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POLYTOPAL KKMS THEOREM AND APPLICATIONS IN GEOMETRIC
TRANSVERSAL THEORY

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PRESENTA:
ARIADNA OLVERA SAMPIERI

TUTOR PRINCIPAL:
DR. LEONARDO IGNACIO MARTÍNEZ SANDOVAL
FACULTAD DE CIENCIAS, UNAM

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«“Qu’est-ce qu’optimisme ? disait Cacambo.” “Hélas! dit Candide, c’est la rage de soutenir que tout est bien quand on est mal.” (...) “Cela est bien dit, répondit Candide, mais il faut cultiver notre jardin.”»

Voltaire in *Candide*

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Introducción

La teoría del punto fijo es una rama de las matemáticas que estudia cuándo una función f tiene por lo menos un punto fijo, es decir, un punto x para el cual $f(x) = x$, bajo ciertas condiciones que pueden ser establecidas sobre f .

La geometría discreta y la geometría combinatoria son ramas que estudian ciertas propiedades combinatorias y métodos constructivos de objetos geométricos discretos. Varios problemas en geometría discreta involucran conjuntos finitos o discretos de objetos geométricos básicos, como puntos, líneas, planos, polígonos o polítopos. Esta disciplina se enfoca en las propiedades combinatorias de estos objetos, como el cómo se intersectan entre sí o cómo pueden acomodarse para cubrir un objeto más grande.

La geometría discreta también se relaciona con la geometría convexa y la geometría computacional, y tiene aplicaciones en diversos otros temas como la optimización combinatoria, la teoría geométrica de gráficas y la topología combinatoria.

En este trabajo estudiamos el Teorema de Knaster-Kuratowski-Mazurkiewicz (KKM) [5] y algunas de sus generalizaciones, así como algunas aplicaciones de este resultado a la teoría geométrica de transversales, basándonos en [8]. El Teorema KKM es un resultado fundamental en topología, teoría de punto fijo y análisis convexo. Este teorema establece condiciones bajo las cuales se garantiza la existencia de un punto en la intersección de una cubierta específica de conjuntos cerrados en un simplejo n -dimensional. Es equivalente a otros resultados en combinatoria y topología algebraica, como el lema de Sperner y el teorema del punto fijo de Brouwer. También tiene diversas aplicaciones en problemas tipo Helly y en teoría geométrica de transversales.

Existen varias generalizaciones de este teorema. Entre ellas, el Teorema KKMS propuesto por Lloyd Shapley generaliza el Teorema KKM en el contexto de economía y juegos cooperativos. Introduce conceptos como el balanceo combinatorio y generaliza la cubierta de cerrados del n -simplejo (véase la Sección 2.2 para definiciones precisas). El Teorema KKMS politopal es una generalización del Teorema KKMS propuesta por Komiya en [6] para polítopos convexos compactos en \mathbb{R}^n .

En el primer capítulo, introducimos conceptos y definiciones básicas que son necesarios para comprender el Teorema KKM y sus generalizaciones. Nos basamos en [7], [4] y [2].

En el segundo capítulo presentamos un examen detallado de las diferentes versiones del Teorema KKM y exponemos una demostración explícita del Teorema KKMS politopal, un resultado que Komiya inicialmente sugirió de forma implícita en [6], utilizando el teorema del punto fijo de Kakutani [4] y el teorema del máximo de Berge [2]. Finalmente, enunciamos y demostramos la versión del Teorema KKMS politopal con una cubierta de conjuntos abiertos en lugar de cerrados. Proporcionamos una demostración original de este resultado utilizando únicamente el Teorema KKMS politopal y la normalidad de \mathbb{R}^n .

El último capítulo está dedicado a aplicaciones de estos teoremas en teoría geométrica de transversales y problemas tipo Helly, basándonos en [8]. Estas aplicaciones muestran la versatilidad y el poder del Teorema KKM y el Teorema KKMS politopal para resolver problemas complejos en geometría discreta.

Introduction

Fixed-point theory is a branch of some fields that studies whether a function f has at least one fixed point, that is, a point x for which $f(x) = x$, under some conditions that can be stated on f .

Discrete geometry and combinatorial geometry are branches of geometry that study combinatorial properties and constructive methods of discrete geometric objects. Many problems in discrete geometry involve finite or discrete sets of basic geometric objects, such as points, lines, planes, polygons or polytopes. The subject focuses on the combinatorial properties of these objects, such as how they intersect one another, or how they may be arranged to cover a larger object.

Discrete geometry is also related to convex geometry and computational geometry, and has applications in many other topics such as combinatorial optimization, geometric graph theory and combinatorial topology.

In this work we study the Knaster-Kuratowski-Mazurkiewicz Theorem (KKM) [6] and some of its generalizations, as well as some applications of this result to geometric transversal theory, based on [8]. The KKM Theorem is a fundamental result in topology, fixed-point theory and convex analysis. This theorem establishes conditions under which the existence of a point in the intersection of a specific cover of closed sets in an n -dimensional simplex is guaranteed. It is equivalent to other results in combinatorics and algebraic topology, such as Sperner's lemma and Brouwer's fixed-point theorem. It also has several applications in Helly-type problems and geometric transversal theory.

There exist many generalizations of this theorem. Among them, the KKMS Theorem proposed by Lloyd Shapley generalizes the KKM Theorem in the context of economics and cooperative games. It introduces concepts like combinatorial balancing and generalizes the closed cover of the n -simplex (see Section 2.2 for precise definitions). The Polytopal KKMS Theorem is a generalization of the KKMS Theorem proposed by Komiya in [6] for compact convex polytopes in \mathbb{R}^n .

In the first chapter, we introduce basic concepts and definitions that are needed to the understanding of the KKM Theorem and its generalizations. We are based on [7], [4] and [2].

In the second chapter we present a detailed examination of the different versions of the KKM Theorem, and we expose an explicit proof of the Polytopal KKMS Theorem, a result that Komiya initially suggested implicitly in [6], using Kakutani fixed point theorem [4] and Berge maximum theorem [2]. Finally, we state and prove the version of the Polytopal KKMS Theorem with a cover of open sets instead of closed sets. We provide an original proof of this result using only the Polytopal KKMS Theorem and the normality of \mathbb{R}^n .

The last chapter is dedicated to applications of these theorems in geometric transversal theory and Helly-type problems, based on [8]. These applications show the versatility and power of the KKM Theorem and the Polytopal KKMS Theorem in solving complex problems in discrete geometry.

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1 Basic definitions

In this chapter, we introduce some basic definitions and results that we use in subsequent chapters. We address fundamental results in general topology, discrete geometry and set-valued functions. We will need this later on, when we discuss the KKM Theorem and its generalizations.

1.1 Topology

We begin by defining basic concepts and terminology in general topology.

Definition 1.1.1. A **topology** τ on a set X is a set of subsets of X such that:

- $\emptyset \in \tau$ and $X \in \tau$.
- If $\{U_i \mid i \in I\} \subseteq \tau$, then $\bigcup_{i \in I} U_i \in \tau$.
- If $U_1, \dots, U_n \subseteq \tau$, then $U_1 \cap \dots \cap U_n \in \tau$.

The elements of τ are called **open sets**.

A subset $Y \subseteq X$ is called **closed** if $X \setminus Y$ is open. The **closure** of a set Y is the smallest inclusion-wise closed set \bar{Y} containing Y .

Remember that given a set X with topology τ , a family $\mathcal{B} \subseteq \tau$ is a **basis** for the topology if for every $U \in \tau$, there exists $\mathcal{G} \subseteq \mathcal{B}$ such that $U = \bigcup_{G \in \mathcal{G}} G$.

Definition 1.1.2. A subset A of a topological space X is **compact** if for every collection $\mathcal{U} = \{U_i \mid i \in I\}$ of open subsets of X such that $A \subseteq \bigcup_{i \in I} U_i$, there is a finite sub-collection $\mathcal{V} = \{U_{i_1}, \dots, U_{i_n}\} \subseteq \mathcal{U}$ such that $A \subseteq \bigcup_{j=1}^n U_{i_j}$.

Definition 1.1.3. Let $\{X_i\}_{i \in I}$ be a family of topological spaces, and let $X = \prod_{i \in I} X_i$. We define the **product topology** in X as the topology that has as basis the elements of the form $U_{i_1} \times U_{i_2} \times \dots \times U_{i_n} \times \prod_{i \neq i_1, \dots, i_n} X_i$, where U_{i_j} is open in X_{i_j} for every $j \in \{1, \dots, n\}$.

Definition 1.1.4. Given a topological space X , we say that X is **normal** if for any closed sets A and B in X such that $A \cap B = \emptyset$, there exist open sets U and V in X such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

Proposition 1.1. If X is a topological space, and U and V are open sets in X such that $U \cap V = \emptyset$, then $\bar{U} \cap V = \emptyset$.

PROOF. Suppose $\bar{U} \cap V \neq \emptyset$, then there is $x \in X$, such that $x \in \bar{U} \cap V$. Since $x \in \bar{U}$, every open neighbourhood of x , intersects U . In particular, since $x \in V$ and V is open, we have $V \cap U \neq \emptyset$, which is a contradiction. \square

In this work we will focus on the Euclidean space \mathbb{R}^n . In \mathbb{R}^n the compact sets are characterized as follows.

Theorem 1. (Heine-Borel) *In \mathbb{R}^n , a subset $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

1.2 Discrete Geometry

In this section we will introduce essential definitions and results in discrete geometry, based on [7], such as convex and affine sets, transversals, and relevant theorems that we will need in subsequent sections for geometric transversal problems and for the KKM Theorem and its generalizations.

Definition 1.2.1. *A set $C \subseteq \mathbb{R}^d$ is **convex** if for every $x, y \in C$ the **segment***

$$xy := \{tx + (1 - t)y \mid t \in [0, 1]\}$$

is also contained in C .

It is easy to see that the arbitrary intersection of convex sets is convex: two points in such intersection are in each convex set, so its interval is contained in each convex set as well.

Definition 1.2.2. *The **convex hull**, $\text{conv}(X)$, of a set $X \subseteq \mathbb{R}^d$ is the intersection of all convex sets in \mathbb{R}^d containing X .*

Definition 1.2.3. *We say that x is a **convex combination** of x_1, x_2, \dots, x_n if there exist points $x_1, x_2, \dots, x_n \in X$ and non-negative real numbers $0 \leq \alpha_1, \alpha_2, \dots, \alpha_n \leq 1$ with $\sum_{i=1}^n \alpha_i = 1$ such that $x = \sum_{i=1}^n \alpha_i x_i$.*

These two definitions as follows: a point x belongs to $\text{conv}(X)$ if and only if x is a convex combination of some points $x_1, x_2, \dots, x_n \in X$. See [7, Claim 1.1.2].

Recall that an **affine subspace** of \mathbb{R}^d is a set $A \subseteq \mathbb{R}^d$ of the form $A = x + L$, where $x \in \mathbb{R}^d$ is some vector and L is a linear subspace of \mathbb{R}^d .

Definition 1.2.4. *The **affine hull** of a set $X \in \mathbb{R}^d$ is the intersection of all affine subspaces of \mathbb{R}^d containing X .*

Definition 1.2.5. *An **affine combination** of points $x_1, \dots, x_n \in \mathbb{R}^d$ is an expression of the form $\alpha_1 x_1 + \dots + \alpha_n x_n$, where $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and $\sum_{i=1}^n \alpha_i = 1$.*

Definition 1.2.6. We say that points $a_1, \dots, a_n \in \mathbb{R}^n$ are **affinely dependent** if one of them can be written as an affine combination of the others. That is, there exist real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, at least one of them non-zero, such that

$$\begin{aligned}\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n &= 0, \text{ and} \\ \lambda_1 + \lambda_2 + \dots + \lambda_n &= 0.\end{aligned}$$

Respectively, we say that $a_1, \dots, a_n \in \mathbb{R}^n$ are **affinely independent** if they are not affinely dependent.

There is a relationship between affine independence and linear independence. To see this, let's state the following proposition.

Proposition 1.2. Let $\bar{x}_1, \dots, \bar{x}_k \in \mathbb{R}^d$ be such that $\bar{x}_i = (x_{i_1}, x_{i_2}, \dots, x_{i_d})$ for each $i \in [k]$, and let $\bar{y}_1, \dots, \bar{y}_k \in \mathbb{R}^{d+1}$ be given by $\bar{y}_i = (x_{i_1}, x_{i_2}, \dots, x_{i_d}, 1)$ for each $i \in [k]$. Then $\bar{x}_1, \dots, \bar{x}_k$ are affinely dependent if and only if $\bar{y}_1, \dots, \bar{y}_k$ are linearly dependent.

