (x, u) and (u, y), where u is a new vertex. Let D and H be a pair of digraphs, the **union** of D and H, denoted by $D \cup H$, is the digraph such that $V(D \cup H) = V(D) \cup V(H)$ and $\mu_{D \cup H}(x, y) = \mu_D(x, y) + \mu_H(x, y)$, for every pair x, y of vertices in $V(D \cup H)$. Here we assume that $\mu_D(x, y) = 0$ ($\mu_H(x, y) = 0$) if at least one of x, y is not in $V(D)$ ($V(H)$). The **underlying graph** of a digraph D is the graph obtained from D by replacing each arc (u, v) or a pair (u, v), (v, u) of arcs by the edge uv.

A **directed walk** in D is a sequence $W = (x_0, a_0, x_1, a_1, x_2, \ldots, x_{k-1}, a_{k-1}, x_k)$ of vertices v_i and arcs a_j from D such that $a_i = (x_i, x_{i+1})$, for every $i \in \{0, \ldots, k-1\}$. A directed walk W is **closed** if $x_1 = x_k$. The set of vertices $\{x_i : i \in \{1, \ldots, k\}\}$ is denoted by $V(W)$; and the set of arcs $\{a_i : i \in \{0, \ldots, k-1\}\}$ is denoted by $A(W)$. We also say that W is a directed walk **from** x_0 **to** x_k or an $x_0 - x_k$ walk. The **length** of a directed walk is $k - 1$ and is denoted by $l(W)$. When the arcs of W are defined from the context or simply unimportant, we will denote W by $(x_0, x_1, \ldots, x_k)$.

A **directed trail** is a walk in which all arcs are distinct. A walk W is a **directed path** whenever the vertices of W are distinct. If the vertices $x_0, x_1, \ldots, x_{k-1}$ are distinct, $k \geq 2$ and $x_1 = x_k$, W is a **directed cycle**. For a closed walk $W = (x_0, x_1, \ldots, x_p, x_0)$, the subscripts are considered modulo $p + 1$. A directed path (cycle) W is a **Hamilton** or **Hamiltonian** path (cycle) if $V(W) = V(D)$. A digraph D is **Hamiltonian** if D contains a Hamiltonian cycle. A directed trail W is an **Euler** or **Eulerian** if W is closed and $A(W) = A(D)$. A digraph D is **Eulerian** if it has an Euler trail.

Let $Q = (x_0, x_1, \ldots, x_k)$ and $W = (y_0, y_1, \ldots, y_p)$ be a pair of directed walks in a digraph D. The directed walks Q and W are **disjoint** if $V(Q) \cap V(W) = \emptyset$ and **arc-disjoint** if $A(Q) \cap A(W) = \emptyset$. Moreover, Q and W are called **internally disjoint** if $\{x_1, \ldots, x_{k-1}\} \cap V(W) = \emptyset$ and $\{y_1, \ldots, y_{p-1}\} \cap V(Q) = \emptyset$.

Consider a directed walk $W = (x_0, x_1, \ldots, x_k)$. We will use the notation (x_i, W, x_j) to refer to the directed walk $(x_i, x_{i+1}, \ldots, x_j)$ of W. If $0 < i \leq k$, then the **predecessor** of x_i on W is the vertex x_{i-1} (and when W is closed, the predecessor of x_0 is x_k). If $0 \leq i < k$, then the **successor** of x_i on W is the vertex x_{i+1} (and when W is closed, the predecessor of x_k is x_0). If $V = (v_0 = x_k, v_1, \ldots, v_n)$ is another directed walk, the **concatenation** of W and V, denoted by $W \cup V$, is the directed walk $(x_0, \ldots, x_k = v_0, v_1, v_2 \ldots, v_k)$.

If there is no confusion we will write walk, trail, path and cycle instead of directed walk, directed trail, directed path and directed cycles, respectively.

In a digraph D a vertex y is **reachable** from a vertex x if D has an $x - y$ walk. In particular, a vertex is reachable from itself. A digraph D is **strongly connected** or **strong** if, for every pair x, y of distinct vertices in D, there exists an $x - y$ walk and a $y - x$ walk. For a strong digraph D, a set $S \subset V(D)$ is a **separator** if $D - S$ is not strong. A digraph D is **k-strongly connected** if $|V(D)| \geq k + 1$ and D has no separator with less than k vertices. For a pair s, t of distinct vertices in $V(D)$, a **(s, t)-separator** is a subset $X \subseteq V(D) \setminus \{s, t\}$ with the property that $D - X$ has no $s - t$ paths.

1.3 Edge-colored (di)graphs

Let G be a multigraph and k a positive integer. A **k-edge-coloring** of G is a function $c : E(G) \to \{1, \ldots, k\}$, which associates each edge to one of the k colors, $1, 2, \ldots, k - 1$ or k,

and the color of an edge e will be denoted by $c(e)$. We will say that G is a **k-edge-colored** multigraph whenever we are take a fixed k-edge-coloring of G. If the number of colors are defined from the context or simply unimportant, we will say that G is an edge-colored multigraph.

Let G be an edge-colored multigraph. A walk W in G is **properly colored** if no two consecutive edges of W have the same color, including the first and the last in a closed walk. A walk (trail, cycle or path) T in G is **monochromatic** if all the edges of T have the same color.

Let G be a k-edge-colored multigraph and v a vertex of G. For $j \in \{1,\ldots,k\}$, the **jth neighbourhood** of v is $N_j(v) = \{u \in V(G) : e = uv \in E(G) \text{ and } c(e) = j\}$. The **$j$th degree** of v, denoted by $d_j(v)$, is the number of edges incident with v with color j. The **minimum monochromatic degree** of G is defined by $\delta_{mon}(G) = \min\{d_j(v) : v \in V(G), 1 \le j \le k\}$.

Let D be a digraph and k a positive integer. A **k-arc-coloring** of D is a function $c : A(D) \to \{1,\ldots,k\}$, which associates each arc to one of the k colors, $1, 2, \ldots, k-1$ or k, and the color of an arc e will be denoted by $c(e)$. We will say that D is a **k-arc-colored** digraph whenever we take a fixed k-arc-coloring of D. If the number of colors are defined from the context or simply unimportant, we will say that D is an arc-colored digraph.

1.4   Computational complexity theory

A **decision problem** is a problem that has a "yes" or a "no" answer. The complexity class **P** contains all the decision problems which can be solve in polynomial time by a deterministic Turin machine. The collection of all the decision problems whose solutions can be verified in polynomial time is denoted by **NP**. It is well-known that the class P is contained in the class NP. One of the best known problems in mathematics is whether every problem in the class NP also belongs to the class P, this problem is called P=NP problem. A problem in NP is called **NP-complete** if a polynomial time algorithm for a solution would result in a polynomial time solution for all problems in NP.

Chapter 2

A historical review

In this chapter we will present a historical introduction to the study of trails, cycles and paths.

In [13], the well-known Königsberg bridge problem is presented as follows: Early in the 18th century, the East Prussian city of Königsberg (now called Kaliningrad) occupied both banks of the River Pregel and the island of Kneiphof, lying in the river at a point where it branches into two parts. There were seven bridges that spanned the various sections of the river. A popular puzzle, called the Königsberg bridge problem, asked whether there was a route that crossed each of these bridges exactly once.

In 1736, Euler [21] gave a solution to this problem. Moreover, Euler posed and solved the general problem which is the following: whatever the arrangement and division of the river into branches, and no matter how many bridges there are and their arrangement, is it possible to find out whether or not it is possible to cross each bridge exactly once?

Euler stated that Königsberg bridge problem and its solution could be part of the geometry of positions, a branch of geometry concerned with position rather than the distance. In terms of graph theory the general problem can be view as follows: Given a multigraph G, when does it contain an Eulerian trail?

Euler's solution to the general Königsberg bridge problem, provides only the necessary conditions for a multigraph to contain an Eulerian trail. It was until 1873 that Hierholzer [35] proved that the condition given by Euler is necessary and sufficient, see Theorem 2.1. A more detailed historical record can be found in [58].

Theorem 2.1. *Let G be a connected multigraph. Then G is Eulerian if and only if every vertex of G has even degree.*

More than a hundred years later after Euler's publication, Hamilton developed a game called *Icosian Game*, which illustrates a non-commutative algebraic structure named *icosian calculus*. The game was played on a board with 20 holes, where some of these holes were connected by a line, and 20 tiles numbered from 1 to 20, the diagram of the board is illustrated in Figure 2.1(a). There were different challenges to solve but we will focus on two of them, the first one is to find a way to put the 20 tiles in the 20 holes with the only condition that consecutive numbers were in holes that had a line between them; for the second challenge was added the condition that the tile with the number 1 and the tile with the number 20 also fulfilled this condition, that is, that they were in holes with a line between them. We can easily construct a graph, namely G_I, from the board of the game as follows: add a vertex to G_I for each hole of the board; and two vertices are adjacent in G_I if and only if their holes are joined by a line on the board, i.e., the holes become the vertices and the lines become the edges of the graph, see Figure 2.1(b). So, in terms of graph theory, the two aforementioned challenges can be seen as finding a Hamiltonian path and a Hamiltonian cycle in the graph G_I, respectively. It is worth mentioning that Kirkman [36] has previously studied the existence of Hamiltonian cycles in polyhedral graphs.

(a) Diagram of the Icosain game's board.

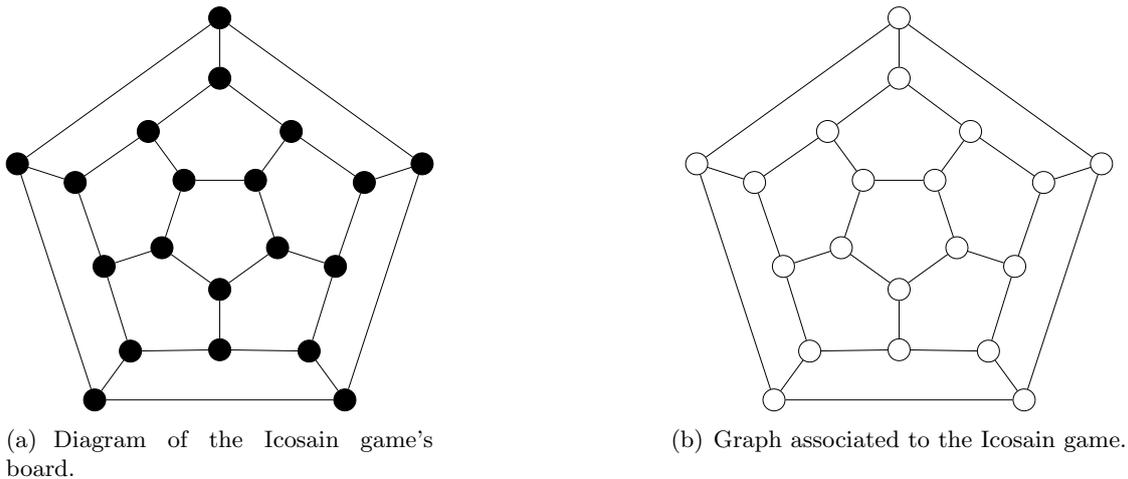(b) Graph associated to the Icosain game.

Figure 2.1: The Icosian game.

Different type of sufficient conditions are well-known for a graph to contain a Hamiltonian cycle. For example, Dirac [18] in 1952 gave a degree condition for a graph to be Hamiltonian.

Theorem 2.2. *Let G be a graph with $n \geq 3$ vertices. If $d(v) \geq n/2$, for every v in $V(G)$, then G is Hamiltonian.*

A few years later, Ore [41] gave the following degree condition that generalizes the one given by Dirac.

Theorem 2.3. *Let G be a graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n$, for every pair of non-adjacent vertices u and v, then G is Hamiltonian.*

As a consequence of the previous result, the following sufficient degree condition, given by Ore [42], for a graph containing Hamiltonian path is obtained.

Theorem 2.4. *Let G be a graph with $n \geq 2$ vertices. If $d(u) + d(v) \geq n - 1$, for every pair of non-adjacent vertices u and v, then G has a Hamiltonian path.*

Ore [43] also studied the existence of a Hamiltonian $x - y$ path, for every pair of different vertices of a graph, and obtained the following result.

Theorem 2.5. *Let G be a graph with $n \geq 4$ vertices. If $d(u) + d(v) \geq n + 1$, for every pair of non-adjacent vertices u and v, then there exists a Hamiltonian $x - y$ path, for every pair of different vertices x and y in $V(G)$.*

In [34], Harary and Nash-Williams introduced the following graphs. Let G be a graph with p vertices and q edges. For $n \geq 2$, let $L_n(G)$ be the graph with nq vertices obtained as follows: for each edge $e = uv$ of G, we take two vertices $f(u, e)$ and $f(v, e)$ in $L_n(G)$ and add a path with $n - 2$ new intermediate vertices connecting $f(u, e)$ and $f(v, e)$. Finally, for each vertex u of G, we add an edge joining $f(u, e)$ and $f(u, g)$, whenever e and g are distinct edges with end-vertex u. They also proved the followings relationships between Eulerian graphs in G and Hamiltonian cycles in $L_n(G)$.

Theorem 2.6 (Harary and Nash-Williams [34])**.** *Let G be a graph. The following assertions holds:*

1. *If G is Eulerian, then $L_n(G)$ is Hamiltonian, for every $n \geq 2$.*

2. *If $L_n(G)$ is Hamiltonian, for some $n \geq 3$, then G is Eulerian.*

3. G is superulerian if and only if $L_2(G)$ is Hamiltonian.

Due to the importance of walks both theoretically and in applications, several generalizations of walks in directed and undirected graphs have been studied. One of the most investigated are properly colored walks, in edge-colored multigraphs. Properly colored walks are of interest as a generalization of walks in undirected and directed graphs, see [5]. In graph theory applications Dorninger [19] studied a model of cell division where a properly colored Hamiltonian cycle of a 2-edge-colored graph is required. As well, Pevzner associated properly colored Eulerian trails, in 2-edge-colored graphs, with the solutions of a small-scale DNA physical mapping problem, called the Double Digest Problem. Further applications can be found in genetic and molecular biology [20, 44, 45], social science [15], channel assignment in wireless networks [3, 46].

In [37], Kotzig studied, from a theoretical point of view, the existence of properly colored Eulerian trails and proved the following characterization of the edge-colored multigraphs containing a properly colored closed Eulerian trail.

Theorem 2.7. *Let G be a c-edge-colored connected multigraph. Then, G has a properly colored closed Eulerian trail if and only if for every vertex v in $V(G)$, $d_i(v) \leq d(v)/2$, for each i in $\{1, \ldots, c\}$.*

Sheng et al. [49] characterized the 2-arc-colored digraphs which contain properly colored closed Eulerian trail. To achieve this, they define the following concept of connectivity: an arc-colored digraph D is **properly colored trail connected** if and only if for every pair of different arcs f_1 and f_2 in $A(D)$, there is a properly colored trail starting with the arc f_1 and ending with the arc f_2.

Theorem 2.8. *Let D be a 2-arc-colored multidigraph. Then, D contains a properly colored closed Eulerian trail if and only if D is properly colored trail connected and for every v in $V(D)$, $d_i^+(v) = d_{3-i}^-(v)$, for each $i \in \{1, 2\}$.*

Sheng et al. [49] showed an example of a digraph D such that: a) D is an 3-arc-colored digraph, b) D is properly colored trail connected, c) for every v in $V(D)$, $d_i^+(v) \leq d_{i-1}^-(v) + d_{i+1}^-(v)$, for each $i \in \{1, 2, 3\}$, (the subindices are taken modulo 3), and d) D does not contain a properly colored closed Eulerian trail, see Figure 2.2.
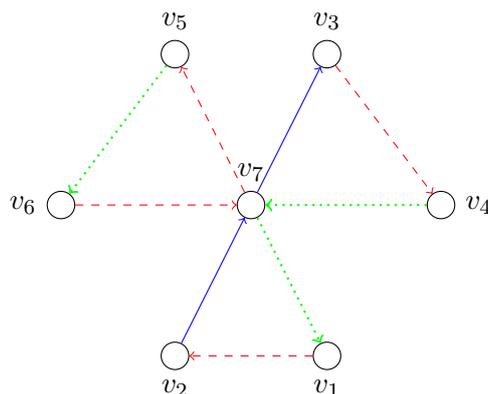

Figure 2.2: The digraph does not contain properly colored closed Eulerian trail.

Several authors have studied the existence and the length of properly colored cycles, paths and trails (not necessarily Eulerian), see [14, 23, 39, 40]. In particular, Grossman and Häggkvist [31] were the first to study the problem of the existence of properly colored cycles in c-edge-colored graphs, and they proved Theorem 2.9 for $c = 2$. Later, Yeo [59] proved it for $c \geq 2$.

Theorem 2.9. *Let G be a c-edge-colored graph, $c \geq 2$, with no properly colored cycle. Then, G has a vertex $z \in V(G)$ such that no connected component of $G - z$ is joined to z with edges of more than one color.*

Abouelaoualim et al. [2] gave the following sufficient degree condition for an edge-colored multigraph to have a properly colored Hamiltonian cycle.

Theorem 2.10. *Let G be a c-edge-colored multigraph of order n, such that no two parallel edges have the same color. Suppose that $\delta_i(x) \geq \lceil (n+1)/2 \rceil$ for every $x \in V(G)$ and $i \in \{1, \ldots, c\}$.*

1. *If $c = 2$, then G has a properly colored Hamiltonian cycle when n is even, and a properly colored cycle of length $n - 1$, when n is odd.*

2. *If $c \geq 3$, then G has a properly colored Hamiltonian cycle.*

Different kinds of edge-colorings in undirected and directed graphs have been studied. For example, Linek and Sands [38] colored the arcs of a tournament with the elements of a partially ordered set P, and defined a **monotone walk** as a walk $(v_1, v_2, \ldots, v_k)$ such that $c(v_i, v_{i+1}) \leq c(v_{i+1}, v_{i+2})$ in P, for each $i \in \{1, \ldots, k-2\}$. They also proposed to color the arcs of a tournament with the vertices of a digraph H with a loop in each vertex, rather than the elements of P, and consider an **H-walk**, that is, a walk $(v_1, v_2, \ldots, v_k)$ such that $(c(v_i, v_{i+1}), c(v_{i+1}, v_{i+2}))$ is an arc of H, for each $i \in \{1, \ldots, k-2\}$. A decade later, Arpin and Linek [4] extended the definition of the arc-coloring with the vertices of a digraph, to digraphs (not only for tournaments) as follows: Let H be a digraph possibly with loops and D be a multidigraph. An **H-coloring** of D is a function $c : A(D) \to V(H)$. We will say that D is an **H-colored multidigraph**, whenever we take a fixed H-coloring of D. A walk $W = (v_0, v_1, v_2, \ldots, v_{k-1}, v_k)$ in D is an **H-walk** if and only if $(c(v_0, v_1), c(v_1, v_2), \ldots, c(v_{k-2}, v_{k-1}), c(v_{k-1}, v_k))$ is a walk in H. We will say that W is **closed** if $v_0 = v_k$ and $c(v_{k-1}, v_k)c(v_0, v_1) \in A(H)$.

Since the beginning, the concepts of H-coloring and H-walks have been studied mainly in the context of kernel theory and related topics, see [16, 17, 28]. Notice that if H is a complete digraph without loops, then a walk W in a multidigraph D is an H-walk if and only if W is a properly colored walk. Moreover, if all the arcs of H are loops, then a walk W in D is an H-walk if and only if W is a monochromatic walk.

Since properly colored walks have been very useful for modeling and solving different problems, as we mentioned above, H-colored digraphs and H-walks can also model interesting problems, for example, routing problems or in communications networks were some transitions are restricted because of natural phenomenon, external attack or failure. Some applications of colored walks with restrictions in color's succession can be found in [50, 51].

An application of these concepts can be found in analyzing different situations in the public transportation network of Mexico City, in particular, we can focus on the metro and metrobus of this city. In Figures 2.3(a) and 2.3(b) we can see the maps of the metro and metrobus of Mexico City, respectively. We can notice that both maps are arc-colored digraphs, the metro-digraph with 12 colors (say $C_1, C_2, \ldots, C_{12}$, one color per line) and the metrobus-digraph with 7 colors (say $c_1, c_2, \ldots, c_7$, one color per line). In Figure 2.3(a), we can see the map of both in a single one, which is a 19-arc-colored digraph, from now on we will call it M. If we only consider monochromatic walks, we can only use one and only one line to move from one place to another, which will not always be what we need. Instead, if we consider properly colored walks, we could not go very far from the starting point, the longest properly colored walk, without repeating arcs, has length five. So, we can consider the digraph M as an H-colored digraph, where $V(H) = \{C_1, \ldots, C_{12}\} \cup \{c_1, \ldots, c_7\}$ and $A(H)$ will change depending on what we want to study. For example, everyday life users seek to get from vertex A to vertex B as fast as possible. In that case we consider H as the complete digraph with loops at each vertex. Sometimes the user has a limited budget to go from A to B, for example, if the user has 5 Mexican pesos, then we would consider $A(H) = \{(C_i, C_j) : i \in \{1, \ldots, 12\} \text{ and } j \in \{1, \ldots, 12\}\}$. If the user has 7 Mexican pesos, then we would consider $A(H) = \{(C_i, C_j) : i \in \{1, \ldots, 12\} \text{ and } j \in \{1, \ldots, 12\}\} \cup \{(c_i, c_j) : i \in \{1, \ldots, 7\} \text{ and } j \in \{1, \ldots, 7\}\}$. Other restrictions can be considered, such as avoiding long transfers between lines, or even adding a new color to avoid passing

through crowded stations or add weight to each arc to take into consideration the time or length of the journey.

(a) Metro of Mexico City.
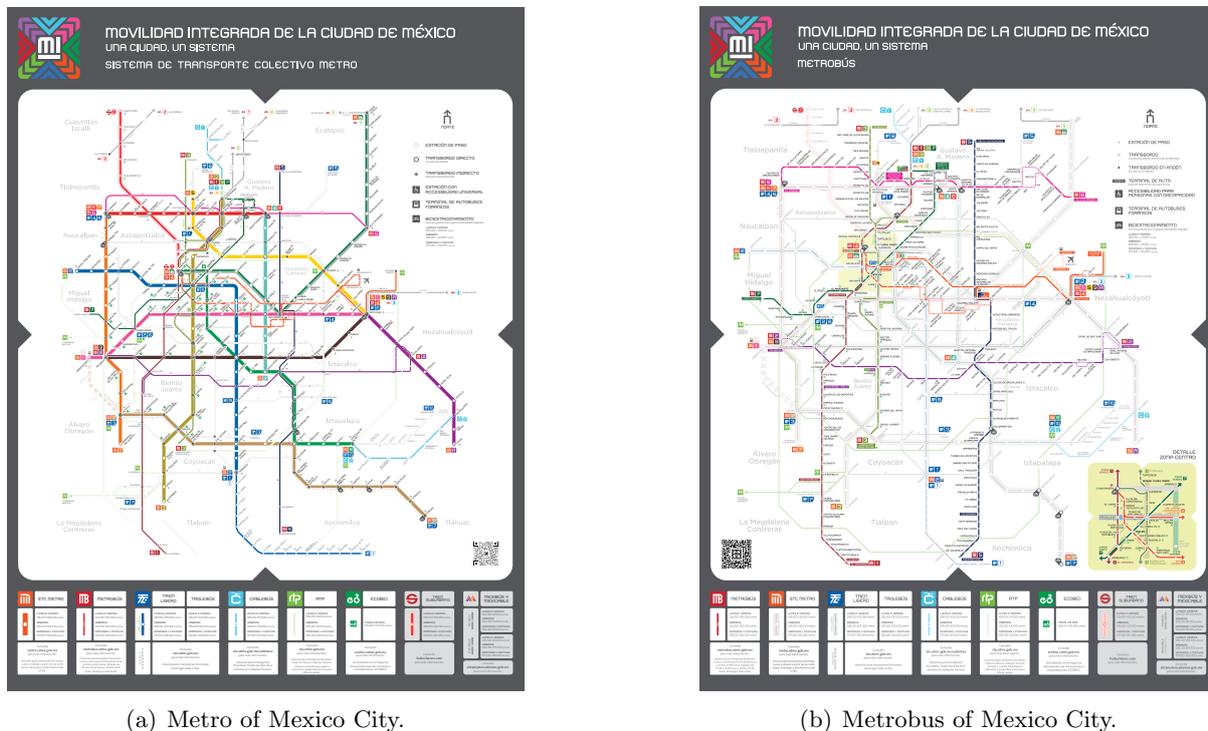

(b) Metrobus of Mexico City.

Figure 2.3: Transportation network maps of Mexico City. Images obtained from [29].

Another application can be found in road maps. Road maps can be easily represented with digraphs placing a vertex at the intersection of two streets, and an arc from a vertex v to a vertex u if and only if there is a street from v to u without passing through any other corner. Due to traffic laws, it is not always possible to turn from one street to another. For example, Figure 2.4(a) shows the intersection of two streets, where no u-turn is allowed, and no left turn is allowed from street B to street A. Figure 2.4(b) shows its digraph representation, namely D. So, if we want to find a walk from one street to another that pass through this corner and respect the traffic law, it is not enough to find walks in a non-arc-colored digraph (since a walk can contain the paths $(v_2, e_D^-, v, e_B^+, v_1)$ or $(v_2, e_D^-, v, e_D^+, v_1)$, that are forbidden by the traffic laws) nor properly colored walks, in a c-arc-colored digraphs (since there is no c-arc-coloring that guarantees that $(v_1, e_A^-, v, e_A^+, v_1)$ and $(v_2, e_D^-, v, e_B^+, v_1)$ are not properly colored walks and $(v_1, e_A^-, v, e_D^+, v_2)$ is a properly colored walk). In Figure 2.4(c), we show an example of an H-coloring of D such that every H-walk represents a route through the corner that respects traffic laws.

Galeana-Sánchez et al. [26] defined the graph G_u, for H-colored multigraphs, as follows: Let G be an H-colored multigraph and u be a vertex of G. Let $\boldsymbol{G_u}$ be the graph such that $V(G_u) = \{e \in E(G) : e \text{ is incident with } u\}$, and two different vertices a and b are joint by only one edge in G_u if and only if $c(a)$ and $c(b)$ are adjacent in H. This graph was introduced by Benkouar et al. [10] for edge-colored graphs, in an algorithmic proof of Theorem 2.7.

Galeana-Sánchez et al. [26] showed necessary and sufficient conditions for the existence of closed Euler H-trails, as follows.

Theorem 2.11 (Galeana-Sánchez et al. [26])**.** *Let H be a graph possibly with loops and G be an H-colored multigraph without loops. Suppose that G is Eulerian and G_u is a complete k_u-partite graph, for every u in $V(G)$ and for some k_u in $\mathbb{N}$. Then G has a closed Euler H-trail if and*

(a) A intersection of four streets in a certain corner.

(b) Digraph representation.

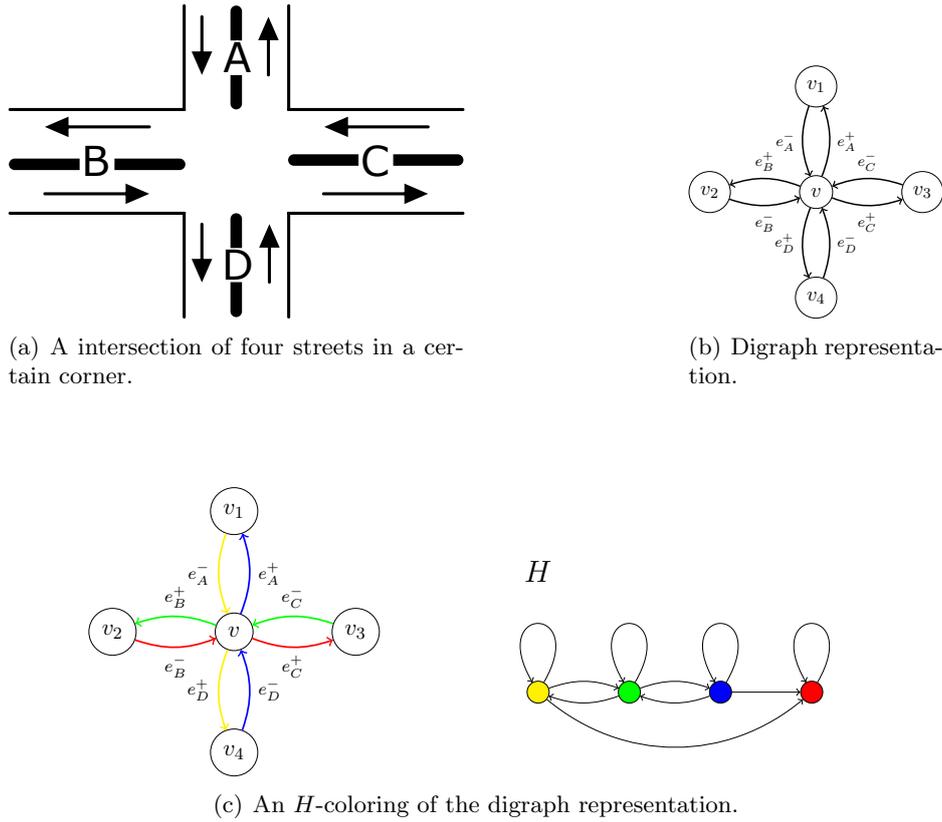(c) An H-coloring of the digraph representation.

Figure 2.4: An H-walk in an H-colored digraph can model in a better way the possible routes of cars, respecting the traffic laws.

only if $|C_i^u| \leq \sum_{j \neq i} |C_j^u|$ for every u in $V(G)$, where $\{C_1^u, ..., C_{k_u}^u\}$ is the partition of $V(G_u)$ into independent sets.

As a first approach to the definition of H-walk in H-colored multidigraphs, Arpin and Linek described the H-walks as those "walks $v_1, v_2, \ldots, v_n$ such that $(\text{color}(v_i, v_{i+1}), \text{color}(v_{i+1}, v_{i+2}))$ is an arc or loop of D for all i" (they used the vertices of a digraph D to color the arcs of a multidigraph G). Inspired by this definition, Benítez-Bobadilla et al. [8] introduced a generalization of H-walk, which uses and takes advantage of the existence of parallel arcs. They allowed "lane changes", i.e., they allowed concatenation of two H-walks as long as the last arc of the first one and the first arc of the second one have the same tail and head. As a result, they defined the following new concept: Let D be an H-colored multidigraph. A **dynamic H-walk** in D is a sequence of vertices $W = (v_0, v_1, \ldots, v_k)$ in D such that for each $i \in \{0, \ldots, k-2\}$ there exists an arc $f_i = (v_i, v_{i+1})$ and there exists an arc $f_{i+1} = (x_{i+1}, x_{i+2})$ such that $(c(f_i), c(f_{i+1}))$ is an arc in H. In Figure 2.5 the sequence $W = (v_3, v_7, v_4, v_5, v_6, v_1)$ is a dynamic H-walk in D, since there exist the arcs $e_8 = (v_3, v_7)$ and $e_{18} = (v_7, v_4)$ such that $(c(e_8), c(e_{18})) = (R, R) \in A(H)$, $e_{18} = (v_3, v_7)$ and $e_{10} = (v_4, v_5)$ such that $(c(e_{18}), c(e_{10})) = (R, R) \in A(H)$, $e_{11} = (v_4, v_5)$ and $e_{13} = (v_5, v_6)$ such that $(c(e_{11}), c(e_{13})) = (B, B) \in A(H)$, and $e_{13} = (v_5, v_6)$ and $e_{15} = (v_6, v_1)$ such that $(c(e_{13}), c(e_{15})) = (B, B) \in A(H)$. Other sequence of vertices that are dynamic H-walks are $W_1 = (v_1, v_2, v_3, v_7, v_4, v_5, v_6, v_1)$ and $W_2 = (v_6, v_1, v_2, v_3, v_2, v_7, v_1)$.

We believe that the study of the dynamic H-walk concept is very interesting as it generalizes previous concepts of walk and it broadens the traditional way of studying the mobility in a given system. Another motivation for the study of dynamic H-walks in H-colored directed and undirected multigraphs, are their possible applications, since directed and undirected multigraphs can model several real world problems in a more natural way than simple graphs, see [22, 47, 48]. Being able to take advantage of the existence of parallel edges can result in the
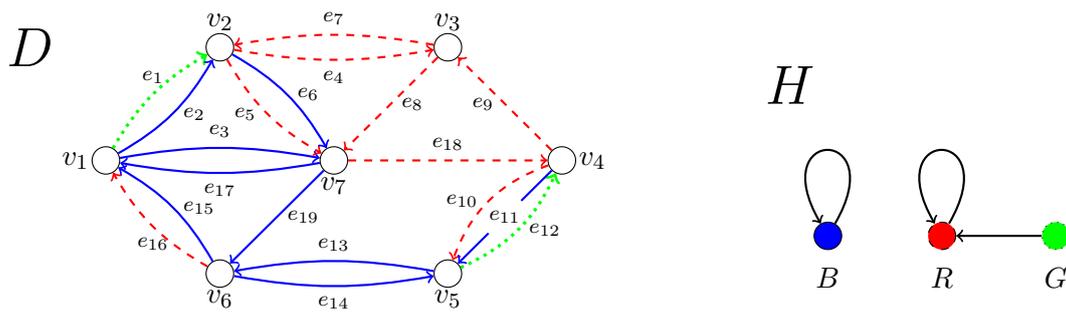
Figure 2.5: The sequence $W_1 = (v_1, v_2, v_3, v_7, v_4, v_5, v_6, v_1)$ is a dynamic H-walk in D and there is no H-walk in D that contains this sequence of vertices.

improvement of a solution, especially when different processes can be done simultaneously, for example, in project scheduling, where each person can perform a different task at the same time and some task cannot be carried out without having finished another before. Another possible application is when a company ships products from one warehouse to another. The use of parallel edges can represent the different types of transportation that exist to ship products between two warehouses, and the a lane change in a dynamic H-walk can mean that different items were shipped on two or more different transports at the same time.

Chapter 3

Dynamic H-trails in multigraphs

Benítez-Bobadilla et al. [8] introduced the concept of dynamic H-walk in the context of kernel theory, where arcs do not play a major role. This is why in their definition they do not mention the arcs that are part of the dynamic H-walk. This chapter is devoted to the study of dynamic H-trails, in H-colored multigraphs. To achieve this we introduce a definition and notation of dynamic H-walk in H-colored multigraphs, that allows us to know the edges that belong to it. Motivated by Theorem 2.11, we think that it is possible to give conditions, similar to those of this theorem, on an H-colored multigraph that guarantee the existence of a closed Euler dynamic H-trail. In the process, we find an auxiliary graph that we call $L_{n,H}^{Dym}(G)$, that allows us to link closed dynamic H-trails in H-colored multigraphs, with cycles in non-colored simple graphs.

3.1 Dynamic H-trails and the auxiliary graph $L_{n,H}^{Dym}(G)$

Let H be a graph possibly with loops and G be an H-colored multigraph. A sequence $W = (v_0, e_0^1, \ldots, e_0^{k_0}, v_1, e_1^1, \ldots, e_1^{k_1}, v_2, \ldots, v_{n-1}, e_{n-1}^{k_{n-1}}, \ldots, e_{n-1}^{k_{n-1}}, v_n)$, where for each $i \in \{0, \ldots, n-1\}$, $k_i \geq 1$ and $e_i^j = v_i v_{i+1}$ for every $j \in \{1, \ldots, k_i\}$, is a **dynamic H-walk** if $c(e_i^{k_i})c(e_{i+1}^1)$ is an edge in H, for each $i \in \{0, \ldots, n-2\}$. If W is a dynamic H-walk that does not repeat edges, then W is a **dynamic H-trail**. We say that a dynamic H-trail, W, is an **Euler** dynamic H-trail if and only if $E(G) = E(W)$. In Figure 3.1 $W_1 = (v_3, e_4, v_4, e_{11}, e_{12}, v_9, e_9, v_{10}, e_8, v_6, e_6, v_5, e_5, e_{11}, v_9, e_{13}, v_8)$ is a dynamic H-walk and $W_2 = (v_1, e_1, v_2, e_3, v_3, e_4, v_4, e_{11}, v_9, e_{13}, v_8, e_{14}, e_{15}, v_7, e_{16}, v_1)$ is a dynamic H-trail.

