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por su amor y apoyo incondicional.
A mi esposa,
por ser mi mejor amiga y mi fortaleza.
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por su cariño y complicidad,
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Chapter 1

Introduction

1.1 Background of the Problem

The notion of quasi-hereditary algebra and highest weight category were introduced and studied by E. Cline, B. Parshall and L. Scott (8; 25; 28). Highest weight categories arise in the representation theory of Lie algebras and algebraic groups. For the setting of finite dimensional algebras, quasi-hereditary algebras were amply studied by V. Dlab and C. M. Ringel in (9; 11; 12; 26). In addition, they introduced the set of standard modules ${}_{\Lambda}\Delta$ associated with an algebra Λ. In (26), C. M. Ringel studied the homological properties of the category $\mathcal{F}({}_{\Lambda}\Delta)$ of ${}_{\Lambda}\Delta$-filtered Λ-modules and constructed the characteristic tilting module ${}_{\Lambda}T$ associated with $\mathcal{F}({}_{\Lambda}\Delta)$. In (30), B. Zhu studied the triangular matrix algebra $\Lambda = \left[\begin{smallmatrix} T & 0 \\ M & U \end{smallmatrix}\right]$ where T and U are quasi-hereditary algebras, and he proved that under suitable conditions on M, Λ is a quasi-hereditary algebra.

On the other hand, B. Mitchell developed the idea that additive categories can be thought as *rings with several objects* and he showed that a substantial amount of non-commutative ring theory is still true in this generality (22).

Recently, R. Martínez-Villa and M. Ortíz studied tilting theory in arbitrary functor categories, in (19; 20). They proved that most of the properties that are satisfied by a tilting module over an artin algebra also hold for functor categories. Following the line of the above mentioned works, M. Ortíz introduced in (24) the concept of quasi-hereditary category to study the Auslander-Reiten components of a finite dimensional algebra Λ. Similarly, as the standard modules appear in the theory of quasi-hereditary algebras, the concept of standard functors appears in this context. We note that the notion of standard functor is a generalization of the notion of standard module. As a consequence, a connection is obtained between highest weight categories and quasi-hereditary categories as stated by H. Krause in (14).

Finally, the concept of the triangular matrix category is introduced in (15; 16), as the analogue of the triangular matrix algebra to the context of *rings with several objects*, and they obtain some applications to path categories given by infinite quivers, the construction of recollements and the study of functorially finite subcategories in functor categories.

The aim of this thesis is to show that the triangular matrix category $\Lambda = \left[\begin{smallmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{smallmatrix}\right]$, where $\mathcal{T}$ and $\mathcal{U}$ are Hom-finite, Krull-Schmidt K-quasi-hereditary categories and M is a $\mathcal{U} \otimes_K \mathcal{T}^{op}$-module that satisfies suitable conditions, is quasi-hereditary in the sense of (15) and (24), generalizing some of the results obtained by B. Zhu in (30).

It is worth mentioning that recently in (18) similar results have been obtained in the context of standardly stratified lower triangular $\mathbb{K}$-algebras with enough idempotents.

1.2 Summary of results

In this thesis, we prove that the lower triangular matrix category $\Lambda = \left[\begin{smallmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{smallmatrix}\right]$, where $\mathcal{T}$ and $\mathcal{U}$ are Hom-finite, Krull-Schmidt K-quasi-hereditary categories and M is an $\mathcal{U} \otimes_K \mathcal{T}^{op}$-module that satisfies suitable conditions, is quasi-hereditary. This result generalizes the work of B. Zhu in his study on triangular matrix algebras over quasi-hereditary algebras. Moreover, we obtain a characterization of the category of the ${}_\Lambda\Delta$-filtered Λ-modules.

1.3 Thesis Outline

We outline the content of the thesis chapter-by-chapter as follows.

In Chapter 2, we recall basic results about $K-$categories, path categories, functor categories, quasi-hereditary categories and triangular matrix categories.

In Chapter 3, that is part of (23), we give the main result of this thesis, see Theorem 3.4.1, which is a generalization of a result given in (30, Theorem 3.1), see 4.1.12, as stated above, see Remark 3.4.7. Moreover, we obtain a characterization of the category of the ${}_\Lambda\Delta$-filtered Λ-modules and we give an example a triangular matrix category Λ which is quasi-hereditary with respect to certain filtration $\{\Lambda_j\}_{j\geq 0}$.

In Chapter 4, we provide a brief overview of quasi-hereditary algebras, outlining their key concepts and significance in representation theory. The objective of this chapter is to present Theorem 4.1.12, given by B. Zhu in (30, Theorem 3.1), which motivated the development of the main results presented in the previous chapter. The results of sections 4.1.1 and 4.1.2 are classical and were developed by C.M. Ringel in (26).

Chapter 2

Preliminaries

In this section, we will present a series of definitions and results, some of which are classical and for which we will not provide a proof, that will be essential for the development of this work.

2.1 K-categories

Let's begin by recalling some basic definitions to approach this work. The reader can consult (1) and (6) for more details.

Let K be a field. A category $\mathcal{C}$ is a **K-Category** if for each pair of objects X and Y in $\mathcal{C}$, the set of morphisms $\mathcal{C}(X,Y)$ is equipped with a K-vector space structure such that the composition $\circ$ of morphisms in $\mathcal{C}$ is a K-bilinear map. A K-category $\mathcal{C}$ is called **Hom-finite** if $\dim_K \mathcal{C}(X,Y) < \infty$.

A **Krull-Schmidt** category is an additive category such that each object decomposes into a finite direct sum of indecomposable objects with local endomorphism rings.

Let $\mathcal{C}$ be an additive K-category. A class I of morphisms of $\mathcal{C}$ is a **two-sided ideal** in $\mathcal{C}$ if:

(a) the zero morphism $0_X \in \mathcal{C}(X,X)$ belongs to I;

(b) if $f, g : X \to Y$ are morphisms in I and λ, $\mu \in K$, then $\lambda f + \mu g \in I$;

(c) if $f \in I$ and g is a morphism in $\mathcal{C}$ that is left-composable with f, then $g \circ f \in I$ and

(d) if $f \in I$ and h is a morphism in $\mathcal{C}$ that is right-composable with f, then $f \circ h \in I$.

Equivalently, a two-sided ideal I of $\mathcal{C}$ can be considered as a subfunctor $I(-,-) \subseteq \mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Mod}\ K$, defined by assigning to each pair (X,Y) of objects X,

Y of $\mathcal{C}$ a K-subspace $I(X,Y)$ of $\mathcal{C}(X,Y)$ such that if $f \in I(X,Y)$, $g \in \mathcal{C}(Y,Z)$ and $h \in \mathcal{C}(U,X)$ then $gfh \in I(U,Z)$.

Given a two-sided ideal I in an additive K-category $\mathcal{C}$, the **quotient category** $\mathcal{C}/I$ is the category whose objects are the same as the objects of $\mathcal{C}$ and the space of morphisms from X to Y in $\mathcal{C}/I$ is the quotient space $(\mathcal{C}/I)(X,Y) = \mathcal{C}(X,Y)/I(X,Y)$ of $\mathcal{C}(X,Y)$. We can see that the quotient category $\mathcal{C}/I$ is an additive K-category, and the projection functor $\pi : \mathcal{C} \to \mathcal{C}/I$ assigning to each $f : X \to Y$ in $\mathcal{C}$ the coset $f + I \in (\mathcal{C}/I)(X,Y)$ is a K-linear functor. Moreover, π is full and dense, and $\mathrm{Ker}(\pi) = I$.

The **(Jacobson) radical** of an additive K-category $\mathcal{C}$ is defined as the two-sided ideal $\mathrm{rad}_{\mathcal{C}}(-,-)$ in $\mathcal{C}$ given by the formula

$$\mathrm{rad}_{\mathcal{C}}(X,Y) = \{h \in \mathcal{C}(X,Y) : 1_X - gh \text{ is invertible for any } g \in \mathcal{C}(Y,X)\},$$

for all objects X and Y of $\mathcal{C}$.