PROOF. Suppose $\bar{x}_1, \dots, \bar{x}_k$ are affinely dependent. Then there exist $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ at least one of them non-zero such that

$$\begin{aligned}\alpha_1 \bar{x}_1 + \dots + \alpha_k \bar{x}_k &= 0, \text{ and} \\ \alpha_1 + \dots + \alpha_k &= 0.\end{aligned}$$

Without loss of generality, suppose that $\alpha_1 \neq 0$. Let $\lambda_i = -\frac{\alpha_i}{\alpha_1}$ for $i \in \{2, \dots, k\}$, then $\bar{x}_1 = \lambda_2 \bar{x}_2 + \dots + \lambda_k \bar{x}_k$, and note that $\lambda_2 + \dots + \lambda_k = 1$, so at least one of them is non-zero. Since $\bar{y}_i = (x_{i_1}, x_{i_2}, \dots, x_{i_d}, 1)$ for each $i \in [k]$, we have that $\bar{y}_1 = \lambda_2 \bar{y}_2 + \dots + \lambda_k \bar{y}_k$, and this means that $\bar{y}_1, \dots, \bar{y}_k$ are linearly dependent.

Reciprocally, if $\bar{y}_1, \dots, \bar{y}_k$ are linearly dependent, there exist $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ at least one of them non-zero such that $\alpha_1 \bar{y}_1 + \dots + \alpha_k \bar{y}_k = 0$. Suppose without loss of generality that $\alpha_1 \neq 0$, then

$$\bar{y}_1 = -\frac{\alpha_2}{\alpha_1} \bar{y}_2 - \dots - \frac{\alpha_k}{\alpha_1} \bar{y}_k.$$

Since $\bar{y}_i = (x_{i_1}, x_{i_2}, \dots, x_{i_d}, 1)$ for each $i \in [k]$, the last coordinate gives us that $-\frac{\alpha_2}{\alpha_1} - \dots - \frac{\alpha_k}{\alpha_1} = 1$, so $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ are such that $\alpha_1 \bar{x}_1 + \dots + \alpha_k \bar{x}_k = 0$ and $\alpha_1 + \dots + \alpha_k = 0$. Thus, $\bar{x}_1, \dots, \bar{x}_k$ are affinely dependent. \square

Therefore, the maximum possible number of affinely independent points in \mathbb{R}^d is $d + 1$, since the maximum possible number of linearly independent points in \mathbb{R}^d is d . Furthermore, $\{0, e_1, \dots, e_d\}$ are $d + 1$ affinely independent elements in \mathbb{R}^d , where e_i is the i -th canonical vector of \mathbb{R}^d .

Definition 1.2.7. One-dimensional affine subspaces are **lines**. The word **plane** usually means a 2-dimensional affine subspace of \mathbb{R}^d for any d . A $(d - 1)$ -dimensional affine subspace of \mathbb{R}^d is called a **hyperplane**, and a k -dimensional affine subspace is often called a **k -flat**.

Note that for $a \in \mathbb{R}^d$, a unit vector and $c \in \mathbb{R}$, the set $H = \{x \in \mathbb{R}^n : a \cdot x = c\}$ is a hyperplane in \mathbb{R}^d .

Definition 1.2.8. For $a \in \mathbb{R}^d$ a unit vector and $c \in \mathbb{R}$, we define $H^+ = \{x \in \mathbb{R}^n : a \cdot x > c\}$ and $H^- = \{x \in \mathbb{R}^n : a \cdot x < c\}$ as the two **open half-spaces** determined by H . Respectively, $\overline{H^+} = \{x \in \mathbb{R}^n : a \cdot x \geq c\}$ and $\overline{H^-} = \{x \in \mathbb{R}^n : a \cdot x \leq c\}$ are the two **closed half-spaces** determined by H .

A fundamental result of convex sets is that when they are disjoint it is possible to separate them with a hyperplane such that each set is contained in each closed half-space determined by the hyperplane. This intuition is formalized by the following result, whose proof can be consulted in [7, Theorem 1.2.4].

Theorem 2. (Separation Theorem for convex sets) Let $C, D \subset \mathbb{R}^d$ be convex sets with $C \cap D = \emptyset$. Then there exist a unit vector $a \in \mathbb{R}^d$ and $c \in \mathbb{R}$ such that the hyperplane $H = \{x \in \mathbb{R}^n : a \cdot x = c\}$ **separates** C and D , that is, $C \subseteq \overline{H^+}$ and $D \subseteq \overline{H^-}$. In addition, if C and D are closed and at least one of them is bounded, then there is a hyperplane H such that the above happens and $C \cap H = \emptyset = D \cap H$.

Geometrical transversal theory is a subfield of convex and discrete geometry that studies the intersections of classes of sets. Helly's theorem belongs to this area. This theorem has a combinatorial nature because it reduces the problem of proving that a family of n convex sets in \mathbb{R}^d has non-empty intersection to doing so for the $\binom{n}{d+1}$ subfamilies of size $d+1$. Further information can be consulted in [7, Theorem 1.3.3].

Theorem 3. (Helly's Theorem). Let C_1, C_2, \dots, C_n be convex sets in \mathbb{R}^d , with $n \geq d+1$, such that the intersection of every $d+1$ of these sets is non-empty. Then $\bigcap_{i=1}^n C_i \neq \emptyset$.

There are many generalizations of this theorem, and it inspired many other problems. Classical geometric transversal theory studies variations of this result, called Helly-type problems. In this kind of problems, we are interested in determining under which conditions a family of geometric sets (i.e. lines, triangles, circles, convex sets, etc.) can be intersected with few points, few lines or few hyperplanes.

Such subspaces, which are called **transversals**, are essential for understanding intersection patterns of geometric objects and their applications in various geometric, computational and combinatorial contexts. By investigating these conditions, we aim to deepen our understanding of the structural aspects of convex sets and their implications in geometric transversal problems.

1.3 Polytopes

Three-dimensional convex polyhedrons, especially the regular ones, have been studied since ancient times. This was a strong motivation defining and studying generalizations of convex polytopes to higher dimensions.

It is even possible to generalize convex polytopes to abstract polytopes in a more combinatorial context using partial orders. This generalization includes objects such

as tessellations, stealled polyhedra and other constructions, but in this work we will only focus on convex polytopes as convex hulls of finite point sets in \mathbb{R}^d . These polytopes constitute an important class of convex sets with an enormous number of applications and connections to other areas such as combinatorial optimization, linear programming, and computational geometry.

In this thesis we are interested in study in convex polytopes in order to state and prove one of the main generalizations of the KKM Theorem, the Polytopal KKMS Theorem, and other applications related to Helly-type problems. So from now on, when we say “polytope” we will refer to convex polytopes. There are two possible ways to define convex polytopes.

Definition 1.3.1. *An **H-polyhedron** in \mathbb{R}^d is an intersection of finitely many closed half-spaces. An **H-polytope** is a bounded H-polyhedron.*

Definition 1.3.2. *A **V-polytope** in \mathbb{R}^d is the convex hull of a finite set of points in \mathbb{R}^d .*

These two definitions are equivalent, as stated by the following result (see [7, Theorem 5.2.2]).

Theorem 4. *Every V-polytope is an H-polytope and every H-polytope is a V-polytope.*

So, from now on, we may say only “polytope” when we refer to an H-polytope, or a V-polytope.

Definition 1.3.3. *A **face** of a convex polytope P is defined as either P itself, or a subset $F \subseteq P$ of the form $F = P \cap H$, where H is an affine hyperplane such that P is fully contained in one of the closed half-spaces determined by H .*

The faces of a polytope P are also polytopes (see [7, Proposition 5.3.2]). The 0-dimensional faces are unitary sets. Each element of a 0-dimensional face is called a **vertex** of the polytope. We sometimes say expressions like *the vertices of a polytope are faces*, by which we mean that their unitary sets are. We use V to denote the set of vertices of a polytope.

The 1-dimensional faces are called **edges**, and the $(d - 1)$ -dimensional faces of a d -dimensional polytope are called **facets**. If the vertices of the face F are v_1, \dots, v_k , then we denote $F = v_1 v_2 \dots v_k$. Note that this introduces the intentional ambiguity that for $v \in V$, the symbol v can mean either the vertex v or the 0-dimensional face $\{v\}$. The context should make this clear.

A polytope that will play a crucial role in the statement of the original KKM Theorem is the simplex, which can be thought of as the polytope that generalizes the ideas of segment, triangle, tetrahedron, etc.

Definition 1.3.4. *A **simplex** is the convex hull of an affinely independent point set in some \mathbb{R}^d .*

These simplices may be irregular. However, by choosing specific coordinates we obtain a canonical simplex.

Remember that $\{e_i \mid i \in [n]\}$ denotes the canonical basis of \mathbb{R}^n where e_i denotes the vector with a 1 in the i th coordinate and 0's elsewhere.

Definition 1.3.5. *The d -dimensional regular simplex in \mathbb{R}^{d+1} , is the set*

$$\Delta^d = \text{conv}\{e_i \mid i \in [d+1]\},$$

where $[d+1] = \{1, 2, \dots, d+1\}$.

Later on we will need the regular simplex on n vertices. Note that this is the $(n-1)$ -dimensional simplex Δ^{n-1} in \mathbb{R}^n .

Example 1. *For each $S \subseteq [n]$, we can define the set $\Delta^S := \text{conv}\{e_i : i \in S\}$. The faces of Δ^{n-1} are precisely the sets Δ^S as S ranges over the non-empty subsets of $[n]$.*

Observation 1. *If $\bar{x} = (x_1, \dots, x_n) \in \Delta^{n-1}$, then $x_1 + \dots + x_n = 1$, since \bar{x} must be a convex combination of $\{e_i : i \in [n]\}$.*

The next result is a consequence of Bauer maximum principle (see [1] for the main result, and the MathSciNet review MR0100774 for a translation). We state and prove a more particular result.

Theorem 5. *Let P be a polytope in \mathbb{R}^d and $l : \mathbb{R}^d \rightarrow \mathbb{R}$ a linear transformation. Then the subset of elements of P that maximize l is a face of P .*

PROOF. Let $l : \mathbb{R}^d \rightarrow \mathbb{R}$ and let $a \in \mathbb{R}^d$ such that $l(x) = a \cdot x$. Let $C = \max\{l(x) \mid x \in P\}$. Note that the sets $\Pi = \{l(x) = c \mid x \in \mathbb{R}^d\}$ are hyperplanes for each $c \in \mathbb{R}$, and note that the set $\{l(x) = C \mid x \in \mathbb{R}^d\}$ must define a hyperplane H such that $P \subseteq \overline{H^-}$, because in other case, if there is $x \in P$ such that $x \in \overline{H^+}$, it would contradict the fact that C is the maximum. So, by Definition 1.3.3, the set $P \cap H = \{l(x) = C \mid x \in P\}$ must define a face of P . \square

1.4 Set-valued functions

Set-valued functions are used in a variety of mathematical fields, including optimization, control theory and game theory. In this section we review set-valued functions and some important results that we will need later for the proof of the Polytopal KKMS Theorem.

Definition 1.4.1. *The **power set** of a set X is the set of all subsets of X . We denote it by 2^X .*

Definition 1.4.2. *A **set-valued function** $f : X \rightrightarrows Y$ is a function $f : X \rightarrow 2^Y$, so it is a function that maps elements from a set X , the **domain of the function**, to subsets of another set Y .*

Definition 1.4.3. The set of non-empty subsets of X is denoted $p(X) := 2^X \setminus \{\emptyset\}$.

Definition 1.4.4. Let $f : X \rightrightarrows X$ be a set-valued function. We say x is a **fixed point** of f if $x \in f(x)$.

Example 2. Let: $f(x) = \begin{cases} \{1/2\} & 0 \leq x < 0.5 \\ [0, 1] & x = 0.5 \\ \{1\} & 0.5 < x \leq 1. \end{cases}$

Then $x = 0.5$ is a fixed point of f since $0.5 \in f(0.5) = [0, 1]$, as shown in Figure 1.1. Note that 1 is also a fixed point since $1 \in f(1) = \{1\}$.