We say that a dynamic H-trail is **closed** whenever a) $v_0 = v_n$ and $c(e_{n-1}^{k_{n-1}})c(e_0^1)$ is an edge in H; or b) $v_1 = v_n$ and $e_{n-1}^{k_{n-1}}$ and e_0^1 are parallel in G. Let $W = (v_0, e_0^1, \ldots, e_0^{k_0}, v_1, e_1^1, \ldots, e_1^{k_1}, v_2, \ldots, v_{n-1} = v_0, e_{n-1}^{k_{n-1}}, \ldots, e_{n-1}^{k_{n-1}}, v_n = v_1)$ is a closed dynamic H -trail, then W can be rewritten as $W = (v_1, e_1^1, \ldots, e_1^{k_1}, v_2, \ldots, v_{n-1} = v_0, e_{n-1}^{k_{n-1}}, \ldots, e_{n-1}^{k_{n-1}}, e_0^1, \ldots, e_0^{k_0}, v_n = v_1)$, i.e., if W satisfies condition b) of the definition of closed dynamic H-trail, then we can rewrite W in a way that satisfies condition a (unless $n = 1$, i.e., W is of the form $(x, e_0, \ldots, e_k, y)$, where $k \geq 1$). In Figure 3.1 $W_1 = (v_1, e_1, v_2, e_3, v_3, e_4, v_4, e_{11}, v_9, e_{13}, v_8, e_{14}, e_{15}, v_7, e_{16}, v_1)$ is not a closed H-trail, because $c(e_{16})c(e_1)$ is not an edge in H. However, $W_3 = (v_1, e_1, v_2, e_3, v_3, e_4, v_4, e_{11}, v_9, e_{13}, v_8, e_{14}, e_{15}, v_7, e_{16}, v_1, e_7, v_2)$ is a closed H-trail (notice that W_3 satisfies condition b of the definition) and can be rewrite as $W_3 = (v_2, e_3, v_3, e_4, v_4, e_{11}, v_9, e_{13}, v_8, e_{14}, e_{15}, v_7, e_{16}, v_1, e_7, e_1, v_2)$ (satisfying condition a of the definition).

In the following definition we introduce an auxiliary graph that will be essential in the rest of this chapter.

Definition 3.1. *Let G be an H-colored multigraph with $|E(G)| = q$. For $n \geq 2$, $\boldsymbol{L_{n,H}^{Dym}(G)}$ is the graph with nq vertices, obtained as follows: for each edge $e = uv$ of G, we take two vertices $f(e, u)$ and $f(e, v)$ in $L_{n,H}^{Dym}(G)$, and adding a path with $n - 2$ new intermediate vertices*
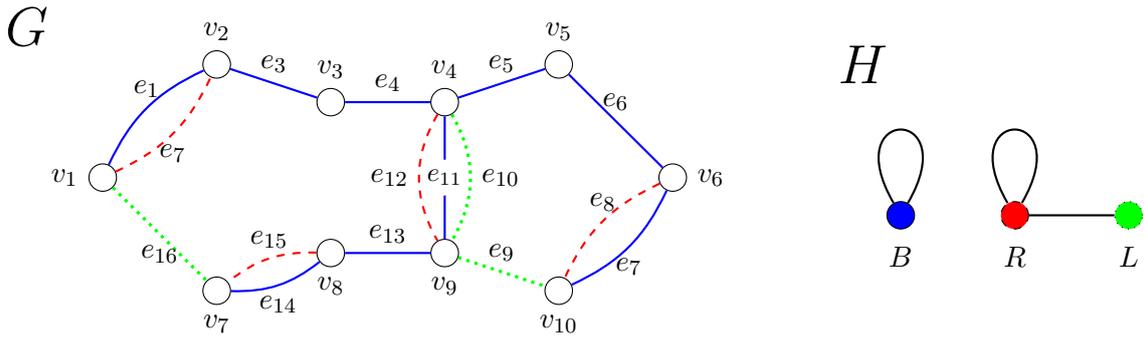
Figure 3.1: The sequence $P = (v_2, e_3, v_3, e_4, v_4, e_5, v_5, e_6, v_6, e_7, e_8, v_{10}, e_9, v_9, e_{12}, v_4, e_{10}, e_{11}, v_9,$ $e_{13}, v_8, e_{14}, e_{15}, v_7, e_{16}, v_1, e_2, e_1, v_2)$ is a closed Euler dynamic H-trail in G and there is no closed Euler H-trail in G.

connecting $f(e,u)$ and $f(e,v)$. The rest of the edges of $L_{n,H}^{Dym}(G)$ are defined as follows: a) $f(e,u)$ and $f(g,u)$ are adjacent if and only if $e \neq g$ and $c(e)c(g) \in E(H)$; b) $f(e,u)$ and $f(g,v)$ are adjacent if and only if $u \neq v$ and e and g are parallel in G.

Observation 1. *It follows from the definition of $L_{n,H}^{Dym}(G)$ that:*

1. *For every $n \geq 2$, $L_{n,H}^{Dym}(G)$ is a simple graph.*

2. *$M_J = \{f(e,x)f(e,y) \in E(L_2^H(G)) : e = xy \in E(G)\}$ is a perfect matching of $L_{2,H}^{Dym}(G)$. This matching will be called the **joint matching of** $L_{2,H}^{Dym}(G)$.*

Figure 3.2 shows an example of the auxiliary graph $L_{2,H}^{Dym}(G)$ an its joint matching.
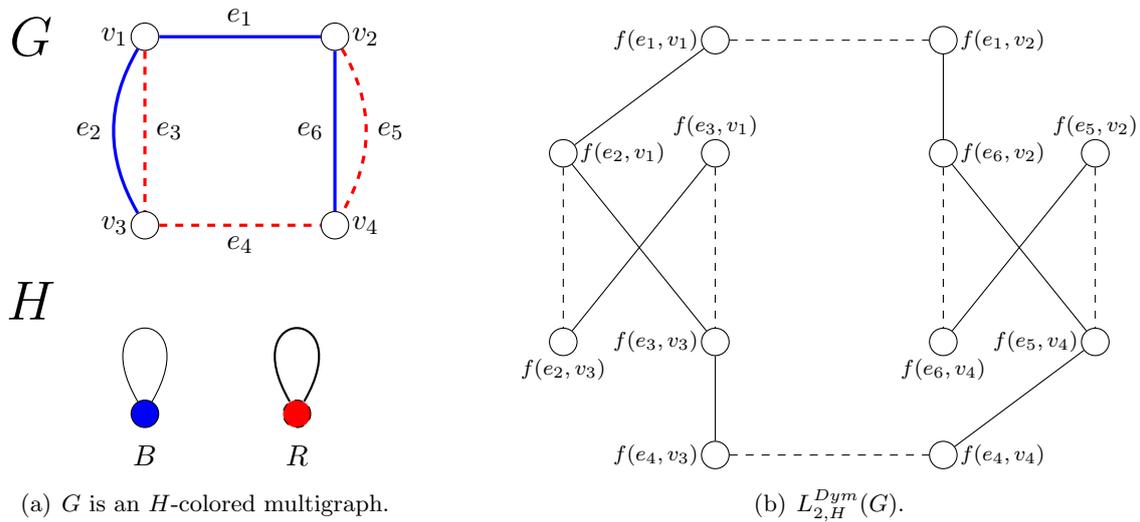
(a) G is an H-colored multigraph.

(b) $L_{2,H}^{Dym}(G)$.

Figure 3.2: In (b), the dashed edges correspond to the joint matching.

Recall that G_u is the graph such that $V(G_u) = \{e \in E(G) : e$ is incident with $u\}$, and two different vertices a and b are joining by only one edge in G_u if and only if $c(a)$ and $c(b)$ are adjacent in H. Notice that for each $e = xy$ in $E(G)$, by the definition of G_x and G_y, we have that $e \in V(G_x)$ and $e \in V(G_y)$. So, we will say that $f(e,x)$ and $f(e,y)$ are the copies of e seen as a vertex in G_x and G_y, respectively. If E_{xy} is the set of all the edges with end-vertices x and y, then we define the set $E_{xy}^x = \{f(e,x) : e \in E_{xy}\}$.

Let H be graph possibly with loops and G an H-colored graph. The graph $L_{n,H}^{Dym}(G)$ can be constructed as follows: First take the disjoint union of G_x, for every x in $V(G)$. Then, for every pair of distinct vertices, x and y in $V(G)$, such that $|E_{xy}| = m \geq 1$, add all possible edges until a complete bipartite graph $K_{m,m}$ between E_{xy}^x and E_{xy}^y is obtained. Finally, for every $e = xy \in E(G)$, change the edge joining $f(e,x)$ and $f(e,y)$ by a path with $n-2$ new intermediate vertices. The construction of $L_{3,H}^{Dym}(G)$ is illustrated in Figure 3.3.

3.2   Euler dynamic *H*-trails

In this section we study the relationship between closed dynamic H-trails in G, an H-colored multigraph, and cycles in the auxiliary graph $L_{n,H}^{Dym}(G)$.

The following two lemmas show how to construct a cycle in $L_{2,H}^{Dym}(G)$, based on the order of the edges in a closed dynamic H-trail in G, and vice versa.

Lemma 3.1. *Let G be an H-colored multigraph. If $P = (x_0, e_1, \ldots, e_{p_0}, x_1, e_{p_0+1}, \ldots, e_{p_0+\ldots+p_1},$ $x_2, \ldots, x_{n-1}, e_{p_0+\ldots+p_{n-2}+1}, \ldots, e_{p_0+\ldots+p_{n-1}}, x_n)$ is a closed dynamic H-trail in G, then $C = (f(e_1, x_0), f(e_1, x_1), \ldots, f(e_{p_0}, x_0), f(e_{p_0}, x_1), f(e_{p_0+1}, x_1), f(e_{p_0+1}, x_2), \ldots, f(e_{p_0+\ldots+p_{n-1}}, x_n),$ $f(e_1, x_0))$ is a cycle in $L_{2,H}^{Dym}(G)$.*

Proof. It follows from the definition of $L_{2,H}^{Dym}(G)$ that $f(e, x_i)f(e, x_{i+1})$ is an edge of $L_{2,H}^{Dym}(G)$, for every $e = x_i x_{i+1} \in E(P)$.

Let e_i and e_{i+1} be consecutive edges in P (if $i = p_0 + \ldots + p_{n-1}$, then $e_{i+1} = e_1$). If e_i and e_{i+1} are parallel, such that x_j and x_{j+1} are the ends of both edges, then $f(e_i, x_{j+1})$ and $f(e_{i+1}, x_j)$ are adjacent in $L_{2,H}^{Dym}(G)$, by the definition of $L_{2,H}^{Dym}(G)$. Otherwise, e_i and e_{i+1} are incident with x_j, for some x_j in $V(P)$. Since P is a closed dynamic H-trail, we have that $c(e_i)c(e_{i+1})$ is an edge in H and $f(e_i, x_j)f(e_{i+1}, x_j) \in E(L_2^H(G))$. Hence, C is a walk in $L_{2,H}^{Dym}(G)$. Since P is a closed dynamic H-trail, it follows that C does not repeat vertices and, so C is a cycle. Moreover, C alternate edges between $E(L_2^H(G)) \setminus M_J$ and M_J.   □

Lemma 3.2. *Let G be an H-colored multigraph and M_J be the joint matching of $L_{2,H}^{Dym}(G)$. If $C = (f(e_1, x_1), f(e_1, y_1), f(e_2, x_2), f(e_2, y_2) \ldots, f(e_q, y_q), f(e_1, x_1))$, where $e_i = x_i y_i \in E(G)$, is a cycle in $L_{2,H}^{Dym}(G)$ that alternates edges between $E(L_2^H(G)) \setminus M_J$ and M_J, then the following steps generate a closed dynamic H-trail P in G.*

1. *Start with $P_1 = (x_1, e_1, y_1)$ and $k = 2$.*

2. *Let e_k, with end-vertices x_k and y_k. If e_{k-1} and e_k are parallel, $x_k = x_{k-1}$ and $y_k = y_{k-1}$, then $P_k = (x_1, P_{k-1}, e_{k-1}, e_k, y_k)$. Otherwise, $P_k = (x_1, P_{k-1}, y_{k-1} = x_k, e_k, y_k)$.*

3. *$k = k + 1$.*

4. *If $k = q + 1$, $P = P_q$. Otherwise, go to step 2.*

Proof. First, we will prove that P is a dynamic H-trail in G.

Let e_{k-1} and e_k be edges of G with $2 \leq k \leq q$.

Since $f(e_{k-1}, y_{k-1})f(e_k, x_k) \in E(C)$, we have that $f(e_{k-1}, y_{k-1})f(e_k, x_k)$ is in $E(L_2^H(G))$. Hence, $y_{k-1} = x_k$ and $c(e_{k-1})c(e_k) \in E(H)$ or $y_{k-1} \neq x_k$ and e_{k-1} and e_k are parallel in G.

If $y_{k-1} = x_k$ and $c(e_{k-1})c(e_k) \in E(H)$, then e_{k-1} and e_k are incident in $y_{k-1} = x_k$ and $P_k = (x_1, P_{k-1}, y_{k-1} = x_k, e_k, y_k)$ is a dynamic H-walk in G.

Otherwise, $y_{k-1} \neq x_k$ and e_{k-1} and e_k are parallel edges in G, and $P_k = (x_1, P_{k-1}, e_{k-1}, e_k, y_{k-1} = y_k)$ is a dynamic H-walk in G.

Hence, every P_k is a dynamic H-walk in G, in particular $P = P_q$.

(a) *G* is an *H*-colored multigraph.

(b) G_{v_1}.

(c) Since $E_{v_1v_3} \neq \emptyset$, make a complete bipartite graph with partition sets $\{f(e_i, v_1) : e_i = v_1v_3\}$ and $\{f(e_i, v_3) : e_i = v_1v_3\}$.

(d) Repeat for every $E_{v_iv_j} \neq \emptyset$.

(e) Change the edge $f(e_1, v_1)f(e_1, v_2)$ by a path with $3 - 2 = 1$ new intermediate vertices.
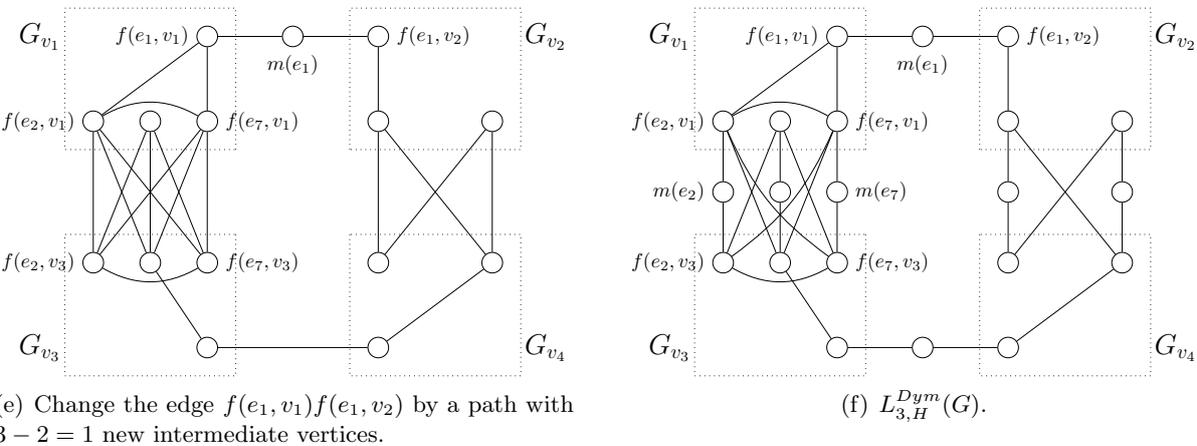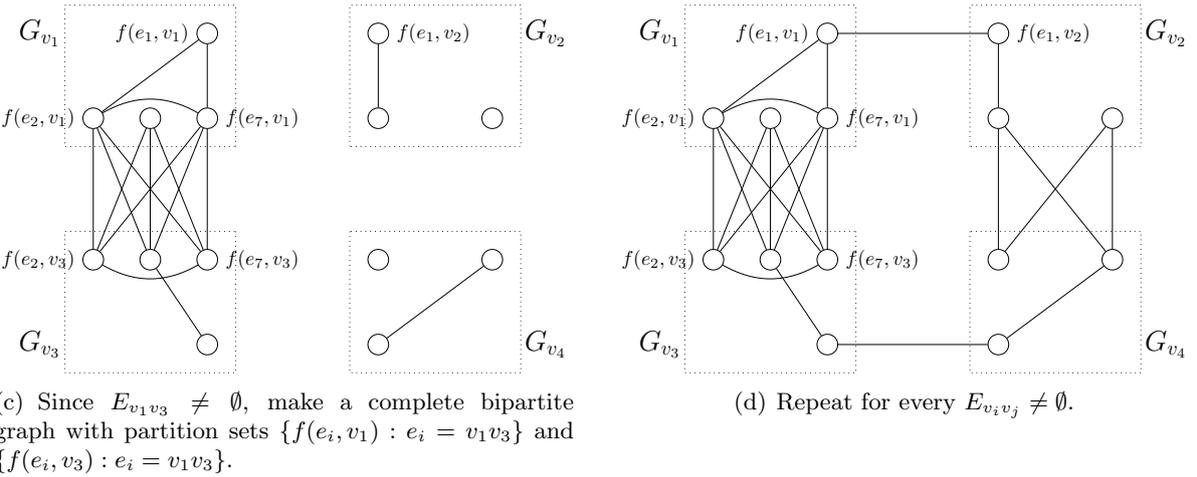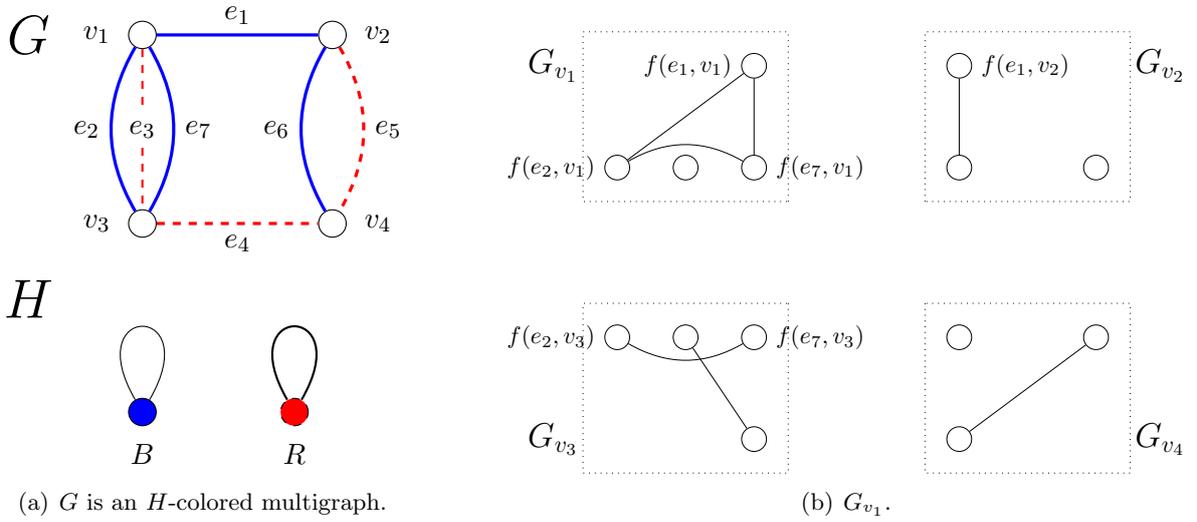
(f) $L_{3,H}^{Dym}(G)$.

Figure 3.3: Procedure to construct the graph $L_{3,H}^{Dym}(G)$.

(a) G is an H-colored multigraph.
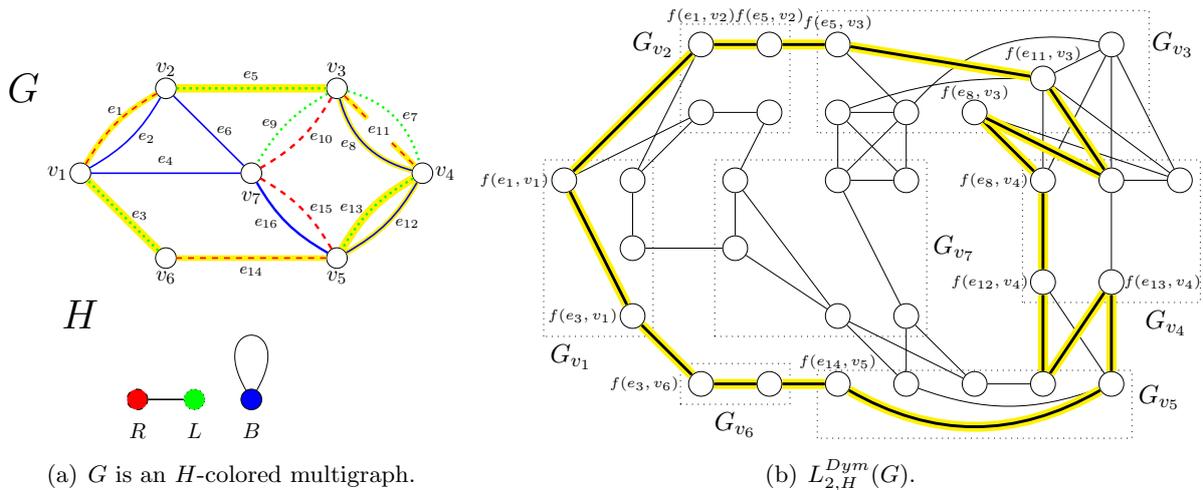
(b) $L_{2,H}^{Dym}(G)$.

Figure 3.4: The $P = (v_1, e_1, v_2, e_5, v_3, e_{11}, e_8, v_4, e_{12}, e_{13}, v_5, e_{14}, v_6, e_3, v_1)$, highlighted in yellow, is a closed H-trail in G. So, Lemma 3.1 ensures that the walk highlighted in (b) is a cycle in $L_{2,H}^{Dym}(G)$.

On the other hand, since C is a cycle, the arc e_i appears exactly once in P, for every $i \in \{1, \ldots, q\}$, and so P is a dynamic H-trail.

Now, we prove that P is closed.

Recall that $e_i = x_i y_i$, for every $i \in \{1, \ldots, q\}$, and $y_i = x_{i+1}$; or $(x_i = x_{i+1}$ and $y_i = y_{i+1})$ (if $i = q + 1$, then $x_{i+1} = x_1$ and $y_{i+1} = y_1$). We will consider two cases.

Case 1. $x_i = x_j$ and $y_i = y_j$, for every pair of distinct elements, $\{i, j\} \subseteq \{1, \ldots, q\}$.

It follows from the construction of P that $P = (x_1, e_1, \ldots, e_q, y_1)$. Since C is a cycle in $L_{2,H}^{Dym}(G)$, we have that $q \geq 2$ and, so P is closed.

Case 2. There exists $i \in \{1, \ldots, q\}$ such that $y_i = x_{i+1}$.

If $y_q = x_1$, then $c(e_q)c(e_1) \in E(H)$. Therefore, P is closed.

Otherwise, e_q and e_1 are parallel and, so P is closed.

Therefore, P is a closed dynamic H-trail in G. □

Figure 3.5 shows the construction Lemma 3.2 established.

It follows from Lemmas 3.1 and 3.2 the following result.

Theorem 3.3. *Let G be an H-colored multigraph and M_J be the joint matching of $L_{2,H}^{Dym}(G)$. Then, there is a bijection between the set of closed dynamic H-trails in G and the set of cycles in $L_{2,H}^{Dym}(G)$ that alternate edges between $E(L_2^H(G)) \setminus M_J$ and M_J.*

Proof. Let G be an H-colored multigraph and M_J the joint matching of $L_{2,H}^{Dym}(G)$. We will denote by $\mathcal{P}$ the set of closed dynamic H-trails in G and by $\mathcal{C}$ the set of cycles in $L_{2,H}^{Dym}(G)$ that alternate edges between $E(L_2^H(G)) \setminus M_J$ and M_J.

Consider the function $T : \mathcal{P} \to \mathcal{C}$ defined by $T(P) = C$, where $P = (x_0, e_1, \ldots, e_{p_0}, x_1, e_{p_0+1}, \ldots, e_{p_1}, x_2, e_{p_1+1}, \ldots, x_{n-1}, e_{p_0+\ldots+p_{n-2}+1}, \ldots, e_{p_0+\ldots+p_{n-1}}, x_n)$, and $C = (f(e_1, x_0), f(e_1, x_1), \ldots, f(e_{p_0}, x_0), f(e_{p_0}, x_1), f(e_{p_0+1}, x_1), \ldots, f(e_{p_0+\ldots+p_{n-1}}, x_n), f(e_1, x_0))$.

Claim 1. T is well-defined.

It follows from Lemma 3.1 that $T(P) \in \mathcal{C}$.

Consider two closed dynamic H-trails in $\mathcal{P}$, say $P_1 = (x_0, e_0, \ldots, e_{p_0}, x_1, e_{p_0+1}, \ldots, e_{p_0+p_1}, x_2, \ldots, x_{n-1}, e_{p_0+\ldots+p_{n-2}+1}, \ldots, e_{p_0+\ldots+p_{n-1}}, x_n)$ and $P_2 = (y_0, f_0, \ldots, f_{q_0}, y_1, f_{q_0+1}, \ldots, f_{q_0+q_1}, y_2, \ldots, y_{n-1}, f_{q_0+\ldots+q_{n-2}+1}, \ldots, f_{q_0+\ldots+q_{n-1}}, y_n)$ such that $P_1 = P_2$.
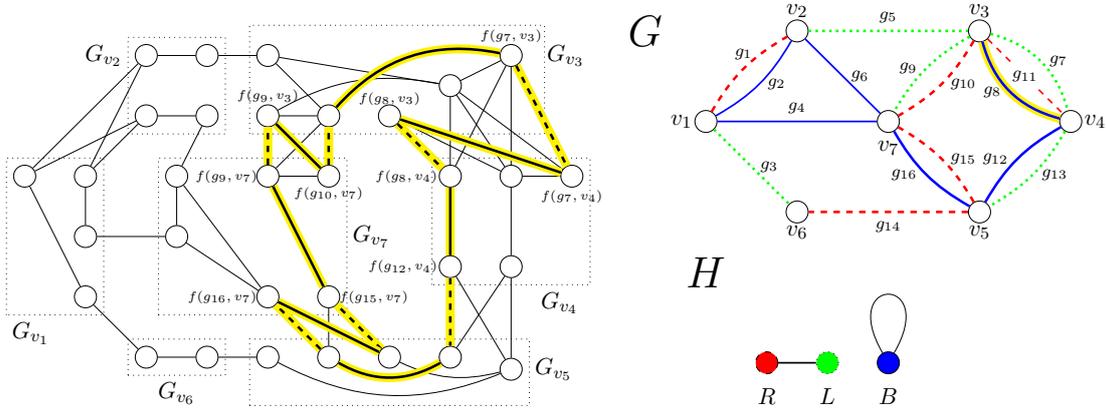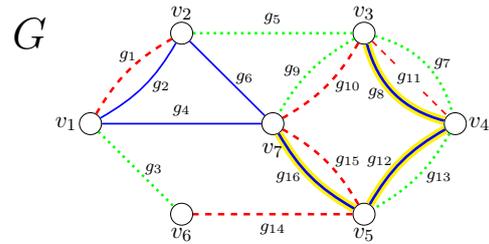
(a) Consider the cycle $C = (f(g_8, v_3), f(g_8, v_4), f(g_{12}, v_4), f(g_{12}, v_5), f(g_{16}, v_5), f(g_{16}, v_7), f(g_{15}, v_5), f(g_{15}, v_7), f(g_9, v_7), f(g_9, v_3), f(g_{10}, v_7), f(g_{10}, v_3), f(g_7, v_3), f(g_7, v_4), f(g_8, v_3))$. Dashed edges of C are in M_J.
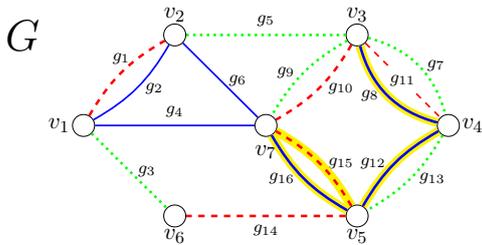
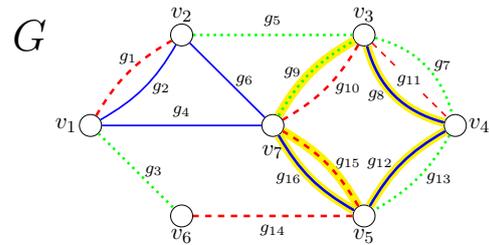(b) Start with $P_1 = (v_3, e_8, v_4)$.
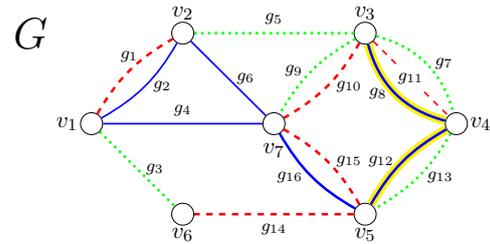
(c) For $k = 2$, $e_2 = g_{12}$, $x_2 = v_4$ and $y_2 = v_5$. So, $P_2 = (v_3, g_8, v_4, g_{12}, v_5)$.

(d) For $k = 3$, $e_3 = g_{16}$, $x_3 = v_5$ and $y_3 = v_7$. So, $P_3 = (v_3, g_8, v_4, g_{12}, v_5, g_{16}, v_7)$.

(e) For $k = 4$, $e_4 = g_{15}$, $x_4 = v_5$ and $y_4 = v_7$. So, $P_4 = (v_3, g_8, v_4, g_{12}, v_5, g_{16}, g_{15}, v_7)$.

(f) For $k = 5$, $e_5 = g_9$, $x_5 = v_7$ and $y_5 = v_3$. So, $P_5 = (v_3, g_8, v_4, g_{12}, v_5, g_{16}, g_{15}, v_7, g_9, v_3)$.
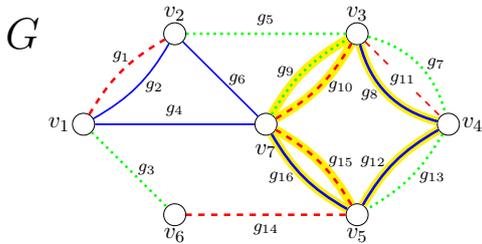
(g) For $k = 6$, $e_6 = g_{10}$, $x_6 = v_7$ and $y_6 = v_3$. So, $P_5 = (v_3, g_8, v_4, g_{12}, v_5, g_{16}, g_{15}, v_7, g_9, g_{10}, v_3)$.
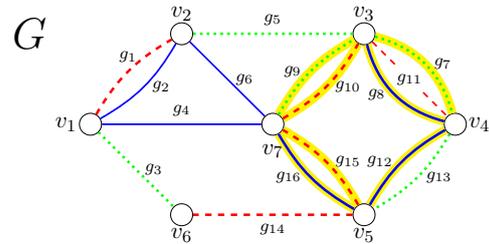
(h) For $k = 7$, $e_7 = g_7$, $x_7 = v_3$ and $y_7 = v_4$. So, $P_5 = (v_3, g_8, v_4, g_{12}, v_5, g_{16}, g_{15}, v_7, g_9, g_{10}, v_3, g_7, v_4)$.

Figure 3.5: Construction of a closed dynamic H-trail in G from a cycle in $L_{2,H}^{Dym}(G)$.

Since $P_1 = P_2$, $E(P_1) = E(P_2)$ and there exists $k \in \mathbb{N}$ such that $e_i = f_{i+k(mod\,p_0+...+p_{n-1})}$ (since the edges are traversed in the same order but the first edge is not the same). It follows from the definition of T that $V(T(P_1)) = V(T(P_2))$ and the order of the vertices are the same. Then, we have that $T(P_1) = T(P_2)$. Therefore, T is well-defined.

Claim 2. T is injective.

Let P_1 and P_2 in $\mathcal{P}$ such that $P_1 \neq P_2$.

If $E(P_1) \neq E(P_2)$, then there is $e = xy \in E(G)$ such that ($e \in E(P_1)$ and $e \notin E(P_2)$) or ($e \notin E(P_1)$ and $e \in E(P_2)$). Without loss of generality, suppose that $e \in E(P_1)$ and $e \notin E(P_2)$. Then, $f(e,x)f(e,y) \in V(P_1)$ and $f(e,x)f(e,y) \notin V(P_1)$, that is $V(T(P_1)) \neq V(T(P_2))$ and, so $T(P_1) \neq T(P_2)$.

If $E(P_1) = E(P_2)$, then there exists $\{e_i, e_{i+1}\} \subseteq E(P_1) = E(P_2)$ such that e_i is the edge preceding e_{i+1} in P_1 and e_i is not the edge preceding e_{i+1} in P_2 (since $P_1 \neq P_2$). Therefore, $T(P_1) \neq T(P_2)$ and T is injective.

Claim 3. T is surjective.

Let C be a cycle in $\mathcal{C}$. Since C alternate edges between $E(L_2^H(G)) \setminus M_J$ and M_J, C must be of the form $C = (f(e_1, x_1), f(e_1, y_1), \ldots, f(e_q, x_q), f(e_q, y_q), f(e_1, x_1))$, where $e_i = x_i y_i \in E(G)$.

It follows from the definition of T that $T(P) = C$, where P is the closed dynamic H-trail obtained by applying Lemma 3.2 to the cycle C. Hence, T is surjective.

Therefore, T is a bijection. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Recall the following classical lemma, which will be useful in what follows.

Lemma 3.4. *Let M and M' be two perfect matchings of a graph G. If $M \cap M' = \emptyset$, then there exists a partition of the vertices of G into vertex-disjoint even cycles. Moreover, every cycle alternates edges between M and M'.*

Theorem 3.5. *Let G be an H-colored multigraph and M_J be the joint matching of $L_{2,H}^{Dym}(G)$. There exists a perfect matching M in $L_{2,H}^{Dym}(G) \setminus M_J$ if and only if there exists a partition of the edges of G into closed dynamic H-trails. (Notice that some of the closed dynamic H-trails can be of the form $(x, e_0, \ldots, e_k, y)$, where $k \geq 1$).*

Proof. Let G be an H-colored multigraph and M_J be the joint matching of $L_{2,H}^{Dym}(G)$.

Suppose that there exists a perfect matching M in $L_{2,H}^{Dym}(G) \setminus M_J$.

By Lemma 3.4, it follows that there exists a partition of the vertices of $L_{2,H}^{Dym}(G)$ into vertex-disjoint cycles, say $\mathfrak{C} = \{C_1, \ldots, C_n\}$. Moreover, each cycle in $\mathfrak{C}$ alternates edges between M_J and M.

Let $\mathfrak{P} = \{P_1, \ldots, P_n\}$ be the set of closed dynamic H-trails in G such that $T(P_i) = C_i$, for every $i \in \{1, \ldots, n\}$.

Claim 1 $E(P_i) \cap E(P_j) = \emptyset$, for every $\{i, j\} \subseteq \{1, \ldots, n\}$.

Suppose to the contrary that there exists a pair of distinct elements, i and j in $\{1, \ldots, n\}$, such that $E(P_i) \cap E(P_j) \neq \emptyset$.