Tensor product of k-categories. Let $\mathcal{C}$ and $\mathcal{C}'$ be K-categories. The tensor product $\mathcal{C} \otimes_K \mathcal{C}'$ is the category whose class of objects is Obj $\mathcal{C} \times$ Obj $\mathcal{C}'$, where the set of morphisms from (p_1, q_1) to (p_2, q_2) is the ordinary tensor product $\mathcal{C}(p_1, p_2) \otimes \mathcal{C}'(q_1, q_2)$. The composition

$$\mathcal{C}(p_1,p_2) \otimes \mathcal{C}'(q_1,q_2) \times \mathcal{C}(p_2,p_3) \otimes \mathcal{C}'(q_2,q_3) \to \mathcal{C}(p_1,p_3) \otimes \mathcal{C}'(q_1,q_3)$$

is given by the rule $((f_1 \otimes g_1), (f_2 \otimes g_2)) \mapsto (f_2 f_1 \otimes g_2 g_1)$. This composition is bilinear; see (22).

2.2 Path Categories and Representations

A **quiver** is an oriented graph, formally denoted by a quadruple $Q = (Q_0, Q_1, s, t)$, with a set of vertices Q_0 and a set of arrows Q_1, and two maps $s, t : Q_1 \to Q_0$, called *source* and *target*, defined by $s(a \to b) = a$ and $t(a \to b) = b$, respectively, if $\alpha : a \to b$ is an arrow in Q_1.

A path of length $l \geq 1$ from a to b in a quiver Q is of the form $(a|\alpha_1, \ldots, \alpha_l|b)$ with arrows α_i satisfying $t(\alpha_i) = s(\alpha_{i+1})$ for all $1 \leq i \leq l$ and $a = s(\alpha_1)$ as well as $b = t(\alpha_l)$. In addition, for any vertex a in Q_0, a path of length 0 from a to itself is denoted by ϵ_a.

Given a quiver Q, its **path category** $K\mathcal{Q}$ is an additive category, with objects being direct sums of indecomposable objects. The indecomposable objects in the path category are given by the set Q_0, and given $a, b \in Q_0$, the set of maps from a to b is given by the K-vector space with basis the set of all paths from a to b. The composition of maps is induced from the usual composition of paths:

$$(a|\alpha_1,\ldots,\alpha_l|b)(b|\beta_1,\ldots,\beta_s|c) = (a|\alpha_1,\ldots,\alpha_l\beta_1,\ldots,\beta_s|c), \tag{2.1}$$

where $(a|\alpha_1,\ldots,\alpha_l|b)$ is a path from a to b and $(b|\beta_1,\ldots,\beta_s|c)$ is a path from b to c.

Similarly, the **path algebra** of Q denoted by KQ, is the K-vector space with basis the set of all paths in Q, and the product of two paths is defined by (2.1) if they are composable, and it is zero if they are non-composable.

In KQ, any ideal I is generated by a set of paths $\{\rho_i|i\}$, that is $I = \langle\rho_i|i\ \rangle$. Let I be an ideal in KQ, then given a pair of finite sets of vertices $\{X_i\}_{i=1}^n$, $\{Y_j\}_{j=1}^m$ we set $\mathcal{I}(\oplus_{i=1}^n X_i, \oplus_{j=1}^m Y_j) = \{(f_{ij}) \in K\mathcal{Q}(\oplus_{i=1}^n X_i, \oplus_{j=1}^m Y_j)|f_{ij} \in I\}$. This allows us to define an ideal $\mathcal{I}$ in $K\mathcal{Q}$, and we refer to it as the ideal generated by I. If $I \subset KQ$ is generated by the set of paths $\{\rho_i|i\}$, we say that $\mathcal{I}$ is generated by the set $\{\rho_i|i\}$.

The ideal generated by all arrows is denoted by KQ^+. Note that $(KQ^+)^n$ is the ideal generated by all paths of length $\geq n$. Given vertices $a, b \in Q_0$, a finite linear combination $\sum_w c_w w$ with $c_w \in K$ where w are paths of lengths ≥ 2 from a to b is called a **relation** on Q. Any ideal $I \subseteq (KQ^+)^2$ can be generated, as an ideal, by relations. An ideal $I \subset KQ$ is called **admisible** if it is generated by a set of relations. We then say that an ideal $\mathcal{I}$ in $K\mathcal{Q}$ is admissible if it is generated by an admissible ideal in KQ.

A **representation of a quiver** Q is a pair $V = ((V_i)_{i\in Q_0}, (V_\alpha)_{\alpha\in Q_1})$, where each element of the family $\{V_i\}_{i\in Q_0}$ is a vector space and $V_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ is a K-linear map. Let V and W be two representations of Q. A morphism from V to W is a family of linear maps $f = (f_i : V_i \to W_i)_{i\in Q_0}$ such that for each arrow $\alpha : i \to j$ we have $f_j V_\alpha = W_\alpha f_i$. We denote by rep Q the abelian category that has as objects the representations of Q and as morphisms just the morphisms of representations.

Let $\rho = (a|\alpha_1,\ldots,\alpha_l|b)$ a path in Q, we set $V_\rho = V_{\alpha_l} \circ \cdots \circ V_{\alpha_1}$. Let $I \subset KQ$ be an ideal, we then say the representation V is **bounded** by I if $V_\rho = 0$ for all $\rho \in I$. The full subcategory of rep Q consisting of representations bounded by I is denoted by rep (Q, I).

Let Q be a quiver. For $x \in Q_0$, we denote by x^+ and x^- the set of arrows starting in x and the set of arrows ending in x, respectively. Recall that x is a **sink vertex** or a **source vertex** if $x^+ = \emptyset$ or $x^- = \emptyset$. One says that Q is **locally finite** if x^+ and x^- are finite sets and **interval finite** if the set of paths from x to y is finite for any $x, y \in Q_0$.

For short, we say that Q is **strongly locally finite** if it is locally finite and interval finite. In particular, Q contains no oriented cycle in case it is interval finite. Note that under these conditions, if Q is a strongly locally finite quiver, the path category $K\mathcal{Q}$ is a Hom-finite Krull-Schmidt K-category; see (7).

2.3 Functor Categories

Recall that a category $\mathcal{C}$ is said to be **skeletally small** if it has a small dense subcategory $\mathcal{C}'$, see (2). Let $\mathcal{C}$ be a Hom-finite Krull-Schmidt and skeletally small $K-$category.

The abelian category $(\mathcal{C}, \mathbf{Ab})$ is the category of all additive covariant functors from $\mathcal{C}$ to the category of abelian groups, which we will call $\mathcal{C}$-modules. Given two $\mathcal{C}$-modules F and G, the set of morphisms $\mathrm{Hom}_{(\mathcal{C},\mathbf{Ab})}(F,G)$ is denoted simply by $\mathrm{Hom}_{\mathcal{C}}(F,G)$. Following (2; 3), $(\mathcal{C}, \mathbf{Ab})$ is denoted by $\mathrm{Mod}(\mathcal{C})$.

A $\mathcal{C}$-module M is **finitely presented** if an exact sequence $P_1 \to P_0 \to M \to 0$ of $\mathcal{C}$-modules exist where P_0 and P_1 are finitely generated projective $\mathcal{C}$-modules. We recall that a $\mathcal{C}$-module P is **finitely generated projective** if P is a direct summand of a finite coproduct of representable functors.