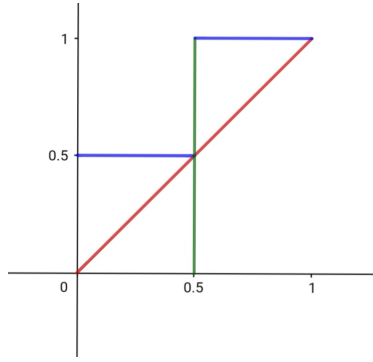


Figure 1.1: Graph of $f(x)$ and $y = x$.

△

The next theorem is a fundamental result in set-valued functions and fixed point theory. The Kakutani fixed point theorem is a generalization of the Brouwer fixed point theorem, which is an important result in topology. This latter result proves the existence of a fixed point for a continuous function defined on a compact, convex subset of an Euclidean space.

Kakutani fixed-point theorem extends this fixed-point theorem for set-valued functions. It provides sufficient conditions for a set-valued function defined on a convex, compact subset of a Euclidean space to have a fixed point as explained in Definition 1.4.4. Kakutani fixed-point theorem has many applications in economy and game theory. Before stating this theorem, we introduce the following definition.

Definition 1.4.5. A set-valued function $f : A \rightrightarrows B$ is said to be **upper semicontinuous** at x if for any open set $V \subseteq B$ such that $f(x) \subseteq V$, there exists an open neighbourhood $U \subseteq A$ with $x \in U$, such that $f(u) \subseteq V$ for every $u \in U$.

Definition 1.4.6. A set-valued function $f : A \rightrightarrows B$ is said to be **lower semicontinuous** at x if for any open set V that intersects $f(x)$, there exists an open neighbourhood U of x such that $f(x)$ intersects V for all $x \in U$.

The following statement is in [4, Theorem 1].

Theorem 6. (*Kakutani fixed point theorem*) *Let S be a non-empty, compact and convex subset of \mathbb{R}^n . Let $\varphi : S \rightrightarrows S$ be an upper semicontinuous set-valued function on S with the property that $\varphi(x)$ is non-empty, closed, and convex for all $x \in S$. Then φ has a fixed point.*

Another relevant result is Berge maximum theorem, which provides conditions for the continuity of an optimized function and the set of its maximizers with respect to its parameters. The theorem is primarily used in mathematical economics and optimal control. Further information can be seen at [2, Section VI.3].

Theorem 7. (*Berge maximum theorem*) *Let X and Y be topological spaces, let $f : X \times Y \rightarrow \mathbb{R}$ be a continuous function on the product $X \times Y$ and let $g : Y \rightrightarrows X$ be a set-valued function such that*

- $g(y)$ is compact for every $y \in Y$ and
- $g(y) \neq \emptyset$ for all $y \in Y$.

Define the value function $F : Y \rightarrow \mathbb{R}$ by $F(y) = \sup\{f(x, y) \mid x \in g(y)\}$ and define the set valued function $G : Y \rightrightarrows X$ by setting for each y the set of maximizers $G(y) = \{x \in g(y) \mid f(x, y) = F(y)\}$. If g is continuous (i.e. both upper and lower semicontinuous) at y , then F is continuous and G is upper-semicontinuous with non-empty and compact values. As a consequence, the sup may be replaced by max.

The terms *value functions* and *set of maximizers* above are motivated by the optimization application of the result. We will use these names in the proof of the polytopal KKMS theorem.

2 Knaster Kuratowski Mazurkiewicz Theorem (KKM)

The Knaster–Kuratowski–Mazurkiewicz (KKM) Theorem is a basic result in topology and fixed-point theory. It was first published in 1929 by B. Knaster, K. Kuratowski and S. Mazurkiewicz. This theorem establishes conditions under which the existence of a point in the intersection of a specific cover of closed sets in an n -dimensional simplex is guaranteed. It is equivalent to other results in combinatorics and algebraic topology, such as Sperner’s lemma and Brouwer’s fixed-point theorem.

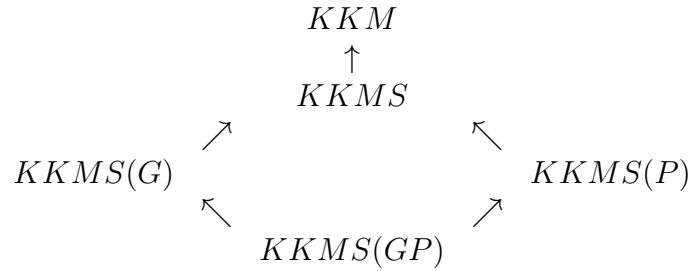
There exist many generalizations of this theorem, such as the KKMS Theorem proposed by Lloyd Shapley in 1991. It generalizes the KKM Theorem in the context of economics and cooperative games. It introduces concepts like combinatorial balancing and generalizes the closed cover of the n -simplex.

The Polytopal KKMS Theorem is a generalization of the KKMS Theorem proposed by Komiya in 1994 for compact convex polytopes in \mathbb{R}^n .

The purpose of this chapter is to state and prove the following theorems:

- Knaster Kuratowski Mazurkiewicz Theorem (KKM)
- Knaster Kuratowski Mazurkiewicz Shapley Theorem (KKMS)
- Knaster Kuratowski Mazurkiewicz with geometric balancedness (KKMS(G))
- Polytopal Knaster Kuratowski Mazurkiewicz Shapley Theorem (KKMS(P))
- Polytopal Knaster Kuratowski Mazurkiewicz Theorem with geometric balancedness (KKMS(GP)),

Each of them is discussed and exemplified in its corresponding section. In these sections we will also prove the implications indicated by the arrows in the following diagram.



The last section contains a full proof of KKMS(GP) Theorem, which together with the partial implications in each section, will prove all the versions of the theorem. The main ideas of the proof follow [6] and [5].

2.1 Knaster Kuratowski Mazurkiewicz Theorem (KKM)

We begin by stating the KKM Theorem for closed sets.

Theorem 8. (KKM). *Let C_1, \dots, C_n be closed sets in Δ^{n-1} such that for every non-empty face $S \subseteq [n]$, we have that $\Delta^S \subseteq \bigcup_{i \in S} C_i$. Then,*

$$\bigcap_{i=1}^n C_i \neq \emptyset.$$

Example 3. *Consider Δ^2 , the 2-dimensional simplex, and let C_1, C_2, C_3 be the closed sets shown in Figure 2.1. Note that $\Delta^{\{1\}} \subseteq C_1$, $\Delta^{\{2\}} \subseteq C_2$, $\Delta^{\{3\}} \subseteq C_3$, $\Delta^{\{1,2\}} \subseteq C_1 \cup C_2$, $\Delta^{\{2,3\}} \subseteq C_2 \cup C_3$, $\Delta^{\{1,3\}} \subseteq C_1 \cup C_3$, $\Delta^{\{1,2,3\}} \subseteq C_1 \cup C_2 \cup C_3$. And indeed, $C_1 \cap C_2 \cap C_3 \neq \emptyset$. \triangle*

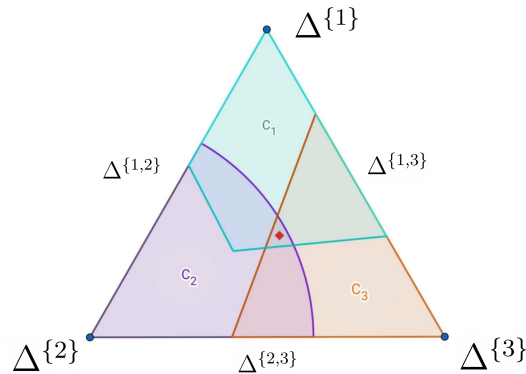


Figure 2.1: Δ^2 and C_1, C_2, C_3 closed sets as in Example 3.

2.2 Knaster Kuratowski Mazurkiewicz Shapley Theorem (KKMS)

In this section we state the KKMS Theorem, but first we introduce some definitions that we will need in order to state this theorem.

Definition 2.2.1. Let $S \in p([n]) = 2^{[n]} \setminus \{\emptyset\}$ and e_1, \dots, e_n the canonical basis of \mathbb{R}^n . We define the **characteristic vector of S** as

$$e^S := \sum_{i \in S} e_i.$$

Note that e^S is the vector whose i -th coordinate is 1 if $i \in S$ and 0 if $i \notin S$.

Definition 2.2.2. We say that a family of non-empty subsets \mathcal{B} of $[n]$ (i.e. a subset of $p([n])$) is **balanced** if there exist non-negative real numbers λ^S , for every $S \in \mathcal{B}$, such that

$$\sum_{S \in \mathcal{B}} \lambda^S e^S = e^{[n]}.$$

We call these real numbers λ^S the **weights**. For each $k \in [n]$ the **total weight** of k is the sum of the weights λ^S for $k \in S$.

Observe that if a collection of non-empty sets \mathcal{B} is balanced, its total weight is equal to 1. This is merely a restatement of the definition, by reading the k -th coordinate. In symbols, for each $k \in [n]$ this coordinate is

$$\sum_{\substack{S \in \mathcal{B} \\ k \in S}} \lambda^S = 1.$$

Example 4. Let $n = 3$ and $\mathcal{B}_1 = \{\{1\}, \{2\}, \{3\}\}$. The family \mathcal{B}_1 is balanced assigning weight 1 to every $B \in \mathcal{B}_1$. Indeed:

- The only set that has 1 is $\{1\}$ and its total weight sum is 1.
- The only set that has 2 is $\{2\}$ and its total weight sum is 1.
- The only set that has 3 is $\{3\}$ and its total weight sum is 1.

Note that in terms of characteristic vectors we verify that:

$$1 \cdot (1, 0, 0) + 1 \cdot (0, 1, 0) + 1 \cdot (0, 0, 1) = (1, 1, 1) = e^{[3]}.$$

The family $\mathcal{B}_2 = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$ is balanced too, assigning weight $1/2$ to every element in \mathcal{B}_2 . Once again, in terms of characteristic vectors we have that:

$$\frac{1}{2} \cdot (1, 1, 0) + \frac{1}{2} \cdot (0, 1, 1) + \frac{1}{2} \cdot (1, 0, 1) = (1, 1, 1) = e^{[3]}.$$

△

Example 5. Let $n = 3$. Then $\mathcal{B}_3 = \{\{1, 2\}, \{1, 3\}\}$ is not balanced. To check this, suppose that $\{1, 2\}$ and $\{1, 3\}$ have respective weights a and b . Then, in terms of characteristic vectors we would have

$$a(1, 1, 0) + b(1, 0, 1) = (1, 1, 1).$$

The second coordinate implies $a = 1$. The third one implies $b = 1$. But then the total weight for 1 (the first coordinate) is $a + b = 2$.

The family $\mathcal{B}_4 = \{\{1\}, \{1, 3\}\}$ is not balanced neither, because there is no possible way that total weight of 2 equals one. △

Example 6. Let $n = 4$, then $\mathcal{B}_5 = \{\{1\}, \{2\}, \{3\}\}$ is not balanced as the total weight for 4 is always 0. \triangle

Observation 2. Note that for each $k \in [n]$, if all subsets of $[n]$ of size k are in \mathcal{B} , then \mathcal{B} is a balanced family, because we can assign weight $1/\binom{n-1}{k-1}$ to each of them, and 0 to the rest of elements in \mathcal{B} . Indeed, each element $m \in [n]$ is in $\binom{n-1}{k-1}$ subsets of size k (after choosing m , we must choose $k-1$ remaining elements from the $n-1$ that are not m). Therefore, with this weight assignment the total weight of m is

$$\binom{n-1}{k-1} / \binom{n-1}{k-1} = 1,$$

for each $m \in [n]$.

Now we present the statement of the KKMS theorem.

Theorem 9. (KKMS). Let $\{C_S \mid S \in p([n])\}$ be a family of closed sets in Δ^{n-1} such that for each $T \in p([n])$, we have that $\Delta^T \subseteq \bigcup_{\substack{S \subseteq T \\ S \neq \emptyset}} C_S$. Then there exists a balanced family \mathcal{B} such that

$$\bigcap_{S \in \mathcal{B}} C_S \neq \emptyset.$$

We will not prove Theorem 9, as we will prove in Section 2.6 a more general result.