Let $e = xy \in E(P_i) \cap E(P_j)$. By the definition of T (see the proof of Theorem 3.3), the vertices $f(e, x)$ and $f(e, y)$ belong to $V(C_i) \cap V(C_j)$, a contradiction.

Claim 2 $\bigcup_{i=1}^n E(P_i) = E(G)$.

By the definition of T, we have that $\bigcup_{i=1}^n E(P_i) \subseteq E(G)$.

On the other hand, let $e \in E(G)$ with end-vertices x and y. Since $\mathfrak{C}$ is a partition of the vertices of $L_{2,H}^{Dym}(G)$, there exists a cycle in $\mathfrak{C}$, say C_j, such that $f(e, x)f(e, y) \in E(C_j)$. By the construction of P_j, it follows that $e \in E(P_j)$. Therefore, $e \in \bigcup_{i=1}^n E(P_i)$ and $E(G) \subseteq \bigcup_{i=1}^n E(P_i)$.

Hence, $\mathfrak{P} = \{P_1, \ldots, P_n\}$ is a partition of the edges of G into closed dynamic H-trails.

Conversely, suppose that there exists a partition of the edges of G into closed dynamic H-trails, say $\mathfrak{P} = \{P_1, \ldots, P_k\}$.

Let $\mathfrak{C} = \{C_1, \ldots, C_k\}$ be a set of cycles in $L_{2,H}^{Dym}(G)$ such that $T(P_i) = C_i$ for every $i \in \{1, \ldots, k\}$.

Claim 3 If $i \neq j$, then $V(C_i) \cap V(C_j) = \emptyset$.

For a contradiction, suppose that there exists $\{i, j\} \subseteq \{1, \ldots, k\}$, such that $V(C_i) \cap V(C_j) \neq \emptyset$. Consider $f(e, x) \in V(C_i) \cap V(C_j)$, for some $e \in E(G)$ and $x \in V(G)$, such that e is incident with x.

By the construction of C_i and C_j, it follows that $e \in E(P_i)$ and $e \in E(P_j)$, a contradiction.

Claim 4 $\bigcup_{i=1}^{k} V(C_i) = V(L_2^H(G))$.

By the construction of C_i, we have that $\bigcup_{i=1}^{k} V(C_i) \subseteq V(L_2^H(G))$.

Let $f(e, x) \in V(L_2^H(G))$, with $e \in E(G)$ and $x \in V(G)$ such that e is incident with x.

Since $\mathfrak{P}$ is a partition of the edges of G, there exists $P_i \in \mathfrak{P}$ such that $e \in E(P_i)$. By the construction of C_i, it follows that $f(e, x) \in V(C_i) \subseteq \bigcup_{j=1}^{k} V(C_j)$. Hence, $V(L_2^H(G)) \subseteq \bigcup_{j=1}^{k} V(C_j)$.

Therefore, $\bigcup_{i=1}^{k} V(C_i) = V(L_2^H(G))$.

Claim 5 $M = \bigcup_{i=1}^{k} E(C_i) \setminus M_J$ is a perfect matching in $L_{2,H}^{Dym}(G)$.

Recall that $M_J = \{f(e, x)f(e, y) \in E(L_2^H(G)) : e = xy \in E(G)\}$.

By the construction of C_i, and Claims 3 and 4, we conclude that M is a perfect matching in $L_{2,H}^{Dym}(G)$.

Therefore, M is a perfect matching in $L_{2,H}^{Dym}(G)$ such that $M \cap M_J = \emptyset$. $\square$

Figure 3.6 shows an example of a partition of $E(G)$ into dynamic H-trails and its corresponding partitioning of $V(L_{2,H}^{Dym}(G))$ into cycles that alternate edges between two perfect matchings, one of them is the joint matching.
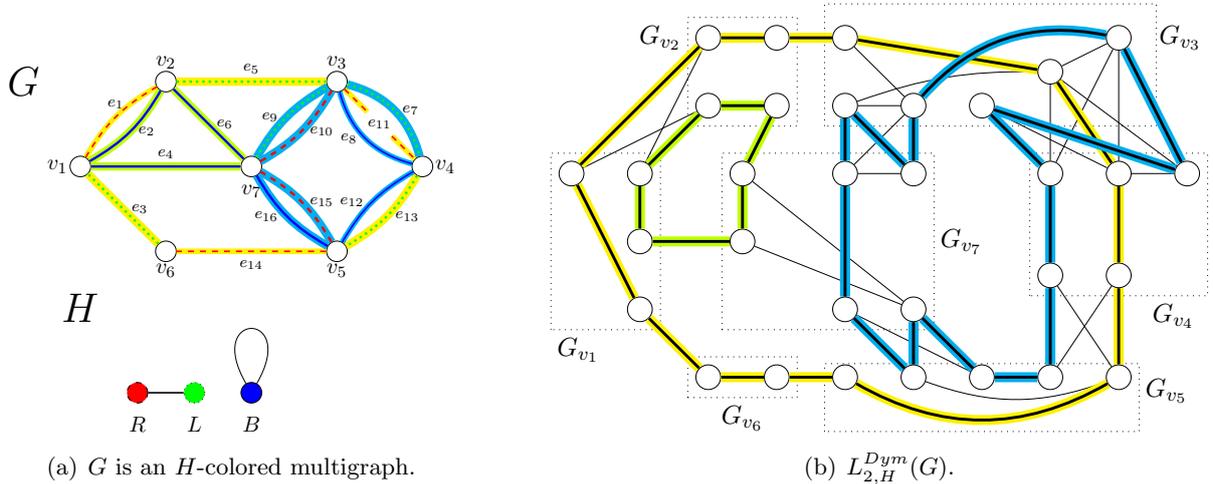

(a) G is an H-colored multigraph.   (b) $L_{2,H}^{Dym}(G)$.

Figure 3.6: In (a) there is a partition of $E(G)$ into three closed dynamic H-trails, each of them is highlighted in a different color. In (b) there is a partition of $V(L_{2,H}^{Dym}(D))$ into three cycles, each of them is obtained by applying the bijection T, which alternate edges between M_J and M, where M is a perfect matching.

Theorem 3.6. *Let G be an H-colored multigraph. Then:*

    a. *If G has a closed Euler dynamic H-trail, then $L_{n,H}^{Dym}(G)$ is Hamiltonian, for every $n \geq 2$.*

    b. *If $L_{n,H}^{Dym}(G)$ is Hamiltonian for some $n \geq 3$, then G has a closed Euler dynamic H-trail.*

  c. *G has a closed Euler dynamic H-trail if and only if $L_{n,H}^{Dym}(G)$ is Hamiltonian, for every* $n \geq 3$.

Proof. Let G be an H-colored multigraph.

  a. It follows from Theorems 3.3 and 3.5 and the definition of $L_{n,H}^{Dym}(G)$.

  b. Suppose that there exist $n \geq 3$ such that $L_{n,H}^{Dym}(G)$ is Hamiltonian. Let C' be a Hamiltonian cycle in $L_{n,H}^{Dym}(G)$.

  By the definition of $L_{n,H}^{Dym}(G)$, every path from $f(e, x)$ to $f(e, y)$ belongs to $V(C')$ in some order for every $e = xy$ in $E(G)$. Hence, we replace the path for the edge $f(e, x)f(e, y)$ in C', and we get a Hamiltonian cycle in $L_{2,H}^{Dym}(G)$, say C.

  Since C is a Hamiltonian cycle, we have that $P = T^{-1}(C)$ is a closed Euler dynamic H-trail in G, by Theorem 3.5.

  c. It follows from a and b. $\square$

  In Theorem 3.6, we cannot change $n \geq 3$ by $n \geq 2$ because there are H-colored multigraphs without closed Euler dynamic H-trails and where $L_{2,H}^{Dym}(G)$ is Hamiltonian, see Figure 3.7.
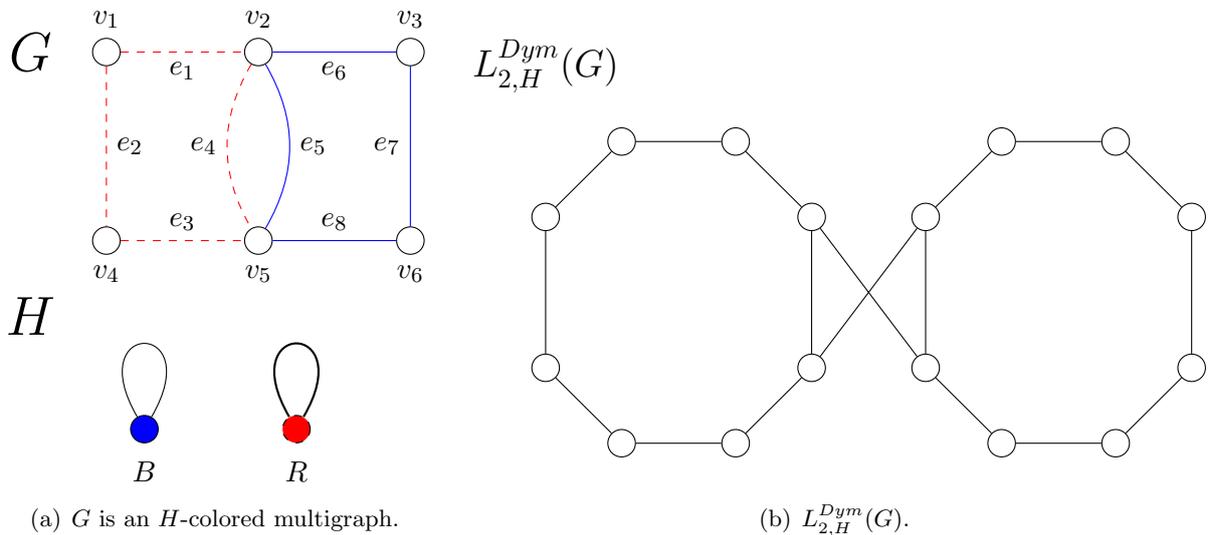

(a) G is an H-colored multigraph.
               (b) $L_{2,H}^{Dym}(G)$.

Figure 3.7: The graph $L_{2,H}^{Dym}(G)$ is Hamiltonian but there is no closed Euler dynamic H-trail in G.

  Recall that E_{xy} is the set of all the edges with end-vertices x and y.

Definition 3.2. *Let G be an H-colored multigraph and M be a matching in $L_{2,H}^{Dym}(G)$. For each x in $V(G)$, we will say that $M_x = \{f(e, x)f(g, x) \in M\}$, i.e., M_x consists of the edges in G_x that are in M. And, for every pair of vertices x and y in $V(G)$, such that $E_{xy} \neq \emptyset$, we will define the following sets: $A_{xy}^M = \{e \in E_{xy} : f(e, x)$ is M_x-saturate and $f(e, y)$ is not M_y-saturate$\}$; $B_{xy}^M = \{e \in E_{xy} : f(e, y)$ is M_y-saturate and $f(e, x)$ is not M_x-saturate$\}$; $C_{xy}^M = \{e \in E_{xy} : f(e, x)$ is M_x-saturate and $f(e, y)$ is M_y-saturate$\}$; and $D_{xy}^M = \{e \in E_{xy} : f(e, x)$ is not M_x-saturate and $f(e, y)$ is not M_y-saturate$\}$.*

  Consider the H-colored multigraph of the Figure 3.2. Note that if in the above definition we take $M = M_J$, then $A_{xy}^{M_J} = B_{xy}^{M_J} = C_{xy}^{M_J} = \emptyset$ and $D_{xy}^{M_J} = E_{xy}$, for every $\{x, y\} \subset V(G)$ such that $E_{xy} \neq \emptyset$, see Figure 3.8(a). However, if we consider $M = M_1$, as shown in Figure 3.8(b), then $A_{v_1v_2}^{M_1} = C_{v_1v_2}^{M_1} = D_{v_1v_2}^{M_1} = \emptyset$ and $B_{v_1v_2}^{M_1} = \{e_1\}$ since $f(e_1, v_2)$ is M_1 saturate with an edge in G_{v_2}; $A_{v_1v_3}^{M_1} = C_{v_1v_2}^{M_1} = \emptyset$, $D_{v_1v_2}^{M_1} = \{e_2\}$ and $B_{v_1v_2}^{M_1} = \{e_3\}$; $A_{v_3v_4}^{M_1} = B_{v_3v_4}^{M_1} = D_{v_3v_4}^{M_1} = \emptyset$ and $C_{v_3v_4}^{M_1} = \{e_4\}$; $A_{v_2v_4}^{M_1} = \{e_6\}$, $B_{v_2v_4}^{M_1} = \{e_5\}$ and $C_{v_2v_4}^{M_1} = D_{v_2v_4}^{M_1} = \emptyset$.
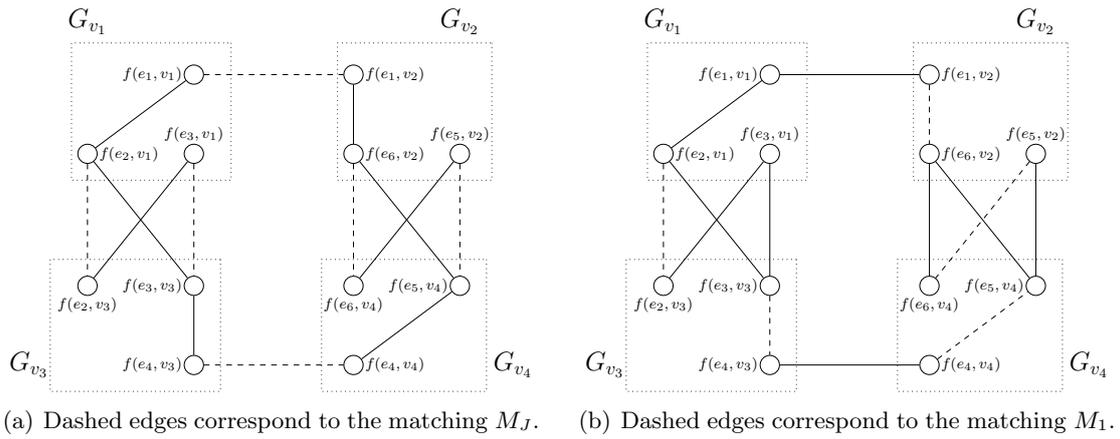
(a) Dashed edges correspond to the matching M_J.   (b) Dashed edges correspond to the matching M_1.

Figure 3.8: Examples for Definition 3.2.

Theorem 3.7. *Let G be an H-colored multigraph. There exists a partition of the edges of G into closed dynamic H-trails, none of them in the form $(x, a_1, \ldots, a_l, y)$, $l \geq 2$, if and only if there exists a non empty matching, say M, in $L_{2,H}^{Dym}(G)$, such that for each set $E_{xy} \neq \emptyset$ one of the following conditions hold: a) $C_{xy}^M = E_{xy}$; or b) $|C_{xy}^M| < |E_{xy}|$ and $1 \leq |A_{xy}^M| = |B_{xy}^M|$.*

Proof. Let G be an H-colored multigraph and M_J be the joint matching of $L_{2,H}^{Dym}(G)$.

Suppose that $\mathfrak{P} = \{P_1, \ldots, P_k\}$ is a partition of the edges of G into closed dynamic H-trails, such that none of them have the form $(x, a_1, \ldots, a_l, y)$, $l \geq 2$.

By Theorem 3.5, there exists a perfect matching in $L_{2,H}^{Dym}(G) \setminus M_J$. Let M be the perfect matching obtained as in the proof of Theorem 3.5, i.e, $M = \bigcup_{i=1}^{k} E(T(P_i)) \setminus M_J$.
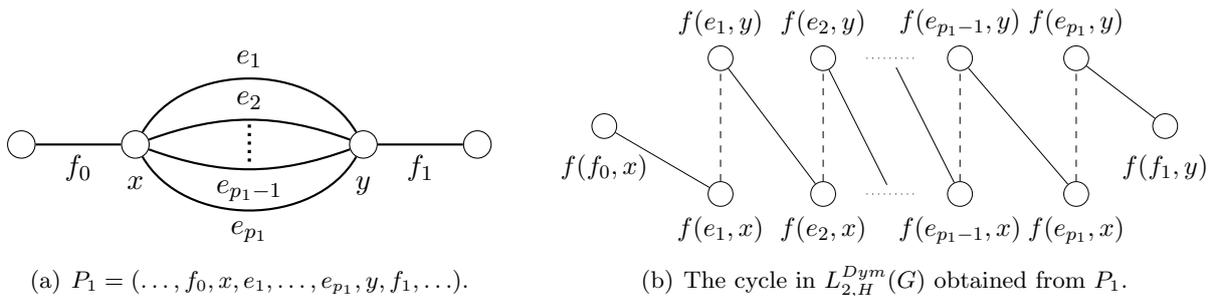
Let x and y be a pair of vertices in G, such that $E_{xy} = \{e_1, \ldots, e_q\}$, with $q \geq 1$. Since $\mathfrak{P}$ is a partition of the edges of G into closed dynamic H-trails, there exists $P_i \in \mathfrak{P}$ such that $E_{xy} \cap E(P_i) \neq \emptyset$, say $P_i = P_1$.

Since P_1 is not of the form $(x, a_1, \ldots, a_l, y)$, $l \geq 2$, and $E_{xy} \cap E(P_1) \neq \emptyset$, we have that $P_1 = (\ldots, f_0, x, e_{t_1}, \ldots, e_{t_{p_1}}, y, f_1, \ldots)$, where $p_1 \leq q$ and $\{f_0, f_1\} \subset E(G)$. (Recall that M_x consists of the edges in G_x that are in M).

If $p_1 = 1$, then $P_1 = (\ldots, f_0, x, e_{t_1}, y, f_1, \ldots)$. Hence, $f(f_0, x)f(e_{t_1}, x)$ and $f(e_{t_1}, y)f(f_1, y)$ are in M.

Therefore, $f(f_0, x)f(e_{t_1}, x) \in M_x$, $f(e_{t_1}, y)f(f_1, y) \in M_y$ and $e_{t_1} \in C_{xy}^M$.

On the other hand, if $p_1 \geq 2$, then $f(f_0, x)f(e_{t_1}, x)$, $f(e_{t_{p_1}}, y)f(f_1, y)$ and $f(e_{t_i}, y)f(e_{t_{i+1}}, x)$ are in M, for every $i \in \{1, \ldots, p_1 - 1\}$, see Figure 3.9. Hence, $e_{t_1} \in A_{xy}^M$, $e_{t_{p_1}} \in B_{xy}^M$ and $e_{t_i} \in D_{xy}^M$, for every $i \in \{2, \ldots, p_1 - 1\}$.

(a) $P_1 = (\ldots, f_0, x, e_1, \ldots, e_{p_1}, y, f_1, \ldots)$.   (b) The cycle in $L_{2,H}^{Dym}(G)$ obtained from P_1.

Figure 3.9: In (b), the continued edges are in M and the dashed edges are in M_J.

If $E_{xy} \setminus \{e_{t_1}, \ldots, e_{t_{p_1}}\} = \emptyset$, then $C_{xy}^M = E_{xy}$ (when $p_1 = 1$ occurs) or $|C_{xy}^M| < |E_{xy}|$ and

$1 \leq |A_{xy}^M| = |B_{xy}^M|$ (when $p_1 \geq 2$). So, suppose that $E_1 = E_{xy} \setminus \{e_{t_1}, \ldots, e_{t_{p_1}}\} \neq \emptyset$. Then there exists $P \in \mathfrak{P}$, such that $E(P) \cap E_1 \neq \emptyset$.

Since P is not of the form $(x, a_1, \ldots, a_l, y)$, $l \geq 2$, and $E_1 \cap E(P) \neq \emptyset$, we have that $P = (\ldots, f_2, x, e_{s_1}, \ldots, e_{s_{p_2}}, y, f_3, \ldots)$, where $\{f_2, f_3\} \subset E(G)$. We will consider two cases.

If $p_2 = 1$, then $P = (\ldots, f_2, x, e_{s_1}, y, f_3, \ldots)$. By the construction of M, we have that the edges $f(f_2, x)f(e_{s_1}, x)$ and $f(e_{s_1}, y)f(f_3, y)$ are in M. Hence, $f(f_2, x)f(e_{s_1}x) \in M_x$ and $f(e_{s_1}, y)f(e_{s_1}, y) \in M_y$, therefore $e_{s_1} \in C_{xy}^M$.

On the other hand, if $p_2 \geq 2$, then $f(f_2, x)f(e_{s_1}, x)$, $f(e_{s_{p_2}}, y)f(f_3, y)$ and $f(e_{s_i}, y)f(e_{s_{i+1}}, x)$ are in M, for every $i \in \{1, \ldots, s_2 - 1\}$. Hence, $e_{s_1} \in A_{xy}^M$, $e_{s_{p_2}} \in B_{xy}^M$ and $e_{s_i} \in D_{xy}^M$, for every $i \in \{2, \ldots, p_2 - 1\}$.

If $E_2 = E_1 \setminus \{e_{s_1}, \ldots, e_{s_{p_2}}\} = \emptyset$, then $C_{xy}^M = E_{xy}$ (when $p_1 = p_2 = 1$) or $|C_{xy}^M| < |E_{xy}|$ and $1 \leq |A_{xy}^M| = |B_{xy}^M|$ (when $p_1 \geq 2$ or $p_2 \geq 2$). Otherwise, there exists $P' \in \mathfrak{P}$, such that $E_2 \cap E(P') \neq \emptyset$ and we can repeat this procedure and after a finite number of steps we obtain that $C_{xy}^M = E_{xy}$ or ($|C_{xy}^M| < |E_{xy}|$ and $1 \leq |A_{xy}^M| = |B_{xy}^M|$).

Now, suppose that there exists M, a matching in $L_{2,H}^{Dym}(G)$, such that for every set $E_{xy} \neq \emptyset$, it holds that: $C_{xy}^M = E_{xy}$ or ($|C_{xy}^M| < |E_{xy}|$ and $1 \leq |A_{xy}^M| = |B_{xy}^M|$).

If M is a perfect matching in $L_{2,H}^{Dym}(G) \setminus M_J$, then there exists a partition of $E(G)$ into closed dynamic H-trails, by Theorem 3.5. So, suppose that M is not a perfect matching in $L_{2,H}^{Dym}(G) \setminus M_J$. Recall that $M_x = M \cap E(G_x)$, for every $x \in V(G)$.

Since $C_{xy}^M = E_{xy} \neq \emptyset$ or $1 \leq |A_{xy}^M|$, it follows that for every $x \in V(G)$, $M_x \neq \emptyset$.

Let $N = \bigcup_{x \in V(G)} M_x$. Notice that N is a matching in $L_{2,H}^{Dym}(G) \setminus M_J$, $A_{xy}^M = A_{xy}^N$, $B_{xy}^M = B_{xy}^N$, $C_{xy}^M = C_{xy}^N$ and $D_{xy}^M = D_{xy}^N$, see Figure 3.10(a). Therefore, for every set $E_{xy} \neq \emptyset$, it holds that: $C_{xy}^N = E_{xy}$ or ($|C_{xy}^N| < |E_{xy}|$ and $1 \leq |A_{xy}^N| = |B_{xy}^N|$).

Let x and y be two different vertices in $V(G)$, such that $E_{xy} \neq \emptyset$.

If $C_{xy}^N = E_{xy}$, then $f(e, x)$ and $f(e, y)$ are N-saturate for every $e \in E_{xy}$. Then, we do not add any edge to N.

On the other hand, if $|C_{xy}^N| < |E_{xy}|$ and $1 \leq |A_{xy}^N| = |B_{xy}^N|$, then we will consider G_{xy}, the subgraph of $L_{2,H}^{Dym}(G)$ induced by the set $\{f(e, x) : e \in E_{xy} \setminus C_{xy}^N\} \cup \{f(e, y) : e \in E_{xy} \setminus C_{xy}^N\}$, see Figure 3.10(b).

Since $|C_{xy}^N| < |E_{xy}|$, $V(G_{xy}) \neq \emptyset$. Let $A_{xy}^N = \{e_1, \ldots, e_t\}$ and $B_{xy}^N = \{f_1, \ldots, f_t\}$. Since $1 \leq |A_{xy}^N| = |B_{xy}^N|$, we have that $t \geq 1$.

If $D_{xy}^N = \emptyset$, then we add the edges $f(e_i, y)f(f_i, x)$ to N for every $i \in \{1, \ldots, t\}$. By the definition of the edges of $L_{2,H}^{Dym}(G)$ and the definition of G_{xy}, we have that $f(e_i, y)f(f_i, x) \in E(G_{xy})$. Moreover, $N \cup \{f(e_i, y)f(f_i, x) : i \in \{1, \ldots, t\}\}$ is still a matching, see Figure 3.11(a).

On the other hand, when $D_{xy}^N \neq \emptyset$, let $D_{xy}^N = \{g_1, \ldots, g_s\}$. We add the edges $f(e_1, y)f(g_1, x)$, $f(f_1, x)f(g_s, y)$, $f(e_i, y)f(f_i, x)$ and $f(g_j, y)f(g_{j+1}, x)$ to N for every $i \in \{2, \ldots, t\}$ and $j \in \{1, \ldots, s - 1\}$, see Figure 3.11(b).

By the definition of the edges of $L_{2,H}^{Dym}(G)$ and the definition of G_{xy}, we have that all the edges we add to N are in $E(G_{xy})$ and N is still a matching.

Thus, we add edges to N in such a way every vertex in G_{xy} is N-saturate.

We repeat the same process for x and y in $V(G)$ such that $E_{xy} \neq \emptyset$.

Therefore, N is a perfect matching in $L_{2,H}^{Dym}(G) \setminus M_J$ and there exists a partition of $E(G)$ into closed dynamic H-trails, by Theorem 3.5 and, by the construction of N, none of them is of the form $(x, a_1, \ldots, a_l, y)$, with $l \geq 2$. $\qquad \square$

The following result shows a condition on the graph G_x that guarantees the existence of closed Euler dynamic H-trail.

Theorem 3.8. *Let G be a connected H-colored multigraph, such that G_u is a complete k_u-partite graph for every $u \in V(G)$ and for some k_u in $\mathbb{N} \setminus \{1\}$. Then G has a closed Euler dynamic*
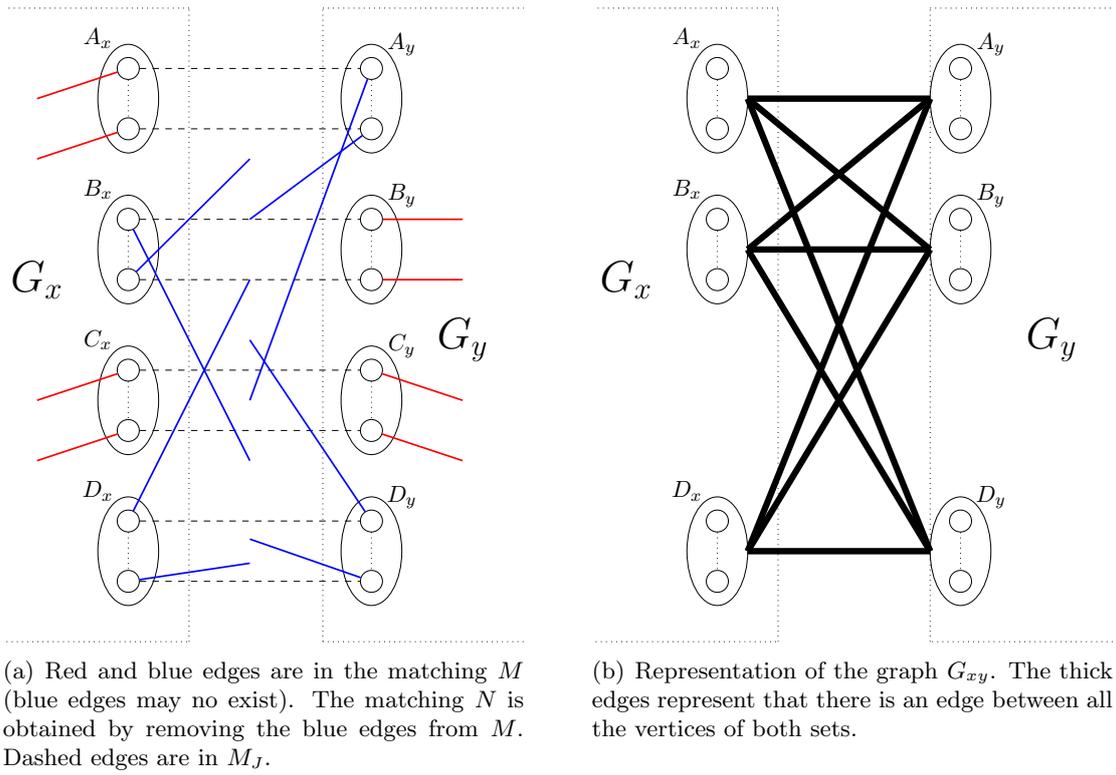
(a) Red and blue edges are in the matching M (blue edges may no exist). The matching N is obtained by removing the blue edges from M. Dashed edges are in M_J.

(b) Representation of the graph G_{xy}. The thick edges represent that there is an edge between all the vertices of both sets.

Figure 3.10

(a) If $D_{xy}^N = \emptyset$.
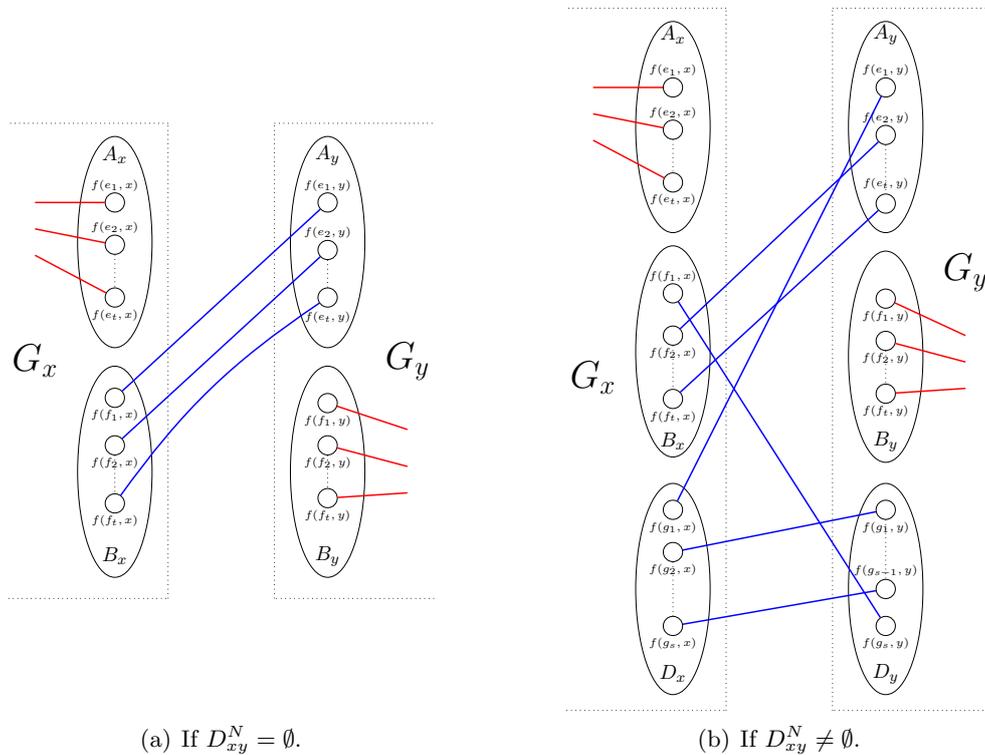
(b) If $D_{xy}^N \neq \emptyset$.

Figure 3.11: The blue edges are added to the matching N.

H-trail if and only if there exists a matching M in $L_{2,H}^{Dym}(G)$, such that for every set $E_{xy} \neq \emptyset$ one of the following conditions holds: a) $C_{xy}^M = E_{xy}$; or b) $|C_{xy}^M| < |E_{xy}|$ and $1 \leq |A_{xy}^M| = |B_{xy}^M|$.

Proof. Let G be a connected H-colored multigraph, such that G_u is a complete k_u-partite graph for every $u \in V(G)$ and for some k_u in $\mathbb{N} \setminus \{1\}$.

Suppose that G has a closed Euler dynamic H-trail, say P.

Case 1. P is not of the form $(x, e_1, \ldots, e_k, y)$, with $k \geq 2$.

In this case, by Theorem 3.7, there exists a matching, say M, in $L_{2,H}^{Dym}(G)$, such that for every set $E_{xy} \neq \emptyset$ one of the following conditions holds: a) $C_{xy}^M = E_{xy}$; or b) $|C_{xy}^M| < |E_{xy}|$ and $1 \leq |A_{xy}^M| = |B_{xy}^M|$.

Case 2. $P = (x, e_1, \ldots, e_k, y)$, with $k \geq 2$.

In this case, since G_x is a complete k_x-partite graph, for some $k_x \geq 2$, it follows that there exists two edges e and f in $E(G) = E(P)$, such that $c(e)c(f) \in E(H)$. Without loss of generality, suppose that $c(e_1)c(e_2) \in E(H)$.

If $k = 2$, then $P' = (x, e_1, y, e_2, x)$ is a closed Euler dynamic H-trail. Otherwise $k \geq 3$, since G_x is a complete k_x-partite graph, we have that $c(e_1)c(e_k) \in E(H)$; or $c(e_2)c(e_k) \in E(H)$.

Hence, $P' = (x, e_1, y, e_2, \ldots, e_k, x)$ or $P' = (x, e_1, e_3, \ldots, e_k, y, e_2, x)$ is a closed Euler dynamic H-trail. By Theorem 3.7, there exists a matching M in $L_{2,H}^{Dym}(G)$, such that for every set $E_{xy} \neq \emptyset$, $C_{xy}^M = E_{xy}$ or $(|C_{xy}^M| < |E_{xy}|$ and $1 \leq |A_{xy}^M| = |B_{xy}^M|)$.

Conversely, suppose that there exists a matching M in $L_{2,H}^{Dym}(G)$, such that for every set $E_{xy} \neq \emptyset$ one of the following conditions holds: a) $C_{xy}^M = E_{xy}$; or b)$|C_{xy}^M| < |E_{xy}|$ and $1 \leq |A_{xy}^M| = |B_{xy}^M|$.

It follows from Theorem 3.7 that there exists a partition of $E(G)$ into closed dynamic H-trails, none of them in the form $(x, e_1, \ldots, e_l, y)$, with $l \geq 2$, say $\mathcal{P} = \{P_1, \ldots, P_k\}$.

If $k = 1$, then $E(P_1) = E(G)$ and P_1 is a closed Euler dynamic H-trail. Otherwise, since G is connected, it follows that there exists $P_i \in \mathcal{P} \setminus \{P_1\}$, say P_2, such that $V(P_1) \cap V(P_2) \neq \emptyset$. Since P_1 and P_2 are closed, we can suppose that $P_1 = (x_0, e_1, \ldots, e_m, x_0)$, where $e_1 = x_0 x_1$ and $e_m = x_{k_1} x_0$, and $P_2 = (y_0, f_1, \ldots, f_n, y_0)$, where $f_1 = y_0 y_1$ and $f_n = y_{k_2} y_0$, such that $x_0 = y_0$. By the definition of dynamic H-trail, we have that $c(e_1)c(e_m)$ and $c(f_1)c(f_n)$ are in $E(H)$. Furthermore, e_1, f_1, e_m and f_n are incident with $x_0 = y_0$. Hence, $e_1 e_m$ and $f_1 f_n$ are in $E(G_{x_0})$.