We denote by $\mathrm{mod}(\mathcal{C})$ the full subcategory of $\mathrm{Mod}(\mathcal{C})$ consisting of finitely presented $\mathcal{C}$-modules. Let M be a $\mathcal{C}$-module, so each C in $\mathcal{C}$ the abelian group $M(C)$ has a structure as a $\mathrm{End}_{\mathcal{C}}(C)$-module and hence as a K-module since $\mathrm{End}_{\mathcal{C}}(C)$ is a K-algebra.

We denote by $(\mathcal{C}, \mathrm{mod}\ K)$ the full subcategory of $\mathrm{Mod}(\mathcal{C})$ of all $\mathcal{C}$-modules such that $M(C)$ is a finitely generated K-module. The category $(\mathcal{C}, \mathrm{mod}\ K)$ is an abelian category with the property that the inclusion $(\mathcal{C}, \mathrm{mod}\ K) \to \mathrm{Mod}(\mathcal{C})$ is exact and contains $\mathrm{mod}(\mathcal{C})$ as a full subcategory.

Let Q be a quiver and I be an ideal $I \subset KQ$. Set $\mathcal{C} = K\mathcal{Q}/\mathcal{I}$. Then each representation $V = ((V_i)_{i\in Q_0}, (V_\alpha)_{\alpha\in Q_1})$ in rep (Q, I) defines a $\mathcal{C}$-module $\tilde{V}$ in $(\mathcal{C}, \mathrm{mod}\ K)$ by setting $\tilde{V}(i) = V_i$ and $\tilde{V}(\alpha) = V_\alpha$.

In general, the functor $D : (\mathcal{C}, \mathrm{mod}\ K) \to (\mathcal{C}^{op}, \mathrm{mod}\ K)$ given by

$$D(M)(X) = \mathrm{Hom}_K(M(X), K)$$

for all X in $\mathcal{C}$ defines a duality between $(\mathcal{C}, \mathrm{mod}\ K)$ and $(\mathcal{C}^{op}, \mathrm{mod}\ K)$, and we refer to it as the **standard duality**.

According to M. Auslander, see (2), there exists a unique functor, up to isomorphism, $- \otimes_{\mathcal{C}} - : Mod(\mathcal{C}^{op}) \times Mod(\mathcal{C}) \to Ab$, called the tensor product, with the following properties:

(a) (i) For each $\mathcal{C}$-module N, the functor $- \otimes_{\mathcal{C}} N : Mod(\mathcal{C}^{op}) \to Ab$, given by $(- \otimes_{\mathcal{C}} N)(M) = M \otimes_{\mathcal{C}} N$ is right exact. For each $\mathcal{C}$-module N, the functor $- \otimes_{\mathcal{C}} N : Mod(\mathcal{C}^{op}) \to Ab$, given by $(- \otimes_{\mathcal{C}} N)(M) = M \otimes_{\mathcal{C}} N$ is right exact.

(ii) For each $\mathcal{C}^{op}$-module M, the functor $M \otimes_{\mathcal{C}} - : Mod(\mathcal{C}) \to Ab$, given by $(M \otimes_{\mathcal{C}} -)(N) = M \otimes_{\mathcal{C}} N$ is right exact.

(b) The functors $-\otimes_{\mathcal{C}} N : Mod(\mathcal{C}^{op}) \to Ab$ and $M \otimes_{\mathcal{C}} - : Mod(\mathcal{C}) \to Ab$ preserve arbitrary sums.

(c) For each object $C \in \mathcal{C}$, $(C,-)\otimes_{\mathcal{C}} N = N(C)$ and $M \otimes_{\mathcal{C}} (-,C) = M(C)$.

Now we will see an important pair of adjoint functors. Given a subcategory $\mathcal{C}'$ of $\mathcal{C}$, the restriction functor is defined as

$$\begin{aligned} res : Mod(\mathcal{C}) &\longrightarrow Mod(\mathcal{C}') \\ res(M) &:= Hom_{Mod(\mathcal{C})}(-,M)\big|_{\mathcal{C}'}, \end{aligned}$$

which has a right adjoint

$$\begin{aligned} F : Mod(\mathcal{C}') &\longrightarrow Mod(\mathcal{C}) \\ F(M)(C) &= (C,-)|_{\mathcal{C}'} \otimes_{\mathcal{C}'} M, \end{aligned}$$

for all $M \in Mod(\mathcal{C}')$ and $C \in \mathcal{C}$, see (2). This is also called tensor product and is denoted by $\mathcal{C}\otimes_{\mathcal{C}'} : Mod(\mathcal{C}') \longrightarrow Mod(\mathcal{C})$.

Proposition 2.3.1. *(2, Proposition 3.1). Let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$. The functor $\mathcal{C}\otimes_{\mathcal{C}'} : Mod(\mathcal{C}') \longrightarrow Mod(\mathcal{C})$ satisfies the following properties:*

(a) $\mathcal{C}\otimes_{\mathcal{C}'}$ is right exact and preserves arbitrary sums.

(b) The composition $Mod(\mathcal{C}') \xrightarrow{\mathcal{C}\otimes_{\mathcal{C}'}} Mod(\mathcal{C}) \xrightarrow{res} Mod(\mathcal{C}')$ is the identity on $Mod(\mathcal{C}')$.

(c) For each object $C' \in \mathcal{C}$, $\mathcal{C}\otimes_{\mathcal{C}'} \mathcal{C}'(-,C') = \mathcal{C}(-,C')$.

(d) $\mathcal{C}\otimes_{\mathcal{C}'}$ is fully faithful.

(e) $\mathcal{C}\otimes_{\mathcal{C}'}$ preserves projective objects.

The functor $M \in Mod(\mathcal{C})$ is said to be projectively presented over $\mathcal{C}'$ if there exists an exact sequence

$$\coprod_{i\in I} \mathcal{C}(-,C'_i) \longrightarrow \coprod_{j\in J} \mathcal{C}(-,C'_j) \longrightarrow M \longrightarrow 0\ ,$$

with $C'_i, C'_j \in \mathcal{C}'$. The category $\mathcal{C}\otimes_{\mathcal{C}'} Mod(\mathcal{C}')$, is the subcategory of $Mod(\mathcal{C})$ whose objects are the functors projectively presented over $\mathcal{C}'$. Moreover, the functors res and $\mathcal{C}\otimes_{\mathcal{C}'}$ induce an equivalence between $Mod(\mathcal{C}')$ and $\mathcal{C}\otimes_{\mathcal{C}'} Mod(\mathcal{C}')$.

Proposition 2.3.2. *(2, Proposition 3.2). Let $\mathcal{C}'$be a full subcategory of the skeletally small category $\mathcal{C}$. The following conditions are equivalent for a $\mathcal{C}-$module M.*

(a) The canonical morphism $\mathcal{C}\otimes_{\mathcal{C}'}(M|_{\mathcal{C}'})\longrightarrow M$ *is an isomorphism.*

(b) There exists an exact sequence

$$\textstyle\coprod_{i\in I}\mathcal{C}(-,C'_i)\longrightarrow\coprod_{j\in J}\mathcal{C}(-,C'_j)\longrightarrow M\longrightarrow 0\,,$$

of $\mathcal{C}-$*modules, with* C'_i *and* C'_j *in the subcategory* $\mathcal{C}'$*.*

2.4 Triangular Matrix Categories

In this section we recall some results from the article (15). The notion of triangular matrix categories is introduced in (15; 16) in order to define the analogous of the triangular matrix algebras to the context of *rings with several objects.*

Definition 2.4.1. (15, Definition 3.5) Given $\mathcal{T}$ and $\mathcal{U}$ additive K-categories and a functor $M:\mathcal{U}\otimes_K\mathcal{T}^{op}\to\mathrm{Mod}(K)$, the triangular matrix category $\Lambda=\begin{pmatrix}\mathcal{T} & 0\\ M & \mathcal{U}\end{pmatrix}$ is the additive K-category whose collection of objects are the *matrices* $\begin{pmatrix}T & 0\\ M & U\end{pmatrix}$, where U and T are objects in $\mathcal{U}$ and $\mathcal{T}$, respectively.