Example 7. Let Δ^2 be the 2-dimensional simplex, and let $\{C_S \mid S \in p([3])\}$ be the family of closed sets of Δ^2 as in Figure 2.2 with $C_{[3]} = \emptyset$. Then, there exists a balanced family $\mathcal{B} \subseteq p([n])$, namely $\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ such that $\bigcap_{S \in \mathcal{B}} C_S \neq \emptyset$. \triangle

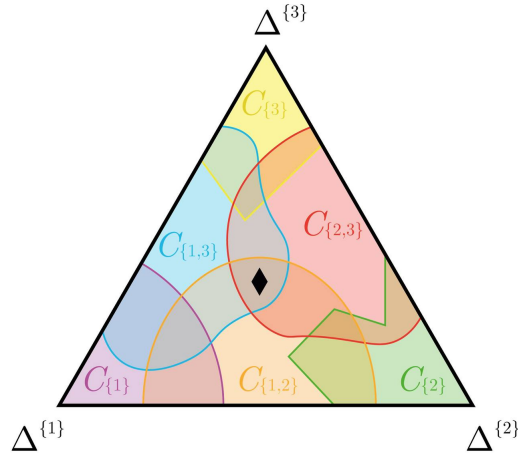


Figure 2.2: Δ^2 and $\{C_S \mid S \in p([3])\}$.

We have that the KKMS Theorem generalizes the KKM Theorem. We see this on the next Proposition.

Proposition 2.1. Theorem 9 implies Theorem 8.

PROOF. On Theorem 8 we have closed sets C_1, \dots, C_n in Δ^{n-1} . We take these as the closed sets $C_{\{1\}}, \dots, C_{\{n\}}$ of Theorem 9 and take $C_S = \emptyset$ if $|S| \geq 2$. Then the family $\{C_S \mid S \in p([n])\} = \{C_{\{1\}}, \dots, C_{\{n\}}\}$ of closed sets satisfies the conditions of Theorem 9. Namely, for each $T = \{k_1, \dots, k_m\} \in p([n])$, with $k_1, \dots, k_m \in [n]$, we have that

$$\Delta^T \subseteq C_{\{k_1\}} \cup \dots \cup C_{\{k_m\}} = \bigcup_{\substack{S \subseteq T \\ |S|=1}} C_S \subseteq \bigcup_{\substack{S \subseteq T \\ S \neq \emptyset}} C_S.$$

Then, due to Theorem 9, there exists a balanced family, say \mathcal{B} such that $\bigcap_{S \in \mathcal{B}} C_S \neq \emptyset$. We claim that $\mathcal{B} = \{\{1\}, \dots, \{n\}\}$ is the only possibility. First, if a family contains a subset $S \in p([n])$ with $|S| \geq 2$ we have that $\bigcap_{S \in \mathcal{B}} C_S = \emptyset$, as $C_S = \emptyset$ for every $|S| \geq 2$. Then, \mathcal{B} must consist of subsets of $[n]$ of size 1. It must contain all of them: if $\{j\}$ is missing for some j , the total weight of j will be 0 and not 1. Because of Observation 2, taking all the subsets of size 1 is indeed a balanced family. Then $\mathcal{B} = \{\{1\}, \dots, \{n\}\}$ is the only balanced family with these conditions. We conclude that

$$C_1 \cap \dots \cap C_n = C_{\{1\}} \cap \dots \cap C_{\{n\}} = \bigcap_{S \in \mathcal{B}} C_S \neq \emptyset.$$

□

2.3 KKMS Theorem with arbitrary points

The KKMS Theorem is also true when we consider balancedness in geometrical terms. To see this, we will first review the following equivalences between balanced families of sets and geometric properties.

Definition 2.3.1. *The **center of gravity** m^{Δ^S} of the face Δ^S is the vector $m^{\Delta^S} = \sum_{i \in S} e^i / |S|$.*

Example 8. *Let $\Delta^{n-1} = \text{conv}\{e_i \mid i \in [n]\}$ be the $(n-1)$ -dimensional simplex. Then the center of gravity of Δ^{n-1} is $m^{\Delta^{n-1}} = (\frac{1}{n}, \dots, \frac{1}{n})$. \triangle*

Proposition 2.2. *A family $\mathcal{B} \subseteq p([n])$ is balanced if and only if*

$$m^{\Delta^{n-1}} \in \text{conv}\{m^{\Delta^S} \mid S \in \mathcal{B}\}.$$

PROOF. Suppose that $\mathcal{B} = \{S_1, \dots, S_r\} \subseteq p([n])$ is a balanced family. So, there exist non-negative real numbers $\lambda^{S_1}, \dots, \lambda^{S_r}$, such that

$$\sum_{i=1}^r \lambda^{S_i} e^{S_i} = e^{[n]} = (1, 1, \dots, 1).$$

As remarked before, this means that for each k the total weight is equal to 1, that is

$$\sum_{\substack{S \in \mathcal{B} \\ k \in S}} \lambda^S = 1.$$

From Example 8, we have that $m^{\Delta^{n-1}} = (\frac{1}{n}, \dots, \frac{1}{n})$. Note that if $|S_i| = k_i$, then $m^{\Delta^{S_i}} = \frac{1}{k_i} e^{S_i}$.

We now show how from the weights λ^{S_i} we obtain the coefficients of the desired convex combination. For every $i \in [r]$, we propose $\alpha_i = \lambda^{S_i} \frac{k_i}{n}$. We now check that these are indeed the desired coefficients.

- For each i , clearly $0 \leq \alpha_i$, and

$$\alpha_i = \lambda^{S_i} \frac{k_i}{n} \leq \frac{k_i}{n} \leq 1.$$

- The coefficients sum to 1, as

$$\sum_{i=1}^r \alpha_i = \frac{1}{n} \sum_{i=1}^r \lambda^{S_i} k_i = \frac{1}{n} \sum_{i=1}^r \sum_{j \in S_i} \lambda^{S_i} = \frac{1}{n} \sum_{j=1}^n \sum_{\substack{S \in \mathcal{B} \\ j \in S}} \lambda^S = \frac{1}{n} \sum_{j=1}^n 1 = 1.$$

We have third equality because on both sides we have a term λ^S in the double sum for each $j \in S$.

- Finally,

$$\sum_{i=1}^r \alpha_i m^{\Delta^{S_i}} = \sum_{i=1}^r \left(\lambda^{S_i} \frac{k_i}{n} \right) \left(\frac{1}{k_i} e^{S_i} \right) = \frac{1}{n} \sum_{i=1}^r \lambda^{S_i} e^{S_i} = \frac{1}{n} e^{[n]} = \left(\frac{1}{n}, \dots, \frac{1}{n} \right) = m^{\Delta^{n-1}}.$$

So $m^{\Delta^{n-1}} \in \text{conv}\{m^{\Delta^S} \mid S \in \mathcal{B}\}$.

Now suppose that $m^{\Delta^{n-1}} \in \text{conv}\{m^{\Delta^S} \mid S \in \mathcal{B}\}$. So, there exist $0 \leq \alpha_1, \dots, \alpha_r \leq 1$, such that $\sum_{i=1}^r \alpha_i = 1$ and

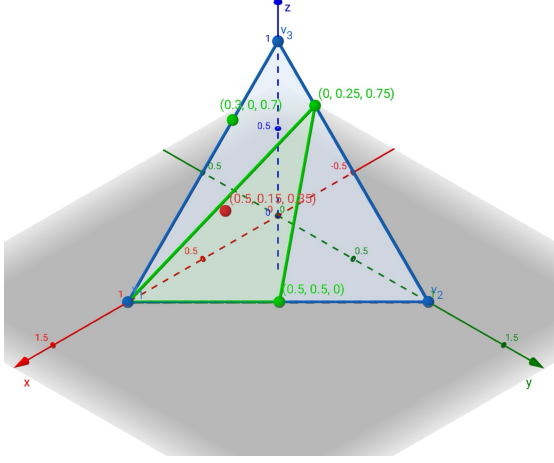
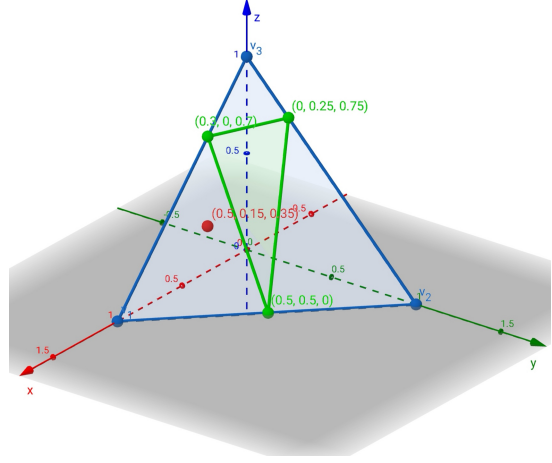
$$m^{\Delta^{n-1}} = \left(\frac{1}{n}, \dots, \frac{1}{n} \right) = \alpha_1 m^{\Delta^{S_1}} + \dots + \alpha_r m^{\Delta^{S_r}} = \sum_{i=1}^r \alpha_i \frac{1}{k_i} e^{S_i}.$$

For every $i \in [r]$, define $\lambda^{S_i} := n \frac{\alpha_i}{k_i}$. So there exist non-negative real numbers $\lambda^{S_1}, \dots, \lambda^{S_r}$, such that

$$\sum_{i=1}^r \lambda^{S_i} e^{S_i} = n \sum_{i=1}^r \frac{\alpha_i}{k_i} e^{S_i} = n m^{\Delta^{n-1}} = n \left(\frac{1}{n}, \dots, \frac{1}{n} \right) = (1, 1, \dots, 1) = e^{[n]},$$

and therefore \mathcal{B} is balanced. □

Definition 2.3.2. Let $\mathcal{F} = \{\Delta^S \subseteq \Delta^{n-1} \mid S \in p([n])\}$ be the set of non-empty faces of Δ^{n-1} and let $\mathcal{P} = \{b^S \in \Delta^S \mid S \in p([n])\}$ be a set of points on the non-empty faces of Δ^{n-1} . We say that a set $\mathcal{B} \subseteq \mathcal{F}$ is **geometrically balanced with respect to \mathcal{P}** if $b^{\Delta^{n-1}} \in \text{conv}\{b^{\Delta^S} \mid S \in \mathcal{B}\}$.


 Figure 2.3: Δ^2 and \mathcal{B}_1 .

 Figure 2.4: Δ^2 and \mathcal{B}_2 .

Example 9. Consider Δ^2 , the 2-dimensional simplex consisting of the points (x, y, z) in \mathbb{R}^3 with $x + y + z = 1$ and $x \geq 0, y \geq 0, z \geq 0$. Its vertices are $v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (0, 0, 1)$.

Consider the points b^S as follows:

$$\begin{aligned} b^{\{1\}} &= v_1, \quad b^{\{2\}} = v_2, \quad b^{\{3\}} = v_3, \\ b^{\{1,2\}} &= (1/2, 1/2, 0), \quad b^{\{2,3\}} = (0, 1/4, 3/4), \quad b^{\{1,3\}} = (0.3, 0, 0.7) \\ b^{\{1,2,3\}} &= (0.5, 0.15, 0.35) \end{aligned}$$

Note that $\mathcal{B}_1 = \{\{1\}, \{1, 2\}, \{1, 3\}\}$ is a geometrically balanced set (see Figure 2.3) as

$$b^{\{1,2,3\}} \in \text{conv}\{b^{\{1\}}, b^{\{1,2\}}, b^{\{1,3\}}\},$$

but $\mathcal{B}_2 = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$ is not balanced (see Figure 2.4), as

$$b^{\{1,2,3\}} \notin \text{conv}\{b^{\{1,2\}}, b^{\{2,3\}}, b^{\{1,3\}}\}.$$

△

The example above also shows that there is no generalization of Proposition 2.2 when we do not use the centers of gravity. Indeed \mathcal{B}_1 is geometrically balanced but not balanced, and \mathcal{B}_2 is balanced but not geometrically balanced.

The definitions above allow us to state the KKMS Theorem with arbitrary points.