We will consider the following: if $\{e_1, f_1\} \subseteq E_j^{x_0}$ or $\{e_m, f_n\} \subseteq E_j^{x_0}$, for some $j \in \{1, \ldots, k_{x_0}\}$, it follows that $e_1 f_n$ and $e_m f_1$ are in $E(G_{x_0})$, because G_{x_0} is a complete k_{x_0}-partite graph. In other case, $e_1 f_1$ and $e_m f_n$ are in $E(G_{x_0})$, see Figure 3.12.

By the definition of G_{x_0}, we have that $c(e_1)c(f_n)$ and $c(e_m)c(f_1)$ are edges of H or $c(e_1)c(f_1)$ and $c(e_m)c(f_n)$ are edges of H.

If $c(e_1)c(f_n)$ and $c(e_m)c(f_1)$ are edges of H, we have that $Q_1 = P_1 \cup P_2$ is a closed dynamic H-trail. And, if $c(e_1)c(f_1)$ and $c(e_m)c(f_n)$ are edges of H, we have that $Q_1 = P_1 \cup P_2^{-1}$ is a closed dynamic H-trail.

If $E(Q_1) = E(G)$, then Q_1 is a closed Euler dynamic H-trail. Otherwise, since G is connected, it follows that there exists $P_i \in \mathcal{P} \setminus \{P_1, P_2\}$, say P_3, such that $V(Q_1) \cap V(P_3) \neq \emptyset$. Since Q_1 and P_3 are closed, we can suppose that $P_1 = (v_1, a_1, \ldots, a_s, v_0)$, where $a_1 = v_0 v_1$ and $a_s = v_{k_3} v_0$, and $P_3 = (u_0, g_1, \ldots, g_t, u_0)$, where $g_1 = u_0 u_1$ and $g_t = u_{k_3} u_0$, such that $v_0 = u_0$. By the definition of dynamic H-trail, we have that $c(a_1)c(a_s)$ and $c(g_1)c(g_t)$ are in $E(H)$. Furthermore, a_1, g_1, a_s and g_t are incident with $u_0 = v_0$. Hence, $a_1 a_s$ and $g_1 g_t$ are in $E(G_{v_0})$.

We will consider the following: if $\{a_1, g_1\} \subseteq E_j^{v_0}$ or $\{a_s, g_t\} \subseteq E_j^{v_0}$, for some $j \in \{1, \ldots, k_{v_0}\}$, it follows that $a_1 g_t$ and $a_s g_1$ are in $E(G_{v_0})$, because G_{v_0} is a complete k_{v_0}-partite graph. In other case, $a_1 g_1$ and $a_s g_t$ are in $E(G_{v_0})$.

By the definition of G_{v_0}, we have that $c(a_1)c(g_t)$ and $c(a_s)c(g_1)$ are edges of H or $c(a_1)c(g_1)$ and $c(a_s)c(g_t)$ are edges of H.
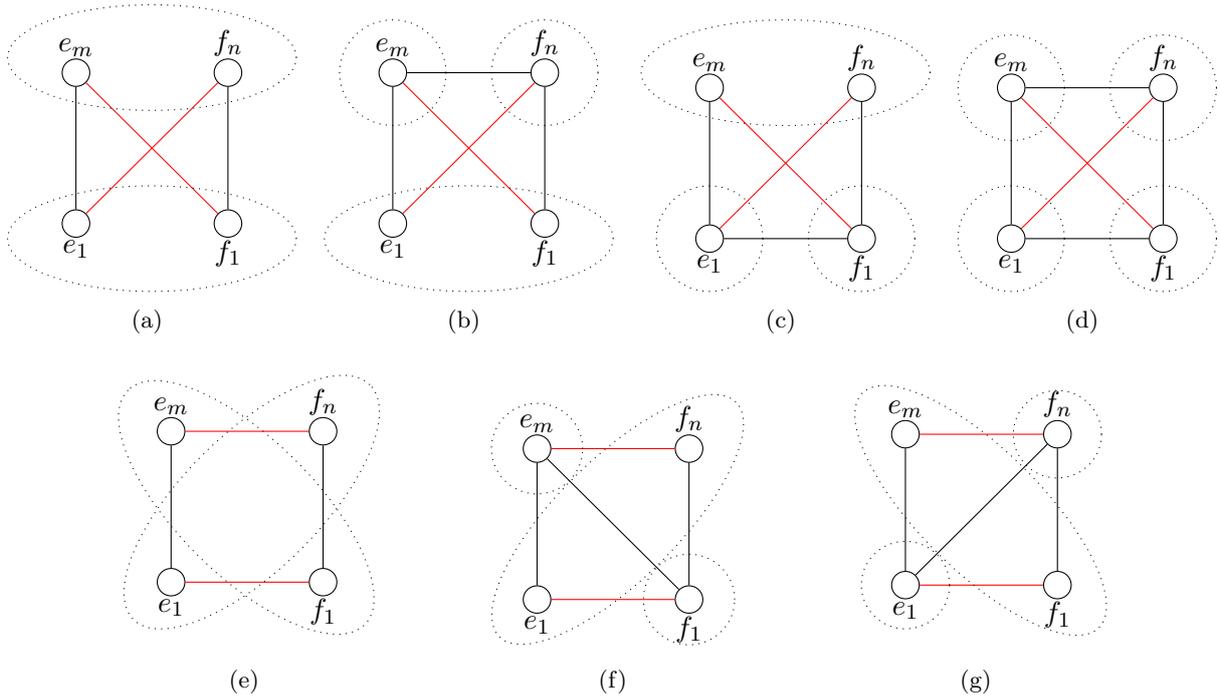
Figure 3.12: Possible adjacencies between edges e_1, e_m, f_1 and f_n in the auxiliary graph G_{x_0}.

If $c(a_1)c(g_t)$ and $c(a_s)c(g_1)$ are edges of H, we have that $Q_2 = Q_1 \cup P_3$ is a closed dynamic H-trail. And, if $c(a_1)c(g_1)$ and $c(a_s)c(g_t)$ are edges of H, we have that $Q_2 = Q_1 \cup P_3^{-1}$ is a closed dynamic H-trail.

If $E(Q_2) = E(G)$, then Q_2 is a closed Euler dynamic H-trail. Otherwise, we repeat the procedure until we get the desired closed Euler dynamic H-trail. $\qquad\square$

In Theorem 3.8 we cannot change the hypothesis "G_u is a complete k_u-partite for every $u \in V(G)$" to "G_u has a complete k_u-partite spanning subgraph", as Figure 3.13 shows.

Corollary 3.9. *Let G be an H-colored multigraph. Then G_x has a perfect matching, for every $x \in V(G)$, if and only if there exists a partition of $E(G)$ into closed H-trails.*

Proof. Let G be an H-colored multigraph and M_J be the joint matching of $L_{2,H}^{Dym}(G)$.

Suppose that G_x has a perfect matching, for every $x \in V(G)$, say M_x. Let $M = \bigcup_{x \in V(G)} M_x$.

It follows from the definition of M_J that $M \cap M_J = \emptyset$. Hence, G has a partition of its edges into closed dynamic H-trails, by Theorem 3.5.

Let $\mathcal{P}$ be the partition that is obtained as in the proof of Theorem 3.5. It follows from the construction of each dynamic H-trail in $\mathcal{P}$ that there is no lane change. So, $\mathcal{P}$ is a partition of $E(G)$ into closed H-trails.

Now, suppose that there exists a partition of $E(G)$ into closed H-trails, say $\mathcal{P}$. By Theorem 3.5, there exists a perfect matching in $L_{2,H}^{Dym}(G) \setminus M_J$. Let M be the perfect matching that is obtained as in the proof of Theorem 3.5.

Let $f(e, x) \in V(L_2^H(G))$. Then, there exists $P \in \mathcal{P}$ such that $e \in E(P)$. Since P is an H-trail and by the construction of M, it follows that $P = (u, g, x, e, v, \dots)$ and $f(g, x)f(e, x) \in M$. Moreover, $f(g, x)f(e, x) \in M \cap E(G_x)$.

Therefore, $M \cap E(G_x)$ is a perfect matching of G_x. $\qquad\square$

The following classical theorem will be useful.

Theorem 3.10 (Tutte [53])**.** *G has a perfect matching if and only if $o(G \setminus S) \leq |S|$ for all proper subset S of $V(G)$.*

(a) G is an H-colored multigraph.
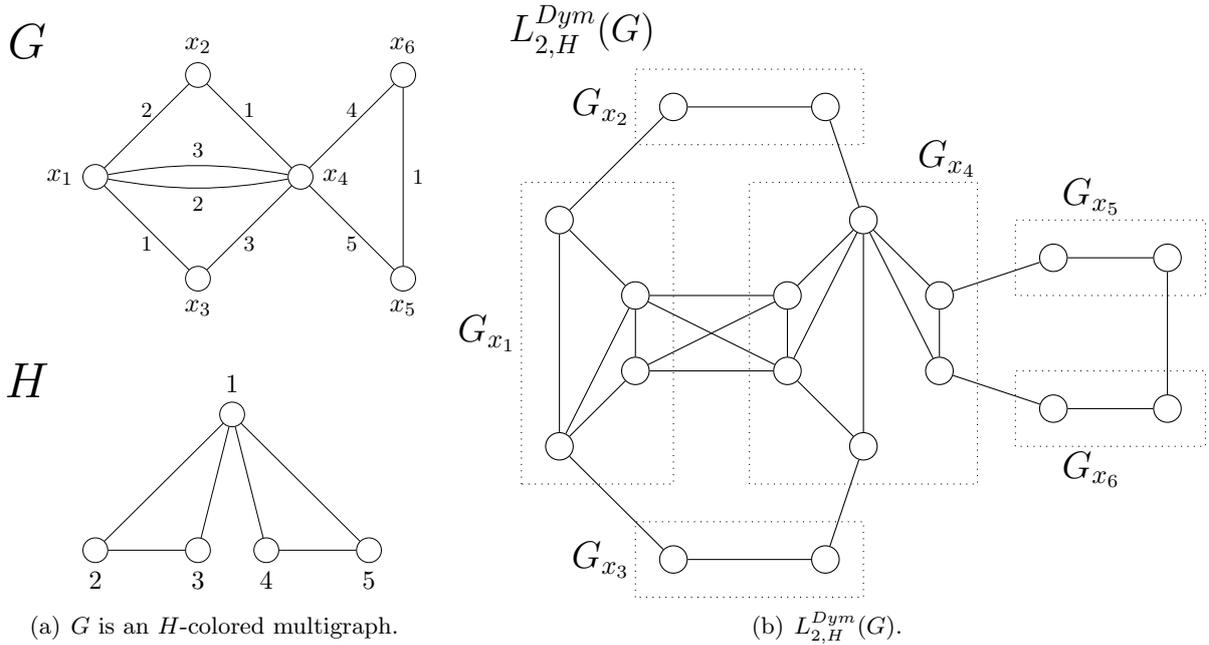
(b) $L_{2,H}^{Dym}(G)$.

Figure 3.13: G_u is a complete k_u-partite graph, for $u \in V(G) \setminus \{x_4\}$, G_{x_1} is a 3-partite graph, $G_{x_2}, G_{x_3}, G_{x_5}$ and G_{x_6} are bipartite graphs and G_{x_4} has a bipartite spanning subgraph but there is no closed Euler dynamic H-trail in G.

Notice that if H is a complete graph without loops and G is an H-colored multigraph, then P is a properly colored closed trail if and only if P is a closed H-trail.

Corollary 3.11 (Kotzig [37])**.** *Let G be a c-edge-colored Eulerian multigraph. Then G has a properly colored closed Euler trail if and only if $\delta_i(x) \leq \sum_{j \neq i} \delta_j(x)$, where $\delta_i(x)$ is the number of edges with color i incident with x, for each vertex x of G.*

Proof. Let G be an H-colored multigraph, where H is a complete graph without loops and $|V(H)| = c$.

Let G_u, with $u \in V(G)$. It follows from the definition of G_u and H is a complete graph that G_u is a complete k_u-paritite graph with independent sets $E_i^u = \{e \in E(G) : e = ux$ for some $x \in V(G)$ and $c(e) = i\}$, for every $i \in \{j : \delta_j(u) > 0\}$. Moreover, $|E_i^u| = \delta_i(u)$.

Suppose that G has a closed Euler H-trail. By Corollary 4.8, G_u has a perfect matching, for every $u \in V(G)$.

Hence, $\delta_i(u) = |E_i^u| = o(G \setminus S) \leq |S| = \sum_{j \neq i} |E_j^u| = \sum_{j \neq i} \delta_i(u)$, where $S = V(G_u) \setminus E_i^u$, by Theorem 3.10.

Conversely, suppose that $|E_i^u| = \delta_i(x) \leq \sum_{j \neq i} \delta_j(x) = \sum_{j \neq i} |E_j^u|$, for each vertex x of G.

Claim 1. G_u has a perfect matching.

Let S be a proper subset of $V(G_u)$. Consider $G_u \setminus S$.

Case 1. $G_u \setminus S$ is connected.

Then $o(G_u \setminus S) \leq 1 \leq |S|$.

Case 2. $G_u \setminus S$ is disconnected.

Since G_u is a complete k_u-partite graph, $V(G_u \setminus S) \subseteq E_i^u$, for some $i \in \{1, \dots, c\}$. Hence $o(G_u \setminus S) = |V(G_u \setminus S)| \leq |E_i^u| \leq \sum_{j \neq i} |E_j^u| \leq |S|$.

Therefore, it follows from Theorem 3.10 that G_u has a perfect matching.

Let $M = \bigcup_{u \in V(G)} M_u$, where M_u is a perfect matching in G_u. So, M is a perfect matching in $L_{2,H}^{Dym}(G)$ and $C_{xy}^M = E_{xy}$, for every $E_{xy} \neq \emptyset$.

It follows from Theorem 3.8 and Corollary 3.9 that G has a closed Euler H-trail. $\qquad\square$

Corollary 3.12 ([26])**.** *Let H be a graph possibly with loops and G be an H-colored multigraph without loops. Suppose that G is Eulerian and G_u is a complete k_u-partite graph, for every u in $V(G)$ and for some k_u in $\mathbb{N}$. Then G has a closed Euler H-trail if and only if $|C_i^u| \leq \sum_{j \neq i} |C_j^u|$ for every u in $V(G)$, where $\{C_1^u, ..., C_{k_u}^u\}$ is the partition of $V(G_u)$ into independent sets.*

Proof. First suppose that G has a closed Euler H-trail. By Corollary 3.9, G_x has a perfect matching, say M_x, for every $x \in V(G)$. Consider C_i^x a part of the partition of $V(G_x)$ into independent sets. Since M_x is a perfect matching in G_x, we have that for every $e \in C_i^x$ there is a vertex f_e in $V(G_x) \setminus C_i^x$ such that $ef_e \in M_x$ and $|\{f_e : e \in C_i^x\}| = |C_i^x|$. Therefore, $|C_i^u| = |\{f_e : e \in C_i^x\}| \leq \sum_{j \neq i} |C_j^u|$, because $\{f_e : e \in C_i^x\} \subseteq \bigcup_{j \neq i} C_j^x$ and $C_j^x \cap C_k^x = \emptyset$, for every $j \neq k$.

Now suppose that $|C_i^u| \leq \sum_{j \neq i} |C_j^u|$ for every u in $V(G)$, where $\{C_1^u, ..., C_{k_u}^u\}$ is the partition of $V(G_u)$ into independent sets. It follows from Theorem 3.10 that G_x has a perfect matching, and by Theorem 3.8 and Corollary 3.9 that G has a closed Euler H-trail. □

Chapter 4

Dynamic H-trails in digraphs

In this chapter we will study the existence of closed Euler dynamic H-trails in H-colored multidigraphs. In order to obtain our results we define the auxiliary directed graphs D_u and $L_{n,H}^{Dym}(D)$. Then we will prove that there exists a bijection between the set of closed Euler dynamic H-trails in D and the set of directed cycles in $L_{2,H}^{Dym}(D)$. As a consequence, D has a closed Euler dynamic H-trail if and only if $L_{n,H}^{Dym}(D)$ is Hamiltonian for every $n \geq 2$.

Since dynamic H-trails generalized the H-trails, in Section 4.3, we study the existence of closed Euler H-trails using the results obtained in Section 4.2. We will show conditions on an H-colored directed graph that guarantee the existence of closed Euler H-trail. Finally, we extend Theorem 2.8 for c-arc-colored digraphs with $c \geq 3$ under an additional condition.

We finish the chapter by showing some H such that for any H-coloring of D, a multidigraph without isolated vertices, satisfies that UD_u, the underlying graph of the digraph D_u, is union of complete bipartite graphs.

4.1   Dynamic H-trails and the auxiliary digraph $L_{n,H}^{Dym}(D)$

Let H be a digraph possibly with loops and G be an H-colored multidigraph. We say that $W = (v_0, e_0^1, \ldots, e_0^{k_0}, v_1, e_1^1, \ldots, e_1^{k_1}, v_2, \ldots, v_{n-1}, e_{n-1}^1, \ldots, e_{n-1}^{k_{n-1}}, v_n)$ is a **dynamic H-trail** if W does not repeat arcs and $(c(e_i^{k_i}), c(e_{i+1}^1))$ is an arc in H for every $i \in \{0, \ldots, n-2\}$, where for $k_i \geq 1$ and $e_i^j = (v_i, v_{i+1})$ for every $j \in \{1, \ldots, k_i\}$ and $i \in \{0, \ldots, n-1\}$. The **dynamic length** of a dynamic H-trail is $\sum_{i=0}^{k_{n-1}} k_i$. We say that a dynamic H-trail, W, is a **Euler dynamic H-trail** if $A(D) = A(W)$. In Figure 4.1, $W_1 = (v_1, e_1, v_2, e_5, e_6, v_7, e_{19}, v_6, e_{14}, v_5, e_{11}, e_{12}, v_4, e_9, v_3, e_7, v_2)$ is a dynamic H-walk which does not repeat arcs, i.e., W_1 is a dynamic H-trail and the dynamic length of W_1 is nine.

We will say that a dynamic H-trail is **closed** whenever a) $v_0 = v_n$ and $(c(e_{n-1}^{k_{n-1}}), c(e_0^1))$ is an arc in H; or b) $v_1 = v_n$ and $e_{n-1}^{k_{n-1}}$ and e_0^1 are parallel in D. Notice that if W is a closed dynamic H-trail that satisfies condition b), then W can be rewritten as $W = (v_1, e_1^1, \ldots, v_{n-1} = v_0, e_{n-1}^1, \ldots, e_{n-1}^{k_{n-1}}, e_0^1, \ldots, e_0^{k_0}, v_n = v_1)$, and W satisfies condition a) (unless $n = 1$, i.e., W is of the form $(x, e_0, \ldots, e_k, y)$, where $k \geq 1$).

Throughout this chapter we will use the auxiliary digraphs D_u and $L_{n,H}^{Dym}(D)$, which are defined below.

Definition 4.1. *Let D be a digraph and u be a vertex of D. The digraph $\mathbf{D_u}$ is defined by the digraph such that $V(D_u) = \{f(e, u) \mid e \in A(D)$ and e is incident with $u\}$, and two different vertices $f(e, u)$ and $f(g, u)$ are joint by only one arc from $f(e, u)$ to $f(g, u)$ in D_u if $e = (x, u)$ and $g = (u, y)$ for some x and y in $V(D)$ and $(c(e), c(g)) \in A(H)$.*
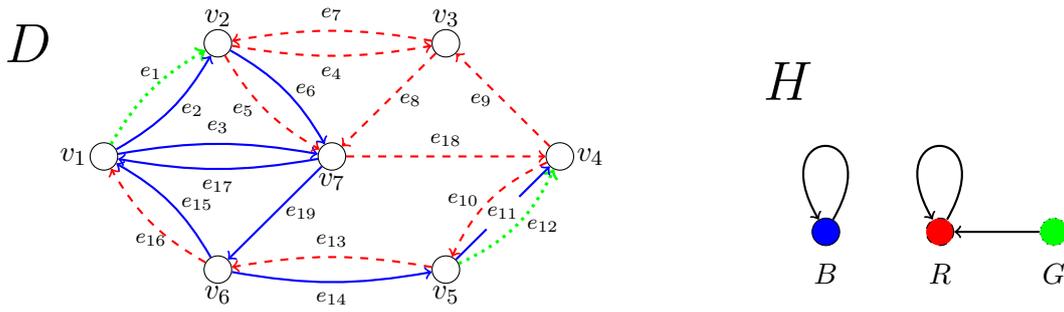
Figure 4.1: The sequence $P = (v_1, e_1, v_2, e_4, v_3, e_8, v_7, e_{18}, v_4, e_{10}, v_5, e_{13}, v_6, e_{16}, e_{15}, v_1, e_3, v_7, e_{19},$ $v_6, e_{14}, v_5, e_{11}, e_{12}, v_4, e_9, v_3, e_7, v_2, e_5, e_6, v_7, e_{17}, v_1, e_2, v_2)$ is a closed Euler dynamic *H*-trail in *D* and there is no closed Euler *H*-trail in *D*.

Observation 2. D_u *is a bipartite digraph with bipartition* (X, Y), *where* $X = \{f(e, u) \in V(D_u) : e = (x, u) \text{ for some } x \in V(D)\}$ *and* $Y = \{f(e, u) \in V(D_u) : e = (u, y) \text{ for some } y \in V(D)\}$. *Moreover, if* $(f(e, u), f(g, u)) \in A(D_u)$, *then* $f(e, u) \in X$ *and* $f(g, u) \in Y$.

Definition 4.2. *Let* D *be an* H-*colored digraph with* $|A(D)| = q$. *For* $n \geq 2$, $L_{n,H}^{Dym}(D)$ *is the digraph with* nq *vertices, obtained as follows: for each arc* $e = (u, v)$ *of* D, *we take two vertices* $f(e, u)$ *and* $f(e, v)$ *in* $L_{n,H}^{Dym}(D)$, *and add a directed path from* $f(e, u)$ *to* $f(e, v)$ *with* $n - 2$ *new intermediate vertices. The rest of the edges of* $L_{n,H}^{Dym}(D)$ *are defined as follows: a)* $f(e, u)$ *and* $f(g, u)$ *are joint by only one arc from* $f(e, u)$ *to* $f(g, u)$ *if* $e = (x, u)$ *and* $g = (u, y)$ *for some* x *and* y *in* $V(D)$ *and* $(c(e), c(g)) \in A(H)$; *b)* $f(e, v)$ *and* $f(g, u)$ *are joint by only one arc from* $f(e, u)$ *to* $f(g, v)$ *if* $e \neq g$, e *and* g *are parallel in* D, *where both arcs have head* v *and tail* u.

Observation 3. *Let* D *be an* H-*colored multidigraph.*

1. $L_{n,H}^{Dym}(D)$ *is a digraph with neither parallel arcs nor symmetry arcs.*

2. *For every* $e = (u, v) \in A(D)$, *we have that* $d^+(f(e, u)) = d^-(f(e, v)) = 1$ *in* $L_{n,H}^{Dym}(D)$ *for each* $n \geq 2$. *Moreover, when* $n = 2$, *we have that* $N^+(f(e, u)) = \{f(e, v)\}$ *and* $N^-(f(e, v)) = \{f(e, u)\}$.

3. *The underlying graph of* $L_{n,H}^{Dym}(D)$ *will be denoted by* $UL_{n,H}^{Dym}(D)$. *It follows from the definition of* $L_{2,H}^{Dym}(D)$ *that* $M_J = \{f(e, x)f(e, y) \in E(UL_{2,H}^{Dym}(D)) : e = (x, y) \in A(D)\}$ *is a perfect matching of* $UL_{2,H}^{Dym}(D)$, *that we will call the **joint matching of** $UL_{2,H}^{Dym}(D)$.*
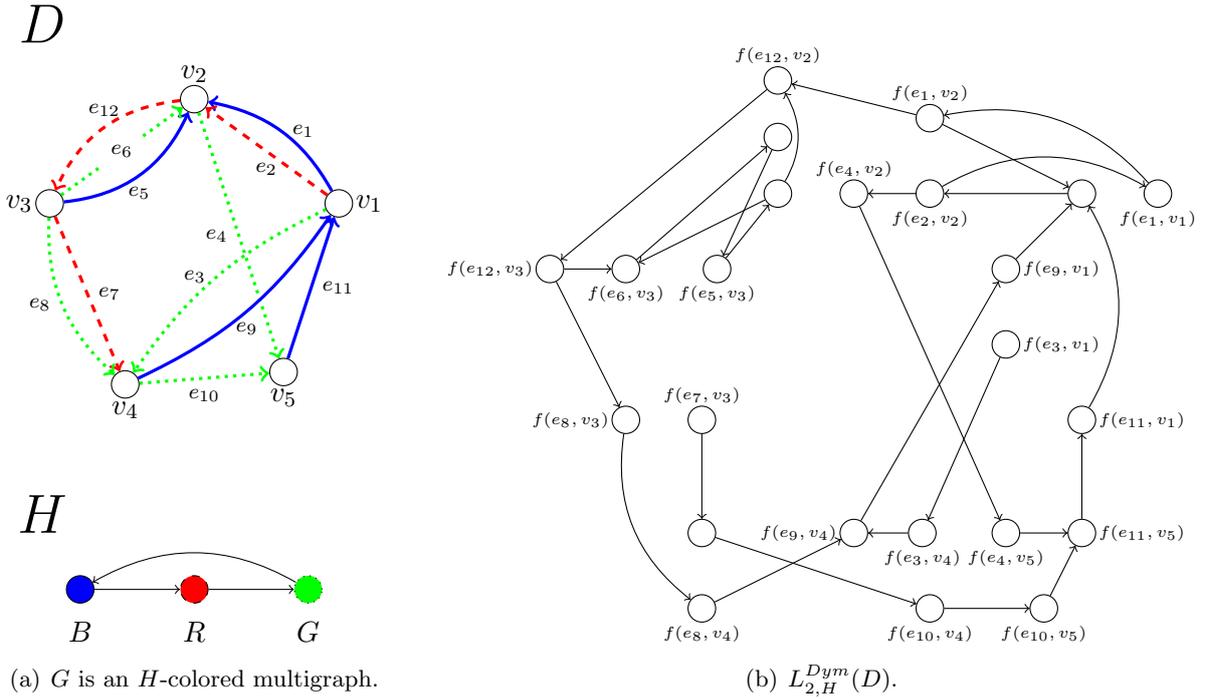
Figure 4.2 shows an example of an H-colored digraph and its auxiliary digraph $L_{2,H}^{Dym}(D)$.

Notice that the digraph $L_{n,H}^{Dym}(D)$ can be constructed as follows: First take the disjoint union of D_x, for every $x \in V(D)$. Then, for every $e = (u, v) \in A(D)$, add a directed path from $f(e, u)$ to $f(e, v)$ with $n - 2$ new intermediate vertices. Finally, add the arcs $(f(e, v), f(g, u))$ and $(f(g, v), f(e, u))$ if and only if $e = (u, v)$ and $g = (u, v)$ in D. The construction of $L_{3,H}^{Dym}(D)$ is illustrated in Figure 4.3.

4.2   Euler dynamic *H*-trails in digraphs

Theorem 4.1. *Let* D *be an* H-*colored digraph. Then there is a bijection between the set of closed dynamic* H-*trails in* D *and the set of directed cycles in* $L_{2,H}^{Dym}(D)$.

Proof. Let D be an H-colored digraph. We denote by $\mathcal{P}$ the set of closed dynamic H-trails in D, and by $\mathcal{C}$ the set of directed cycles in $L_{2,H}^{Dym}(D)$.

D

H

(a) G is an H-colored multigraph.

(b) $L_{2,H}^{Dym}(D)$.

Figure 4.2: Example of the auxiliary digraph $L_{2,H}^{Dym}(D)$ of an H-colored digraph.

Let $T : \mathcal{P} \to \mathcal{C}$ be a function, defined by $T(P) = C$, where $P = (x_0, e_1, \ldots, e_{p_0}, x_1, e_{p_0+1}, \ldots,$ $x_{n-1}, e_{p_0+\ldots+p_{n-2}+1}, \ldots, e_{p_0+\ldots+p_{n-1}}, x_n)$ and $C = (f(e_1, x_0), f(e_1, x_1), \ldots, f(e_{p_0}, x_0), f(e_{p_0}, x_1),$ $f(e_{p_0+1}, x_1), \ldots, f(e_{p_0+\ldots+p_{n-1}}, x_n), f(e_1, x_0))$, see Figure 4.4.

Claim 1. T is well-defined.

First we will prove that $T(P) = C \in \mathcal{C}$.

It follows from the definition of $L_{2,H}^{Dym}(D)$ that $(f(e, x_i), f(e, x_{i+1}))$ is an arc of $L_{2,H}^{Dym}(D)$, for every $e = (x_i, x_{i+1}) \in A(P)$.

Let e_i and e_{i+1} be consecutive arcs in P (if $i = p_0 + \ldots, p_{n-1}$, then $e_{i+1} = e_1$). If e_i and e_{i+1} are parallel, such that x_j and x_{j+1} are the tail and the head of both arcs, then $f(e_i, x_{j+1})$ is adjacent to $f(e_{i+1}, x_j)$ in $L_{2,H}^{Dym}(D)$, by the definition of $L_{2,H}^{Dym}(D)$. Otherwise, e_i and e_{i+1} are incident with x_j, for some x_j in $V(P)$. Since P is a closed dynamic H-trail, it follows that $(c(e_i), c(e_{i+1})) \in A(H)$ and $(f(e_i, x_j), f(e_{i+1}, x_j)) \in A(L_{2,H}^{Dym}(D))$. Hence, C is a walk in $L_{2,H}^{Dym}(D)$. Since P is a closed dynamic H-trail, C does not repeat vertex and C is a cycle, i.e., $T(P) = C \in \mathcal{C}$.

Let $P_1 = (x_0, e_0, \ldots, e_{p_0}, x_1, e_{p_0+1}, \ldots, e_{p_0+p_1}, x_2, \ldots, x_{n-1}, e_{p_0+\ldots+p_{n-2}+1}, \ldots, e_{p_0+\ldots+p_{n-1}},$ $x_n)$ and $P_2 = (y_0, f_0, \ldots, f_{q_0}, y_1, f_{q_0+1}, \ldots, f_{q_0+q_1}, y_2, \ldots, y_{n-1}, f_{q_0+\ldots+q_{n-2}+1}, \ldots, f_{q_0+\ldots+q_{m-1}},$ $y_m)$ in $\mathcal{P}$ such that $P_1 = P_2$.

Since $P_1 = P_2$, $A(P_1) = A(P_2)$ and there exists $k \in \mathbb{N}$ such that $e_i = f_{i+k(mod \, p_0+\ldots+p_{n-1})}$ (because the arcs are traversed in the same order but the first arc may not be the same). It follows from the definition of T that $V(T(P_1)) = V(T(P_2))$ and the order of the vertices are the same. Thus, we have that $T(P_1) = T(P_2)$. Therefore, T is well-defined.
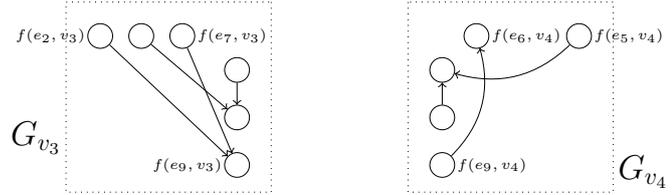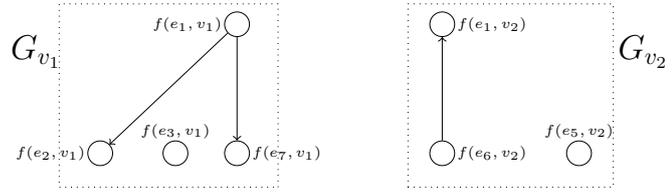
Claim 2. T is injective.

Let P_1 and P_2 in $\mathcal{P}$ such that $P_1 \neq P_2$.

If $A(P_1) \neq A(P_2)$, then $V(T(P_1)) \neq V(T(P_2))$ and $T(P_1) \neq T(P_2)$. Otherwise, since $P_1 \neq P_2$, there exists $\{e_j, e_k\} \subseteq A(P_1) = A(P_2)$ such that $e_j = (x_j, y_j)$ is the arc preceding $e_k = (x_k, y_k)$ at P_1 and e_j is not the arc preceding e_k at P_2. Hence, $(f(e_j, y_j), f(e_k, x_k)) \in A(T(P_1))$ and $(f(e_j, y_j), f(e_k, x_k)) \notin A(T(P_2))$. Therefore, $T(P_1) \neq T(P_2)$ and T is injective.
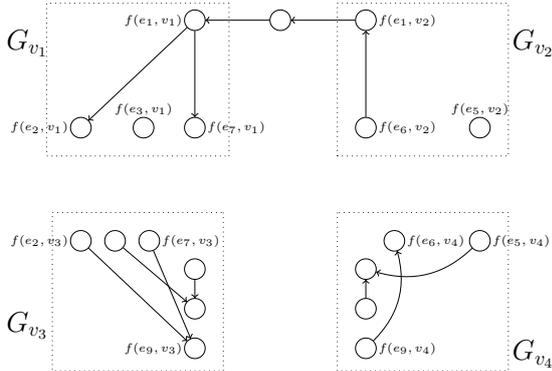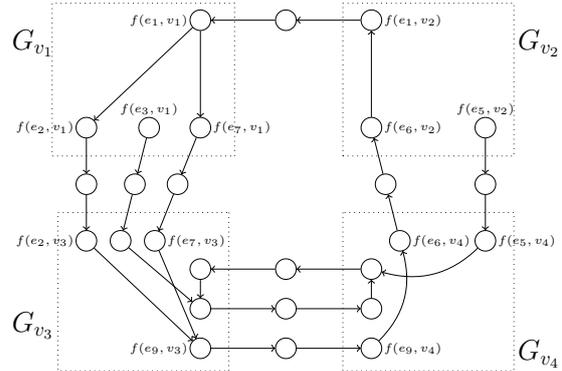
Claim 3. T is surjective.
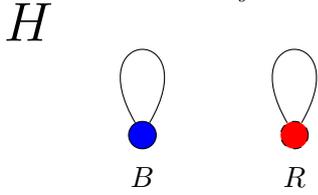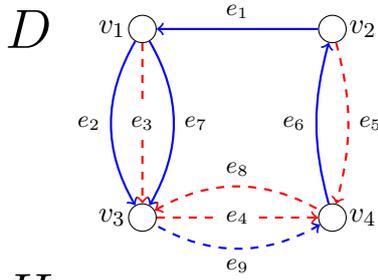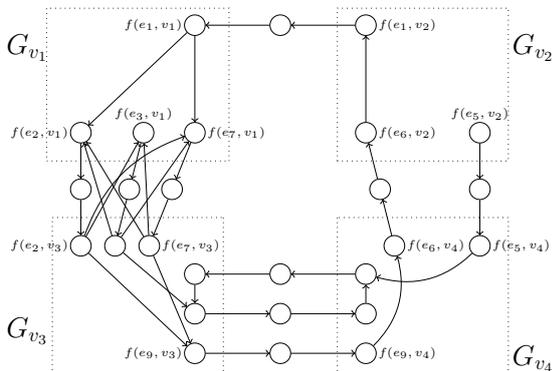
(a) *D* is an *H*-colored multigraph.

(b) The disjoint union of D_{v_i}, for every $i \in \{1, 2, 3, 4\}$.

(c) Since $e_1 = (v_2, v_1)$ in D, add a path from $f(e_1, v_2)$ to $f(e_1, v_1)$ with a new intermediate vertex, namely m_1.

(d) Repeat for every $e \in A(D)$.