Let $X=\begin{pmatrix}T & 0\\ M & U\end{pmatrix}$, $X'=\begin{pmatrix}T' & 0\\ M & U'\end{pmatrix}$ and $X''=\begin{pmatrix}T'' & 0\\ M & U''\end{pmatrix}$ be objects in Λ. The set of morphisms from X to X' is $\Lambda(X,X')=\begin{pmatrix}\mathcal{T}(T,T') & 0\\ M(U',T) & \mathcal{U}(U,U')\end{pmatrix}$, where

$$\begin{pmatrix}\mathcal{T}(T,T') & 0\\ M(U',T) & \mathcal{U}(U,U')\end{pmatrix}:=\left\{\begin{pmatrix}f & 0\\ m & g\end{pmatrix}:f\in\mathcal{T}(T,T'),g\in\mathcal{U}(U,U'),m\in M(U',T)\right\},$$

and the composition $\Lambda(X',X'')\times\Lambda(X,X')\to\Lambda(X,X'')$ is defined by

$$\left(\begin{pmatrix}f_2 & 0\\ m_2 & g_2\end{pmatrix},\begin{pmatrix}f_1 & 0\\ m_1 & g_1\end{pmatrix}\right)\mapsto\begin{pmatrix}f_2\circ f_1 & 0\\ m_2 let f_1+g_2\bullet m_1 & g_2\circ g_1\end{pmatrix}$$

with $m_2\bullet f_1:=M(1_{U''}\otimes f_1^{op})(m_2)$ and $g_2\bullet m_1:=M(g_2\otimes 1_T)(m_1)$.

We note that

$$M(1_{U''}\otimes f_1^{op}):M(U'',T')\to M(U'',T),$$

$$M(g_2 \otimes 1_T) : M(U', T) \to M(U'', T)$$

are morphisms in $\mathrm{Mod}(K)$.

The triangular matrix category Λ is a preadditive category and, furthermore, it satisfies the following property.

Proposition 2.4.2. *(15, Proposition 3.7) If $\mathfrak{U}$ and $\mathfrak{T}$ have finite coproducts, then the triangular matrix category $\Lambda = \begin{pmatrix} \mathfrak{T} & 0 \\ M & \mathfrak{U} \end{pmatrix}$ has finite coproducts.*

Moreover, in (15), the authors compute the radical of the category Λ.

Proposition 2.4.3. *(15, Proposition 3.8) Let $\mathfrak{U}$ and $\mathfrak{T}$ be preadditive categories and $M \in Mod(\mathfrak{U} \otimes \mathfrak{T}^{op})$. Consider the category of triangular matrices $\Lambda = \begin{pmatrix} \mathfrak{T} & 0 \\ M & \mathfrak{U} \end{pmatrix}$. It follows that*

$$rad_\Lambda\left(\begin{pmatrix} T & 0 \\ M & U \end{pmatrix}, \begin{pmatrix} T' & 0 \\ M & U' \end{pmatrix}\right) = \begin{pmatrix} rad_{\mathfrak{T}}(T, T') & 0 \\ M(U', T) & rad_{\mathfrak{U}}(U, U') \end{pmatrix}.$$

Proof:
Let

$$\begin{pmatrix} t & 0 \\ m & u \end{pmatrix} \in rad_\Lambda\left(\begin{pmatrix} T & 0 \\ M & U \end{pmatrix}, \begin{pmatrix} T' & 0 \\ M & U' \end{pmatrix}\right),$$

and let $t' : T' \longrightarrow T$ and $u' : U' \longrightarrow U$ be morphisms in $\mathfrak{T}$ and $\mathfrak{U}$ espectively. Consider

$$\begin{pmatrix} t' & 0 \\ 0 & u' \end{pmatrix} \in Hom_\Lambda\left(\begin{pmatrix} T' & 0 \\ M & U' \end{pmatrix}, \begin{pmatrix} T & 0 \\ M & U \end{pmatrix}\right).$$

Then $\begin{pmatrix} 1_T & 0 \\ 0 & 1_U \end{pmatrix} - \begin{pmatrix} t' & 0 \\ 0 & u' \end{pmatrix}\begin{pmatrix} t & 0 \\ m & u \end{pmatrix} = \begin{pmatrix} 1_T - t't & 0 \\ -u' \bullet m & 1_U - u'u \end{pmatrix}$ is invertible in Λ and therefore $1_T - t't$ and $1_U - u'u$ are invertible in $\mathfrak{T}$ and $\mathfrak{U}$ respectively. From this, we conclude that $t \in rad_{\mathfrak{T}}(T, T')$ and $u \in rad_{\mathfrak{U}}(U, U')$.

For the other inclusion, let $t \in rad_{\mathfrak{T}}(T, T')$, $u \in rad_{\mathfrak{U}}(U, U')$ and $m \in M(U', T)$. We assert that

$$\begin{pmatrix} t & 0 \\ m & u \end{pmatrix} \in rad_\Lambda\left(\begin{pmatrix} T & 0 \\ M & U \end{pmatrix}, \begin{pmatrix} T' & 0 \\ M & U' \end{pmatrix}\right).$$

Indeed, let

$$\begin{pmatrix} t' & 0 \\ m' & u' \end{pmatrix} \in Hom_\Lambda\left(\begin{pmatrix} T' & 0 \\ M & U' \end{pmatrix}, \begin{pmatrix} T & 0 \\ M & U \end{pmatrix}\right).$$

Since $1_T - t't$ and $1_U - u'u$ are invertible, there exist $t'' \in Hom_{\mathcal{T}}(T,T)$ and $U'' \in Hom_{\mathcal{U}}(U,U)$ such that

$$(1_T - t't)t'' = t''(1_T - t't) = 1_T, \qquad \text{and} \qquad (1_U - u'u)u'' = u''(1_U - u'u) = 1_U.$$

Let $m'' := u'' \bullet (m' \bullet t + u' \bullet m) \bullet t'' \in M(U,T)$. Since $(1_U - u'u)u'' = 1_U$ and $t''(1_T - t't) = 1_T$ we have that

$$\left(\begin{pmatrix}1_T & 0\\ 0 & 1_U\end{pmatrix} - \begin{pmatrix}t' & 0\\ m' & u'\end{pmatrix}\begin{pmatrix}t & 0\\ m & u\end{pmatrix}\right)\left(\begin{pmatrix}t'' & 0\\ m'' & u''\end{pmatrix}\right) =$$

$$\begin{aligned}
&= \left(\begin{pmatrix}1_T & 0\\ 0 & 1_U\end{pmatrix} - \begin{pmatrix}t't & 0\\ m' \bullet t + u' \bullet m & u'u\end{pmatrix}\right)\left(\begin{pmatrix}t'' & 0\\ m'' & u''\end{pmatrix}\right)\\
&= \begin{pmatrix}1_T - t't & 0\\ -(m' \bullet t + u' \bullet m) & 1_U - u'u\end{pmatrix}\left(\begin{pmatrix}t'' & 0\\ m'' & u''\end{pmatrix}\right)\\
&= \begin{pmatrix}(1_T - t't)t'' & 0\\ -(m' \bullet t + u' \bullet m) \bullet t'' + (1_U - u'u) \bullet m'' & (1_U - u'u)u''\end{pmatrix}\\
&= \begin{pmatrix}1_T & 0\\ -(m' \bullet t + u' \bullet m) \bullet t'' + (1_U - u'u) \bullet m'' & 1_U\end{pmatrix}\\
&= \begin{pmatrix}1_T & 0\\ 0 & 1_U\end{pmatrix},
\end{aligned}$$

where the final equality is due to

$$\begin{aligned}
(m' \bullet t + u' \bullet m) \bullet t'' &= 1_U \bullet ((m' \bullet t + u' \bullet m) \bullet t'')\\
&= \big((1_U - u'u)u''\big) \bullet ((m' \bullet t + u' \bullet m) \bullet t'')\\
&= (1_U - u'u) \bullet \big(u'' \bullet (m' \bullet t + u' \bullet m) \bullet t''\big)\\
&= (1_U - u'u) \bullet m''.
\end{aligned}$$