Theorem 10. (KKMS with arbitrary points). Let $\mathcal{P} = \{b^S \in \Delta^S \mid S \in p([n])\}$ be a set of arbitrary points on the non-empty faces of Δ^{n-1} . Let $\{C_S \mid S \in p([n])\}$ be closed sets in Δ^{n-1} such that for each $T \in p([n])$, we have that $\Delta^T \subseteq \bigcup_{\substack{S \subseteq T \\ S \neq \emptyset}} C_S$. Then there exists a geometrically balanced family \mathcal{B} with respect to \mathcal{P} such that

$$\bigcap_{S \in \mathcal{B}} C_S \neq \emptyset.$$

of each face. That is, the vertices $v_1 = (0, 0)$, $v_2 = (1, 0)$, $v_3 = (1, 1)$, $v_4 = (0, 1)$, the midpoints $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$, $(\frac{1}{2}, 1)$, $(1, \frac{1}{2})$ of each side, and $m^P = (\frac{1}{2}, \frac{1}{2})$, are the centers of gravity of faces of dimension 0, 1 and 2, respectively. See Figure 2.6.

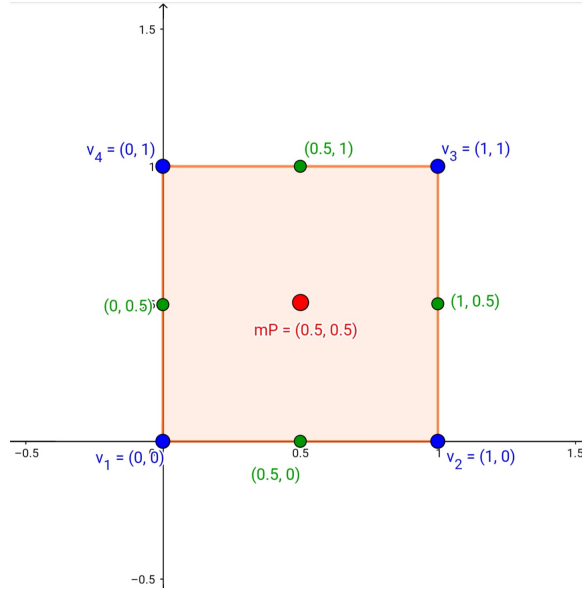


Figure 2.6: $P = [0, 1] \times [0, 1]$ and its centers of gravity.

△

Theorem 11. (Polytopal KKMS with centers of gravity). Let P be a compact convex polytope in \mathbb{R}^n , and let \mathcal{F} be the family of all non-empty faces of P . Let $\{C_F \mid F \in \mathcal{F}\}$ be a family of closed sets of P such that for every $F \in \mathcal{F}$, we have that $F \subseteq \bigcup\{C_G \mid G \in \mathcal{F}, G \subseteq F\}$. Then, there exists $\mathcal{B} \subseteq \mathcal{F}$ such that

$$m^P \in \text{conv}\{m^F \mid F \in \mathcal{B}\}$$

and

$$\bigcap_{F \in \mathcal{B}} C_F \neq \emptyset.$$

Example 12. Let $P = [0, 1] \times [0, 1]$ be the square of side 1, and let $\{C_F \mid F \in \mathcal{F}\}$ be the family of closed sets of P as in Figure 2.7, with $C_P = \emptyset$. Then, there exists a balanced family $\mathcal{B} \subseteq \mathcal{F}$ such that $m^P \in \text{conv}\{m^F \mid F \in \mathcal{B}\}$ and $\bigcap_{F \in \mathcal{B}} C_F \neq \emptyset$. In this case, $\mathcal{B} = \{v_2, v_3, v_4\}$. △

We also have the following implication between Polytopal KKMS Theorem and the KKMS Theorem.

Proposition 2.4. *Theorem 11 implies Theorem 9.*

PROOF. It follows from the fact that Δ^{n-1} is a compact convex polytope in \mathbb{R}^n and the fact that Proposition 2.2 implies that the geometrically balanced family of the conclusion is balanced. □

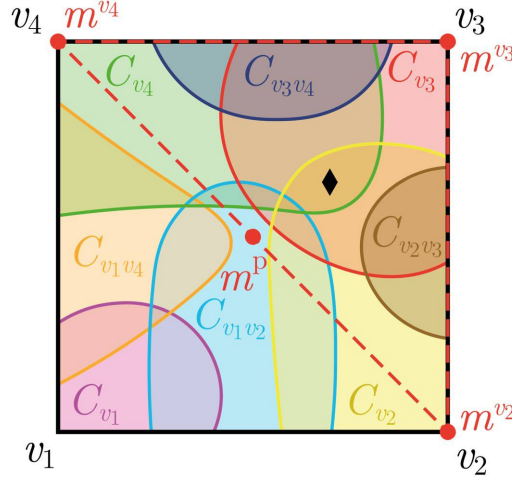


Figure 2.7: $P = [0, 1] \times [0, 1]$, $\{C_F \mid F \in \mathcal{F}\}$, and the family \mathcal{B} .

2.5 Polytopal KKMMS Theorem with arbitrary points

In this section we state the Polytopal KKMMS Theorem with arbitrary points (KKMS(GP)), that is a generalization of Theorem 9 in a convex compact polytope instead of a n -dimensional simplex, it generalises all the previous theorems.

Definition 2.5.1. Let P be a compact convex polytope in \mathbb{R}^n and let \mathcal{F} be the family of non-empty faces of P . Consider a set of points $\mathcal{P} = \{b^F \in F \mid F \in \mathcal{F}\}$ on the faces of P . We say that a subset $\mathcal{B} \subseteq \mathcal{F}$ is **geometrically balanced with respect to \mathcal{P}** if $b^P \in \text{conv}\{b^F \mid F \in \mathcal{B}\}$.

Example 13. Let $P = [0, 1] \times [0, 1]$ be again be the square of side 1 with vertices $v_1 = (0, 0)$, $v_2 = (1, 0)$, $v_3 = (1, 1)$, $v_4 = (0, 1)$.

Consider

$$\begin{aligned} b^{v_1} &= v_1, b^{v_2} = v_2, b^{v_3} = v_3, b^{v_4} = v_4 \\ b^{v_1v_2} &= \left(\frac{1}{2}, 0\right), b^{v_2v_3} = \left(1, \frac{1}{5}\right), b^{v_3v_4} = \left(\frac{1}{3}, 1\right), b^{v_1v_4} = \left(0, \frac{3}{4}\right), \\ b^{v_1v_2v_3v_4} &= \left(\frac{3}{4}, \frac{3}{4}\right) \end{aligned}$$

Take $\mathcal{P} = \{b^F \in F \mid F \in \mathcal{F}\}$. The family $\mathcal{B}_1 = \{v_3, v_1v_2, v_1v_4\}$ is geometrically balanced, as $b^P = b^{v_1v_2v_3v_4} = \left(\frac{3}{4}, \frac{3}{4}\right) \in \text{conv}\left\{\left(1, 1\right), \left(\frac{1}{2}, 0\right), \left(0, \frac{3}{4}\right)\right\} = \text{conv}\{b^{v_3}, b^{v_1v_2}, b^{v_1v_4}\}$, see Figure 2.8. But the family $\mathcal{B}_2 = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$ is not balanced as $b^P = b^{v_1v_2v_3v_4} = \left(\frac{3}{4}, \frac{3}{4}\right) \notin \text{conv}\left\{\left(\frac{1}{2}, 0\right), \left(1, \frac{1}{5}\right), \left(\frac{1}{3}, 1\right), \left(0, \frac{3}{4}\right)\right\} = \text{conv}\{b^{v_1v_2}, b^{v_2v_3}, b^{v_3v_4}, b^{v_4v_1}\}$, see Figure 2.9. \triangle

In the context of polytopes we do not have a generalization of Proposition 2.2. The following example shows a family of faces that is geometrically balanced but not balanced.

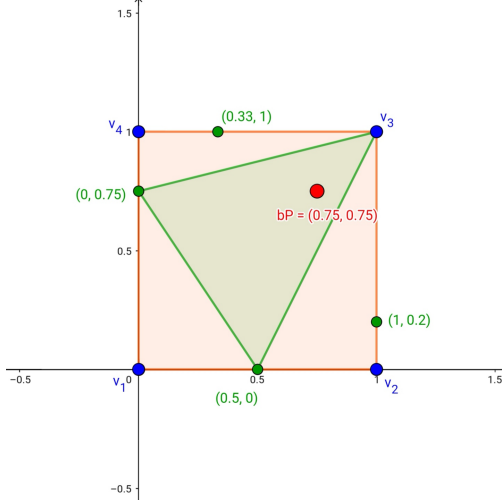


Figure 2.8: $P = [0, 1] \times [0, 1]$ and \mathcal{B}_1 .

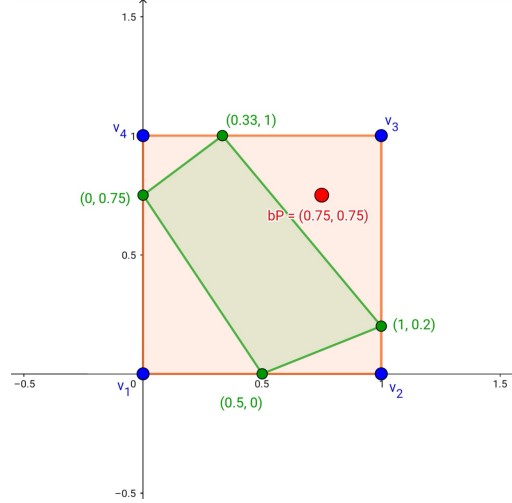


Figure 2.9: $P = [0, 1] \times [0, 1]$ and \mathcal{B}_2 .

Example 14. Let $P = [0, 1] \times [0, 1]$ be the square of side 1. If we take $\mathcal{P} = \{m^F \in F \mid F \in \mathcal{F}\}$ (i.e. using the centers of gravity of each face), we have that $\mathcal{B} = \{v_1, v_2, v_3\}$ is geometrically balanced since $m^P = (\frac{1}{2}, \frac{1}{2})$ is in $\text{conv}(m^{v_1}, m^{v_2}, m^{v_3}) = \text{conv}(v_1, v_2, v_3)$. But note that $\{\{1\}, \{2\}, \{3\}\} \subseteq p([4])$ is not a balanced family of $\{1, 2, 3, 4\}$, due to Example 6. See also Figure 2.10. \triangle

So, now we can state the Polytopal KKMS Theorem with arbitrary points.

Theorem 12. (Polytopal KKMS Theorem). [6]

Let P be a compact convex polytope in \mathbb{R}^n and let \mathcal{F} be the family of non-empty faces of P . Let $\{C_F \mid F \in \mathcal{F}\}$ be a family of closed sets of P such that for every $F \in \mathcal{F}$, we have that $F \subseteq \cup\{C_G \mid G \in \mathcal{F}, G \subseteq F\}$. Let $\mathcal{P} = \{b^F \in F \mid F \in \mathcal{F}\}$ any choice of points on the faces of P .

Then there exists a subfamily \mathcal{B} of faces of P geometrically balanced with respect to \mathcal{P} such that

$$\bigcap_{F \in \mathcal{B}} C_F \neq \emptyset.$$

Example 15. Let $P = [0, 1] \times [0, 1]$ be the square of side 1, and let $\{C_F \mid F \in \mathcal{F}\}$ be the family of closed sets of P as in Figure 2.11, let $C_P = \emptyset$. Then, the family $\mathcal{B} \subseteq \mathcal{F}$, given by $\mathcal{B} = \{v_1v_2, v_3v_4, v_1v_4\}$ is a balanced family such that $b^P \in \text{conv}\{b^F \mid F \in \mathcal{B}\}$ and $\bigcap_{F \in \mathcal{B}} C_F \neq \emptyset$. \triangle

This Theorem implies KKMS Theorem with arbitrary points and the KKMS Theorem with centers of gravity, as we will show in the following propositions. This, in combination with Proposition 2.3, proves that this theorem implies all the previous versions of the KKM Theorem.

Proposition 2.5. Theorem 12 implies Theorem 10.