(e) Since e_2, e_3 and e_7 are parallel in D, add the arcs $(f(e_2, v_3), f(e_3, v_1)), (f(e_2, v_3), f(e_7, v_1)),$ $(f(e_3, v_3), f(e_2, v_1)), (f(e_3, v_3), f(e_7, v_1)),$ $(f(e_7, v_3), f(e_2, v_1))$ and $(f(e_7, v_3), f(e_3, v_1))$.
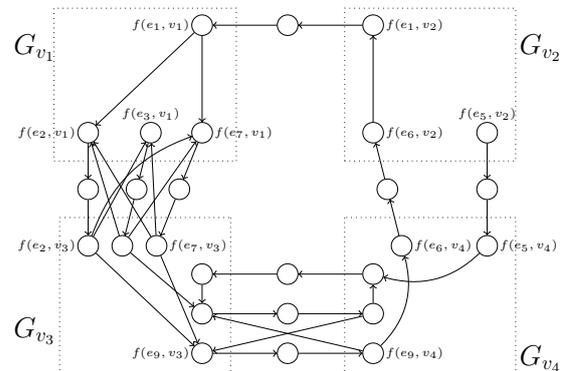
(f) Repeat for every set of parallel arcs and we obtained $L_{3,H}^{Dym}(D)$.

Figure 4.3: Procedure to construct the auxiliary digraph $L_{3,H}^{Dym}(D)$.

(a) D is an H-colored multigraph.

(b) $L_{2,H}^{Dym}(D)$.

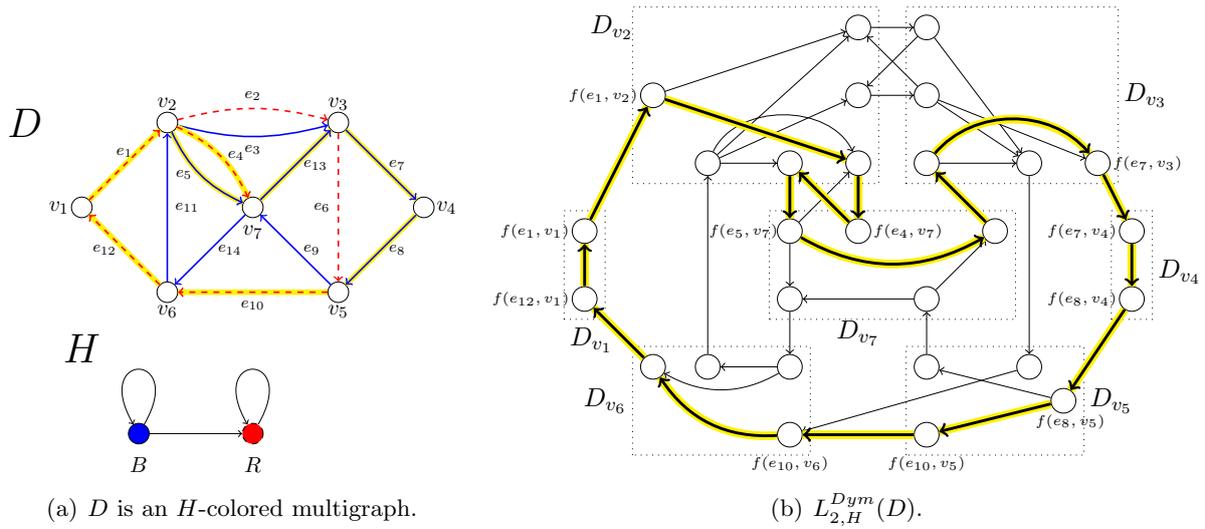Figure 4.4:   The trail $P = (v_1, e_1, v_2, e_4, e_5, v_7, e_{13}, v_3, e_7, v_4, e_8, v_5, e_{10}, v_6, e_{12}, v_1)$, that is highlighted in (a), is a closed dynamic H-trail in D. Then, $T(P) = (f(e_1, v_2), f(e_1, v_2),$ $f(e_4, v_2), f(e_4, v_7), f(e_5, v_2), f(e_5, v_7), f(e_{13}, v_7), f(e_{13}, v_3), f(e_7, v_3), f(e_7, v_4), f(e_8, v_4), f(e_8, v_5),$ $f(e_{10}, v_5), f(e_{10}, v_6), f(e_{12}, v_6), f(e_{12}, v_1), f(e_1, v_1))$, that is highlighted in (b), is a cycle in $L_{2,H}^{Dym}(D)$.

Let C be a cycle in $L_{2,H}^{Dym}(D)$. It follows from the fact that $(f(e, x), f(e, y)) \in A(L_{2,H}^{Dym}(D))$ and Observation 3 .2, for every $e = (x, y) \in A(D)$, that C must be of the form $C = (f(e_1, x_1),$ $f(e_1, y_1), \ldots, f(e_q, x_q), f(e_q, y_q), f(e_1, x_1))$, where $e_i = (x_i, y_i) \in A(D)$.

We construct a closed dynamic H-trail in D from the vertices of C, considering the following steps.

1. Start with $P_1 = (x_1, e_1, y_1)$ and $k = 2$.

2. Let e_k, with tail x_k and head y_k. If e_{k-1} and e_k are parallel, $x_k = x_{k-1}$ and $y_k = y_{k-1}$, then $P_i = (x_1, P_{i-1}, e_{k-1}, e_k, y_k)$. Otherwise, $P_i = (x_1, P_{i-1}, y_{k-1} = x_k, e_k, y_k)$.

3. $k = k + 1$.

4. If $k = q + 1$, then $P = P_q$ and finish. Otherwise, go to step 2.

Claim 4. P is a dynamic H-trail in D.

Let e_{k-1} and e_k be arcs of D with $2 \leq k \leq q$.

Since $(f(e_{k-1}, y_{k-1}), f(e_k, x_k)) \in A(C)$, it follows that $(f(e_{k-1}, y_{k-1}), f(e_k, x_k))$ is an arc of $L_{2,H}^{Dym}(D)$. Hence, by the definition of $L_{2,H}^{Dym}(D)$, $(y_{k-1} = x_k$ and $(c(e_{k-1}), c(e_k)) \in A(H))$ or $(y_{k-1} \neq x_k$ and e_{k-1} and e_k are parallel in $D)$.

If $y_{k-1} = x_k$ and $(c(e_{k-1}), c(e_k)) \in A(H)$, then e_{k-1} and e_k are incident in $y_{k-1} = x_k$ and $P_k = (e_1, P_{k-1}, y_{k-1} = x_k, e_k, y_k)$ is a dynamic H-walk in D. Otherwise, $y_{k-1} \neq x_k$ and e_{k-1} and e_k are parallel in D, hence $P_k = (x_1, P_{k-1}, e_{k-1}, e_k, y_{k-1} = y_k)$ is a dynamic H-walk in D.

Hence, P is a dynamic H-walk in D.

On the other hand, since C is a cycle, e_i appears once in P, for every $i \in \{1, \ldots, q_1\}$, and P is a dynamic H-trail.

Claim 5. P is closed.

Recall that $e_i = (x_i, y_i)$, for every $i \in \{1, \ldots, q\}$, and $y_i = x_{i+1}$ or $(x_i = x_{i+1}$ and $y_i = y_{i+1})$ (if $i = q + 1$, then $x_{i+1} = x_1$ and $y_{i+1} = y_1$). We will consider two cases.

Case 1. $x_i = x_j$ and $y_i = y_j$, for every pair of distinct elements, $\{i, j\} \subseteq \{1, \ldots, q\}$.

It follows from the construction of P that $P = (x_1, e_1, \ldots, e_q, y_1)$. Since C is a directed cycle in $L_{2,H}^{Dym}(D)$, we have that $q \geq 2$ and P is closed.

Case 2. There exists $i \in \{1, \ldots, q\}$ such that $y_i = x_{i+1}$.

If $y_q = x_1$, then $(c(e_q), c(e_1)) \in A(H)$. Therefore, P is closed.

Otherwise, e_q and e_1 are parallel and P is closed.

Therefore, P is a closed dynamic H-trail in D.

It follows from the definition of T that $T(P) = C$. Therefore, T is surjective.

By Claims 2 and 3, we have that T is a bijection. $\square$

Theorem 4.2. *Let D be an H-colored digraph. There exists a partition of the arcs of D into closed dynamic H-trails if and only if $UL_{2,H}^{Dym}(D) \setminus M_J$ has a perfect matching, where M_J is the joint matching of $UL_{2,H}^{Dym}(D)$.*

Proof. Let D be an H-colored digraph and M_J the joint matching of $L_{2,H}^{Dym}(D)$.

Suppose that there exists a partition of the arcs of D into closed dynamic H-trails, say $P = \{P_1, \ldots, P_k\}$.

By Theorem 4.1, it follows that $C = \{C_1, \ldots, C_k\}$ is a set of cycles in $L_{2,H}^{Dym}(D)$, where $C_i = T(P_i)$. Since $P = \{P_1, \ldots, P_k\}$ is a partition of the arcs of D, we have that $V(C_i) \cap V(C_j) = \emptyset$ (because $A(P_i) \cap A(P_j) = \emptyset$) and $V(L_{2,H}^{Dym}(D)) = \bigcup_{i=1}^{k} V(C_i)$ (because $A(D) = \bigcup_{i=1}^{k} A(P_i)$). Therefore, C is a partition of the vertices of $L_{2,H}^{Dym}(D)$ into cycles. Moreover, $C' = \{C_1', \ldots, C_k'\}$ is a partition of the vertices of $UL_{2,H}^{Dym}(D)$ into cycles, where C_i' is the underlying cycle of C_i, for each $C_i \in C$.

By construction of C_i', we have that C_i' alternate edges between M_J and $E(UL_{2,H}^{Dym}(D)) \setminus M_J$. Hence, $M = \bigcup_{i=1}^{k} E(C_i) \setminus M_J$ is a perfect matching of $UL_{2,H}^{Dym}(D) \setminus M_J$.

Conversely, suppose that M is a perfect matching of $UL_{2,H}^{Dym}(D) \setminus M_J$. By Lemma 3.4, we have that there exists a partition of G into even cycles, say $C' = \{C_1', \ldots, C_k'\}$, such that every cycle alternate edges between M and M_J.

Consider $C_i' \in C'$. Since C_i' alternate edges between M_J and M, C_i' must be of the form $C_i' = (f(e_1^i, x_1^i), f(e_1^i, x_2^i), \ldots, f(e_n^i, x_n^i), f(e_n^i, x_1^i), f(e_1^i, x_1^i))$.

By Observation 3.2, it follows that if $(f(e_1^i, x_1^i), f(e_1^i, x_2^i)) \in A(L_{2,H}^{Dym}(D))$, then C_i' is a directed cycle in $L_{2,H}^{Dym}(D)$. Otherwise, $(f(e_1^i, x_2^i), f(e_1^i, x_1^i)) \in A(L_{2,H}^{Dym}(D))$ and $C_i'^{-1}$ is a directed cycle in $L_{2,H}^{Dym}(D)$. So, C_i will be the directed cycle, i.e., $C_i = C_i'$ or $C_i = C_i'^{-1}$, for every $i \in \{1, \ldots, k\}$.

Consider the set $C = \{C_1, \ldots, C_k\}$. Since C' is a partition of the vertices of $UL_{2,H}^{Dym}(D)$ into cycles, we have that C is a partition of the vertices of $L_{2,H}^{Dym}(D)$ into directed cycles.

By Theorem 4.1, it follows that $P = \{P_1, \ldots, P_k\}$ is a set of closed dynamic H-trails, where $P_i = T^{-1}(C_i)$. Since C is a partition of the vertices of $L_{2,H}^{Dym}(D)$, we have that P is a partition of the arcs of D into closed dynamic H-trails. $\square$

Definition 4.3. *Let D be an H-colored digraph. The dynamic H-line digraph, denoted by $L_{1,H}^{Dym}(D)$, is the digraph such that $V(L_{1,H}^{Dym}(D)) = A(D)$, and two different vertices $e = (x, y)$ and $g = (u, v)$ are joining by only one arc from e to g in $L_{1,H}^{Dym}(D)$ whenever $y = u$ and $(c(e), c(g)) \in A(H)$. Finally, we add a symmetric arc between the vertices e and g if and only if e and g are parallel in D.*

Observation 4. *The digraph $L_{1,H}^{Dym}(D)$ can be obtained from $L_{2,H}^{Dym}(D)$ by contracting the arc $(f(e, u), f(e, v))$, for each $e = (u, v) \in A(D)$, in $L_{2,H}^{Dym}(D)$. The construction is illustrated in Figure 4.5.*
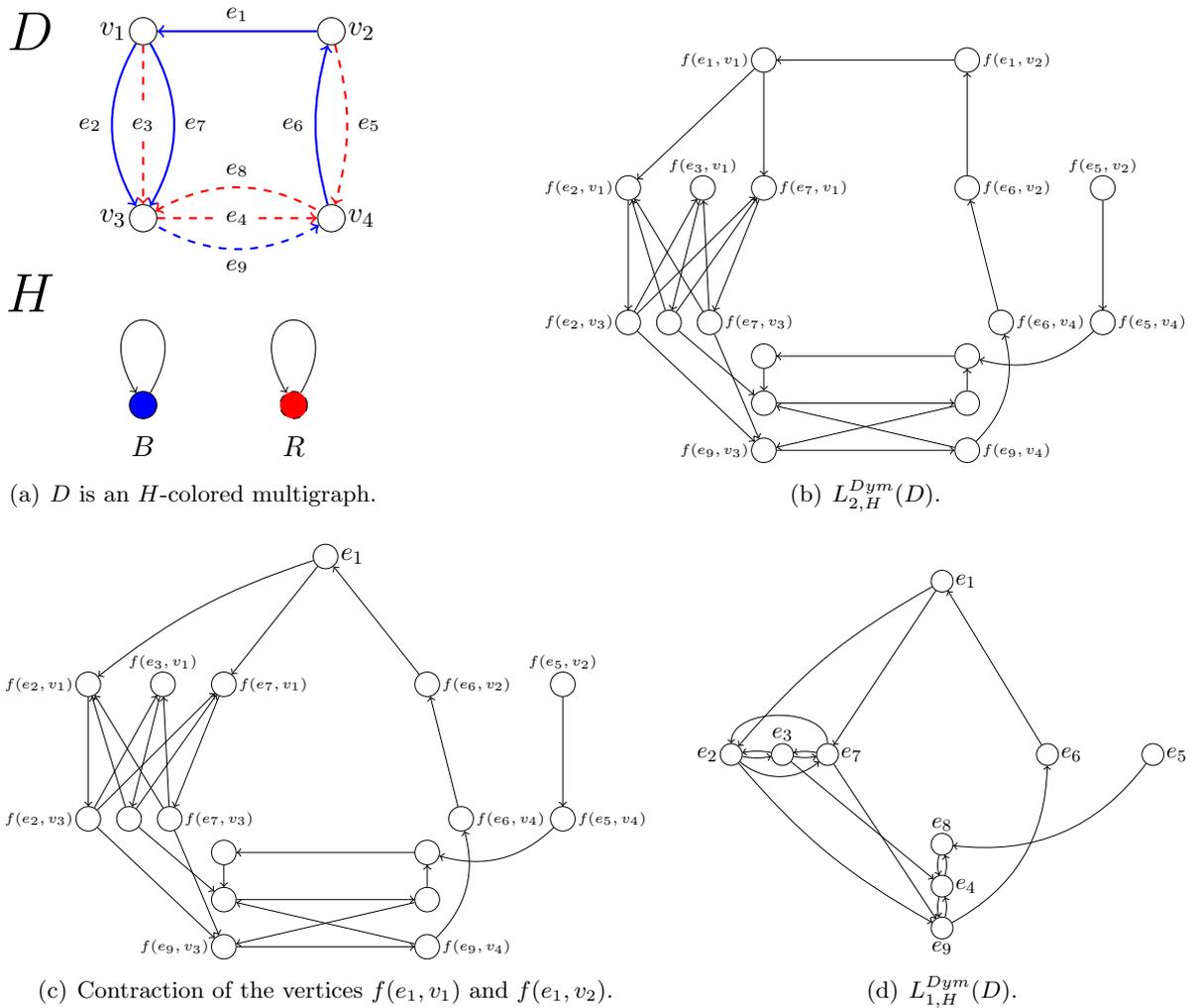
(a) D is an H-colored multigraph.

(b) $L_{2,H}^{Dym}(D)$.

(c) Contraction of the vertices $f(e_1, v_1)$ and $f(e_1, v_2)$.

(d) $L_{1,H}^{Dym}(D)$.

Figure 4.5: Procedure to construct the auxiliary digraph $L_{1,H}^{Dym}(D)$.

Theorem 4.3. *Let D be an H-colored digraph. Then:*

a. *If D has a closed Euler dynamic H-trail, then $L_{n,H}^{Dym}(D)$ is Hamiltonian, for every $n \geq 1$.*

b. *If $L_{n,H}^{Dym}(D)$ is Hamiltonian for some $n \geq 1$, then D has a closed Euler dynamic H-trail.*

c. *D has a closed Euler dynamic H-trail if and only if $L_{n,H}^{Dym}(D)$ is Hamiltonian, for every $n \geq 1$.*

Proof. **a.** Suppose that D has a closed Euler dynamic H-trail, say P. Let $C = T(P)$, we have by Theorem 4.2 that C is a Hamiltonian cycle in $L_{2,H}^{Dym}(D)$.

On the other hand, by Observation 3.2, we have that $(f(e, u), f(e, v)) \in A(C)$, for every $e = (u, v) \in A(D)$. We construct a cycle C_n in $L_{n,H}^{Dym}(D)$ replacing the arc $(f(e, u), f(e, v))$ in C by the directed path joining $f(e, u)$ with $f(e, v)$ in $L_{n,H}^{Dym}(D)$, for every $e = (u, v) \in A(D)$, then C_n is a Hamiltonian cycle in $L_{n,H}^{Dym}(D)$.

By Observation 4, if we contract the arc $(f(e, u), f(e, v))$ and delete the corresponding loop for each $e = (u, v) \in A(D)$, in C, then we obtain a Hamiltonian cycle in $L_{1,H}^{Dym}(D)$. Therefore, $L_{n,H}^{Dym}(D)$ is Hamiltonian, for every $n \geq 1$.

b. Suppose that there exist $n \geq 1$ such that $L_{n,H}^{Dym}(D)$ is Hamiltonian. Let C' be a Hamiltonian cycle in $L_{n,H}^{Dym}(G)$.

Case 1. $n = 1$.

Let $C' = (e_1, a_1, e_3, a_2, \ldots, e_{2k+1}, a_{k+1}, e_1)$, where $e_{2i+1} = (x_{2i+1}, x_{2i+2}) \in V(L_{1,H}^{Dym}(D)) = A(D)$, for each $i = \{0, \ldots, k\}$. So, we construct a cycle C in $L_{2,H}^{Dym}(D)$ as follows: $C = (f(e_1, x_1), f(e_1, x_2), f(e_3, x_3), \ldots, f(e_{2k+1}, x_{2k+1}), f(e_{2k+1}, x_{2k+2}), f(e_1, x_1))$. By the definition of $L_{2,H}^{Dym}(D)$, it follows that C is a Hamilton cycle in $L_{2,H}^{Dym}(D)$ and, by Theorem 4.2, $P = T^{-1}(C)$ is a closed dynamic H-trail in D.

Case 2. $n \geq 2$.

By the definition of $L_{n,H}^{Dym}(D)$, we have that the directed path from $f(e, x)$ to $f(e, y)$ is in $V(C')$, for every $e = (x, y)$ in $A(D)$. Hence, we construct a cycle in $L_{2,H}^{Dym}(D)$ by replacing the path from $f(e, x)$ to $f(e, y)$ for the arc $(f(e, x), f(e, y))$ in C', and we get a Hamilton cycle C in $L_{2,H}^{Dym}(D)$. Hence, $P = T^{-1}(C)$ is a closed dynamic H-trail. Since C is a Hamiltonian cycle, we have that P is a closed Euler dynamic H-trail in D, by Theorem 4.2.

c. It follows from a and b. $\square$

(a) D is an H-colored multigraph.   (b) A Hamiltonian cycle C in $L_{2,H}^{Dym}(D)$ is highlighted in yellow.
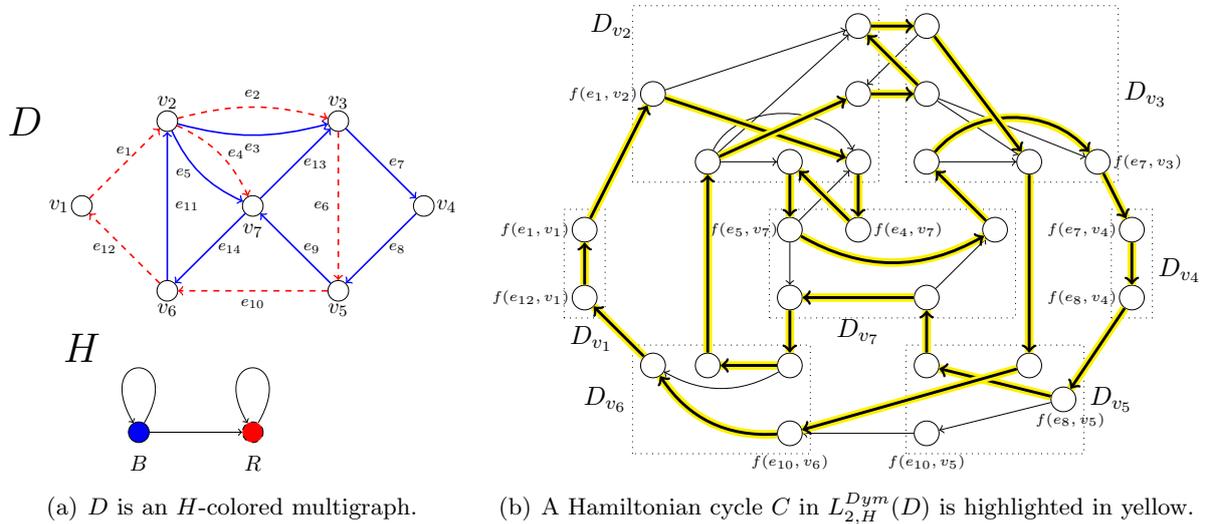
Figure 4.6: $T^{-1}(C) = (v_1, e_1, v_2, e_4, e_5, v_7, e_{13}, v_3, e_7, v_4, e_8, v_5, e_9, v_7, e_{14}, v_6, e_{11}, v_2, e_3, e_2, v_3, e_6, v_5, e_{10}, v_6, e_{12}, v_1)$ is a closed Eulerian H-trail in D.

Let D be an H-colored digraph with q arcs. We will say that D is a **dynamic H-pancircular** digraph whenever it contains a closed dynamic H-trail of dynamic length L, for each $2 \leq L \leq q$.

Corollary 4.4. *Let D be an H-colored digraph with q arcs. Then, D is dynamic H-pancircular digraph if and only if $L_{1,H}^{Dym}(D)$ is pancyclic.*

4.3   Euler H-trails in digraphs

In this section we will show a characterization of those digraphs containing a closed Euler H-trail, and as a consequence, we will give conditions to know if a 2-arc-colored digraph has a closed Euler H-trail. We begin by taking a subdigraph of $L_{2,H}^{Dym}(D)$ to obtain similar results to those of the previous section.

Definition 4.4. *Let D be an H-colored multigraph with $|A(D)| = q$. For $n \geq 2$, $L_n^H(D)$ is the digraph with nq vertices, obtained as follows: for each arc $e = (u, v)$ of D, we take two vertices $f(e, u)$ and $f(e, v)$ in $L_n^H(D)$, and we add a directed path from $f(e, u)$ to $f(e, v)$ with $n - 2$ new intermediate vertices. And the rest of the arcs of $L_n^H(D)$ are defined as follows:*

$(f(e, u), f(g, u)) \in A(L_n^H(D))$ if and only if $e = (x, u)$ and $g = (u, y)$ in D, for some x and y in $V(D)$, and $(c(e), c(g)) \in A(H)$.

Notice that the digraph $L_n^H(D)$ is a subdigraph of $L_{n,H}^{Dym}(D)$ and can be constructed as follows: take the disjoint union of D_x, for every $x \in V(D)$, and for every $e = (x, y) \in A(D)$, add a directed path from $f(e, u)$ to $f(e, v)$ with $n - 2$ new intermediate vertices. Figure 4.7 shows an example of the digraph $L_2^H(D)$.
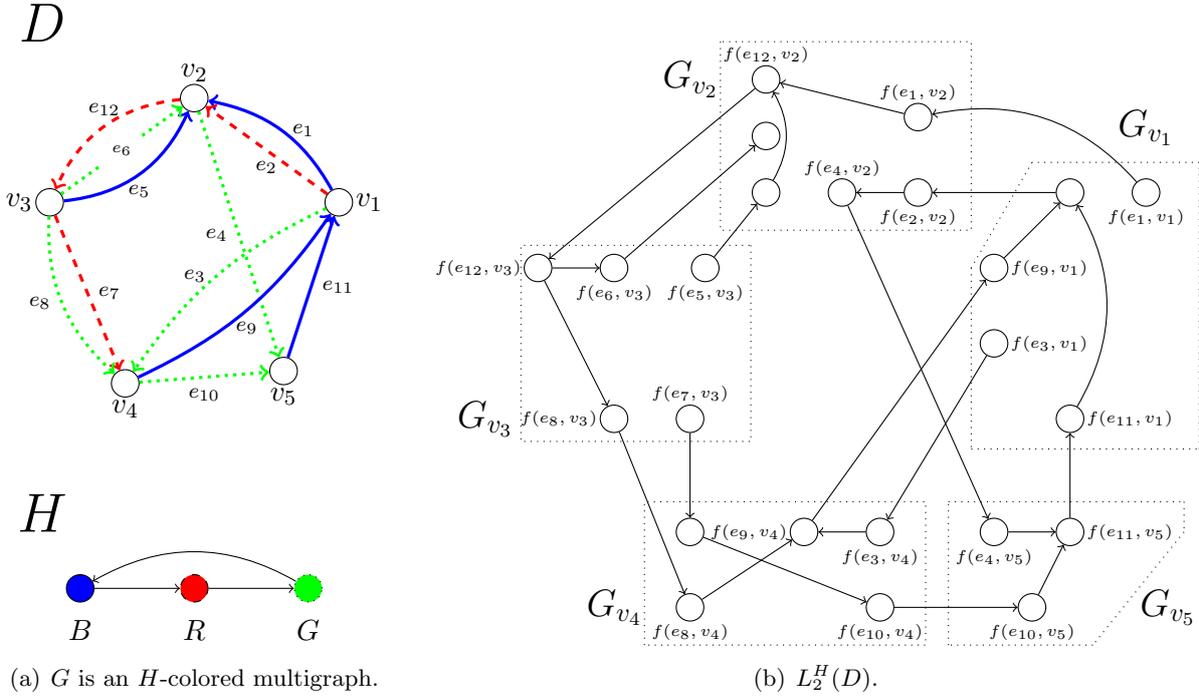

(a) G is an H-colored multigraph.

(b) $L_2^H(D)$.

Figure 4.7: Note that $L_2^H(D)$ is a subdigraph of $L_{2,H}^{Dym}(D)$, that can be found in Figure 4.2(b).

Theorem 4.5. *Let D be an H-colored digraph. Then, there is a bijection between the set of closed H-trails in D and the set of directed cycles in $L_2^H(D)$.*

Proof. Let D be an H-colored digraph, $\mathcal{Q}$ the set of closed H-trails in D and $\mathcal{C}'$ the set of directed cycles in $L_2^H(D)$.

It follows from the definition of dynamic H-trail that $\mathcal{Q} \subseteq \mathcal{P}$, recall that $\mathcal{P}$ is the set of closed dynamic H-trails in D.

Consider $T' = T|_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{C}$. By Theorem 4.1, we just need to verify that $T'(P) \in C'$, for every $P \in \mathcal{Q}$.

Consider $P = (x_1, e_1, x_2, e_2, \ldots, x_n, e_n, x_1)$ a closed H-trail in D. Then, $T'(P) = (f(e_1, x_1), f(e_1, x_2), f(e_2, x_2), \ldots, f(e_n, x_n), f(e_n, x_1), f(e_1, x_1))$.

By the definition of the graph $L_2^H(D)$ and the fact that P is an H-trail, we have that $(f(e_i, x_{i+1}), f(e_{i+1}, x_{i+1})) \in A(L_2^H(D))$, for every $i \in \{1, \ldots, n\}$ (if $i = n$, then $e_{n+1} = e_1$ and $x_{i+1} = x_1$), and $T'(P)$ is a walk in $L_2^H(D)$. Since P is a closed H-trail, it follows that $T'(P)$ is a cycle in $L_2^H(D)$. Therefore T' is a bijection. $\square$

The following result is obtained from Theorems 4.2 and 4.5.

Theorem 4.6. *Let D be an H-colored digraph. There exists a partition of the arcs of D into closed H-trails if and only if UD_x has a perfect matching, for every $x \in V(D)$.*

Definition 4.5. *Let D be an H-colored digraph. The H-line digraph, denoted by $L_1^H(D)$, is the digraph such that $V(L_1^H(D)) = A(D)$, and for two different vertices $e = (x, y)$ and $g = (u, v)$, there is an arc with tail e and head g in $L_1^H(D)$ if and only if $y = u$ and $(c(e), c(g)) \in A(H)$.*

Notice that $L_1^H(D)$ is a subdigraph of $L_{1,H}^{Dym}(D)$, and it can be obtained from $L_2^H(D)$ by contracting the arc $(f(e,u), f(e,v))$ and deleting the corresponding loop, for each $e = (u,v) \in A(D)$, in $L_2^H(D)$.

The following result is the H-colored version in H-colored digraphs of Theorem 2.6.

Theorem 4.7. *Let D be an H-colored digraph. Then,*

    a. If D has a closed Euler H-trail, then $L_n^H(D)$ is Hamiltonian, for every $n \geq 1$.

    b. If $L_n^H(D)$ is Hamiltonian for some $n \geq 1$, then D has a closed Euler H-trail.

    c. D has a closed Euler H-trail if and only if $L_n^H(D)$ is Hamiltonian, for every $n \geq 1$.

Let D be an H-colored digraph with q arcs. We will say that D is an **H-pancircular** digraph if and only if D contains a closed H-trail of length L, for each $2 \leq L \leq q$.

Corollary 4.8. *Let D be an H-colored digraph with q arcs. Then, D is H-pancircular digraph if and only if $L_1^H(D)$ is pancyclic.*

An H-colored digraph D is **H-trail connected** if and only if there is an H-trail starting with arc f_1 and ending with arc f_2, for any pair of distinct arcs f_1 and f_2 in D.

Recall the following classical theorem, which will be useful for what follows.

Theorem 4.9 (Hall's Theorem). *A bipartite graph with bipartition (X, Y) has a matching that saturates every vertex in X if and only if $|N(S)| \geq |S|$, for every $S \subseteq X$.*

Theorem 4.10. *Let D be an H-colored digraph such that $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every u in $V(D)$ and some $k_u \geq 1$. Then D has a closed Euler H-trail if and only if D is H-trail connected and, for every $u \in V(D)$, $n_i^u = m_i^u$ for each $i \in \{1, \ldots, k_u\}$.*

Proof. Let D be an H-colored digraph and $D_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every u in $V(D)$ and some $k_u \geq 1$.

Suppose that D has a closed Euler H-trail. Then, D is H-trail connected and by Theorem 4.6, UD_u has a perfect matching, for every $u \in V(D)$, say M_u. Since $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$ and M_u is a perfect matching, we have that $n_i^u = m_i^u$, for every $i \in \{1, \ldots, k_u\}$.

Conversely, suppose that D is H-trail connected and, for every $u \in V(D)$, $n_i^u = m_i^u$ for every $i \in \{1, \ldots, k_u\}$. Hence, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, n_i^u}$.

By Theorem 4.9, it follows that UD_u has a perfect matching, for every $u \in V(D)$. So, by Theorem 4.6, D has a partition of the arcs of D into closed H-trails, say $\mathcal{P} = \{P_1, \ldots, P_k\}$.

If $k = 1$, then D has a closed Euler H-trail. Otherwise, since D is H-trail connected, there is an H-trail in D starting with arc $e_1 \in A(P_1)$ and ending with arc $e_2 \in A(D) \setminus A(P_1)$, say Q_1.

Let $g_1 = (x_1, x_2) \in A(Q_1)$ be the first arc in $A(Q_1) \setminus A(P_1)$ and $g_0 = (x_0, x_1)$ the arc prior to g_1 in Q_1. Hence, $g_0 \in A(P_1)$ and $(c(g_0), c(g_1)) \in A(H)$. Without loss of generality, suppose that $g_1 \in A(P_2)$.

Let $P_1 = (x_1, f_1, \ldots, x_0, g_0, x_1)$ and $P_2 = (x_1, g_1, x_2, \ldots, f_0, x_1)$. By the definition of closed H-trail, we have that $\{(c(g_0), c(f_1)), (c(f_0), c(g_1))\} \subseteq A(H)$. By the definition of D_{x_1}, we have that (g_0, f_1), (f_0, g_1) and (g_0, g_1) are arcs in $A(D_{x_1})$. By the Observation 3.2 and the fact that $UD_{x_1} = \bigcup_{i=1}^{k_{x_1}} K_{n_i^{x_1}, n_i^{x_1}}$, we have that $(f_0, f_1) \in A(D_{x_1})$. Therefore, $(c(f_0), c(f_1)) \in A(H)$ and $T_1 = P_1 \cup P_2 = (x_1, f_1, \ldots, x_0, g_0, x_1, g_1, x_2, \ldots, f_0, x_1)$ is a closed H-trail.

If $A(T_1) = A(D)$, then T_1 is a closed Euler H-trail. Otherwise, there is an H-trail in D starting with arc $e_3 \in A(T_1)$ and ending with arc $e_4 \in A(D) \setminus A(T_1)$, say Q_2.

Let $g_3 = (v_1, v_2) \in A(Q_2)$ be the first arc in $A(Q_2) \setminus A(T_1)$ and $g_2 = (v_0, v_1)$ the arc prior to g_3 in Q_2. Hence, $g_2 \in A(T_1)$ and $(c(g_2), c(g_3)) \in A(H)$. Without loss of generality, suppose that $g_3 \in A(P_3)$.

Let $T_1 = (v_1, f_3, \ldots, g_2, v_1)$ and $P_3 = (v_1, g_3, v_2, \ldots, f_2, v_1)$. By the definition of closed H-trail, we have that $\{(c(g_2), c(f_3)), (c(f_2), c(g_3))\} \subseteq A(H)$. By the definition of D_{v_1}, we have that $(f(g_2, v_1), f(f_3, v_1))$, $(f(f_2, v_1), f(g_3, v_1))$ and $(f(g_2, v_1), f(g_3, v_1))$ are arcs in $A(D_{v_1})$. By the Observation 3.2 and the fact that $UD_{v_1} = \bigcup_{i=1}^{k_{v_1}} K_{n_i^{v_1}, n_i^{v_1}}$, we have that $(f(f_2, v_1), f(f_3, v_1)) \in A(D_{v_1})$. Therefore, $(c(f_2), c(f_3)) \in A(H)$ and $T_2 = T_1 \cup P_3 = (v_1, f_3, \ldots, g_2, v_1, g_3, v_2, \ldots, f_2, v_1)$ is a closed H-trail.