On the other hand,

$$\left(\begin{pmatrix}t'' & 0\\ m'' & u''\end{pmatrix}\right)\left(\begin{pmatrix}1_T & 0\\ 0 & 1_U\end{pmatrix} - \begin{pmatrix}t' & 0\\ m' & u'\end{pmatrix}\begin{pmatrix}t & 0\\ m & u\end{pmatrix}\right) =$$

$$\begin{aligned}
&= \left(\begin{pmatrix}t'' & 0\\ m'' & u''\end{pmatrix}\right)\left(\begin{pmatrix}1_T & 0\\ 0 & 1_U\end{pmatrix} - \begin{pmatrix}t't & 0\\ m' \bullet t + u' \bullet m & u'u\end{pmatrix}\right)\\
&= \left(\begin{pmatrix}t'' & 0\\ m'' & u''\end{pmatrix}\right)\begin{pmatrix}1_T - t't & 0\\ -(m' \bullet t + u' \bullet m) & 1_U - u'u\end{pmatrix}\\
&= \begin{pmatrix}t''(1_T - t't) & 0\\ m'' \bullet (1_T - t't) - u'' \bullet (m' \bullet t + u' \bullet m) & u''(1_U - u'u)\end{pmatrix}\\
&= \begin{pmatrix}1_T & 0\\ m'' \bullet (1_T - t't) - u'' \bullet (m' \bullet t + u' \bullet m) & 1_U\end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} 1_T & 0 \\ 0 & 1_U \end{pmatrix},$$

where the last equality is a consequence of

$$\begin{aligned} u'' \bullet (m' \bullet t + u' \bullet m) &= \big(u'' \bullet (m' \bullet t + u' \bullet m)\big) \bullet 1_T \\ &= \big(u'' \bullet (m' \bullet t + u' \bullet m)\big) \bullet \big(t''(1_T - t't)\big) \\ &= \big(u'' \bullet (m' \bullet t + u' \bullet m) \bullet t''\big) \bullet (1_T - t't) \\ &= m'' \bullet (1_T - t't). \end{aligned}$$

Thus, $\begin{pmatrix} t & 0 \\ m & u \end{pmatrix} \in rad_\Lambda \left(\begin{pmatrix} T & 0 \\ M & U \end{pmatrix}, \begin{pmatrix} T' & 0 \\ M & U' \end{pmatrix} \right)$ as we wanted. ■

Let $M \in \mathrm{Mod}(\mathcal{U} \otimes_K \mathcal{T}^{op})$. Then, there exists a covariant functor

$$E : \mathcal{T} \longrightarrow \mathrm{Mod}(\mathcal{U})^{op},$$

defined as follows:

(a). For $T \in \mathcal{T}$, we have a covariant functor $E(T) := M_T : \mathcal{U} \longrightarrow \mathbf{Ab}$ given as:

(i) $M_T(U) := M(U, T)$, for all $U \in \mathcal{U}$.

(ii) $M_T(u) := M(u \otimes 1_T)$, for all $u \in \mathrm{Hom}_{\mathcal{U}}(U, U')$.

(b). Now, given a morphism $t : T \longrightarrow T'$ in $\mathcal{T}$ we define the natural transformation $E(t) := \bar{t} : M_{T'} \longrightarrow M_T$ where $\bar{t} = \{[\bar{t}]_U : M_{T'}(U) \longrightarrow M_T(U)\}_{U \in \mathcal{U}}$ with $[\bar{t}]_U = M(1_U \otimes t^{op}) : M(U, T') \longrightarrow M(U, T)$.

Now, let

$$Y : \mathrm{Mod}(\mathcal{U}) \longrightarrow \mathrm{Mod}\Big(\mathrm{Mod}(\mathcal{U})^{op}\Big)$$

be the Yoneda functor given as follows $Y(B) := \mathrm{Hom}_{\mathrm{Mod}(\mathcal{U})}(-, B)$; and consider the following functor

$$I : \mathrm{Mod}\Big(\mathrm{Mod}(\mathcal{U})^{op}\Big) \longrightarrow \mathrm{Mod}(\mathcal{T}),$$

which is induced by composing with $E : \mathcal{T} \longrightarrow \mathrm{Mod}(\mathcal{U})^{op}$. We set

$$\mathbb{G} := I \circ Y : \mathrm{Mod}(\mathcal{U}) \longrightarrow \mathrm{Mod}(\mathcal{T}).$$

Hence we have the comma category $\Big(\mathrm{Mod}(\mathcal{T}), \mathbb{G}(\mathrm{Mod}(\mathcal{U}))\Big)$;

i) whose objects are the triples (A, f, B) with $A \in \mathrm{Mod}(\mathcal{T}), B \in \mathrm{Mod}(\mathcal{U})$, and $f : A \longrightarrow \mathbb{G}(B)$ is a morphism of $\mathcal{T}$-modules.

ii) A morphism between two objects (A, f, B) and (A', f', B') is a pair of morphism (α, β) where$\alpha : A \longrightarrow A'$ is a morphism of $\mathfrak{T}$-modules and $\beta : B \longrightarrow B'$ is a morphism of $\mathfrak{U}$-modules such that the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ \mathbb{G}(B) \arrow[r, "\mathbb{G}(\beta)"'] & \mathbb{G}(B') \end{tikzcd}$$

commutes.

Now, given $(A, f, B) \in \Big(\mathrm{Mod}(\mathfrak{T}), \mathbb{G}\mathrm{Mod}(\mathfrak{U})\Big)$, we can construct a functor

$$A \amalg_f B : \mathbf{\Lambda} \to \mathbf{Ab}$$

defined as follows:

(a) For $\begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \in \mathbf{\Lambda}$ we set $\Big(A \amalg_f B\Big)\left(\begin{pmatrix} T & 0 \\ M & U \end{pmatrix}\right) := A(T) \amalg B(U) \in \mathbf{Ab}$.

(b) If $\begin{pmatrix} t & 0 \\ m & u \end{pmatrix} \in \mathrm{Hom}_{\mathbf{\Lambda}}\left(\begin{pmatrix} T & 0 \\ M & U \end{pmatrix}, \begin{pmatrix} T' & 0 \\ M & U' \end{pmatrix}\right)$ we define the map

$$\Big(A \amalg_f B\Big)\left(\begin{pmatrix} t & 0 \\ m & u \end{pmatrix}\right) := \begin{pmatrix} A(t) & 0 \\ m & B(u) \end{pmatrix} : A(T) \amalg B(U) \to A(T') \amalg B(U')$$

given by $\begin{pmatrix} A(t) & 0 \\ m & B(u) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A(t)(x) \\ m \cdot x + B(u)(y) \end{pmatrix}$ for $(x, y) \in A(T) \amalg B(U)$, where $m \cdot x := [f_T(x)]_{U'}(m) \in B(U')$.

We define $\mathfrak{f} : (\mathrm{Mod}(\mathfrak{T}), \mathbb{G}(\mathrm{Mod}(\mathfrak{U}))) \to \mathrm{Mod}(\Lambda)$ as $\mathfrak{f}\Big((A, f, B)\Big) = A \amalg_f B$ and it is a covariant functor (see (15, Proposition 3.12)). We have the following Theorem.