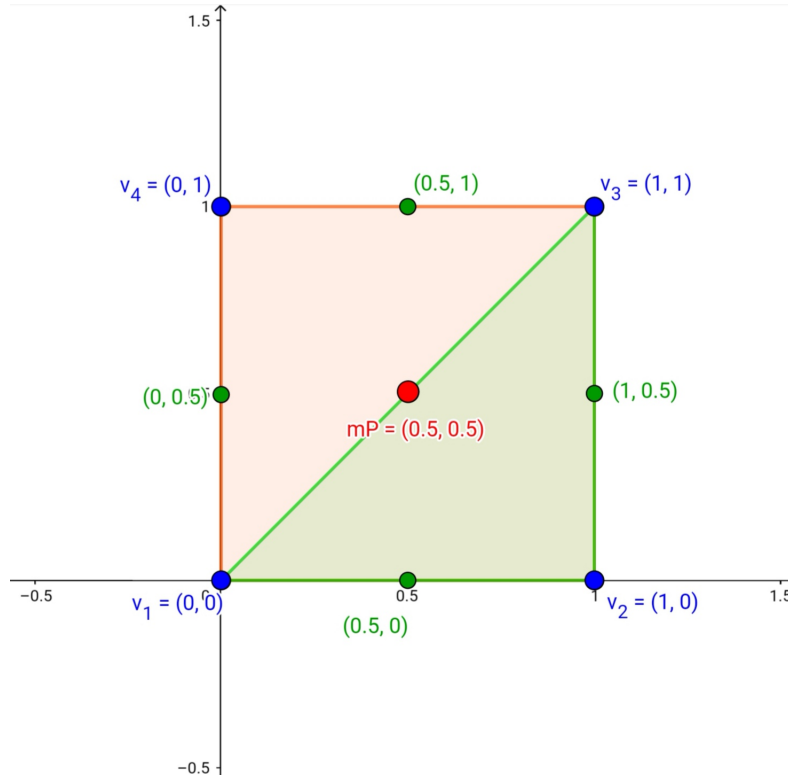


Figure 2.10: $P = [0, 1] \times [0, 1]$ and its family \mathcal{B} .

PROOF. This follows from the fact that Δ^{n-1} is a compact convex polytope in \mathbb{R}^n . \square

Proposition 2.6. *Theorem 12 implies Theorem 11.*

PROOF. Since the polytope is convex, then all its faces are also convex, and then the centers of gravity are a valid choice of points on every face. \square

In this way, by proving Theorem 12, we will have all the previous variants.

2.6 Proof of the polytopal KKMS Theorem with arbitrary points

In this section we prove Theorem 12, the KKMS theorem with arbitrary points using Theorem 6 (Kakutani fixed point theorem), and Theorem 7 (Berge maximum theorem).

Let P be a compact convex polytope in \mathbb{R}^n and let \mathcal{F} be the family of non-empty faces of P . Let $\{C_F \mid F \in \mathcal{F}\}$ be a family of closed sets of P such that for every $F \in \mathcal{F}$, we have that $F \subseteq \bigcup \{C_G \mid G \in \mathcal{F}, G \subseteq F\}$. Let $\mathcal{P} = \{b^F \in F \mid F \in \mathcal{F}\}$ be any choice of points ¹ on the faces of P . Also, let V be the vertex set of P .

Definition 2.6.1. *Given the above, for every $x \in P$, we define the **family of faces whose closed sets have x** and we denote it $I(x) = \{F \in \mathcal{F} : x \in C_F\}$.*

¹As we have done before, for 0-dimensional faces $\{v\}$ we will simplify $C_{\{v\}}$ to C_v , and $b^{\{v\}}$ to b^v .

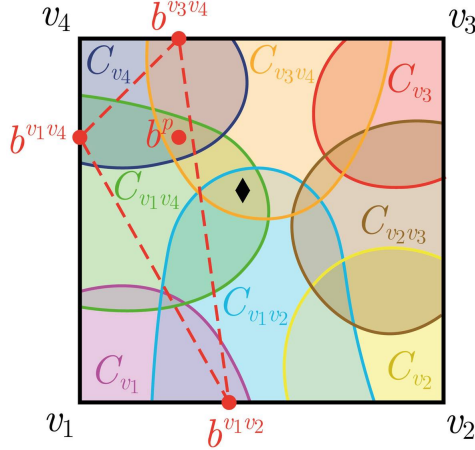


Figure 2.11: $P = [0, 1] \times [0, 1]$, $\{C_F \mid F \in \mathcal{F}\}$ and the family \mathcal{B} .

Definition 2.6.2. We define \mathfrak{B} as the family of all possible sets obtained from Definition 2.6.1, that is $\mathfrak{B} = \{I(x) : x \in P\}$.

Observation 3. The family $\mathfrak{B} = \{I(x) : x \in P\}$ is finite because it is an element of $\mathcal{P}(\mathcal{P}(\mathcal{F}))$ and \mathcal{F} is finite. Thus, let $\mathfrak{B} = \{\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_p\}$.

Lemma 1. If for some $k \in \{1, \dots, p\}$, we have that the face $\{v\}$ is in \mathcal{B}_k for every vertex $v \in V$, then the conclusion of Theorem 12 is true.

PROOF. Suppose that the face $\{v\}$ is in \mathcal{B}_k for every vertex $v \in V$ and that $\mathcal{B}_k = I(x)$ for some $x \in P$. Then $x \in C_v$ for every $v \in V$ and thus

$$\bigcap_{v \in V} C_v \neq \emptyset.$$

Also, $\{\{v\} : v \in V\}$ is geometrically balanced, as for every $v \in V$, we have that b^v is in $\{v\}$, and thus is equal to v . Then $\text{conv}\{b^v \mid v \in V\} = P$, from where

$$b^P \in \text{conv}\{b^v \mid v \in V\} = P.$$

□

Lemma 2. If $\mathcal{B}_k = \mathcal{F}$ for some $k \in \{1, \dots, p\}$, then the conclusion of Theorem 12 is true.

PROOF. This follows from Lemma 1 since if $\mathcal{B}_k = \mathcal{F}$, in particular $\{v\} \in \mathcal{B}_k$ for every $v \in V$. □

To reach a contradiction, we will now suppose that the conclusion of the theorem does not hold. If any of the conditions of Lemma 1 or Lemma 2 hold, then we have the conclusion of the theorem. Thus, we may assume from now on that no \mathcal{B}_k has $\{v\}$ for every $v \in V$. Therefore, for every k , there is at least one $v \in V$ such that $\{v\} \notin \mathcal{B}_k$. For this v , it happens that $C_v \subseteq \bigcup_{F \notin \mathcal{B}_k} C_F$ (the set on the left is one of the sets in the union).

Now, by the hypothesis applied to the face $\{v\}$ of P , we have that

$$\{v\} \subseteq \bigcup_{F \subseteq \{v\}} C_F = C_v.$$

This and the comment above shows that $v \in \bigcup_{F \notin \mathcal{B}_k} C_F$. The fact that this last set is non-empty allows us to give the following definition.

Definition 2.6.3. For every $k \in \{1, \dots, p\}$ we define the function $\beta_k : P \rightarrow \mathbb{R}^+ \cup \{0\}$ given by

$$\beta_k(x) = d \left(x, \bigcup_{F \notin \mathcal{B}_k} C_F \right),$$

where $d(x, A)$ denotes the distance between a point x and the set A .

Proposition 2.7. If the conclusion of the theorem is not true, then for every $k \in \{1, \dots, p\}$ there exist $n_k \in \mathbb{R}^n$ such that

$$b^P \cdot n_k > b^F \cdot n_k$$

for every $F \in \mathcal{B}_k$.

PROOF. Note that for every $k \in \{1, \dots, p\}$, we have $\mathcal{B}_k = I(x)$ for some $x \in P$, and then $\bigcap_{F \in \mathcal{B}_k} C_F$ is never empty. Since the conclusion of the theorem is not true, \mathcal{B}_k is never geometrically balanced. Therefore, for every $k \in \{1, \dots, p\}$, we have that $b^P \notin \text{conv}\{b^F \mid F \in \mathcal{B}_k\}$. By Theorem 2 (separation of convex sets), we may find a hyperplane Π separating b^P and $\text{conv}\{b^F \mid F \in \mathcal{B}_k\}$, and thus separating b^P and each b^F for $F \in \mathcal{B}_k$. Let us assume that

$$\Pi = \{x \in \mathbb{R}^n : x \cdot n_k = c\}.$$

Without loss of generality, b^P satisfies $b^P \cdot n_k > c$ and then $b^F \cdot n_k < c$ for each $F \in \mathcal{B}_k$. Then,

$$b^P \cdot n_k > c > b^F \cdot n_k$$

for any $F \in \mathcal{B}_k$, as desired. \square

The fact that this n_k exists for every $k \in \{1, \dots, p\}$ allows us to give the following definition.

Definition 2.6.4. When the conclusion of the theorem does not hold, we define the function $f : P \rightarrow \mathbb{R}^n$ given by

$$f(x) = \sum_{k=1}^p \beta_k(x) n_k,$$

where β_k and n_k are defined as above.

Note that f is continuous as it is a finite sum of continuous functions.

Definition 2.6.5. When f can be defined as above, we define the set-valued function $F : P \rightarrow \mathcal{P}(P)$, given by $F(x) = \{y \in P \mid y \cdot f(x) = \max\{z \cdot f(x) \mid z \in P\}\}$.

This function F also satisfies the following important properties.

Proposition 2.8. When F can be defined as above, F is upper hemicontinuous, closed valued, convex valued and non-empty valued.

PROOF. Consider the function $G : P \times P \rightarrow \mathbb{R}$ given by $G(z, x) = z \cdot f(x)$, and the correspondence $C : P \rightrightarrows P$ given by $C(x) = P$. Since P is compact and non-empty, C is a valid correspondence for the Berge Maximum theorem (Theorem 7). Also, since C is constant, it is continuous.

The marginal function of G is

$$G^*(z) = \sup\{G(z, x) : x \in P\} = \max\{z \cdot f(x) : x \in P\},$$

where we replaced the supremum by the maximum because P is compact.

Therefore the set of maximizers is precisely $F(x)$. By the Berge Maximum theorem, F is upper-hemicontinuous, non-empty and compact-valued. In particular, F is closed-valued.

To see that F is convex valued, note that for each x , we have that $F(x)$ is the subset of P consisting of the points that maximize the linear form $z \mapsto z \cdot f(x)$. By Theorem 5, this set must be a face of P , so F is convex valued. \square

The previous proposition allows us to apply Kakutani fixed point theorem (Theorem 6) to $F(x)$.

Proposition 2.9. If the conclusion of the theorem does not hold, then there exists $x_0 \in P$ such that x_0 is in the face $F(x_0)$ of P . In consequence, there is a subface $S \subseteq F(x_0)$ such that $x_0 \in C_S$.

PROOF. By Proposition 2.8, we may apply Kakutani fixed point theorem (Theorem 6) to F , so there exists $x_0 \in P$ such that $x_0 \in F(x_0)$, where $F(x_0)$ is a face of P .

The hypothesis of the problem states that $F(x_0) \subseteq \bigcup_{S \subseteq F(x_0)} C_S$, so then there is a subface $S \subseteq F(x_0)$ such that $x_0 \in C_S$. \square

We have all the ingredients to state our final contradiction. Let k be such that $I(x_0) = \mathcal{B}_k$. Let S be the face given by the previous proposition, so $S \in \mathcal{B}_k$. By definition of $I(x_0)$, there cannot be $H \in \mathcal{F}$ such that $x_0 \in C_H$ and $H \notin \mathcal{B}_k$. Therefore,

$$\beta_k(x_0) = d\left(x_0, \bigcup_{H \notin \mathcal{B}_k} C_H\right) > 0.$$

By definition of n_k , we thus have

$$\beta_k(x_0)(b^P \cdot n_k) > \beta_k(x_0)(b^S \cdot n_k).$$

This strict inequality allows us to obtain the strict inequality in the following chain:

$$b^P \cdot f(x_0) = \sum_{j=1}^p \beta_j(x_0)(b^P \cdot n_j) > \sum_{j=1}^p \beta_j(x_0)(b^S \cdot n_j) = b^S \cdot f(x_0).$$

But on the other hand, $b^S \in S \subseteq F(x_0)$, so $b^S \cdot f(x_0) = \max\{z \cdot f(x_0) \mid Z \in P\}$. Then, we have

$$b^S \cdot f(x_0) \geq b^P \cdot f(x_0).$$

This together with the strict inequality above yields the desired final contradiction. \square

2.7 Polytopal KKMS Theorem for open sets

The Polytopal KKMS Theorem is also true for open sets. To see this, we will first need the following proposition:

Proposition 2.10. *Let X be a normal space, and let C_1, \dots, C_n be closed sets in X such that $\bigcap_{i=1}^n C_i = \emptyset$. Then there exist open sets U_1, \dots, U_n such that $C_i \subseteq U_i$ for every $i \in [n]$, and $U_1 \cap \dots \cap U_n = \emptyset$.*

PROOF. Since C_1, \dots, C_n are closed sets in X , then $(C_2 \cap \dots \cap C_n)$ is closed too. We have that

$$C_1 \cap (C_2 \cap \dots \cap C_n) = \emptyset,$$

and since X is a normal space, we can find disjoint open sets U_1 and V_1 such that $C_1 \subseteq U_1$ and $(C_2 \cap \dots \cap C_n) \subseteq V_1$. Due to Proposition 1.1,

$$\overline{U_1} \cap (C_2 \cap \dots \cap C_n) \subseteq \overline{U_1} \cap V_1 = \emptyset.$$

And then $\overline{U_1}, C_2, \dots, C_n$ are closed sets in X with empty intersection. So again, since X is normal, we can find open sets U_2 and V_2 such that $C_2 \subseteq U_2$ and

$$\overline{U_1} \cap C_3 \cap \dots \cap C_n \subseteq V_2,$$

with $\overline{U_1} \cap \overline{U_2} \cap C_3 \cap \dots \cap C_n = \emptyset$. Repeating the same process for every $1 \leq j \leq n-1$ with $\{U_i\}_{i \leq j} \cup \{C_i\}_{i > j}$, we can find sets U_1, \dots, U_n such that $C_i \subseteq U_i$ for every $i \in [n]$, and

$$(U_1 \cap \dots \cap U_n) \subseteq (\overline{U_1} \cap \dots \cap \overline{U_n}) = \emptyset.$$

\square

Now we will see the statement and proof of the Polytopal KKMS Theorem for open sets. This is an original contribution of this work.