If $A(T_2) = A(D)$, then T_2 is a closed Euler H-trail. Otherwise, we can repeat this procedure and after a finite number of steps we obtain a closed Euler H-trail in D.   □

Although in the proof of Theorem 4.10 the importance of the hypothesis "D is H-trail connected" is evident, Figure 4.8 shows that this hypothesis cannot be removed.
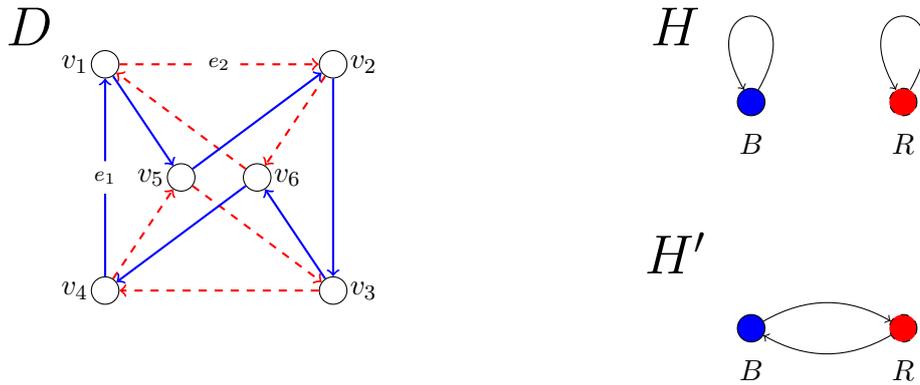

Figure 4.8: If D is an H-colored digraph, then there is no H-trail starting with arc e_1 and ending with arc e_2, so D is not an H-trail connected. However, if D is an H'-colored graph, then D is H'-trail connected and, by Theorem 4.10, D has a closed Euler H'-trail.

Let D be a k-arc-colored digraph and $v \in V(D)$. We say that $F_v = \{e \in A(D) : e \text{ is incident with } v\}$, and $c(F_v) = \{i \in \{1, \ldots, k\} : \text{there exists } e \in F_v \text{ such that } c(e) = i\}$. We say that D is **properly colored trail connected** if and only if there is a properly colored trail starting with arc f_1 and ending with arc f_2, for any pair of different arcs f_1 and f_2 of D.

Corollary 4.11. *Let D be a k-arc-colored digraph such that $c(F_v) = \{c_1^v, c_2^v\}$, for every $v \in V(D)$. Then, D is properly colored Euler if and only if D is properly colored trail connected and for every $v \in V(D)$, $d_{c_i^v}^+(v) = d_{c_{3-i}^v}^-(v)$, for $i \in \{1, 2\}$.*

Proof. Suppose that D is a k-arc-colored digraph. Then, D is an H-colored digraph, where H is the complete digraph without loops and $V(H) = \{1, 2, \ldots, k\}$. Notice that if P is a (closed) trail in D, then P is a properly colored (closed) trail if and only if P is a (closed) H-trail.

Suppose that $c(F_v) = \{c_1^v, c_2^v\}$, for every $v \in V(D)$. Let u be a vertex in $V(D)$.

Claim 1. $UD_u = K_{n_1, m_1} \cup K_{n_2, m_2}$, where $n_i = d_{c_i^u}^-(u)$ and $m_i = d_{c_{3-i}^u}^+(u)$.

Let $E_u^+ = \{f = (u, x) \in A(D)\}$ and $E_u^- = \{g = (x, u) \in A(D)\}$. Hence, $F_{c_i^u}^+ = \{f \in E_u^+ : c(f) = c_i^u\}$ and $F_{c_i^u}^- = \{f \in E_u^- : c(f) = c_i^u\}$ are independent sets in UD_u, for every $i \in \{1, 2\}$. Moreover, for every $i \in \{1, 2\}$, $fg \in E(UD_u)$ if and only if $f \in F_{c_i^u}^-$ and $g \in F_{c_{3-i}^u}^+$.

Notice that $|F_{c_i^u}^+| = d_{c_i^u}^+(u)$ and $|F_{c_i^u}^-| = d_{c_i^u}^-(u)$. Therefore, $UD_u = K_{n_1, m_1} \cup K_{n_2, m_2}$, where $n_i = d_{c_i^u}^-(u)$ and $m_i = d_{c_{3-i}^u}^+(u)$.

Suppose that D is properly colored Euler. Then, D has closed Euler H-trail. By Theorem 4.10 and Claim 1, it follows that D is H-trail connected and $d_{c_i^u}^-(u) = d_{c_{3-i}^u}^+(u)$, for every $i \in \{1, 2\}$. Therefore, D is properly colored trail connected and for every $v \in V(D)$, $d_{c_i^u}^+(u) = d_{c_{3-i}^u}^-(u)$, for $i \in \{1, 2\}$.

Conversely, supposed that D is properly colored trail connected and for every $v \in V(D)$, $d^+_{c^v_i}(v) = d^-_{c^v_{3-i}}(v)$, for $i \in \{1,2\}$. Since D is properly colored trail connected, we have that D is H-trail connected.

Since $d^+_{c^u_i}(u) = d^-_{c^u_{3-i}}(u)$, for $i \in \{1,2\}$, it follows from Claim 1 that $UD_u = K_{n_1,n_1} \cup K_{n_2,n_2}$. Hence, by Theorem 4.10, D has a closed Euler H-trail, i.e., D is properly colored Euler.   □

Figure 4.9(b) shows an example of a 3-arc-colored digraph that fulfill the hypothesis of Corollary 4.11.

The following result is a direct consequence of Corollary 4.11.

Corollary 4.12 ([49])**.** *Let D be a 2-arc-colored digraph. Then, D is properly colored Euler if and only if D is properly colored trail connected and for every $v \in V(D)$, $d^+_i(v) = d^-_{3-i}(v)$, for each $i \in \{1,2\}$.*

Figure 4.9(a) shows an example of a 2-arc-colored digraph with a partition of its arcs into closed properly colored trails that does not have a closed properly colored Eulerian trail.
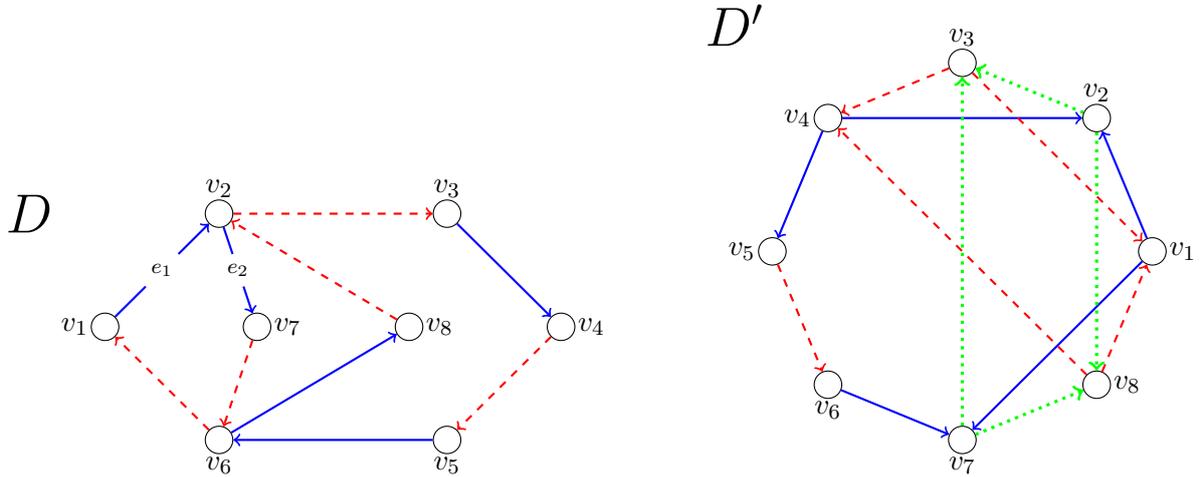
(a) D is a 2-arc-colored digraph that is not properly colored trail connected since there is no properly colored trail starting with arc e_1 and ending with arc e_2.

(b) D' is a 3-arc colored multigraph such that every vertex in D' has incident arcs of just two colors and is properly colored trail connected.

Figure 4.9: (a) D has a partition of its arcs into two closed properly colored trails, namely $W_1 = (v_2, v_7, v_6, v_8, v_2)$ and $W_2 = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$, but D is not properly colored Eulerian; (b) $W = (v_1, v_2, v_8, v_4, v_2, v_3, v_1, v_7, v_3, v_4, v_5, v_6, v_7, v_8, v_1)$ is a closed properly colored Eulerian trail in D'.

Theorem 4.13. *Let D be an H-colored digraph. Then, D is H-trail connected if and only if $L^H_2(D)$ is strongly connected.*

Proof. Let D be an H-colored digraph.

Suppose that D is H-trail connected. Let $f(e,u)$ and $f(g,x)$ be two vertices in $V(L^H_2(D))$.

Since D is H-trail connected, there exists an H-trail such that $e = (u_1, u_2)$ is the first arc and $g = (x_1, x_2)$ is the last arc, say $P = (u_1, e, u_2, e_1, \ldots, x_1, g, x_2)$. Notice that $T'(P) = (f(e,u_1), f(e,u_2), f(e_1,u_2), \ldots, f(g,x_1), f(g,x_2))$ is a path.

Since $f(e,u)$ and $f(g,x)$ are in $V(L^H_2(D))$, it follows that ($u = u_1$ or $u = u_2$) and ($x = x_1$ or $x = x_2$). Therefore, there exists a path from $f(e,u)$ to $f(g,x)$ and $L^H_2(D)$ is strongly connected.

Conversely, suppose that $L^H_2(D)$ is strongly connected. Let $e = (x,y)$ and $g = (u,v)$ be two arcs in $A(D)$.

Since $L_2^H(D)$ is strongly connected, there exists a path from $f(e, x)$ to $f(g, v)$, say P. By Observation 3.2, it follows that P must be of the form $P = (f(e, x), f(e, y), f(e_1, y), \ldots, f(g, u), f(g, v))$. So, $T'^{-1}(P) = (x, e, y, e_1, \ldots, u, g, v)$ is an H-trail such that its first arc is e and its last arc is g. Therefore, D is H-trail connected. $\square$

The next assertion follows directly from the previous results.

Corollary 4.14. *Given a 2-arc-colored digraph, we can check in polynomial time whether D has a close properly colored Eulerian trail.*

4.4 The auxiliary digraph D_u

This section is devoted to answering the question, which arises naturally after reading Theorem 4.10, when does a digraph satisfy the hypothesis that $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every u in $V(D)$?

Recall that $V(D_u)$ is the set of arcs in D incident with u, so in this section we will use a simplified notation for the vertices of D_u, the vertex $f(e, u)$ will denote by e.

In the remainder of this section we will denote the empty graph with n vertices by $K_{n,0}$.

Theorem 4.15. *Let H be a digraph, possibly with loops, such that for every pair of vertices u and v of H, $N_H^+(u) = N_H^+(v)$ or $N_H^+(u) \cap N_H^+(v) = \emptyset$. Then, for every multidigraph D without isolated vertices, and every H-coloring of D, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$ and for some $k_u \geq 1$.*

Proof. Suppose that H is a digraph, possibly with loops, such that for every pair of vertices u and v of H, $N_H^+(u) = N_H^+(v)$ or $N_H^+(u) \cap N_H^+(v) = \emptyset$.

Let D be an H-colored digraph without isolated vertices and $u \in V(D)$.

We define $V_0 = \{e \in V(UD_u) : d_{UD_u}(e) = 0\}$. Notice that $UD_u[V_0] = K_{m,0}$, for a non-negative integer m. Thus, we only need to prove that each component in $UD_u \setminus V_0$ is a complete bipartite graph.

Let B be a component of UD_u with at least two vertices. By Observation 2, we have that B is a bipartite graph with partition sets $X_B = X \cap V(B)$ and $Y_B = Y \cap V(B)$, where X and Y are the partition set of D_u defined in Observation 2.

Now we will prove that every vertex in X is adjacent to every vertex in Y. So, let $x \in X_B$ and $y \in Y_B$. Since B is a component of UD_u, there is an xy-path in B, namely $T = (x = e_1, e_2, \ldots, e_{2p} = y)$.

Notice that for every $i \in \{1, \ldots, p\}$, $e_{2i-1} \in X_B$ and $e_{2i} \in Y_B$. By the definition of D_u, it follows that $(c(e_{2i-1}), c(e_{2i})) \in A(H)$ and $(c(e_{2(i+1)-1}), c(e_{2i})) \in A(H)$, i.e., $N_H^+(c(e_{2i-1})) \cap N_H^+(c(e_{2(i+1)-1})) \neq \emptyset$. By hypothesis, we have that $N_H^+(c(e_{2i-1})) = N_H^+(c(e_{2(i+1)-1}))$. Thus, $N_H^+(c(e_1)) = N_H^+(c(e_{2p-1}))$. Moreover, $c(e_{2p}) \in N_H^+(c(e_{2p-1})) = N_H^+(c(e_1))$. Hence, $x = e_1$ and $y = e_{2p}$ are adjacent in B and B is a complete bipartite graph.

Thus, every component in $UD_u \setminus V_0$ is a complete bipartite graph and $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$ and for some $k_u \geq 1$. $\square$

In Figure 4.10 we show an example of a digraph H such that for every pair of vertices u and v of H, $N_H^+(u) = N_H^+(v)$ or $N_H^+(u) \cap N_H^+(v) = \emptyset$. Moreover, we H-color the digraph D in such a way that for every u in $V(D)$, D_u has no isolated vertices. So, by Theorem 4.15, we have that D_u is a complete bipartite digraph or a union of two complete bipartite digraphs.

Proposition 4.16. *Let H be a complete digraph with a loop in each vertex. Then, for every multidigraph D without isolated vertices, and every H-coloring of D, UD_u is complete bipartite graph or an empty graph, for every $u \in V(D)$.*
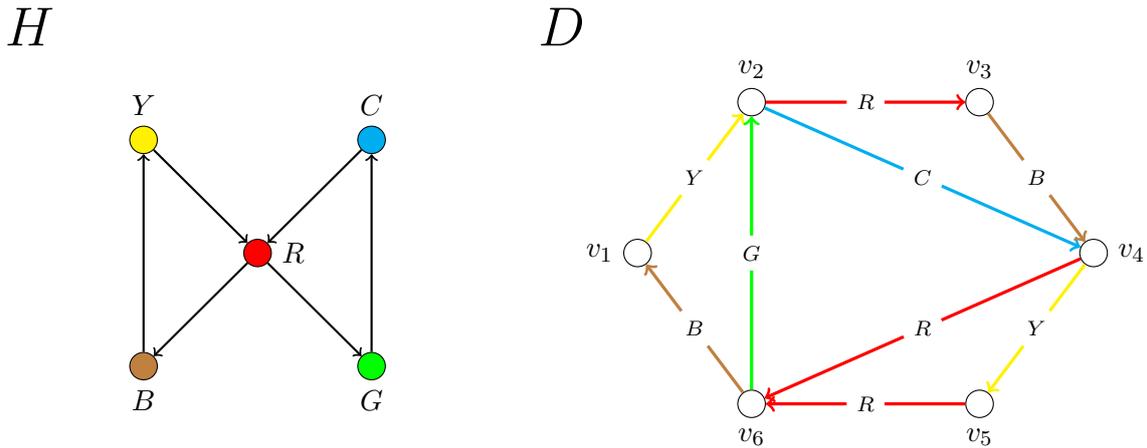
Figure 4.10: The digraph D is an H-colored digraph. By Theorem 4.10, we have that D has a closed Euler H-trail, namely $P = (v_1, v_2, v_3, v_4, v_5, v_6, v_2, v_4, v_6, v_1)$.

Proof. Suppose that H is a complete digraph with a loop in each vertex. Let D be an H-colored multidigraph without isolated vertices, and u a vertex of D.

Case 1. $d^-(u) = 0$ or $d^+(u) = 0$.

By Observation 2, it follows that UD_u is an empty graph.

Case 2. $d^-(u) \neq 0$ and $d^+(u) \neq 0$.

Let $X = \{e \in A(D) : e = (x, u) \text{ for some } x \in V(D)\}$ and $Y = \{f \in A(D) : f = (u, x) \text{ for some } x \in V(D)\}$. Since $d^-(u) \neq 0$ and $d^+(u) \neq 0$, we have that $X \neq \emptyset$ and $Y \neq \emptyset$.

Consider $e \in X$ and $f \in Y$. Given that H is a complete graph with loop at each vertex, it follows that $(c(e), c(f))$ is an arc in H. Therefore, (e, f) is an arc in D_u.

By Observation 2, it follows that $V(G) = X \cup Y$. Therefore, UD_u is a complete bipartite graph. $\qquad\qquad\square$

Corollary 4.17. *Let H be a digraph such that $d^+(x) = 1$, for every vertex u in $V(H)$. Then, for every multidigraph D without isolated vertices, and every H-coloring of D, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$ and for some $k_u \geq 1$.*

Proof. Let H be a digraph such that $d^+(x) = 1$, for every vertex u in $V(H)$. Hence, for every pair of vertices, u and v, in H, we have that $N_H^+(u) = N_H^+(v)$ or $N_H^+(u) \cap N_H^+(v) = \emptyset$.

Therefore, by Theorem 4.15, every multidigraph D without isolated vertices, and every H-coloring of D, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$ and for some $k_u \geq 1$. $\qquad\square$

Corollary 4.18. *If H is a digraph with only loops, then for every multidigraph D without isolated vertices, and every H-coloring of D, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$ and for some $k_u \geq 1$.*

Corollary 4.19. *If H is a cycle, then for every multidigraph D without isolated vertices, and every H-coloring of D, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$ and for some $k_u \geq 1$.*

Corollary 4.20. *If H is a path with a loop only in the end-vertices, then for every multidigraph D without isolated vertices, and every H-coloring of D, $UD_u = \bigcup_{i=1}^{k_u} K_{n_i^u, m_i^u}$, for every $u \in V(D)$ and for some $k_u \geq 1$.*

In Figure 4.11, we show an example of a digraph H_1 and an H_1-colored digraph such that every UD_u is the union of complete bipartite graphs.
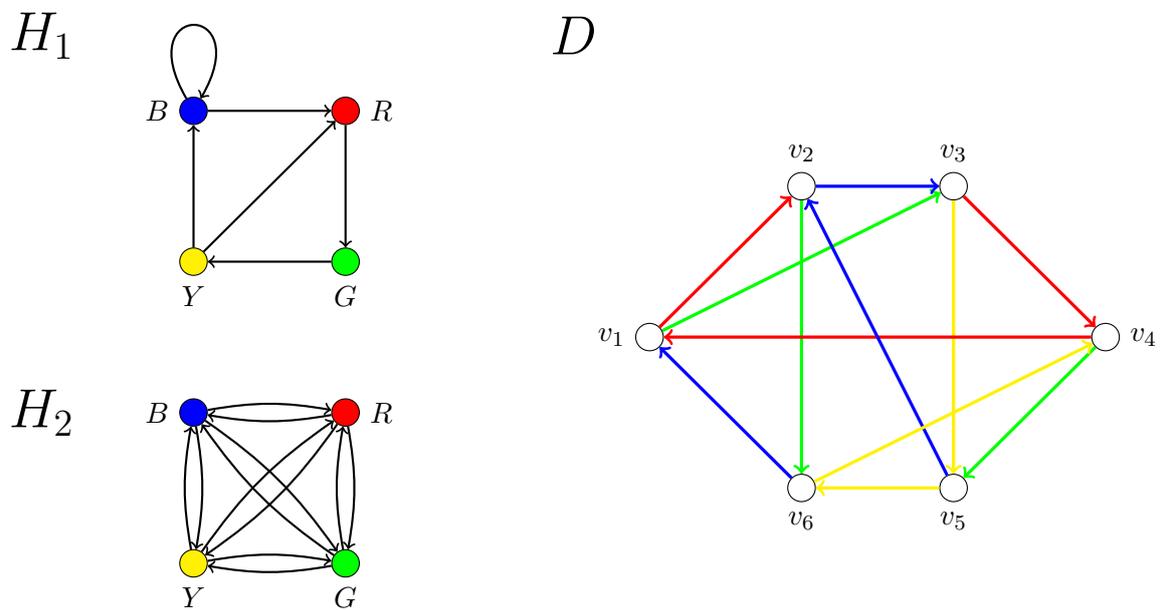
Figure 4.11: The trail $P_1 = (v_2, v_3, v_4, v_5, v_6, v_1, v_2, v_6, v_4, v_1, v_3, v_5, v_2)$ is a closed Eulerian H_1-trail but is not an H_2-trail, and $P_2 = (v_2, v_3, v_5, v_2, v_6, v_4, v_1, v_3, v_4, v_5, v_6, v_1, v_2)$ is a closed Eulerian H_2-trail but is not an H_1-trail.

Chapter 5

Finding H-trails in H-colored digraphs

5.1 Introduction

In view of the relevance in applications of finding properly colored walks, the problem of finding properly colored trails and paths, between two given vertices, has been studied from an algorithmic perspective. For example, Abouelaoualim et al. [1] proved that finding the shortest properly colored trail between two vertices can be done in polynomial time. Szeider [52] proved that given a c-edge-colored graph, $c \geq 2$, it can be found a properly colored path between two vertices in linear time on the number of edges of the graph.

Theorem 5.1 (Abouelaoualim et al. [1])**.** *Let G be a c-edge-colored graph, with $c \geq 2$. The problem of finding the shortest properly colored trail in G (if any) can be solved in polynomial time.*

Theorem 5.2 (Szeider [52])**.** *Let s and t be two vertices in a c-edge-colored graph G, with $c \geq 2$. Then, either we can find a properly edge-colored path between s and t or else decide that such a path does not exist in G in linear time on the size of the graph.*

Gourvès et al. [30] proved that decided whether a planar c-arc-colored digraph with no properly colored cycle contains a properly colored $s - t$ path is NP-complete. However, they also proved that the problem of finding a directed properly colored trails from a vertex s to a vertex t in a c-arc-colored digraph can be done within polynomial time. Moreover, they proved that the problem of maximizing the number of arc disjoint properly colored trails between two vertices can be solved in polynomial time.

Theorem 5.3 (Gourvès et al. [30])**.** *Deciding whether a planar c-arc-colored digraph with no properly colored cycle contains a properly colored $s - t$ path is NP-complete.*

Theorem 5.4 (Gourvès et al. [30])**.** *Let D be a c-arc-colored digraph, with $c \geq 2$. The problem of finding a directed properly colored trail in D (if any) can be solved in polynomial time.*

Theorem 5.5 (Gourvès et al. [30])**.** *The problem of maximizing the number of arc disjoint properly colored trails from s to t in D can be solved in polynomial time.*

A **transition** in a digraph D is a pair of adjacent arcs of D such that the head of one is the tail of the other one. A **transition system** of a digraph D is a set of transitions in D. Let T be a transition system, we say that a transition is **permitted** if it is in T and it is **forbidden** otherwise. We say that a walk is **T-compatible** whenever all its transitions are permitted. For every vertex $v \in V(D)$, the set of allowed transitions define a digraph $T(v)$, called **transition**

digraph of v, with vertex set the set of arcs incident with v, and there is an arc from e to f in $T(v)$ if and only if $\{e, f\} \in T$, where v is the head of e and the tail of f.

Szeider [52] studied the computational complexity of finding T-compatible paths between two given vertices of a graph, with a transition system T, proving that the problem is in NP, but find a class of transition systems that can be solve in linear time, for example, if $T(v)$ are complete graphs, for every $v \in V(G)$. Several authors have studied the existence of T-compatible trails, paths and cycles in graphs with a given transition system, from an algorithmic point of view (see, [6, 7]).

Note that H-walks in H-colored digraphs and T-compatibles walks in a digraph with a transition system T are equivalent. To see this equivalence consider the following constructions.

Let D be an H-colored digraph. We define the transition system of $v \in V(D)$ as the set $T(v) = \{(e, f) : e = (x, v), f = (v, y),$ for some $x, y \in V(D)$ and $(c(e), c(f)) \in A(H)\}$. Hence, if $T = \{T(v) : v \in V(D)\}$, then every T-compatible walk is an H-walk.

One the other hand, let D be a digraph with arc set $A(D) = \{f_1, \ldots, f_m\}$ and T a transition system of D. Consider the digraph H defined as follows: $V(H) = \{c_1, \ldots, c_m\}$; and $(c_i, c_j) \in A(H)$ if and only if $\{f_i, f_j\} \in T$. If we color the arc f_i with the vertex c_i, then every H-walk is a T-compatible walk. (In a natural way it arises the following question. Given a digraph D with transition system T, find a digraph H with the minimum number of vertices such that there exists an H-coloring of D where a walk is an H-walk if and only if is a T-compatible walk. Notice that such a digraph H exists and $|V(H)| \leq |A(D)|$).

It is important to notice that transition systems provide local information about the allowed transitions at each vertex. On the other hand, H-colorings provide global information about the allowed transitions.

In this chapter we study the problem of finding H-trail between two different vertices, in H-colored digraphs. We proved that determine if there exists an H-trail starting with the arc e and ending at arc f can be done in polynomial time. As a consequence, we give a polynomial time algorithm to find (if any exists) the shortest H-trail from the vertex s to the vertex t. Moreover, we show that the problem of maximizing the number of arc disjoint $s - t$ H-trails in D can be solved in polynomial time. We also study the computational complexity of finding an H-path between two given vertices of an H-colored digraph in terms of the digraph H.

5.2   Complexity of finding *H*-trails

In this section, we are interested in the complexity of finding H-trails in H-colored digraphs. To achieve this we will use the auxiliary digraph $L_2^H(D)$. The following observations follow as a direct consequence of the definition.

Observation 5. *Let D be an H-colored digraph. Then:*

1. *For every $e = (x, y)$ in $A(D)$, $d^+(f(e, x)) = d^-(f(e, y)) = 1$ in $L_2^H(D)$. Moreover, $N^+(f(e, x)) = \{f(e, y)\}$ and $N^-(f(e, y)) = \{f(e, x)\}$.*

2. *For every $u \in V(D)$, D_u is a bipartite digraph with partition $\{A_u^+, A_u^-\}$, where A_u^+ (respectively, A_u^-) is the set of arcs in D with head u (respectively, with tail u). Moreover, every arc in D_u has tail in A_u^- and head in A_u^+.*

3. *$L_2^H(D)$ is a bipartite digraph.*

The Algorithm 1 is a linear time algorithm that starting with a path in $L_2^H(D)$ obtains an H-trail in D.

Lemma 5.6. *Let H be a digraph possibly with loops and D an H-colored digraph. Given a path $T = (f(e_1, x_1), f(e_2, x_2), \ldots, f(e_k, x_k))$ in $L_2^H(D)$ such that $x_1 \neq x_2$ and $x_{k-1} \neq x_k$, Algorithm 1 returns an $x_1 - x_k$ H-trail in G.*

Algorithm 1 Path to *H*-trail

Require: A path $T = (f(e_1, x_1), f(e_2, x_2), \ldots, f(e_k, x_k))$ in $L_2^H(D)$ such that $x_1 \neq x_2$ and $x_{k-1} \neq x_k$.
Ensure: An *H*-trail in *D*.
  $i \leftarrow 1$
  P
  while $i \neq k/2 + 1$ **do**
    if $i = 1$ **then**
      $P \leftarrow P = (x_1, e_1, x_2)$
    else
      $P \leftarrow P = P \cup (x_{2i-1}, e_{2i-1}, x_{2i})$.
    end if
  end while

Proof. Let $T = (f(e_1, x_1), f(e_2, x_2), \ldots, f(e_k, x_k))$ be a path in $L_2^H(D)$ such that $x_1 \neq x_2$ and $x_{k-1} \neq x_k$.

By the definition of $L_2^H(D)$ and the fact that $x_1 \neq x_2$, we have that $(f(e_1, x_1), f(e_2, x_2))$ is not an edge of G_{x_1}. So, by Observation 5.1, it follows that $e_1 = e_2$ and $e_1 = (x_1, x_2)$. Hence, $P = (x_1, e_1, x_2)$ is an *H*-path in *D*. By the definition of $L_2^H(D)$ and Observation 5.1, we have that $x_2 = x_3$ and $(f(e_2, x_2), f(e_3, x_3)) \in A(G_{x_2})$, that is, $(c(e_1), c(e_3)) \in A(H)$. By Observation 5.1, it follows that $x_3 \neq x_4$, hence $e_3 = e_4$ and $e_3 = (x_3, x_4)$. So, $P = (x_1, e_1, x_2 = x_3, e_3, x_4)$ is an *H*-trail (notice that $e_1 \neq e_3$ otherwise $f(e_1, x_1) = f(e_3, x_3)$ which is impossible since $x_1 \neq x_2$ and $x_2 = x_3$).

By Observation 5.1, we have that $x_4 = x_5$ and $(f(e_4, x_4), f(e_5, x_5)) \in A(G_{x_4})$, that is, $(c(e_3), c(e_5)) \in A(H)$. By Observation 5.1, it follows that $x_5 \neq x_6$, hence $e_5 = e_6$ and $e_5 = (x_5, x_6)$. So, $P = (x_1, e_1, x_2 = x_3, e_3, x_4 = x_5, e_5, x_6)$ is an *H*-trail (notice that $e_5 \notin \{e_1, e_3\}$ otherwise *T* is not a path).

Following this reasoning, we have that $x_{2i} = x_{2i+1}$, $x_{2i+1} \neq x_{2i+2}$, $(c(e_{2i-1}), c(e_{2i+1})) \in A(H)$ and $e_{2i+1} = (x_{2i+1}, x_{2i+2})$, for every $i \in \{1, \ldots, k/2 - 1\}$. (Notice that *k* is even since $x_{k-1} \neq x_k$).

Hence, $P = (x_1, e_1, x_2 = x_3, e_3, x_4 = x_5, \ldots, x_{k-1}, e_{k-1}, x_k)$ is an *H*-walk in *D*. Moreover, since *T* is a path, it follows that *P* is an *H*-trail.

Therefore, Algorithm 1 returns an *H*-trail in *D* in linear time. $\square$

Lemma 5.7. *Let D be an H-colored digraph and* $e = (x_1, y_1)$, $g = (x_2, y_2)$ *different arcs of D. If there is no* $f(e, y_1) - f(g, x_2)$ *path in* $L_2^H(D)$, *then there is no H-trail in D starting in e and ending in g.*

Proof. Proceeding by contradiction, assume that there exists an *H*-trail in *D* starting in *e* and ending in *g*, namely $P = (x_1, e, y_1, e_1, v_2, \ldots, v_k, e_k, x_2, g, y_2)$.

Hence, $T = (f(e, x_1), f(e, y_1), f(e_1, v_2), \ldots, f(e_k, v_k), f(e_k, x_2), f(g, x_2), f(g, y_2))$ is a path in $L_2^H(D)$.

Therefore, there exist an $f(e, y_1) - f(g, x_2)$ path in $L_2^H(D)$, a contradiction. $\square$

Theorem 5.8. *Given an arbitrary H-colored digraph D, finding an H-trail starting with the arc e and ending with the arc g (if any) can be done within polynomial time.*

Proof. Let *D* be an *H*-colored digraph, $e = (x_1, y_1)$ and $g = (x_q, y_q)$ two arcs in *D*. Construct the auxiliary digraph $L_2^H(D)$.

Consider the vertices $f(e, y_1)$ and $f(g, x_q)$ in $L_2^H(D)$. Find a path from $f(e, y_1)$ to $f(g, x_q)$, namely *P*. (If there is no $f(e, y_1) - f(g, x_q)$ path in $L_2^H(D)$, then there is no *H*-trail starting in *e* and ending in *g* in *D*).

It follows from Observation 5.1 that $P' = (f(e, x_1), f(e, y_1), P, f(g, x_q), f(g, y_q))$ is also a path in $L_2^H(D)$.

Therefore, by Algorithm 1, there is an H-trail in D starting with the arc e and ending with the arc g. Notice that each step can be done in polynomial time.   □

Corollary 5.9. *Given an arbitrary H-colored digraph D, finding a closed H-trail containing the arc e (if any) can be done within polynomial time.*

Proof. Let D be an H-colored digraph and $e = (x, y)$ an arc in $A(D)$. Construct the auxiliary digraph $L_2^H(D)$.

Consider the vertices $f(e, y)$ and $f(e, x)$ in $L_2^H(D)$. Find a path from $f(e, y)$ to $f(e, x)$, namely C. (If there is no $f(e, y) - f(e, x)$ path in $L_2^H(D)$, then there is no closed H-trail containing the arc e in D).

It follows directly from Observation 5.1 that $C' = (f(e, y), C, f(e, x), f(e, y))$ is a cycle in $L_2^H(D)$.

Therefore, by Theorem 4.5, there is a closed H-trail in D that contains the arc e.   □

Theorem 5.10. *Let D be an H-colored digraph. It can be find the shortest H-trail between any pair of arcs in polynomial time, if any exists.*

Proof. Let D be an H-colored digraph, $e = (x_1, y_1)$ and $g = (x_q, y_q)$ two arcs in D. Construct the auxiliary digraph $L_2^H(D)$.

Consider the vertices $f(e, y_1)$ and $f(g, x_q)$ in $L_2^H(D)$. Find the shortest path from $f(e, y_1)$ to $f(g, x_q)$, say P. (If there is no $f(e, y_1) - f(g, x_q)$ path in $L_2^H(D)$, then there is no H-trail starting in e and ending in g).

Then, $P' = (f(e, x_1), f(e, y_1), P, f(g, x_q), f(g, y_q))$ is also a path in $L_2^H(D)$ because of the Observation 5.1.

Therefore, by Algorithm 1, there is an H-trail in D starting with the arc e and ending with the arc g, namely T. Notice that each step can be done in polynomial time.

Notice that if there is a shorter H-trail from e to g in D than T, namely $Q = (v_1 = x_1, g_1 = e, v_2 = y_1, g_2, v_3, \ldots, v_{j-1} = x_q, g_{j-1} = g, v_j = y_q)$, then $W = (f(g_1, v_2), f(g_2, v_2), f(g_2, v_3), \ldots, f(g_{j-1}, v_{j-1}))$ is a shorter $f(e, y_1) - f(g, x_q)$ path than P in $L_2^H(D)$, a contradiction.   □

It follows from Theorem 5.8 that finding an $s - t$ H-trail in D (if any) can be done within polynomial time. This can be done using Theorem 5.8 with all the possible pairs of arcs, one with tail s and the other one with head t. This method can be improved using the following variation of the auxiliary digraph $L_2^H(D)$.

Definition 5.1. *Let D be an H-colored digraph and s, t a pair of distinct vertices in $V(D)$. The digraph $L_2^H(s, t)$ is the digraph with vertex set $V(L_2^H(s, t)) = V(L_2^H(D)) \cup \{x_s, x_t\}$; and arc set $A(L_2^H(s, t)) = A(L_2^H(D)) \cup \{(x_s, f(e, s)) : e = (s, u) \in A(D)\} \cup \{(f(e, t), x_t) : e = (u, t) \in A(D)\}$.*

Theorem 5.11. *Given an arbitrary H-colored digraph D, finding an $s - t$ H-trail in D (if any) can be done within polynomial time.*

Proof. Let D be an H-colored digraph and s and t two vertices of D. Construct the auxiliary digraph $L_2^H(s, t)$.

Find a path from x_s to x_t in the digraph $L_2^H(s, t)$ (if any), namely P. (If there is no $x_s - x_t$ path in $L_2^H(s, t)$, then there is no $s - t$ H-trail in D).