Theorem 2.4.1. *(15, Theorem 3.17) There exists an equivalence of categories*

$$\mathfrak{f} : (\mathrm{Mod}(\mathfrak{T}), \mathbb{G}(\mathrm{Mod}(\mathfrak{U}))) \to \mathrm{Mod}(\Lambda).$$

Hence, given a Λ-module C, there exists a pair of functors $C^{(1)} : \mathfrak{T} \to \mathbf{Ab}$, $C^{(2)} : \mathfrak{U} \to \mathbf{Ab}$ and $C^{(1)} \xrightarrow{f} \mathbb{G}(C^{(2)})$ such that $\mathfrak{f}((C^{(1)}, f, C^{(2)})) \cong C$, where we have that $C^{(1)}(T') := C\left(\begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix}\right)$ and $C^{(2)}(U') := C\left(\begin{pmatrix} 0 & 0 \\ M & U' \end{pmatrix}\right)$.

Now, we have the following Proposition.

Proposition 2.4.4. *Let* $C \in \mathrm{Mod}(\Lambda)$ *and consider* $C^{(1)} : \mathcal{T} \to \mathbf{Ab}$ *and* $C^{(2)} : \mathcal{U} \to \mathbf{Ab}$ *and* $C^{(1)} \xrightarrow{f} \mathbb{G}(C^{(2)}\ )$ *such that* $\mathfrak{f}((C^{(1)}, f, C^{(2)})) \cong C$*. Suppose that there exists exact sequences*

$$\mathcal{T}(T_1, -) \longrightarrow \mathcal{T}(T_0, -) \longrightarrow C^{(1)} \longrightarrow 0\ , \quad T_0, T_1 \in \mathcal{T} \quad \textit{and}$$

$$\mathcal{U}(U_1, -) \longrightarrow \mathcal{U}(U_0, -) \longrightarrow C^{(2)} \longrightarrow 0\ , \quad U_0, U_1 \in \mathcal{U}.$$

Then there exists an exact sequence in $\mathrm{Mod}(\Lambda)$

$$\Lambda\left(\begin{pmatrix} T_1 & 0 \\ M & U_1 \end{pmatrix}, -\right) \longrightarrow \Lambda\left(\begin{pmatrix} T_0 & 0 \\ M & U_0 \end{pmatrix}, -\right) \longrightarrow \mathfrak{f}\left((C^{(1)}, f, C^{(2)})\right) \longrightarrow 0,$$

where $P_j := \Lambda\left(\begin{pmatrix} T_j & 0 \\ M & U_j \end{pmatrix}, -\right)$ *is a projective* $\Lambda-$*module for* $j = 0, 1$.

Proof:

It is similar to the proof of (15, Proposition 6.3). ■

Proposition 2.4.5. *(15, Proposition 5.5) The subcategory* $\mathrm{proj}(\mathrm{Mod}(\Lambda))$ *of* $\mathrm{Mod}(\Lambda)$ *consisting of finitely generated projective* Λ*-modules is equivalent to the subcategory of* $\Big(\mathrm{Mod}(\mathcal{T}), \mathbb{G}\mathrm{Mod}(\mathcal{U})\Big)$ *consisting of morphism of* $\mathcal{T}$*-modules*

$$g : \mathcal{T}(T, -) \longrightarrow \mathbb{G}\Big(M_T \amalg \mathcal{U}(U, -)\Big)$$

given by $g := \Big\{[g]_{T'} : \mathcal{T}(T, T') \longrightarrow \mathrm{Hom}_{\mathrm{Mod}(\mathcal{U})}\big(M_{T'}, M_T \amalg \mathcal{U}(U, -)\big)\Big\}_{T' \in \mathcal{T}}$*, with* $[g]_{T'}(t) := \begin{bmatrix} \overline{t} \\ 0 \end{bmatrix} : M_{T'} \to M_T \amalg \mathcal{U}(U, -)$ *for all* $t \in \mathcal{T}(T, T')$*, where for* $U' \in \mathcal{U}$*, the map* $\left[\begin{bmatrix} \overline{t} \\ 0 \end{bmatrix}\right]_{U'} : M(U', T') \to M(U', T) \amalg \mathcal{U}(U, U')$ *is defined by* $\left[\begin{bmatrix} \overline{t} \\ 0 \end{bmatrix}\right]_{U'}(m) := (M(1_U \otimes t^{op})(m), 0) = (m \bullet t, 0), \forall m \in M(U', T)$.

Proposition 2.4.6. *(15, Proposition 5.6) A sequence of maps*

$$0 \longrightarrow (A, f, B) \xrightarrow{(\alpha, \beta)} (A', f', B') \xrightarrow{(\alpha', \beta')} (A'', f'', B'') \longrightarrow 0$$

is exact in $\Big(\mathrm{Mod}(\mathcal{T}), \mathbb{G}(\mathrm{Mod}(\mathcal{U}))\Big)$ *if and only if the following are exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] & A' \arrow[r, "\alpha"] & A \arrow[r] & 0 \end{tikzcd},$$

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "\beta"] & B' \arrow[r, "\beta"] & B'' \arrow[r] & 0 \end{tikzcd}$$

in $\mathrm{Mod}(\mathcal{T})$ *and* $\mathrm{Mod}(\mathcal{U})$ *respectively.*

The following results are also available.

Lemma 2.4.7. *(15, Lemma 6.6) Let* $\mathcal{U}$ *and* $\mathcal{T}$ *be* $K-$*varieties that are Hom−finite, and suppose that* $M \in Mod(\mathcal{U} \otimes_K \mathcal{T}^{op})$ *satisfies* $M_T \in mod(\mathcal{U})$*, for all* $t \in \mathcal{T}$*. Then*

$$\Gamma = End_\Lambda\left(\begin{pmatrix} T & 0 \\ M & U \end{pmatrix}\right) := \begin{pmatrix} Hom_{\mathcal{T}}(T,T) & 0 \\ M(U,T) & Hom_{\mathcal{U}}(U,U) \end{pmatrix}$$

is an Artin $K-$*algebra.*

Proposition 2.4.8. *(15, Proposition 6.8) Assume that* $\mathcal{T}$ *and* $\mathcal{U}$ *are additive categories with splitting idempotents. Then* Λ *is also an additive category with splitting idempotents.*

Proof:

Let

$$\begin{pmatrix} t & 0 \\ m & u \end{pmatrix} : \begin{pmatrix} T & 0 \\ M & U \end{pmatrix} \longrightarrow \begin{pmatrix} T & 0 \\ M & U \end{pmatrix}$$

be an idempotent morphism with $t \in Hom_{\mathcal{T}}(T,T)$, $u \in Hom_{\mathcal{U}}(U,U)$ and $m \in M(U,T)$. Then,

$$\begin{pmatrix} t & 0 \\ m & u \end{pmatrix} = \begin{pmatrix} t & 0 \\ m & u \end{pmatrix}\begin{pmatrix} t & 0 \\ m & u \end{pmatrix} = \begin{pmatrix} t^2 & 0 \\ m \bullet t + u \bullet m & u^2 \end{pmatrix}.$$

Thus, $m = m \bullet t + u \bullet m = M(1_U \otimes t^{op})(m) + M(u \otimes 1_T)(m)$, $t^2 = t$ and $u^2 = u$.

Let $\mu : L \longrightarrow T$ be the kernel of the endomorphism $t : T \longrightarrow T$ and $\nu : K \longrightarrow U$ be the kernel of the endomorphism $u : U \longrightarrow U$, which exist since $\mathcal{U}$ and $\mathcal{T}$ have split idempotents.