Theorem 13. (Polytopal KKMS Theorem for open sets) Let P be a compact convex polytope in \mathbb{R}^n and let \mathcal{F} be the family of non-empty faces of P . Let $\{A_F \mid F \in \mathcal{F}\}$ be a family of open sets of P such that for every $F \in \mathcal{F}$, we have that $F \subseteq \bigcup\{A_G \mid G \in \mathcal{F}, G \subseteq F\}$. Let $\mathcal{P} = \{b^F \in F \mid F \in \mathcal{F}\}$ be any choice of points on the faces of P .

Then there exists a subfamily \mathcal{B} of faces of P geometrically balanced with respect to \mathcal{P} such that

$$\bigcap_{F \in \mathcal{B}} A_F \neq \emptyset.$$

PROOF. We explain how to build sets $\{C_F \mid F \in \mathcal{F}\}$ from $\{A_F \mid F \in \mathcal{F}\}$ such that

1. C_F is closed for every $F \in \mathcal{F}$.
2. For every $F \in \mathcal{F}$, we have $F \subseteq \bigcup\{C_G \mid G \in \mathcal{F}, G \subseteq F\}$.
3. $C_F \subseteq A_F$.

If we achieve this, we can apply Theorem 12 (Polytopal KKMS Theorem), and then argue why the conclusion follows.

Let $F \in \mathcal{F}$. Since $F \subseteq \bigcup_{\substack{G \in \mathcal{F}, \\ G \subseteq F}} A_G$ for every $F \in \mathcal{F}$, then

$$F \cap \bigcap_{\substack{G \in \mathcal{F}, \\ G \subseteq F}} (A_G)^c = \emptyset.$$

Since F and $(A_G)^c$ are closed in Δ^{n-1} , due to Proposition 2.10, there exists an open set F' with $F \subset F'$, and open sets $\{B_{G,F} \mid G \in \mathcal{F}, G \subseteq F\}$ with $(A_G)^c \subset B_{G,F}$ for every $G \subseteq F$, such that

$$F' \cap \bigcap_{\substack{G \in \mathcal{F}, \\ G \subseteq F}} B_{G,F} = \emptyset. \tag{2.1}$$

We define $B_F = \bigcap_{\substack{H \in \mathcal{F}, \\ F \subseteq H}} B_{F,H}$. And $C_F = (B_F)^c$ for every $F \in \mathcal{F}$.

We claim that the sets C_F satisfy the properties (1), (2) and (3) above.

To see (1), note that since each $B_{F,H}$ is open, the set

$$B_F = \bigcap_{\substack{H \in \mathcal{F}, \\ F \subseteq H}} B_{F,H}$$

is open, and therefore $C_F = (B_F)^c$ is closed, for every $F \in \mathcal{F}$.

On the other hand, we have (2) i.e.

$$F \subseteq \bigcup_{\substack{G \in \mathcal{F}, \\ G \subseteq F}} C_G$$

for every $F \in \mathcal{F}$. Indeed, suppose there is $F \in \mathcal{F}$ such that there is $x \in F$ with

$$x \notin \bigcup_{\substack{G \in \mathcal{F}, \\ G \subseteq F}} C_G.$$

Then $x \notin C_G$ for every $G \subseteq F$, then

$$x \in (C_G)^c = B_G = \bigcap_{\substack{H \in \mathcal{F}, \\ G \subseteq H}} B_{G,H}.$$

In particular, since $G \subseteq F$, we have $x \in B_{G,F}$ for every $G \subseteq F$, that is

$$x \in \bigcap_{\substack{G \in \mathcal{F}, \\ G \subseteq F}} B_{G,F}.$$

But also $x \in F \subseteq F'$, so that means

$$F' \cap \bigcap_{\substack{G \in \mathcal{F}, \\ G \subseteq F}} B_{G,F} \neq \emptyset,$$

which is a contradiction to Equation (2.1).

It remains to prove (3), namely, that $C_F \subset A_F$ for every $F \in \mathcal{F}$. To see this, let $F \in \mathcal{F}$ and let $x \in C_F = (B_F)^c$. Then

$$x \notin B_F = \bigcap_{\substack{H \in \mathcal{F}, \\ F \subseteq H}} B_{F,H},$$

and therefore $x \notin B_{F,H}$ for some $H \in \mathcal{F}$ such that $F \subseteq H$. This means that $x \in (B_{F,H})^c$, and since $(A_F)^c \subset B_{F,H}$, we have $(B_{F,H})^c \subseteq A_F$, which implies $x \in A_F$.

So, the sets C_F satisfy (1), (2) and (3) so due to Theorem 12 (Polytopal KKMS Theorem), there exists a subfamily \mathcal{B} of faces of P geometrically balanced with respect to \mathcal{P} such that

$$\bigcap_{F \in \mathcal{B}} A_F \supseteq \bigcap_{F \in \mathcal{B}} C_F \neq \emptyset.$$

This concludes the proof. □

In particular, this implies open set versions for all the KKM variants in this chapter. For example.

Theorem 14. (KKM for open sets). *Let A_1, \dots, A_n be open sets in Δ^{n-1} such that for every non-empty face $S \subseteq [n]$, we have that $\Delta^S \subseteq \bigcup_{i \in S} A_i$. Then,*

$$\bigcap_{i=1}^n A_i \neq \emptyset.$$

PROOF. We follow the steps in the proofs of Proposition 2.5, Proposition 2.3 and Proposition 2.1. □

3 KKM Theorem Applications

3.1 Line transversals in families of compact connected sets in the plane

The problem of bounding the number of points or lines that can intersect certain families of sets in the plane has been studied since the mid-20th century. In 1969, Eckhoff proved that if a family of compact convex sets in the plane has the property that any four of them can be intersected by a line, then it is possible to intersect all of them with at most two lines. Due to a result of Santaló from 1940, this result is best possible in the sense that some families with this property cannot be intersected by a single line. In 1974, he showed that there exists a family of compact convex sets in the plane such that any three of them can be intersected by a line, but it is not possible to intersect all of them with two lines. This motivated the following definitions.

Definition 3.1.1. *Let \mathcal{F} be a family of sets in the plane. We say that \mathcal{F} has **property $T(r)$** if there exists a line intersecting every subfamily of \mathcal{F} with r or fewer sets.*

Definition 3.1.2. *We say that \mathcal{F} is **pierced** by k lines if there are k lines in the plane whose union intersects all the sets in \mathcal{F} .*

*We also define the **line-piercing number** of the family \mathcal{F} as the minimum k such that \mathcal{F} is pierced by k lines.*

For a time, it was unknown whether there was an upper bound on the line-piercing number of families of convex sets with the $T(3)$ property. However, in 1975, Kramer proved that in any family of compact convex sets in the plane with the $T(3)$ property, the line-piercing number was at most 5. In 1993, Eckhoff improved this bound to 4 and questioned whether it could be reduced to three lines.

Finally, in 2022, McGinnis and Zerbib proved that any family of compact convex sets in the plane with the $T(3)$ property has line piercing number at most 3. Actually, this follows from a much more general result of them ([8, Theorem 1.1]), which we state below.

Theorem 15. *Let $\mathcal{F}_1, \dots, \mathcal{F}_6$ be families of compact connected sets in \mathbb{R}^2 . If every three sets $A_1 \in \mathcal{F}_{i_1}$, $A_2 \in \mathcal{F}_{i_2}$, $A_3 \in \mathcal{F}_{i_3}$, $1 \leq i_1 < i_2 < i_3 \leq 6$, form a tight triple, then there exists $i \in [6]$ such that the line-piercing number of \mathcal{F}_i is at most 3.*

This result is a generalization in many directions. First, the result is extended from convex sets to connected sets. Then, the $T(3)$ property is weakened to the notion of forming a tight triples, which is out of the scope of this work. Finally, this is a *colorful* result, in the sense that several families of convex sets are studied at once.

The breakthrough of McGinnis and Zerbib was realizing that the following colorful variant of the KKM theorem due to Gale ([3]) could be used to prove Theorem 15.

Theorem 16. *Let $A_1^i, \dots, A_n^i, i \in [n]$ be open sets of Δ^{n-1} , such that for every $i \in [n]$ and for every face σ of Δ^{n-1} we have $\sigma \subseteq \bigcup_{e_j \in \sigma} A_j^i$. Then there exists a permutation π of $[n]$ such that $\bigcap_{i=1}^n A_i^{\pi i} \neq \emptyset$.*

We will not discuss the colorful version of Theorem 15. However, we present here the core ideas of their proof in a simplified version of the technique of McGinnis and Zerbib, for when the convex sets are not colored. Before stating the result that we will prove, we will show how in problems about line transversals it is equivalent to study convex sets or connected sets.

Proposition 3.1. *If S is a connected set in \mathbb{R}^2 and ℓ is a line intersecting $\text{conv}(S)$, then ℓ must intersect S .*

PROOF. In the aim of reaching a contradiction, let's suppose that ℓ does not intersect S . Observe that ℓ divides \mathbb{R}^2 in two open half-planes H^+ and H^- .

If $S \cap H^+ = \emptyset$, then $S \subseteq H^-$ and thus $\text{conv}(S) \subseteq H^-$. Since $H^- \cap \ell = \emptyset$, then $\text{conv}(S) \cap \ell = \emptyset$, which contradicts that ℓ intersects $\text{conv}(S)$. Therefore, S intersects H^+ . Analogously, S intersects H^- .

So S must intersect both H^+ and H^- , but then H^+ and H^- are two open sets such that:

- $H^+ \cap H^- = \emptyset$,
- $S \subseteq H^+ \cup H^-$,
- $S \cap H^+ \neq \emptyset$ and $S \cap H^- \neq \emptyset$.

So S is disconnected, which is a contradiction. □

Theorem 17. *A finite family \mathcal{F} of compact, connected sets in the plane with the $T(3)$ property has a line-piercing number at most 3.*

PROOF. Since the family \mathcal{F} is finite, we may rescale the plane so that every set in \mathcal{F} is contained in the unit disk. Let $f : \mathbb{R} \rightarrow S^1$ be the parametrization of S^1 defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$.

To each point $x = (x_1, \dots, x_6) \in \Delta^5$ we assign six points on S^1 given by $f_i(x) = f(\sum_{j=1}^i x_j)$, for every $i \in [6]$. We define the following lines:

- $\ell_1(x) = \ell_4(x)$ is the line connecting the points $f_1(x)$ and $f_4(x)$,
- $\ell_2(x) = \ell_5(x)$ is the line connecting the points $f_2(x)$ and $f_5(x)$,

- $\ell_3(x) = \ell_6(x)$ is the line connecting the points $f_3(x)$ and $f_6(x)$.

Now, for every $i \in [6]$, let R_x^i be the interior of the region bounded by $\ell_{i-1}(x)$, $\ell_i(x)$ and the arc on S^1 connecting $f_{i-1}(x)$ and $f_i(x)$ clockwise. See Figure 3.1. Note for every $i \in [6]$, if $x_i = 0$, then $f(x_{i-1}) = f(x_i)$, and then $R_x^i = \emptyset$. Also notice that it is possible that some of the regions R_x^i have non-empty intersection. See Figure 3.2.