Hence, $P' = P \setminus \{x_s, x_t\}$ is a path in $L_2^H(D)$, and by Algorithm 1, there is an $s - t$ H-trail in D.   □

Corollary 5.12. *Let D be an H-colored digraph. It can be found the shortest H-trail between any pair of distinct vertices in polynomial time, if any exists.*
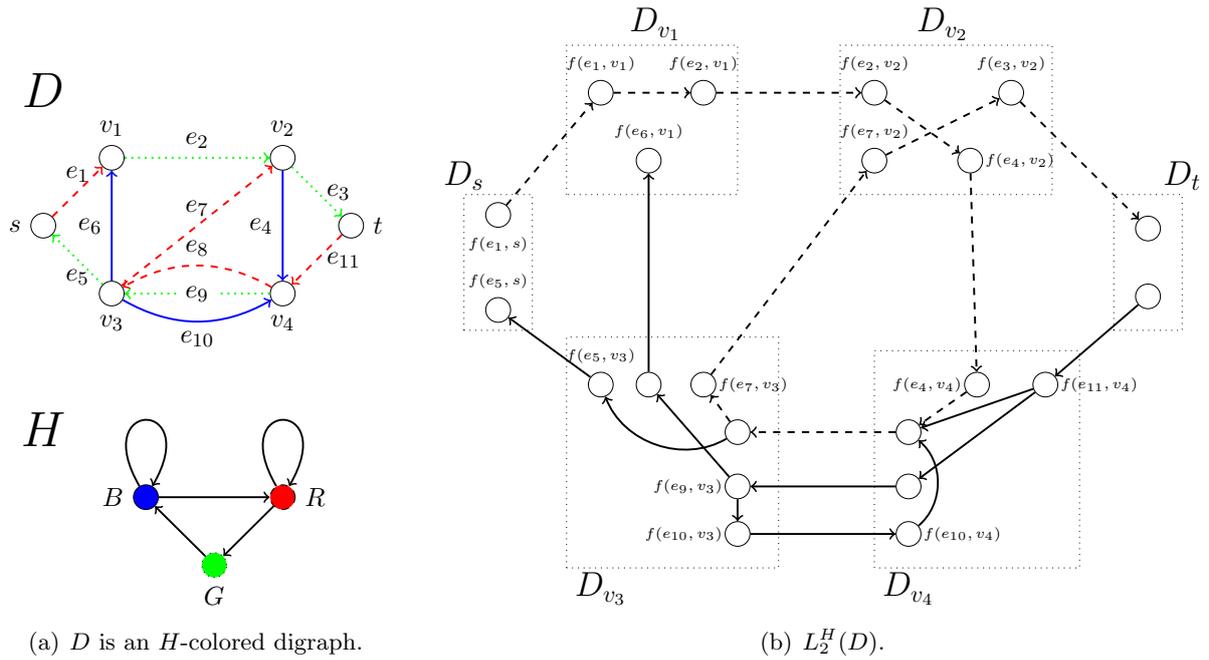
(a) D is an H-colored digraph.

(b) $L_2^H(D)$.

Figure 5.1: $P = (s, e_1, v_1, e_2, v_2, e_4, v_4, e_8, v_3, e_7, v_2, e_3, t)$ is an $s - t$ H-trail in D but there is no H-path from s to t in D. Moreover, P is obtained by applying Algorithm 1 to the dashed path of $L_2^H(D)$.

If D is an H-colored digraph, we say that an (s, t)-H-**trails-separator by arcs** is a subset $X \subseteq A(D)$ with the property that $D - X$ has no $s - t$ H-trails.

Theorem 5.13. *Let D be an H-colored digraph and s, t a pair of different vertices in $V(D)$. Then, the following assertions are equivalent.*

a. *The maximum number of arc-disjoint $s - t$ H-trail in D is equal to k.*

b. *The maximum number of internally disjoint $x_s - x_t$-paths in $L_2^H(s, t)$ is equal to k.*

c. *The minimum number of vertices in an (x_s, x_t)-separator in $L_2^H(s, t)$ is equal to k.*

d. *The minimum number of arc in an (s, t)-H-trails-separator by arcs in D is equal to k.*

Proof. Let k_1 be the maximum number of arc-disjoint $s - t$ H-trail in D, k_2 the maximum number of internally disjoint $x_s - x_t$-paths in $L_2^H(s, t)$, k_3 the minimum number of vertices in an (x_s, x_t)-separator in $L_2^H(s, t)$ and k_4 the minimum number of arcs in an (s, t)-H-trails-separator by arcs in D. We will prove that $k_1 = k_2 = k_3 = k_4$.

Claim 1. $k_1 = k_2$.

It follows by Algorithm 1 that $k_2 \leq k_1$. Let $\{P_1, \ldots, P_{k_1}\}$ a set of k_1 arc-disjoint $s-t$ H-trails in D. For each i in $\{1, \ldots, k_1\}$, we can construct an $x_s - x_t$-path in $L_2^H(s, t)$ from the H-trail P_i as follows: Let $P_i = (s, e_0^i, x_1^i, e_1^i, x_2^i, \ldots, x_{j_i}^i, e_{j_i}^i, t)$, so $T_1 = (x_s, f(e_0^i, s), f(e_0^i, x_1^i), f(e_1^i, x_1^i), f(e_1^i, x_2^i), \ldots, f(e_{j_i}^i, x_{j_i}^i), f(e_{j_i}^i, t), x_t)$ is a path in $L_2^H(s, t)$.

It follows from the construction of each T_i and Observation 5.1 that T_i and T_j are internally disjoint $x_s - x_t$-paths in $L_2^H(s, t)$, for every $\{i, j\} \subseteq \{1, \ldots, k_1\}$. Hence, $k_1 \leq k_2$ and the Claim 1 holds.

Claim 2. $k_3 \leq k_4$.

Let $A = \{e_i = (x_i, y_i) \in A(D) : i \in \{1, \ldots, k_4\}\}$ be an (s, t)-H-trails-separator by arcs in D with k_4 arcs. Consider $B = \{f(e_i, y_i) \in L_2^H(s, t)\}$.

Suppose that there exists an $x_s - x_t$-path in $L_2^H(s,t) \setminus B$, namely P. Hence, by Algorithm 1 applied to the path P, there is T an H-trail from s to t in D such that $e_i \notin A(T)$, a contradiction. Therefore, B is an (x_s, x_t)-separator in $L_2^H(s,t)$ with k_4 vertices, and the claim holds.

Claim 3. $k_4 \leq k_3$.

Let $A = \{f(e_i, x_i) : i \in \{1, \ldots, k_3\}\}$ be an (x_s, x_t)-separator in $L_2^H(s,t)$.

It follows by Observation 5.1 that for every $e = (x, y) \in A(D)$, at most one of the vertices $f(e, x)$ and $f(e, y)$ is in a minimum (x_s, x_t)-separator in $L_2^H(s,t)$. Hence, $B = \{e_i \in A(D) : f(e_i, x_i) \in A\}$ has k_3 arcs.

Notice that B is an (s,t)-H-trails-separator by arcs in D. Otherwise, there is an (s,t) H-trail in D, say P, and we can found T an $x_s - x_t$-path in $L_2^H(s,t)$ from P (as in Claim 1) such that $V(T) \cap A = \emptyset$, a contradiction. Therefore, $k_4 \leq k_3$ and the claim holds.

Notice that $k_2 = k_3$ follows by Menger's Theorem. Therefore, $k_1 = k_2 = k_3 = k_4$. □

Corollary 5.14. *The problem of maximizing the number of arc disjoint $s - t$ H-trails in D can be solved in polynomial time.*

Recall that if H is a complete digraph without loops, then every H-trail is a properly colored trail.

Corollary 5.15. *Given an arbitrary c-arc-colored digraph D, finding an properly colored trail starting with the arc e and ending with the arc f (if any) can be done within polynomial time.*

Corollary 5.16. *Given an arbitrary c-arc-colored digraph D and s and t vertices of D, finding a properly colored $s - t$ trail in D (if any) can be done within polynomial time.*

Corollary 5.17. *The problem of maximizing the number of arc disjoint properly colored $s - t$ trails in D can be solved in polynomial time.*

Recall that if H is a complete digraph without loops, then every H-path is a properly colored path. So, the following result follows immediately from Theorem 5.3.

Corollary 5.18. *Deciding whether an H-colored digraph contains an $s - t$ H-path is NP-complete.*

Benítez-Bobadilla et al. [9] gave a characterization of the digraphs H such that for every digraph D and every H-coloring of D, it follows that every H-walk between two vertices in D contains an H-path with the same endpoints.

Theorem 5.19 (Benítez-Bobadilla et al. [9])**.** *Let H be a reflexive digraph. H is transitive if and only if for every H-colored digraph D, and every pair of different vertices s and t of D, every $s - t$ H-walk in D contains an $s - t$ H-path in D.*

The following results are a direct consequence of Corollary 5.11 and Theorem 5.19.

Corollary 5.20. *Let H be a transitive and reflexive digraph. Given an arbitrary H-colored digraph D, finding an $s - t$ H-path in D (if any) can be done within polynomial time.*

Corollary 5.21. *Given an arc-colored digraph D, finding a monochromatic $s - t$ path in D (if any) can be done within polynomial time.*

Chapter 6

Dynamic H-cycles in H-colored multigraphs

In this chapter, we study the existence of dynamic H-cycles, and the length of dynamic H-cycles and dynamic H-paths in H-colored multigraphs. To accomplish this, we introduce a new concept of color degree that allows us to extend some classic results, such as, Ore's Theorems, for H-colored multigraphs. Also, we give sufficient conditions for the existence of Hamiltonian dynamic H-cycles in H-colored multigraphs with at most one "lane change", and as a consequence, we obtain sufficient conditions for the existence of properly colored Hamiltonian cycle in c-edge-colored multigraphs, with $c \geq 3$. Moreover, we improve the conditions given in Theorem 2.10 b) for an infinitely family of multigraphs. In most of the results in this chapter we use as a hypothesis that the auxiliary graph G_u is a complete k_u-partite graph, so we finish the chapter by giving examples of H-colored multigraphs that fulfill this hypothesis.

6.1   Notation and Preliminaries

Let G be an H-colored multigraph, and $W = (v_0, e_0^1, \ldots, e_0^{k_0}, v_1, e_1^1, \ldots, e_1^{k_1}, v_2, \ldots, v_{n-1}, e_{n-1}^1, \ldots, e_{n-1}^{k_{n-1}}, v_n)$ a dynamic H-walk in G. We define the **length** of W, denoted by $l(W)$, as the number n. We will say that the dynamic H-walk W has $k_i - 1$ **changes from** v_i **to** v_{i+1}, and W has $\sum_{i=1}^{n-1}(k_i - 1)$ **changes**. Notice that if W is a dynamic H-walk with zero changes, then W is an H-walk. In Figure 6.1, $T = (v_1, e_4, v_4, e_{10}, v_5, e_{14}, e_{13}, v_6, e_{15}, v_7)$ is a dynamic H-path of length 4 with one change, and $C = (v_4, e_9, v_5, e_{14}, e_{13}, v_6, e_{15}, v_7, e_{11}, v_4)$ is a dynamic H-cycle of length 4 with one change.

Let G be an H-colored multigraph and $\{u, v\} \subseteq V(G)$. We will say that E_{uv} is a **dynamic edge set** if and only if there exists $\{e, f\} \subseteq E_{uv}$ such that $N_H(c(e)) \neq N_H(c(f))$ and neither of them is subset of the other. The **dynamic degree** of u, denoted by $\delta_{dym}(u)$, is the number of vertices v such that E_{uv} is a dynamic edge set. In Figure 6.1, the set $E_{v_1 v_4}$ is a dynamic edge set but $E_{v_1 v_2}$ is not a dynamic edge set, since $N_H(c(e_1)) = N_H(c(e_2)) = \{R\}$.

Observation 6. *Let G be an H-colored multigraph such that G_u is a complete k_u-partite graph, for some $k_u \geq 2$. If E_{uv} is a dynamic edge set, then there exist e and f, in E_{uv}, in different partite sets of $V(G_u)$.*

Proposition 6.1. *Let G be an H-colored multigraph such that G_u is a complete k_u-partite graph, for every u in $V(G)$ and for some k_u, $k_u \geq 2$. If $T = (x_0, x_1, \ldots, x_p)$ is a walk in G such that for each $i \in \{0, \ldots, p-1\}$, $E_{x_i x_{i+1}}$ is a dynamic edge set, then there exist $e_i \in E_{x_i x_{i+1}}$, for every $i \in \{0, \ldots, p-1\}$, such that $T' = (x_0, e_0, x_1, \ldots, x_{p-1}, e_{p-1}, x_p)$ is an H-walk. Moreover, if $E_{x_0 x_p}$ is a dynamic edge set, then there exists $\{e_p, e_{p+1}\} \subseteq E_{x_0 x_p}$ such that $C = (x_0, e_0, x_1, \ldots, x_{p-1}, e_{p-1},$*
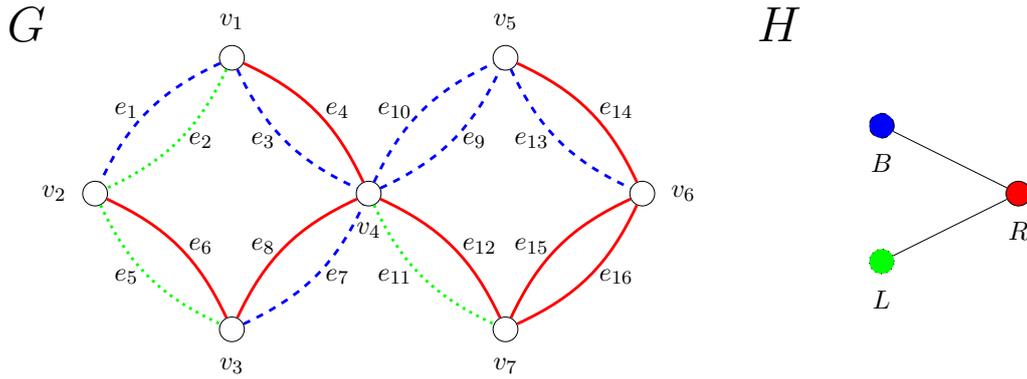
Figure 6.1: The sequence $P = (v_4, e_9, v_5, e_{14}, e_{13}, v_6, e_{16}, v_7, e_{11}, e_{12}, v_4)$ is a dynamic H-cycle in G and there is no H-cycle of length greater than 2 containing v_5, v_6 or v_7.

$x_p, e_p, x_0)$ is a closed H-walk or $C = (x_0, e_0, x_1, \ldots, x_{p-1}, e_{p-1}, x_p, e_p, e_{p+1}, x_0)$ is a closed dynamic H-walk, i.e., there exists a closed dynamic H-walk with at most one change.

Proof. Suppose that G is an H-colored multigraph such that G_u is a complete k_u-partite graph, for every u in $V(G)$ and for some $k_u \geq 2$. Let $T = (x_0, x_1, \ldots, x_p)$ be a walk in G such that $E_{x_i x_{i+1}}$ is a dynamic edge set.

Consider the edge $e_0 = x_0 x_1$ in $E(G)$, since $E_{x_1 x_2}$ is a dynamic edge set, by Observation 6, we have that there exist $f_1 = x_1 x_2$ and $f_2 = x_1 x_2$ in different partite sets of $V(G_{x_1})$. It follows from the fact that $e_0 \in V(G_{x_1})$ and G_{x_1} is a complete k_{x_1}-partite graph, that $e_0 f_1 \in E(G_{x_1})$ or $e_0 f_2 \in E(G_{x_1})$. Let e_1 be the edge such that $e_0 e_1 \in E(G_{x_1})$, i.e., $e_1 = f_1$ or $e_1 = f_2$ (in case that both edges are adjacent to e_0, we take $e_1 = f_1$).

Since $E_{x_2 x_3}$ is a dynamic edge set, by Observation 6, we have that there exist $g_1 = x_2 x_3$ and $g_2 = x_2 x_3$ in different partite sets of $V(G_{x_2})$. It follows from the fact that $e_1 \in V(G_{x_2})$ and G_{x_2} is a complete k_{x_2}-partite graph, that $e_1 g_1 \in E(G_{x_2})$ or $e_1 g_2 \in E(G_{x_2})$. Let e_2 be the edge such that $e_1 e_2 \in E(G_{x_2})$, i.e., $e_2 = g_1$ or $e_2 = g_2$ (in case that both edges are adjacent to e_1, we take $e_2 = g_1$). Repeating this procedure, we have that for every $i \in \{0, \ldots, n-1\}$, there exist $e_i \in E_{x_i x_{i+1}}$ such that $T' = (x_0, e_0, x_1, e_1, x_2, \ldots, x_{p-1}, e_{p-1}, x_p)$ is an H-walk.

Now, suppose that $E_{x_0 x_p}$ is a dynamic edge set, since G_{x_p} is a complete k_{x_p}-partite graph, there exists $e_p = x_p x_0$ such that $e_{p-1} e_p \in E(G_{x_p})$.

If $e_p e_0 \in E(G_{x_0})$, then $C = (x_0, e_0, x_1, \ldots, x_p, e_p, x_0)$ is a closed H-walk.

Otherwise, $e_p e_0 \notin E(G_{x_0})$. Since $E_{x_p x_0}$ is a dynamic edge set, there exist an edge $e_{p+1} = x_p x_0$ such that e_p and e_{p+1} are in different partite sets of $V(G_{x_0})$. Now, since G_{x_0} is a complete k_{x_0}-partite graph and $e_p e_0 \notin E(G_{x_0})$, we have that $e_{p+1} e_0 \in E(G_{x_0})$. Therefore, $C = (x_0, T', x_p, e_p, e_{p+1}, x_0)$ is a closed dynamic H-walk. $\square$

The following result is a direct consequence of Proposition 6.1 and the definition of path, cycle, H-path and dynamic H-cycle.

Corollary 6.2. *Let G be an H-colored multigraph such that G_u is a complete k_u-partite graph, for every u in $V(G)$ and for some k_u, $k_u \geq 2$. If $T = (x_0, x_1, \ldots, x_p)$ is a path in G such that for every $i \in \{0, \ldots, p-1\}$, $E_{x_i x_{i+1}}$ is a dynamic edge set, then there exist $e_i \in E_{x_i x_{i+1}}$, for every $i \in \{0, \ldots, p-1\}$, such that $T' = (x_0, e_0, x_1, \ldots, x_{p-1}, e_{p-1}, x_p)$ is an H-path. Moreover, if $E_{x_0 x_p}$ is a dynamic edge set, then there exist $\{e_p, e_{p+1}\} \subseteq E_{x_0 x_p}$ such that $C = (x_0, e_0, x_1, \ldots, x_{p-1}, e_{p-1}, x_p, e_p, x_0)$ is an H-cycle or $C = (x_0, e_0, x_1, \ldots, x_{p-1}, e_{p-1}, x_p, e_p, e_{p+1}, x_0)$ is a dynamic H-cycle, i.e., there exists a dynamic H-cycle with at most one change.*

The multigraph G, in Figure 6.2, is an H-colored multigraph such that G_{v_i} is a complete bipartite graph, for every $i \in \{1, 2, 3, 4, 5, 6, 7\}$. Moreover, $E_{v_i v_j}$ is a dynamic edge set if and

only if $E_{v_iv_j} \neq \emptyset$. So, the path $P = (v_1, v_2, v_7, v_4, v_5)$ and the cycle $C = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$ fulfill the hypothesis of Corollary 6.2. Therefore, there exist an H-path and a dynamic H-cycle with at most one change with the same order of the vertices of P and C, respectively, namely $P' = (v_1, e_1, v_2, , e_8, v_7, e_{13}, v_4, e_{12}, v_5)$ and $C' = (v_1, e_2, v_2, e_5, v_3, e_{10}, v_4, e_{11}, v_5, e_{16}, v_6, e_3, v_1)$.
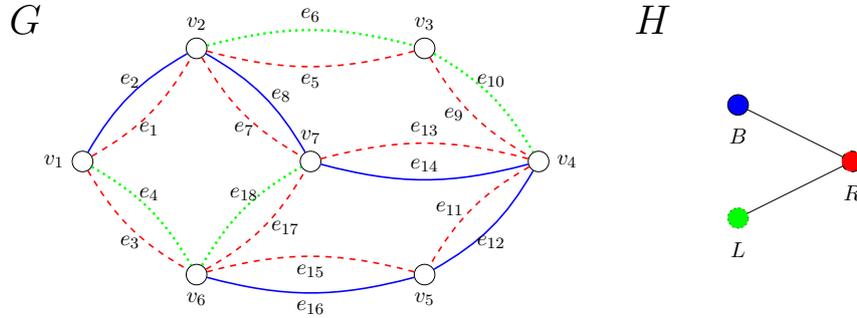

Figure 6.2: Notice that $T = (v_6, v_7, v_2, v_1, v_6, v_7, v_4)$ is a trail that fulfill the hypothesis of Proposition 6.1 but there is no H-trail with the same order of vertices. Although, $T' = (v_6, e_{17}, v_7, e_8, v_2, e_1, v_1, e_4, v_6, e_{17}, v_7, e_{14}, v_4)$ is the H-walk that Proposition 6.1 states that exists.

6.2 Dynamic *H*-cycles and *H*-paths

Theorem 6.3. *Let G be an H-colored multigraph such that G_u is a complete k_u-partite graph, for every u in $V(G)$ and for some k_u, $k_u \geq 2$. If $\delta_{dym}(u) \geq d \geq 2$ for every $u \in V(G)$, then G has a dynamic H-cycle of length at least $d+1$ and with at most one change.*

Proof. Let $T = (x_0, x_1, \ldots, x_k)$ be a path of maximum length such that $E_{x_ix_{i+1}}$ is a dynamic edge set, for every $i \in \{0, \ldots, k-1\}$.

Claim 1. T has length at least d.

Suppose that T has length at most $d-1$. Since $\delta_{dym}(x_k) \geq d$, there exists a vertex $x_{k+1} \in V(G) \setminus V(T)$ such that $E_{x_kx_{k+1}}$ is a dynamic edge set. Hence, $T' = (x_0, x_1, \ldots, x_k, x_{k+1})$ is a path of length $k+1$ such that $E_{x_ix_{i+1}}$ is a dynamic edge set, for every $i \in \{0, \ldots, k\}$, a contradiction. Therefore, T has length at least d.

Since T is of maximum length, if E_{x_0u} is a dynamic edge set, then $u \in V(T)$ (otherwise we can extend T). Let $j = max\{i : E_{x_0x_i}$ is a dynamic edge set$\}$. Since $\delta_{dym}(x_0) \geq d$, we have that $j \geq d$. Therefore, $C' = (x_0, x_1, \ldots, x_j, x_0)$ is a cycle such that $E_{x_ix_{i+1}}$ is a dynamic edge set, for every $i \in \{0, \ldots, j\}$, see Figure 6.3. Hence by Corollary 6.2, we have that there exists a dynamic H-cycle of length $j + 1 \geq d + 1$ with at most one change. $\square$

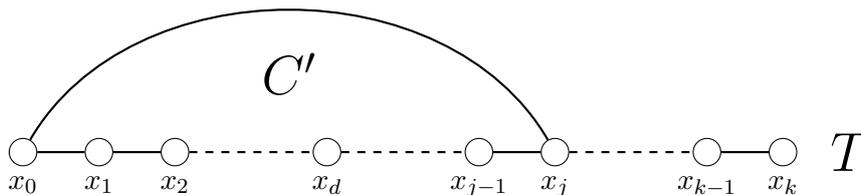
Figure 6.3: For every $i \in \{1, \ldots, k-1\}$, $E_{x_ix_{i+1}}$ is a dynamic edge set. Moreover, $E_{x_0x_j}$ is also a dynamic edge set and there is no x_q, with $q > j$, such that $E_{x_0x_q}$ is a dynamic edge set.

Let K_n^2 be a complete multigraph with $|E_{uv}| = 2$, for every $\{u, v\} \subseteq V(K_n^2)$, and H be a complete simple graph with $k \geq 2$ vertices. Consider the graph G that is the union of two K_n^2

that share a unique vertex. If we H-color G in such a way that every pair of parallel edges has different color, then G has $2n-1$ vertices, $\delta_{dym}(x) \geq n-1$, for every $x \in V(G)$, and the length of the maximum dynamic H-cycle in G is n, see Figure 6.4. So, we cannot improve the length of the dynamic H-cycle in Theorem 6.3.
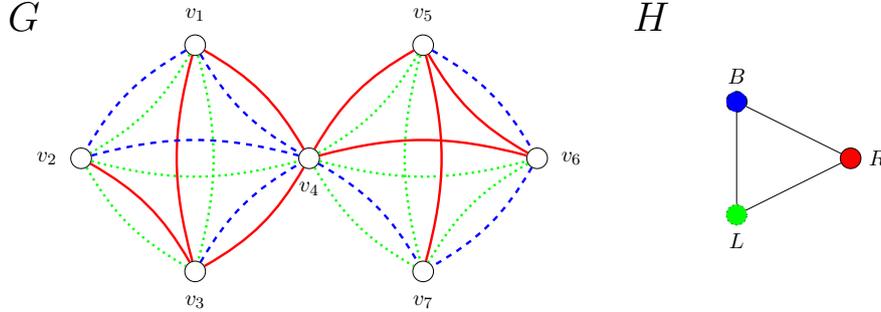

Figure 6.4: G is an H-colored graph with $n = 7$ vertices such that $\delta_{dym}(x) \geq 3$ and there is no dynamic H-cycle of length greater than 4.

Corollary 6.4. *Let G be an H-colored complete multigraph such that G_u is a complete k_u-partite graph, for every $u \in V(G)$ and for some k_u, $k_u \geq 2$. If E_{xy} is a dynamic edge set, for every $\{x, y\} \subseteq V(G)$, then G has a Hamiltonian dynamic H-cycle with at most one change.*

Theorem 6.5. *Let G be an H-colored multigraph such that G_u is a complete k_u-partite graph, for every u in $V(G)$ and for some k_u, $k_u \geq 3$. If $\delta_{dym}(u) \geq d \geq 2$, for every $u \in V(G)$, then G has an H-path of length at least $\min\{2d, n\}$, or G has an H-cycle of length at least $d + 1$.*

Proof. Suppose that G is an H-colored multigraph such that for every $u \in V(G)$, we have that $\delta_{dym}(u) \geq d \geq 2$, and G_u is a complete k_u-partite graph, for some $k_u \geq 3$.

Then, for every $x \in V(G)$, there exist $e_x, f_x, g_x \in E(G)$ such that $\{e_x, f_x\} \subseteq E_{xv_x}$, for some $v_x \in V(G)$, $g_x = xy_x \notin E_{xv_x}$ and e_x, f_x and g_x are in different partite sets of G_x (it is possible by Observation 6 and the fact that $\delta_{dym}(x) \geq 2$).

Let $T = (u_0, u_1, \ldots, u_{j-1} = v_x, u_j = x, u_{j+1} = y_x, \ldots, u_k)$ the longest path such that $E_{u_i u_{i+1}}$ is a dynamic edge set, for every $i \in \{0, \ldots, j-1, j+1, \ldots, k-1\}$, i.e., E_{xy_x} is not necessarily a dynamic edge set in T.

Notice that $e_x \in E_{u_{j-1} u_j}$, $g_x \in E_{u_j u_{j+1}}$ and $l(T)$ is at least d since $\delta_{dym}(u_0) \geq d$ and T is a path of maximum length, it follows that $k = l(T) \geq d$. We divide the rest of the proof in two cases.

Case 1. $k \geq 2d$.

Since $E_{u_{j-2} u_{j-1}}$ is a dynamic edge set, it follows that there is an edge $e_{j-2} \in E_{u_{j-2} u_{j-1}}$ such that $c(e_{j-2})c(g_x) \in E(H)$. So, following the same procedure as in the proof of Proposition 6.1, we can construct the following: 1) An H-path from u_{j-1} to u_0 starting with the edge e_{j-2}, say $T_0 = (u_{j-1}, e_{j-2}, u_{j-2}, \ldots, u_1, e_0, u_0)$; and 2) an H-path from u_j to u_k starting with the edge e_x, say $T_1 = (u_j, e_x, u_{j+1}, \ldots, u_{k-1}, e_k, u_k)$.

Hence, $T' = (u_0, T_0^{-1}, u_{j-1}, g_x, u_j, T_1, u_k)$ is an H-path of length $k \geq 2d$.

Case 2. $k \leq 2d - 1$.

Subcase 2.1 $j + 1 \leq d$.

Notice that if $E_{u_0 v}$ is a dynamic edge set, then $v \in V(T)$. Otherwise, $T' = (v, u_0, u_1, \ldots, u_k)$ is a path of length $k + 1$ such that $E_{v_i v_{i+1}}$ is a dynamic edge set, for every $i \in \{0, \ldots, j-1, j+1, \ldots, k-1\}$, contradicting the choice of T. Therefore, $v \in V(T)$.

On the other hand, since $\delta_{dym}(u_0) \geq d$, there is $u_p \in V(T)$, where $d \leq p \leq k$ such that $E_{u_0 u_p}$ is a dynamic edge set. Hence, $C = (u_0, u_1, \ldots, u_j, u_{j+1}, \ldots, u_p, u_0)$ is a cycle in G such that $E_{u_i u_{i+1}}$ is a dynamic edge set, for every $i \in \{0, \ldots, j-1, j+1, \ldots, p-1\}$. Hence, by Corollary 6.2, there is an H-path $T' = (x, g_x, y_x, e_{j+1}, u_{j+2}, \ldots, u_{j-2}, e_{j-2}, u_{j-1} = v_x)$.

Since e_x and f_x are incident with v_x, we have that e_{j+2}, e_x and f_x are vertices of G_{v_x}. We know that e_x and f_x are in different partite sets of $V(G_{v_x})$, thus $e_{j+2}e_x \in E(G_{v_x})$ or $e_{j+2}f_x \in E(G_{v_x})$. Hence, $C' = (x, T', v_x, e_x, x)$ or $C' = (x, T', v_x, f_x, x)$ is an H-cycle (because, pairwise e_x, f_x and g_x are in different partite sets of G_x) and C' has length at least $d+1$, see Figure 6.5.
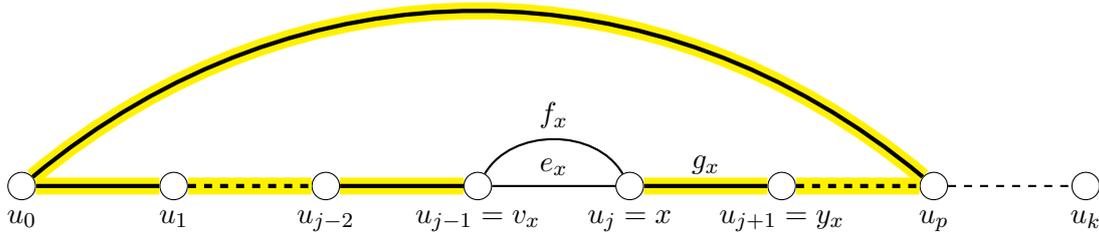

Figure 6.5: T' is highlighted in yellow. Since e_x, f_x and g_x are adjacent in G_x, we can choose an edge between e_x and f_x to obtain an H-cycle.

Subcase 2.2 $j + 1 > d$.
Notice that if $E_{u_k v}$ is a dynamic edge set, then $v \in V(T)$. Otherwise, $T' = (u_0, u_1, \ldots, u_k, u_{k+1} = v)$ is a path of length $k + 1$ such that $E_{v_i v_{i+1}}$ is a dynamic edge set, for every $i \in \{0, \ldots, j-1, j+1, \ldots, k\}$, contradicting the choice of T. Therefore, $v \in V(T)$.

On the other hand, since $\delta_{dym}(u_k) \geq d$, there is $u_p \in V(T)$, where $p \leq 2d - d - 1 = d - 1$, such that $E_{u_k u_p}$ is a dynamic edge set. Hence, by Corollary 6.2, there is an H-path $T' = (x, g_x, y_x, e_{j+1}, u_{j+2}, \ldots, u_k, e_k, u_p, \ldots, u_{j-1} = v_x)$.

Since e_x and f_x are incident with v_x, we have that e_{j+2}, e_x and f_x are vertices of G_{v_x}. We know that e_x and f_x are in different partite sets of $V(G_{v_x})$, thus $e_{j+2}e_x \in E(G_{v_x})$ or $e_{j+2}f_x \in E(G_{v_x})$. Hence, $C' = (x, T', v_x, e_x, x)$ or $C' = (x, T', v_x, f_x, x)$ is an H-cycle (because, pairwise e_x, f_x and g_x are in different parts of the partition of G_x) and C' has length at least $d+1$, see Figure 6.6.

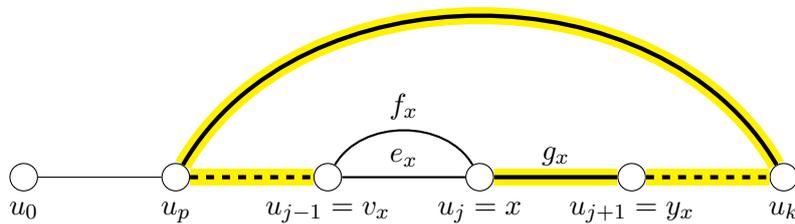Therefore, G has an H-path of length at least $\min\{2d, n\}$, or G has an H-cycle of length at least $d+1$. $\qquad\square$

Figure 6.6: T' is highlighted in yellow. Since e_x, f_x and g_x are adjacent in G_x, we can choose an edge between e_x and f_x to obtain an H-cycle.

Let G be an H-colored multigraph. We will say that the **dynamic graph of** G, denoted by G_{dym}, is the simple graph such that $V(G_{dym}) = V(G)$ and two different vertices u and v are adjacent, with only one edge in G_{dym}, if and only if E_{uv} is a dynamic edge set in G. An example of an H-colored graph and its dynamic graphs is shown in Figure 6.7.

Theorem 6.6. *Let G be an H-colored multigraph such that G_u is a complete k_u-partite graph, for every u in $V(G)$ and for some k_u, $k_u \geq 2$. If G_{dym} is connected and $\delta_{dym}(x) = 2p_x$, where $p_x \geq 1$, then G has a spanning closed dynamic H-trail with at most one change.*

(a) G is an H-colored multigraph.

(b) G_{dym}.
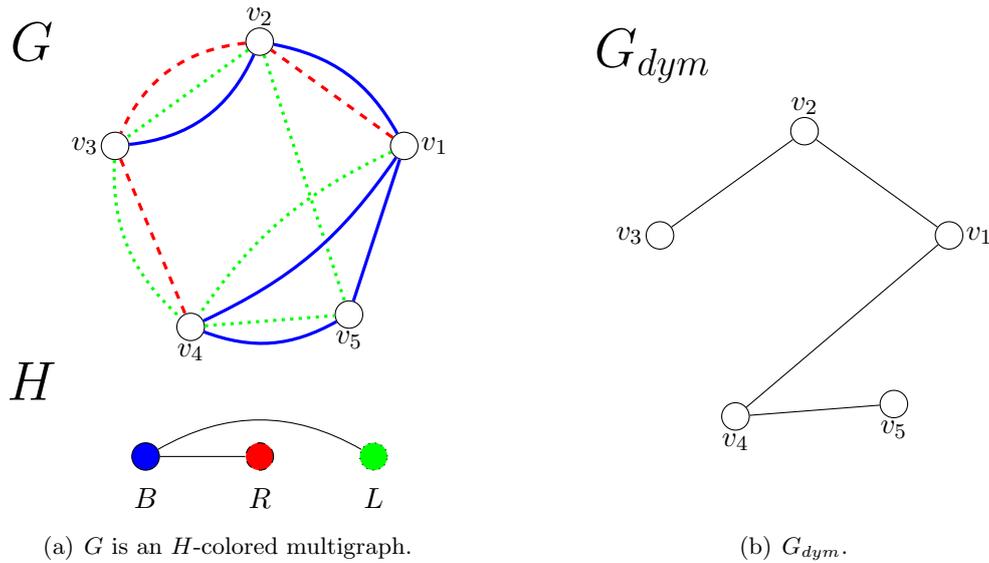
Figure 6.7: The edge sets $E_{v_1v_2}, E_{v_1v_4}, E_{v_2v_3}$ and $E_{v_4v_5}$ are dynamic edge set but $E_{v_3v_4}$ is not a dynamic edge set.

Proof. Suppose that G_{dym} is connected and $\delta_{dym}(x) = 2p_x$, where $p_x \geq 1$, then we have that $d_{G_{dym}}(x) = 2p_x$, for every $x \in V(G_{dym})$. So, G_{dym} has a closed Euler trail, say $T = (x_0, x_1, \ldots, x_n, x_0)$.