Then $0 = t\mu$ and therefore $0 = M(1_U \otimes (t\mu)^{op}) = M(1_U \otimes \mu^{op}) \circ M(1_U \otimes t^{op})$. From which we have

$$0 = \left(M(1_U \otimes \mu^{op}) \circ M(1_U \otimes t^{op})\right)(m) = (m \bullet t) \bullet \mu.$$

Therefore,

$$m \bullet \mu = (m \bullet t + u \bullet m) \bullet \mu = (m \bullet t) \bullet \mu + (u \bullet m) \bullet \mu = (u \bullet m) \bullet \mu \in M(U, L).$$

We assert that $\begin{pmatrix} \mu & 0 \\ -m \bullet \mu & \nu \end{pmatrix} : \begin{pmatrix} L & 0 \\ M & K \end{pmatrix} \longrightarrow \begin{pmatrix} T & 0 \\ M & U \end{pmatrix}$ is the kernel of

$$\begin{pmatrix} t & 0 \\ m & u \end{pmatrix} : \begin{pmatrix} T & 0 \\ M & U \end{pmatrix} \longrightarrow \begin{pmatrix} T & 0 \\ M & U \end{pmatrix}.$$

Indeed, first let's see that

$$\begin{pmatrix} t & 0 \\ m & u \end{pmatrix} \begin{pmatrix} \mu & 0 \\ -m \bullet \mu & \nu \end{pmatrix} = \begin{pmatrix} t\mu & 0 \\ m \bullet \mu + u \bullet (-m \bullet \mu) & u\nu \end{pmatrix} = \begin{pmatrix} t\mu & 0 \\ m \bullet \mu - u \bullet (m \bullet \mu) & u\nu \end{pmatrix}$$

$$= \begin{pmatrix} t\mu & 0 \\ m \bullet \mu - (u \bullet m) \bullet \mu & u\nu \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let's consider now $\begin{pmatrix} \alpha & 0 \\ n & \beta \end{pmatrix} : \begin{pmatrix} T' & 0 \\ M & U' \end{pmatrix} \longrightarrow \begin{pmatrix} T & 0 \\ M & U \end{pmatrix}$ with $n \in M(U, T')$, $\alpha : T' \longrightarrow T$ and $\beta : U' \longrightarrow U$, such that

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} t & 0 \\ m & u \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ n & \beta \end{pmatrix} = \begin{pmatrix} t\alpha & 0 \\ m \bullet \alpha + u \bullet n & u\beta \end{pmatrix}.$$

Then, $m \bullet \alpha = -u \bullet n \in M(U, T')$.

Let's take $\nu : K \longrightarrow U$, and we have $\nu \bullet (m \bullet \alpha) = \nu \bullet (-u \bullet n) \in M(U, T')$.

We want a morphism $\begin{pmatrix} \alpha' & 0 \\ m' & \beta' \end{pmatrix} : \begin{pmatrix} T' & 0 \\ M & U' \end{pmatrix} \longrightarrow \begin{pmatrix} L & 0 \\ M & K \end{pmatrix}$ with $m' \in M(K, T')$, such that

$$\begin{pmatrix} \alpha & 0 \\ n & \beta \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ -m \bullet \mu & \nu \end{pmatrix} \begin{pmatrix} \alpha' & 0 \\ m' & \beta' \end{pmatrix} = \begin{pmatrix} \mu\alpha' & 0 \\ (-(m \bullet \mu)) \bullet \alpha' + \nu \bullet m' & \nu\beta' \end{pmatrix}.$$

For this, we consider $\nu' : K' \longrightarrow U$ the kernel of $1_U - u$. Since idempotents split, we have $U \simeq K' \otimes K$ with $\nu : K \longrightarrow U$ and $\nu' : K' \longrightarrow U$ being the natural inclusions. In particular, there exists $p : U \longrightarrow K$ and $p' : U \longrightarrow K'$ such that

$$1_U = \nu p + \nu' p', \;\; p\nu = 1_K, \;\; p'\nu' = 1_{K'}.$$

It can be seen that $u = \nu' p'$. From this we have that $n = \nu(p \bullet n) + \nu'(p' \bullet n)$. We let $m' := p \bullet n \in M(K, T')$. Then we have

$$(-(m \bullet \mu)) \bullet \alpha' = -(m \bullet (\mu \circ \alpha')) = -(m \bullet \alpha) = u \bullet n.$$

On the other hand,

$$\nu \bullet m' = \nu \bullet (p \bullet n) = n - \nu' \bullet (p' \bullet n) = n - (\nu' \bullet p') \bullet n = n - u \bullet n.$$

Therefore,

$$((-(m \bullet \mu)) \bullet \alpha') + \nu \bullet m' = n,$$

and $\alpha = \mu\alpha'$ and $\beta = \nu\beta'$.Thus, we prove that $\begin{pmatrix} \alpha' & 0 \\ m' & \beta' \end{pmatrix} : \begin{pmatrix} T' & 0 \\ M & U' \end{pmatrix} \longrightarrow \begin{pmatrix} L & 0 \\ M & K \end{pmatrix}$ is the desired morphism.

Now, for uniqueness, suppose that $\begin{pmatrix} \alpha'' & 0 \\ m'' & \beta'' \end{pmatrix}$ is such that

$$\begin{pmatrix} \mu\alpha' & 0 \\ (-(m \bullet \mu)) \bullet \alpha' + \nu \bullet m' & \nu\beta' \end{pmatrix} = \begin{pmatrix} \mu\alpha'' & 0 \\ (-(m \bullet \mu)) \bullet \alpha'' + \nu \bullet m'' & \nu\beta'' \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ n & \beta \end{pmatrix}.$$

We then have $\alpha' = \alpha''$ and $\beta' = \beta''$ and thus $\nu \bullet m' = \nu \bullet m''$. Composing with $p : U \longrightarrow K$ we have

$$m' = 1_K \bullet m' = (p \circ \nu) \bullet m' = p \bullet (\nu \bullet m') = p \bullet (\nu \bullet m'') = (p \circ \nu) \bullet m'' = 1_K \bullet m'' = m''.$$

This proves uniqueness, thereby concluding the proof, i.e., Λ is a category with splitting idempotents. ■

Proposition 2.4.9. *(15, Proposition 6.9) Let $\mathcal{U}$ and $\mathcal{T}$ be Hom−finite K−varieties wich are Krull−Schmidt and $M \in Mod(\mathcal{U} \otimes_K \mathcal{T}^{op})$ satisfies that $M_T \in mod(\mathcal{U})$ for all $T \in \mathcal{T}$. Then,*

(a) $\Lambda = \begin{pmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{pmatrix}$ is a Hom−finite K−variety and Krull−Schmidt, and

(b) $mod(\Lambda)$ is also a Hom−finite K−variety and Krull−Schmidt.

Proof:

(a) By Lemma 2.4.7, we have that

$$\Gamma = End_\Lambda\left(\begin{pmatrix} T & 0 \\ M & U \end{pmatrix}\right) := \begin{pmatrix} Hom_{\mathcal{T}}(T,T) & 0 \\ M(U,T) & Hom_{\mathcal{U}}(U,U) \end{pmatrix},$$

is an Artin K−algebra for every $\begin{pmatrix} T & 0 \\ M & U \end{pmatrix} \in \Lambda$ and therefore semiperfect.

Since $\mathcal{U}$ and $\mathcal{T}$ are Krull−Schmidt, we have by (13, Corollary 4.4), that $\mathcal{U}$ and $\mathcal{T}$ have splitting idempotents. Furthermore, by Propositions 2.4.2 and 2.4.8, we have that Λ is an additive category with splitting idempotents. Therefore, also by Corollary (13, Corolario 4.4), $\Lambda = \begin{pmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{pmatrix}$ is Krull−Schmidt.

(b) This follows from part (*a*) and from (21, Proposition 2.4).

■

2.5 Quasi-hereditary Categories

In this section, we provide some results given by M. Ortiz in (24) that will be essential for the development of the main results that we will obtain in this thesis.

From now on, we will consider K to be an algebraically closed field. We will also assume that every subcategory $\mathcal{B}$ of $\mathcal{C}$ is closed under finite direct sums and isomorphisms.