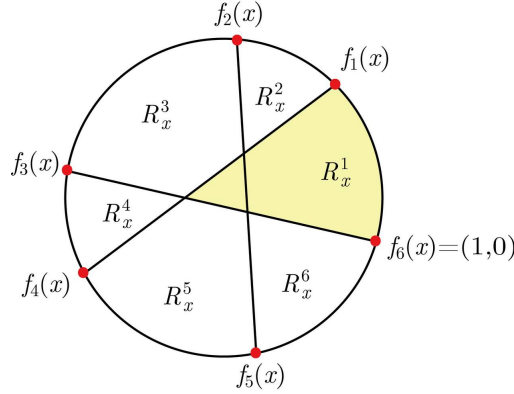


Figure 3.1: S^1 and $R_1^{x_0}$.

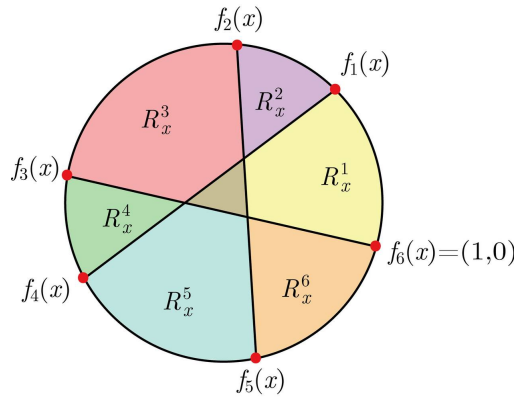


Figure 3.2: S^1 and regions R_x^i , $i \in [6]$.

For $i \in [6]$, let

$$A_i := \{x \in \Delta^5 \mid \exists F \in \mathcal{F} \text{ such that } F \subset R_x^i\}.$$

Since the sets $F \in \mathcal{F}$ are closed, A_i is open¹

¹To see this, let $F \in \mathcal{F}$, $i \in [6]$, and let $h_F^i : \Delta^5 \rightarrow \mathbb{R}$ given by $h_F^i(x) = d(F, R_x^i) = \inf\{d(a, r) \mid a \in F, r \in R_x^i\} = \min\{d(a, r) \mid a \in F, r \in R_x^i\}$. Where inf can be replaced by min since the sets are compact. For each $F \in \mathcal{F}$, let $A_i^F = \{x \in \Delta^5 \mid F \subset R_x^i\}$, and note that $A_i = \bigcup_{F \in \mathcal{F}} A_i^F$. Note that $h_F^i(x) > 0$ if only if $F \subset R_x^i$. So $A_i^F = h_F^i{}^{-1}[(0, \infty)]$. Then, A_i^F is the pre-image of an open set with respect to a continuous function, and A_i is union of open sets, so A_i is open.

If there is some $x \in \Delta^5$ for which $x \notin \bigcup_{i=1}^6 A_i$, since the sets in \mathcal{F} are connected, then $\ell_1(x) \cup \ell_2(x) \cup \ell_3(x)$ must intersect every set in \mathcal{F} . So, in the aim of reaching for a contradiction, suppose that $\Delta^5 = \bigcup_{i=1}^6 A_i$. Note that for each non-empty set $I \subseteq [6]$, if $x \in \Delta^I = \text{conv}\{e_i \mid i \in I\}$, then $x_k = 0$ for $k \notin I$, so by the remark above we have $R_x^k = \emptyset$ for $k \notin I$. By the definition of A_i , we thus have $x \in \bigcup_{i \in I} A_i$. Therefore, $\Delta^I \subseteq \bigcup_{i=1}^6 A_i$. That is, the sets A_i form a KKM cover of Δ^5 .

So we may apply Theorem 14 (KKM Theorem for open sets) to Δ^5 and the KKM cover $\{A_i \mid i \in [6]\}$, and therefore $\bigcap_{i=1}^6 A_i \neq \emptyset$. That is, there exists a point $x_0 = (x_1, \dots, x_6) \in \Delta^5$ such that $x_0 \in \bigcap_{i=1}^6 A_i$. Then, each of the open regions $R_{x_0}^i$ for every $i \in [6]$, contains a set $F_i \in \mathcal{F}$. In particular, $R_{x_0}^i \neq \emptyset$ for every $i \in [6]$, then $x_i \neq 0$ for every $i \in [6]$. But then, we claim that \mathcal{F} does not satisfy the $T(3)$ property, since either F_1, F_3, F_5 or F_2, F_4, F_6 cannot be pierced with one line. To see this, note that either $R_1^{x_0}, R_3^{x_0}, R_5^{x_0}$ or $R_2^{x_0}, R_4^{x_0}, R_6^{x_0}$ are pairwise disjoint, and no line intersects the three of them. See Figure 3.3. Suppose without loss of generality, that $R_1^{x_0}, R_3^{x_0}, R_5^{x_0}$ are pairwise disjoint and cannot be intersected by a line. Then F_1, F_3, F_5 cannot be pierced with a line. This contradicts the fact that \mathcal{F} has the $T(3)$ property.

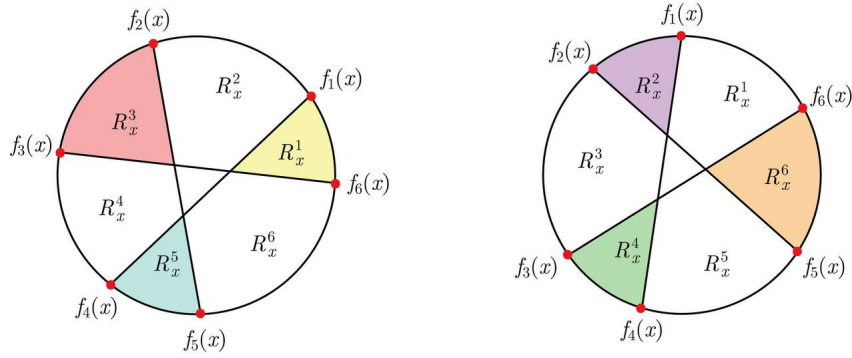


Figure 3.3: $R_1^{x_0}, R_3^{x_0}, R_5^{x_0}$ (left) or $R_2^{x_0}, R_4^{x_0}, R_6^{x_0}$ (right) are pairwise disjoint, and in triangular position.

□

3.2 Line transversals with directions

We have already seen an application of Theorem 8 (KKM Theorem) to transversals. Now we conclude our work with an original application of Theorem 12 (Polytopal KKMS Theorem).

Proposition 3.2. *Let \mathcal{F} be a finite family of compact convex sets in \mathbb{R}^2 such that any two of them can be pierced by a horizontal line or a vertical line. Then there is a vertical and a horizontal line that intersect all sets in \mathcal{F} .*

PROOF. Since the family \mathcal{F} is finite, we may rescale the plane so that every set in \mathcal{F} is contained in the centered square $C_2 = [-1, 1] \times [-1, 1]$. Note that every point $x \in C_2$ defines a horizontal line $\ell_1(x)$ and a vertical line $\ell_2(x)$ such that $\ell_1(x) \cap \ell_2(x) = x$ and

$\ell_1(x)$ and $\ell_2(x)$ are parallel to the x and y axes, respectively. Let $x \in C_2$, for every $i \in [4]$, let R_x^i be the interior of the region bounded by $\ell_1(x)$, $\ell_2(x)$ and the corner opposite to v_i in C_2 as in Figure 3.4. Note that $R_x^i = \emptyset$ if x is on an edge that does not contain v_i . And if $x = v_i$, then the only possible region is R_x^i .

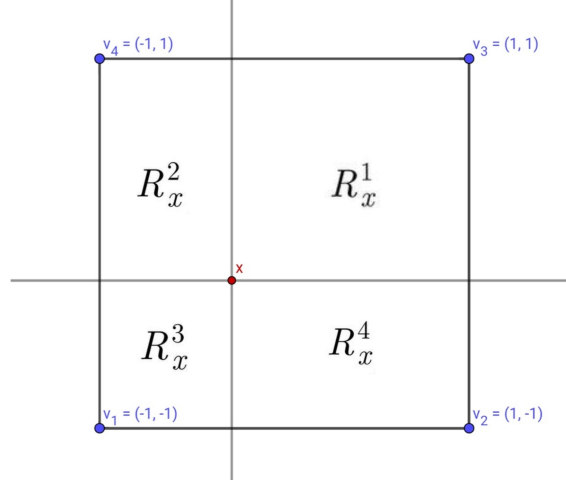


Figure 3.4: C_2 and regions R_x^i .

For $i \in [4]$, let

$$A_i := \{x \in C_2 \mid \exists F \in \mathcal{F} \text{ such that } F \subset R_x^i\}.$$

Since the sets $F \in \mathcal{F}$ are closed, A_i is open. This follows from the same idea in the proof of Theorem 17.

If there is some $x \in C_2$ for which $x \notin \bigcup_{i=1}^4 A_i$, since the sets in \mathcal{F} are connected, then $\ell_1(x) \cup \ell_2(x)$ must intersect every set in \mathcal{F} . So, in the aim of reaching for a contradiction, suppose that $C_2 = \bigcup_{i=1}^4 A_i$.

Note that if $x \in H$ where H is a face of C_2 and $v_i \notin H$, by the remark above we have $R_x^i = \emptyset$. By the definition of A_i , we thus have

$$x \in \bigcup_{i \in I} A_i.$$

Then,

$$H \subseteq \bigcup_{\substack{i \in [4] \\ v_i \in H}} A_i.$$

That is, the sets A_i form a *KKM* cover of C_2 .

So we can apply Theorem 13 (Polytopal KKM's Theorem for open sets) to C_2 and $\{A_i \mid i \in [4]\}$. Therefore there exist a balanced family $\mathcal{B} \subseteq \{v_1, v_2, v_3, v_4\}$ such that

$$\bigcap_{\substack{i \in [4] \\ v_i \in \mathcal{B}}} A_i \neq \emptyset.$$

But note that if \mathcal{B} is balanced, it must contain at least two opposite vertices, either v_1, v_3 , or v_2, v_4 .

Let

$$x_0 \in \bigcap_{\substack{i \in [4] \\ v_i \in \mathcal{B}}} A_i,$$

then each of the open regions $R_{x_0}^i$ for each $i \in [4]$ such that $v_i \in \mathcal{B}$, contains a set $F_i \in \mathcal{F}$. But this means either $R_{x_0}^1$ and $R_{x_0}^3$ contains a set F_1 and F_3 each, or $R_{x_0}^2$ and $R_{x_0}^4$ contains a set F_2 and F_4 each. But this contradicts the hypothesis that any pair of sets can be pierced by a horizontal or a vertical line, since F_1 and F_3 , or either F_2 and F_4 can't be pierced by a horizontal or a vertical line since they are in opposite regions. \square

This result is part of a larger work that we are developing, which involves transversals in higher dimensions and colorful variants.

4 Conclusions

In this text we have worked on a deep study of the Knaster-Kuratowski-Mazurkiewicz Theorem (KKM) and some of its significant generalizations, including the KKMS Theorem and the Polytopal KKMS Theorem.

We began by reviewing essential definitions and concepts in topology and discrete geometry, in order to provide a comprehensive examination of various versions of the KKM Theorem, culminating in an explicit proof of the Polytopal KKMS Theorem. We also presented an original proof for the version of the Polytopal KKMS Theorem where the closed sets of the KKM cover are replaced by open sets.

In the final chapter, we explored the applications of these theorems to problems in geometric transversal theory and Helly-type problems. Our results highlight the versatility and far-reaching implications of the Polytopal KKMS Theorem, particularly in discrete geometry. Specifically, we showed how these theorems can be used to address complex questions in transversal theory, providing new insights and expanding the current understanding of these problems.

Moreover, the Polytopal KKMS Theorem and some of its generalizations, such as the colorful version, have the potential to open new paths for many new results in geometric transversal theory. These extensions open up a range of possibilities for further research, suggesting that the interplay between convex analysis and discrete geometry is rich and with untapped potential.

In conclusion, this thesis has not only deepened the understanding of the KKM and KKMS theorems but also showcased their power in addressing challenging problems in discrete geometry and geometric transversal theory. The results obtained here set the stage for future explorations, and we anticipate that the methods and ideas presented will continue to inspire new developments in the field.

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