Then, T is a spanning closed dynamic H-trail in G such that $E_{x_ix_{i+1}}$ is a dynamic edge set, for every $i \in \{0, 1, 2, \ldots, n\}$ (if $i = n$, then $x_{i+1} = x_0$). Therefore, by Proposition 6.1, there exist a spanning closed dynamic H-walk in G with at most one change, namely T'. Moreover, since G_{dym} is a simple graph, we have that $E_{x_ix_{i+1}} \neq E_{x_jx_{j+1}}$, for every $i \neq j$. Therefore, T' does not repeat edge and T' is a spanning closed dynamic H-trail. $\square$

Theorem 6.7. *Let G be an H-colored multigraph with n vertices such that G_u is a complete k_u-partite graph, for every u in $V(G)$ and for some k_u, $k_u \geq 2$.*

1. *If $d_{G_{dym}}(u) + d_{G_{dym}}(v) \geq n$, for every $\{u, v\} \subseteq V(G)$ such that E_{uv} is not a dynamic edge set, then G has a Hamiltonian dynamic H-cycle with at most one change.*

2. *If there is $x_0 \in V(G)$ such that $k_{x_0} \geq 3$, and $\delta_{dym}(x) + \delta_{dym}(y) \geq n + 1$, for every $\{x, y\} \subseteq V(G)$, such that E_{xy} is not a dynamic edge set, then G has a Hamiltonian H-cycle.*

Proof. **1.** Notice that $d_{G_{dym}}(u) + d_{G_{dym}}(v) \geq n$ for every pair of non-adjacent vertices u and v in G_{dym}. Hence, by Ore's Theorem, we have that G_{dym} has a Hamiltonian cycle, say $C = (x_1, x_2, \ldots, x_n, x_1)$.

Then, C is a Hamiltonian cycle in G such that $E_{x_ix_{i+1}}$ is a dynamic edge set, for every $i \in \{1, 2, \ldots, n\}$ (if $i = n$, then $x_{i+1} = x_1$). Therefore, by Corollary 6.2, there exist a Hamiltonian dynamic H-cycle with at most one change.

2. Suppose that G is an H-colored multigraph such that G_u is a complete k_u-partite graph, for some $k_u \geq 2$, for every $u \in V(G)$, and $\delta_{dym}(x) + \delta_{dym}(y) \geq n + 1$, for every $\{x, y\} \subseteq V(G)$ such that E_{xy} is not a dynamic edge set and there is $x_0 \in V(G)$ such that $k_{x_0} \geq 3$.

It follows from 1 that G has a Hamiltonian cycle such that $E_{x_ix_{i+1}}$ is a dynamic edge set, for every $i \in \{0, \ldots, n\}$ (if $i = n$, then $x_{i+1} = x_0$), say $C = (x_0, x_1, \ldots, x_n, x_0)$.

Notice that $G_{x_i}[E_{x_ix_{i-1}} \cup E_{x_ix_{i+1}}]$ is a complete k'_{x_i}-partite graph, where $2 \leq k'_{x_i} \leq k_{x_i}$, for every $i \in \{0, \ldots, n\}$ (we take subindex modulo $n + 1$).

Case 1. There exists $i \in \{0, \ldots, n\}$ such that $k'_{x_i} \geq 3$.

Then there exist $e \in E_{x_{i-1}x_i}$, $g \in E_{x_ix_{i+1}}$ and $f \in E_{x_ix_{i-1}} \cup E_{x_ix_{i+1}}$ which are in different partite sets of $G_{x_i}[E_{x_ix_{i-1}} \cup E_{x_ix_{i+1}}]$ (it is possible by Observation 6).

If $f \in E_{x_ix_{i-1}}$. By Corollary 6.2, there is an H-path $T_1 = (x_i, g, x_{i+1}, \ldots, x_{i-2}, e_{i-2}, x_{i-1})$. Hence, $C = (x_i, T_1, x_{i-1}, e, x_i)$ or $C' = (x_i, T_1, x_{i-1}, f, x_i)$ is a Hamiltonian H-cycle in G, see Figure 6.8(a).

If $f \in E_{x_ix_{i+1}}$. By Corollary 6.2, there is an H-path $T_2 = (x_i, e, x_{i-1}, \ldots, x_{i+2}, e_{i+2}, x_{i+1})$. Hence, $C = (x_i, T_2, x_{i+1}, f, x_i)$ or $C' = (x_i, T_2, x_{i-1}, g, x_i)$ is a Hamiltonian H-cycle in G, see Figure 6.8(b).

(a) If $f \in E_{x_ix_{i-1}}$.   (b) If $f \in E_{x_ix_{i+1}}$.
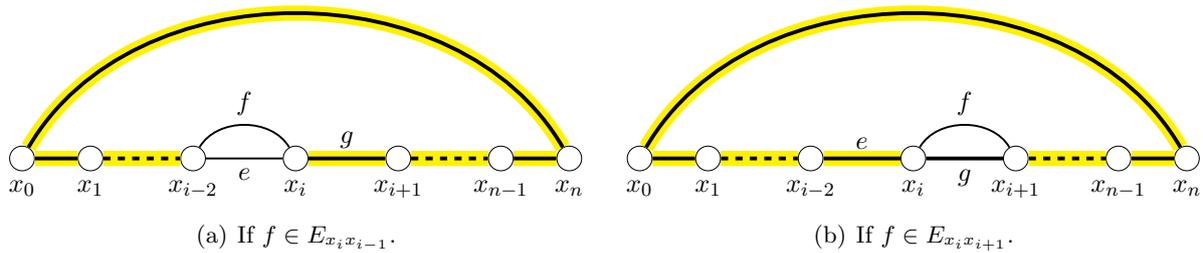
Figure 6.8: Highlighted in yellow are the H-paths T_1 and T_2, respectively. Since e, f and g are adjacent in G_{x_i}, we can choose an edge to obtain an H-cycle.

Case 2. $k'_{x_i} = 2$, for every $x \in V(G)$, i.e., $G_{x_i}[E_{x_ix_{i-1}} \cup E_{x_ix_{i+1}}]$ is a complete bipartite graph.

Let $A = \{g \in V(G_{x_0}) : ge \in E(G_{x_0})$ for every $e \in V(G_{x_0}[E_{x_0x_1} \cup E_{x_0x_n}])\}$. Since G_{x_0} is a complete k_{x_0}-partite graph with $k_{x_0} \geq 3$, it follows that $A \neq \emptyset$. Let $p = max\{i : E_{x_0x_i} \cap A \neq \emptyset\}$. Notice that $p \notin \{0, 1, n\}$ because of the condition of the case and G contains no loops.

Subcase 2.1. $E_{x_1x_{p+1}}$ is a dynamic edge set.

By Corollary 6.2 and $E_{x_px_{p-1}}$ is a dynamic edge set, there is an H-path $T_3 = (x_p, e_p, x_{p-1}, \ldots, x_1, e_1, x_{p+1}, e_{p+1}, x_{p+2}, \ldots, x_n)$ such that $ge_p \in E(G_{v_p})$. Since $E_{x_nx_0}$ is a dynamic edge set and $g \in A$, we have that there is an edge $e_n \in E_{x_nx_0}$ such that $C = (x_0, g, x_p, T_3, x_n, e_n, x_0)$ is a Hamiltonian H-cycle in G, see Figure 6.9.
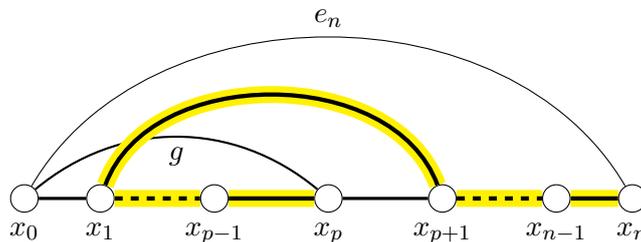

Figure 6.9: The H-path T_3 is highlighted in yellow. By the property of the edge g, we can find an edge e_n in $E_{x_0x_n}$ such that adding the edges g and e_n to T_3, we obtain a Hamiltonian H-cycle.

Subcase 2.2 $E_{x_1x_{p+1}}$ is not a dynamic edge set.

By the hypothesis $\delta_{dym}(x_1) + \delta_{dym}(x_{p+1}) \geq n + 1$.

Subcase 2.2.1. There is j, where $2 < j \leq p$, such that $E_{x_1x_j}$ and $E_{x_{p+1}x_{j-1}}$ are dynamic edge sets.

When $j = p$, Corollary 6.2 and the fact that $E_{x_1x_p}$ is a dynamic edge set, imply that $T_4 = (x_0, g, x_p, e_p, x_1, \ldots, x_{p-1}, e_{p-1}, x_{p+1}, \ldots, x_n)$ is an H-path. Since $E_{x_nx_0}$ is a dynamic

edge set and $g \in A$, there is an edge $e_n \in E_{x_n x_0}$ such that $C = (x_0, g, x_p, T_4, x_n, e_n, x_0)$ is a Hamiltonian H-cycle, see Figure 6.10(a).
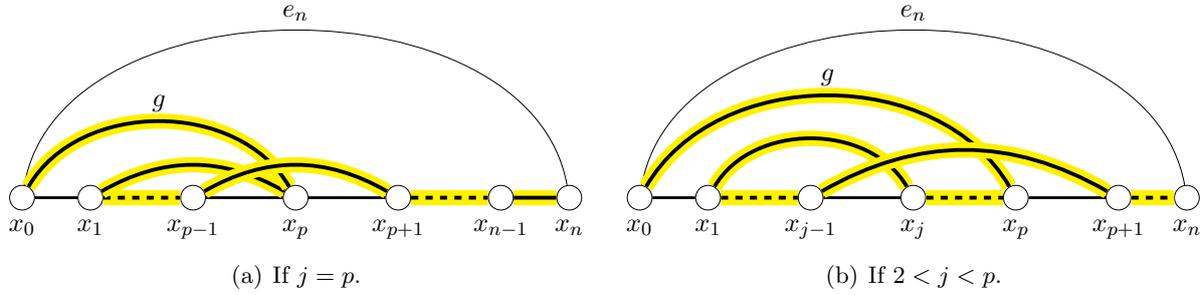
Otherwise, the path $T_5 = (x_0, g, x_p, e_p, x_{p-1}, \ldots, x_j, e_j, x_1, e_1, x_2, \ldots, x_{j-1}, e_{j-1}, x_{p+1}, e_{p+1}, x_{p+2}, \ldots, x_n)$ is an H-path. Since $E_{x_n x_0}$ is a dynamic edge set and $g \in A$, there is an edge $e_n \in E_{x_n x_0}$ such that $C = (x_0, g, x_p, T_5, x_n, e_n, x_0)$ is a Hamiltonian H-cycle, see Figure 6.10(b).


(a) If $j = p$.   (b) If $2 < j < p$.

Figure 6.10: Highlighted in yellow are the H-paths T_4 and T_5, respectively.

Subcase 2.2.2. There is j, where $p + 2 \leq j \leq n$, such that $E_{x_1 x_j}$ and $E_{x_{p+1} x_{j+1}}$ are dynamic edge sets.

When $j = n$, Corollary 6.2 and the fact that $E_{x_p x_{p-1}}$ is a dynamic edge set, imply that $T_6 = (x_0, g, x_p, e_p, x_{p-1}, \ldots, x_1, e_1, x_n, e_n, x_{n-1}, \ldots, x_{p+1})$ is an H-path. Since $E_{x_0 x_{p+1}}$ is a dynamic edge set, $g \in A$ and by the maximality of g; there is an edge $e_{p+1} \in E_{x_{p+1} x_0}$ such that $C = (x_0, T_6, x_{p+1}, e_{p+1}, x_0)$ is a Hamiltonian H-cycle, see Figure 6.11(a).

If $p + 2 \leq j < n$, then $T_7 = (x_0, g, x_p, e_p, x_{p-1}, \ldots, x_1, e_1, x_j, e_j, x_{j-1}, \ldots, x_{p+1}, e_{p+1}, x_{j+1}, e_{j+1}, x_{j+2}, \ldots, x_n)$ is an H-path. Since $E_{x_n x_0}$ is a dynamic edge set and $g \in A$, there is an edge $e_n \in E_{x_n x_0}$ such that $C = (x_0, g, x_p, T_7, x_n, e_n, x_0)$ is a Hamiltonian H-cycle, see Figure 6.11(b).

(a) If $j = n$.   (b) If $p + 2 \leq j < n$.
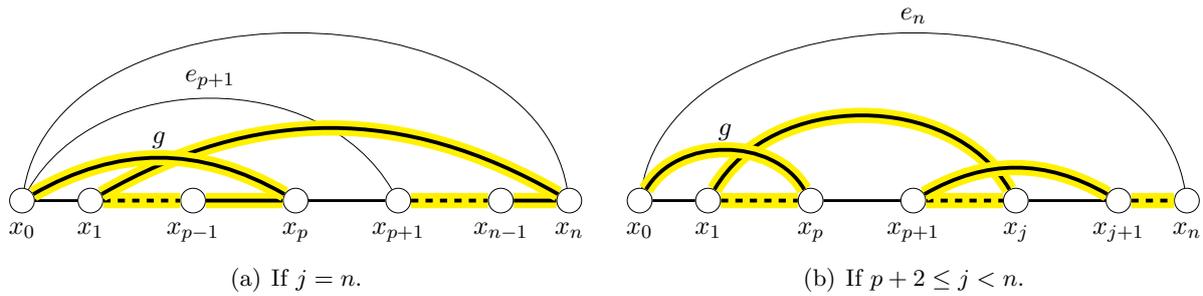
Figure 6.11: Highlighted in yellow are the H-paths T_6 and T_7, respectively.

Subcase 2.2.3. For each j, where $2 < j \leq p$, at least one of $E_{x_1 x_j}$ or $E_{x_{p+1} x_{j-1}}$ is not a dynamic edge set, and for each k, where $p + 2 \leq k \leq n$, at least one of $E_{x_1 x_k}$ or $E_{x_{p+1} x_{k+1}}$ is not a dynamic edge sets.

In this case, $\delta_{dym}(x_{p+1}) \leq (n-2) - (\delta_{dym}(x_1) - 2) = n - \delta_{dym}(x_1)$. So, $\delta_{dym}(x_1) + \delta_{dym}(x_{p+1}) \leq n$, a contradiction.

Therefore, G has a Hamiltonian H-cycle. $\square$

We think (but still we cannot prove) that the statement of Theorem 6.72 remains true if we replace the condition $\delta_{dym}(x) + \delta_{dym}(y) \geq n + 1$ by $\delta_{dym}(x) + \delta_{dym}(y) \geq n$. Moreover, we cannot replace it by $\delta_{dym}(x) + \delta_{dym}(y) \geq n - 1$, since we can H-color the multigraph G resulting from the union of two K_n^3 that share a unique vertex, in such a way that every pair of parallel edges

has different color, where H is a complete simple graph with at least three vertices. Thus, G has no Hamiltonian H-cycle. Figure 6.12 shows an example of this construction.
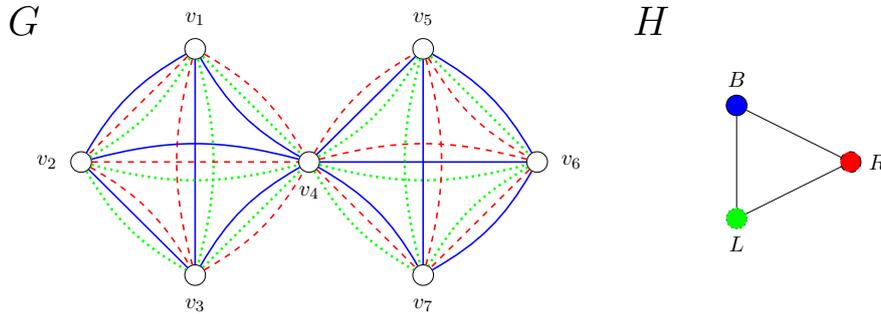

Figure 6.12: G is an H-colored graph with $n = 7$ vertices such that $\delta_{dym}(x) + \delta_{dym}(y) = 6 = n - 1$, for every $x \in \{v_1, v_2, v_3\}$ and $y \in \{v_5, v_6, v_7\}$, and there is no dynamic H-cycle of length greater than 4.

Theorem 6.8. *Let G be an H-colored multigraph with n vertices such that G_u is a complete k_u-partite graph, for every u in $V(G)$ and for some k_u, $k_u \geq 2$. If $\delta_{dym}(u) + \delta_{dym}(v) \geq n - 1$, for every pair of distinct vertices u and v of G such that E_{uv} is not a dynamic edge set, then G has a Hamiltonian H-path.*

Proof. Suppose that G is an H-colored multigraph with n vertices such that G_u is a complete k_u-partite graph, for every u in $V(G)$ and for some k_u, $k_u \geq 2$, and $\delta_{dym}(u) + \delta_{dym}(v) \geq n - 1$, for every pair of distinct vertices u and v of G such that E_{uv} is not a dynamic edge set.

Hence, $d_{G_{dym}}(u) + d_{G_{dym}}(v) \geq n - 1$, for every pair of non-adjacent vertices u and v in G_{dym}. By Theorem 2.4, we have that G_{dym} has a Hamiltonian path, say $P = (x_1, x_2, \ldots, x_n)$.

Then, P is a Hamiltonian path in G such that $E_{x_i x_{i+1}}$ is a dynamic edge set, for every $i \in \{1, 2, \ldots, n-1\}$. Therefore, by Corollary 6.2, there exists a Hamiltonian H-path. $\qquad\square$

Theorem 6.9. *Let G be an H-colored multigraph with n vertices such that G_u is a complete k_u-partite graph, for every u in $V(G)$ and for some k_u, $k_u \geq 2$. If $\delta_{dym}(u) + \delta_{dym}(v) \geq n + 1$, for every pair of distinct vertices u and v of G such that E_{uv} is not a dynamic edge set, then for every pair of distinct vertices x and y, there is a Hamiltonian H-path between x and y.*

Proof. Suppose that G is an H-colored multigraph with n vertices such that G_u is a complete k_u-partite graph, for every u in $V(G)$ and for some k_u, $k_u \geq 2$, and $\delta_{dym}(u) + \delta_{dym}(v) \geq n + 1$, for every pair of distinct vertices u and v of G such that E_{uv} is not a dynamic edge set.

Hence, $d_{G_{dym}}(u) + d_{G_{dym}}(v) \geq n + 1$, for every pair of non-adjacent vertices u and v in G_{dym}. By Theorem 2.5, we have that G_{dym} has a Hamiltonian path between any pair of different vertices.

Therefore, by Corollary 6.2, there exists a Hamiltonian H-path between any pair of different vertices in G. $\qquad\square$

Corollary 6.10. *Let G be an H-colored multigraph such that G_u is a complete k_u-partite graph, for every u in $V(G)$ and for some k_u, $k_u \geq 2$. If $\delta_{dym}(u) \geq n/2$, for every $u \in V(G)$, then G has a Hamiltonian dynamic H-cycle with at most one change.*

Corollary 6.11. *Let G be an H-colored multigraph such that G_u is a complete k_u-partite graph, for every $u \in V(G)$, and for some k_u, $k_u \geq 3$. If $\delta_{dym}(x) \geq (n+1)/2$, for every $x \in V(G)$, then G has a Hamiltonian H-cycle.*

Recall that a c-edge-colored multigraph can be represented as an H-colored multigraph if H is a complete graph with c vertices and without loops. Moreover, if $\{e, f\} \subseteq E_{xy}$, for some

$\{x, y\} \subseteq V(G)$, such that $c(e) \neq c(f)$ (i.e., e and f are parallel edges of different color), then E_{xy} is a dynamic edge set, by Observation 6.

Corollary 6.12. *Let G be a c-edge-colored multigraph such that every vertex is incident to at least two edges of different color. If at least one vertex is incident to at least three edges of different color and, for every pair of distinct vertices x and y, $\delta_{dym}(x) + \delta_{dym}(y) \geq n + 1$, then G has a properly colored Hamiltonian cycle.*

Corollary 6.13. *Let G be a c-edge-colored multigraph such that every vertex is incident to at least two edges of different color. If $\delta_{dym}(x) \geq (n + 1)/2$, for every $x \in V(G)$, and at least one vertex is incident to at least three edges of different color, then G has a properly colored Hamiltonian cycle.*

Recall that in a c-edge-colored multigraph, say G, we say that $N_i^G(x)$ denotes the set of vertices of G that are adjacent to x with an edge of color i. The ith degree of x, $x \in V(G)$, denoted by $\delta_i^G(x)$, is equal to $|N_i^G(x)|$, i.e., the cardinality of $N_i^G(x)$. When there is no confusion, for simplicity, we will write $N_i(x)$ and $\delta_i(x)$ instead of $N_i^G(x)$ and $\delta_i^G(x)$, respectively.

By the definition of $\delta_i^G(x)$ it follows that if $\{e, f\} \subseteq E_{xu}$ for some $u \in V(G)$, such that e and f have the same color, then $\delta_i^{G-e}(x) = \delta_i^{G-f}(x) = \delta_i^G(x)$. So, in what follows, we will consider edge-colored multigraphs with no parallel edges with the same color. Therefore, if G is an c-edge-colored multigraph, then $|E_{uv}| \leq c$, for every $\{u, v\} \subseteq V(G)$.

Theorem 6.14. *Let G be a c-edge-colored multigraph, $c \geq 3$, with n vertices and $|E_{uv}| \leq c - 1$, for every $\{u, v\} \subseteq V(G)$. If for every $x \in V(G)$, $\delta_i(x) \geq n/2$, for every $i \in \{1, \ldots, c\}$, then G has properly colored Hamiltonian cycle.*

Proof. Suppose that G is a c-edge-colored multigraph, $c \geq 3$, with n vertices, $|E_{uv}| \leq c - 1$, for every $\{u, v\} \subseteq V(G)$, and $\delta_i(x) \geq n/2$, for every $x \in V(G)$ and for every $i \in \{1, \ldots, c\}$.

Claim. $\delta_{dym}(x) \geq (n + 1)/2$, for every $x \in V(G)$.

Proceeding by contradiction, suppose that there is a vertex $u \in V(G)$ such that $\delta_{dym}(u) = k < (n + 1)/2$, then $\delta_{dym}(u) = k \leq (n + 1)/2 - 1/2 = n/2$.

On the one hand, since $\delta_i(u) \geq n/2$, for every $i \in \{1, \ldots, c\}$, we have that $d(u) \geq cn/2$, i.e., the number of edges incident to x is at least $cn/2$.

On the other hand, if E_{uy} is a dynamic edge set, then $2 \leq |E_{uy}| \leq (c - 1)$. Otherwise, E_{uy} is not a dynamic edge set and $0 \leq |E_{uy}| \leq 1$. Then, $d(u) \leq (n - 1 - k) + (c - 1)k = n - 1 + (c - 2)k$.

Therefore, $d(u) \leq n - 1 + (c - 2)k \leq n - 1 + (c - 2)(n/2) = n - 1 + cn/2 - n = cn/2 - 1 < d(u)$, a contradiction.

Therefore, $\delta_{dym}(x) \geq (n + 1)/2$, for every $x \in V(G)$, and by Corollary 6.11, G has properly colored Hamiltonian cycle. $\qquad \square$

We think (but still we cannot prove) that the previous theorem remains true, if we remove the condition "$|E_{uv}| \leq c - 1$, for every $\{u, v\} \subset V(G)$".

6.3   The auxiliary graph G_u

Throughout this chapter, as in Chapter 3, some theorems require that the H-colored multigraph G satisfy the condition that G_u has to be a complete multipartite graphs, for every u in $V(G)$. Therefore, in this section we present different results and examples where this hypothesis is fulfilled.

The following results follows immediately from the definition of G_u.

Proposition 6.15. *If H is a complete multipartite graph and G is a multigraph without isolated vertices, then for every H-coloring of G, G_x is a complete multipartite graph or it is an empty graph, for every x in $V(G)$.*

Proposition 6.16. *If H is a complete multipartite graph with loop at each vertex, and G is a multigraph without isolated vertices, then for every H-coloring of G, G_x is a complete multipartite graph whenever $d_G(x) \geq 2$.*

Corollary 6.17. *If H is a complete graph with loop at each vertex and G is a multigraph without isolated vertices, then for every H-coloring of G, G_x is a complete graph, for each $x \in V(G)$.*

The following construction given by Galeana-Sánchez et al. [25] shows how to obtain an H-coloring of a complete simple graph K_{2n+1}, where H is not a complete multipartite graph and, for every $x \in V(K_{2n+1})$, G_x is a complete multipartite graph.

Construction 1 (Galeana-Sánchez et al. [25])**.** *Let $n = 2k+1$, K_n be the complete graph of order n with set vertex $\{v_1, \ldots, v_n\}$, and H_n a graph defined as follows: $V(H_n) = \{x_1, x_2, \ldots, x_n\}$, and x_j is adjacent to x_i, $i < j$, if and only if $i = 1$ or $j - i \leq k$. Color the edges of K_n with the following H-coloring:*

$$c(v_i v_j) = \begin{cases} x_1 & if \quad v_i v_j = v_1 v_n \\ \\ x_{\lceil \frac{i+j}{2} \rceil} & otherwise. \end{cases}$$

An example of this construction is illustrated in Figure 6.13.

(a) G is an H_{13}-colored complete graph.   (b) H_{13}.
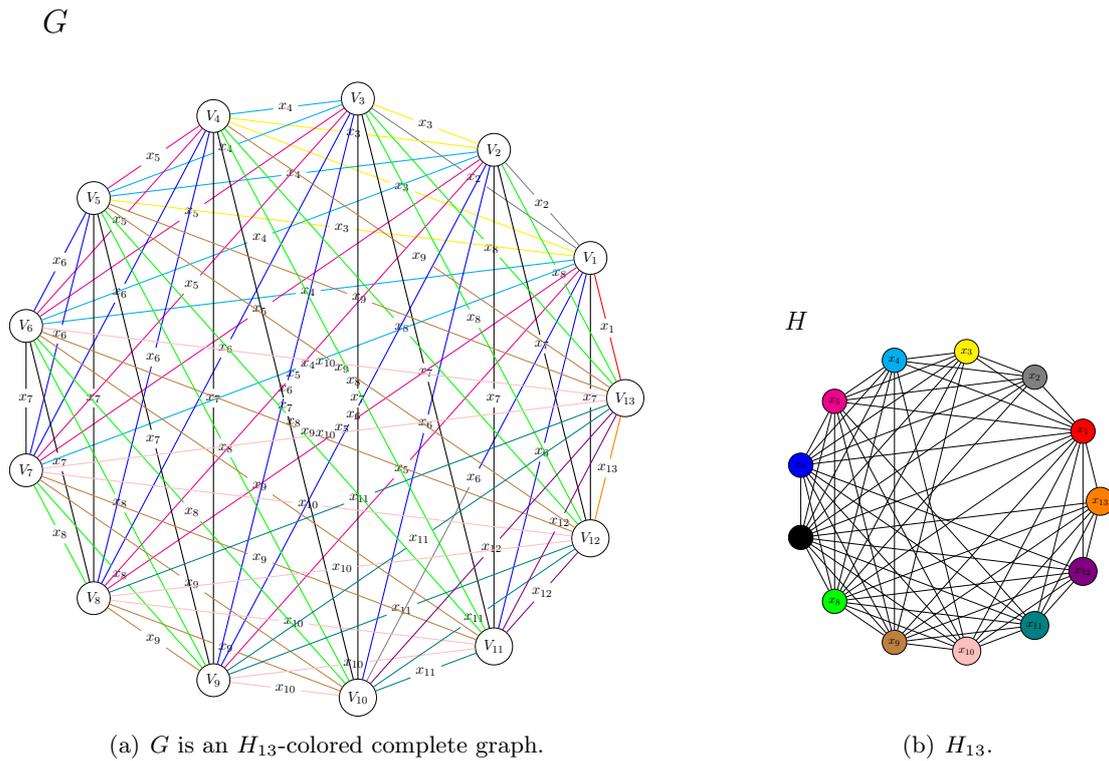
Figure 6.13: An example of an H-colored complete graph.

The following result states that the edges of all graphs, except for odd cycles, can be colored with two colors such that every vertex of degree at least 2 has at least one incident edge of each color.

Lemma 6.18 (Bondy and Murphy [11])**.** *Let G be a connected multigraph that is not an odd cycle. Then G has a 2-edge-coloring in which both colors are represented at each vertex of degree at least two.*

Corollary 6.19. *Let G be a connected multigraph. Then G has a 3-edge-coloring in which at least two colors are represented at each vertex of degree at least two.*

Galena-Sánchez et al. [27] studied the existence of H-cycles in H-colored multigraphs, where one of the hypotheses of the main theorem is that for every u in $V(G)$, G_u is a complete multipartite graph. They proved that for any multigraph G without isolated vertices and any graph H with at least one edge, it is always possible to H-color G such that G_u is either complete bipartite or an empty graph, for every u in $V(G)$.

Theorem 6.20 (Galeana-Sánchez et al. [27])**.** *Let H be a graph possibly with loops and G a multigraph without isolated vertices. If $E(H) \neq \emptyset$, then there exists an H-coloring $c : E(G) \to V(H)$ such that G_x is a complete graph or it is a complete bipartite graph or it is an empty graph for every x in $V(G)$.*

Corollary 6.21 (Galeana-Sánchez et al. [27])**.** *Let H be a graph, without loops, such that $E(H) \neq \emptyset$ and G a multigraph without isolated vertices. If G is not an odd cycle, then there exists an H-coloring of G such that G_x is a complete bipartite graph whenever $d_G(x) \geq 2$ for every x in $V(G)$.*

In Figure 6.14, if we consider G as an H_2-colored multigraph, then G_{v_i} is a complete bipartite graph, for every $i \in \{1, \ldots, 7\}$. On the other hand, if we consider G as an H_1-colored multigraph, then G_{v_i} is a complete k_{v_i}-partite graph, with $k_{v_i} \geq 3$, for every $i \in \{1, \ldots, 7\}$.
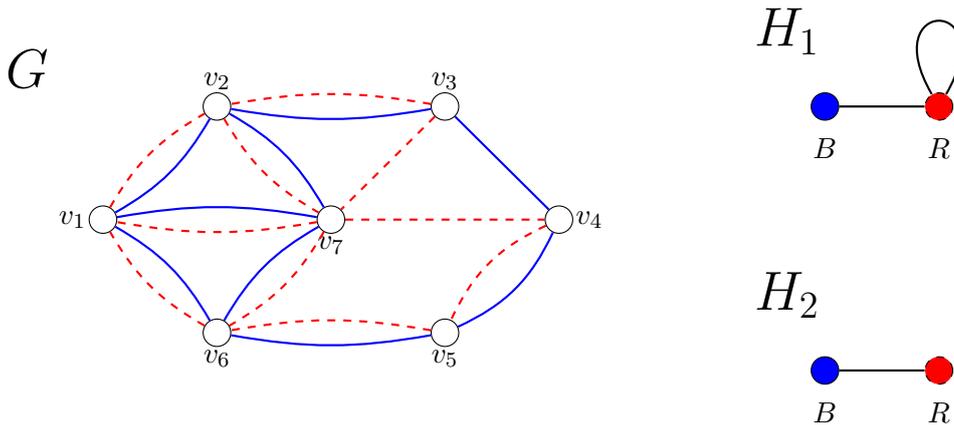
Figure 6.14

Let H be a graph possibly with loops, G a multigraph and $c : E(G) \to V(H)$ an H-coloring of G. We will say that c is a ***p-H-coloring*** of G whenever $p = |Im(c)|$, where $Im(c)$ is the image of c. The number of edges in a maximum matching of a multigraph G is called the **matching number** of G, denoted by $\beta(G)$.

Theorem 6.22 (Galeana-Sánchez et al. [27])**.** *Let H be a connected graph, of order at least two, without loops and G a connected multigraph of order at least two. For every p in $\{2, \ldots, z+1\}$, with $z = min\{\Delta(H), \beta(G)\}$, there exists a p-H-coloring of G such that either G_x is a complete bipartite graph or $E(G_x) = \emptyset$; for every x in $V(G)$.*

Theorem 6.23 (Galeana-Sánchez et al. [27])**.** *Let H be a connected graph without loops and G a connected multigraph, both of order at least two. If $\beta(G) \geq 3$ and $max\{|N_H(e)| : e \in E(H)\} + 1 > \Delta(H)$, then for every p in $\{3, 4, \ldots, w+2\}$, with $w = min\{max\{|N_H(e)| : e \in E(H)\}, \beta(G)\}$, there exists a p-H-coloring for G such that either G_x is a complete bipartite graph or $E(G_x) = \emptyset$ for every x in $V(G)$.*

Summary

The research work contained in this dissertation explores the existence of trails, cycles and paths with restrictions in the color transitions and taking advantage of the existence of parallel edges. The research started with the intention of finding some characterization of H-colored multigraphs containing closed Eulerian dynamic H-trails with conditions similar to those in Theorem 2.11. Although we did not succeed in obtaining such a characterization, we were able to obtain several results such as a bijection between the set of closed dynamic H-trails in the original graph and a subset of the cycles in $L_{n,H}^{Dym}(G)$.

Subsequently, we carried out the previous study to a quite wide class of objects, called digraphs, obtaining some similar results and some exclusive results for this class. For example, the one-to-one correspondence is between the set of closed dynamic H-trails in an H-colored digraph D and the set (not a subset) of directed cycles in $L_{n,H}^{Dym}(D)$. This allowed us to characterize a family of H-colored digraphs containing closed Eulerian H-trail. As a direct consequence of this characterization, we obtained Theorem 4.10 and a generalization for c-arc-colored digraphs. We also proved that the problem of determining the existence of an H-trail starting with a given arc e and ending in a given arc f can be done in polynomial time. As a consequence, we gave a polynomial time algorithm to find (if any exists) the shortest H-trail from a vertex s to a vertex t in an H-colored digraph. Moreover, we showed that the problem of maximizing the number of arc disjoint $s - t$ H-trails in D can be solved in polynomial time.

Finally, we studied the existence and length of dynamic H-cycles in H-colored multigraphs using the dynamic degree, that allowed us to extend some classic results for H-colored multigraphs. Also, we gave sufficient conditions for the existence of Hamiltonian dynamic H-cycles in H-colored multigraphs with at most one "lane change", and as a consequence, we obtained sufficient conditions for the existence of properly colored Hamiltonian cycle in c-edge-colored multigraphs, with $c \geq 3$. Moreover, we improved the conditions given in Theorem 2.10 b) for an infinitely family of multigraphs.

The main objective of this dissertation was to study of dynamic H-walks. In that sense, this work opens a wider panorama where there are even more topics to be researched. Below, we list a series of questions and open problems, some of them were stated throughout the text.

- Find a characterization of the H-colored multigraphs containing closed Eulerian dynamic H-trails.

- In this work we considered dynamic H-trails without a restriction on the number of changes between each pair of vertices. So, what happens if we ask that at most k changes can be made?

- If we remove the condition "$|E_{uv}| \leq c - 1$, for every $\{u, v\} \subset V(G)$", is Theorem 6.14 still true?

- Study variations on the definition of H-walk, for example, ask that the succession of the colors of the edges of the walk be a trail, a cycle or a path instead of a walk in H. Note that if $W = (x_0, x_1, \ldots, x_n)$ is a walk, in an H-colored graph, such that $(c(x_0x_1), c(x_1x_2), \ldots,$

$c(x_{n-1}x_n))$ is a path in H, then W is a rainbow walk (that is a walk that has no two edges of the same color).

As mentioned in Sections 4.4 and 6.3, several results presented in this dissertation are only valid under the hypothesis that G_u (UD_u) is a complete multipartite (union of complete bipartite) graph.

- If we remove that hypothesis, what other conditions should we ask for H and G to obtain the same conclusion?

- Given a multigraph G, a graph H and an integer p. Is there a possible way to H-color G such that G_u is complete k_u-partite graph, with $k_u \geq p$?

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Index