For each $E \in \mathcal{B}$, a function $k_E : Ob\mathcal{C} \to \mathbb{N}$ is defined by $k_E(X) := \dim_K \mathcal{C}(E, X)$. Let $\{g_1, \cdots, g_{k_E(X)}\}$ be a K-basis of $\mathcal{C}(E, X)$. Ortiz defines the following morphism of $\mathcal{C}$-modules for every $C \in \mathcal{C}$,

$$\varphi_C^{(E,X)} : \mathcal{C}(C, E)^{k_E(X)} \to \mathcal{C}(C, X),$$

$$(g_1, \cdots, g_{k_E(X)}) \mapsto \sum_{i=1}^{k_E(X)} f_i g_i.$$

Then, the induced morphism is given by

$$\left(\varphi^{(E,X)}\right)_{E\in\mathcal{B}} : \coprod_{E\in\mathcal{B}} \mathcal{C}(-, E)^{k_E(X)} \to \mathcal{C}(-, X).$$

Recall that

$$Tr_{\{\mathcal{C}(-,E)\}_{E\in\mathcal{B}}}\mathcal{C}(-, X) := \sum_{\{\mathcal{C}(-,E)\}_{E\in\mathcal{B}}} \{Im\psi \mid \psi \in Hom_{Mod(\mathcal{C})}(\mathcal{C}(-, E), \mathcal{C}(-, X))\}$$

is the largest subfunctor of $\mathcal{C}(-, X)$ generated by $\{\mathcal{C}(-, E)\}_{E\in\mathcal{B}}$.

Let $\mathcal{B}$ be a full additive subcategory of $\mathcal{C}$. Given $C, C' \in \mathcal{C}$, we denote by $I_{\mathcal{B}}(C, C')$ the subset of $\mathcal{C}(C, C')$ consisting of morphisms which factor through some object in $\mathcal{B}$. Then, we can define a two-sided ideal $I_{\mathcal{B}}(-, ?)$. The following lemma relates the ideal $I_{\mathcal{B}}(-, ?)$ with the trace $Tr_{\{\mathcal{C}(-,E)\}_{E\in\mathcal{B}}}\mathcal{C}(-, X)$ of the family of projective $\mathcal{C}$-modules $\{\mathcal{C}(-, E)\}_{E\in\mathcal{B}}$ in $\mathcal{C}(-, X)$.

Lemma 2.5.1. *(24, Lemma 3.1) Let* $\left(\varphi^{(E,X)}\right)_{E\in\mathcal{B}} : \coprod_{E\in\mathcal{B}} \mathcal{C}(-, E)^{k_E(X)} \to \mathcal{C}(-, X)$ *be the morphism defined earlier. It follows that,*

$$Im\left(\varphi^{(E,X)}\right)_{E\in\mathcal{B}} = Tr_{\{\mathcal{C}(-,E)\}_{E\in\mathcal{B}}}\mathcal{C}(-, X) = I_{\mathcal{B}}(-, X).$$

Proof:
For every $E \in \mathcal{B}$ we assert that $Im\left(\left(\varphi^{(E,X)}\right)_{E\in\mathcal{B}}\right) = \sum\limits_{\substack{\widehat{f}\in\mathcal{C}(E,X)\\ E\in\mathcal{B}}} Im(\mathcal{C}(-,\widehat{f}))$.

Let $\{f_1,\cdots,f_{k_E(X)}\}$ be a K-basis of $\mathcal{C}(E,X)$. Then,

$$Im\left(\left(\varphi_C^{(E,X)}\right)_{E\in\mathcal{B}}\right) = \left\{\sum_{i=1}^{k_E(X)} f_i g_i \mid g_i \in \mathcal{C}(C,E),\ E\in\mathcal{B}\right\}.$$

On the other hand, we have that

$$\sum_{\substack{\widehat{f}\in\mathcal{C}(E,X)\\ E\in\mathcal{B}}} Im\left(\mathcal{C}(-,\widehat{f})_C\right) = \sum_{\substack{\widehat{f}\in\mathcal{C}(E,X)\\ E\in\mathcal{B}}} Im\left(\mathcal{C}(C,\widehat{f})\right)$$

$$= \left\{\sum_{s=1}^{m} \widehat{f}_s g_s \mid \widehat{f}_s \in \mathcal{C}(E,X), g_s\in\mathcal{C}(C,E),\ E\in\mathcal{B},\ m\geq 1\right\}.$$

It follows that $Im\left(\left(\varphi^{(E,X)}\right)_{E\in\mathcal{B}}\right) \subseteq \sum\limits_{\substack{\widehat{f}\in\mathcal{C}(E,X)\\ E\in\mathcal{B}}} Im(\mathcal{C}(-,\widehat{f}))$. For the other inclusion, let us consider the morphism $\widehat{f}_s = \sum\limits_{i=1}^{k_E(X)} c_{is}f_i$, $c_{is}\in K$ en $\mathcal{C}(E,S)$. Then it follows that

$$\sum_{s=1}^{m}\widehat{f}_s g_s = \sum_{s=1}^{m}\left(\sum_{i=1}^{k_E(X)} c_{is}f_i\right) g_s = \sum_{i=1}^{k_E(X)} f_i \left(\sum_{s=1}^{m} c_{is}\right).$$

From which it follows that $\sum\limits_{\substack{\widehat{f}\in\mathcal{C}(E,X)\\ E\in\mathcal{B}}} Im(\mathcal{C}(-,\widehat{f})) \subseteq Im\left(\left(\varphi^{(E,X)}\right)_{E\in\mathcal{B}}\right)$, as we wanted.

We assert that for every $E\in\mathcal{B}$, $Im\left(\left(\varphi^{(E,X)}\right)_{E\in\mathcal{B}}\right) = Tr_{\{\mathcal{C}(-,E)\}_{E\in\mathcal{B}}}\mathcal{C}(-,X)$. Indeed, by the Yoneda lemma and the previous assertion, we have that

$$Tr_{\{\mathcal{C}(-,E)\}_{E\in\mathcal{B}}}\mathcal{C}(-,X) = \sum_{\substack{\psi\in Hom_{Mod(\mathcal{C})}(\mathcal{C}(-,E),\mathcal{C}(-,X))\\ \{\mathcal{C}(-,E)\}_{E\in\mathcal{B}}}} Im(\psi)$$

$$= \sum_{\substack{\widehat{f}\in\mathcal{C}(E,X)\\ E\in\mathcal{B}}} Im(\mathcal{C}(-,\widehat{f})) = Im\left(\left(\varphi^{(E,X)}\right)_{E\in\mathcal{B}}\right).$$

Finally, we observe that $f\in Im\left(\left(\varphi^{(E,X)}\right)_{E\in\mathcal{B}}\right)(C)$ if and only if $f\in I_{\mathcal{B}}(C,X)$. ■

Chapter 3

Triangular Matrix Categories Over Quasi-hereditary Categories

The class of quasi-hereditary algebras was introduced by E. Cline, B.J. Parshall, and L.L. Scott, and is defined through a given order on the indecomposable projective modules via a very special chain of ideals. They have been extensively studied since the late 1980s; see (8; 25; 28). They also appear in the study of Algebraic Groups, Lie Algebras, Highest-Weight Categories, Invariant Theory, Algebraic Geometry, and certain homological properties in finite-dimensional associative algebras over a field.

Another way to view this class of algebras was provided by V. Dlab and C.M. Ringel (9; 11; 26), who first introduced the class of standard modules associated with a finite-dimensional K-algebra Λ. This class was fundamental in the application of quasi-hereditary algebras in Representation Theory.

C.M. Ringel also studied the homological properties of the category of filtered Λ-modules, $F({}_{\Lambda}\Delta)$, for a quasi-hereditary algebra Λ. In this way, he establishes a relationship between quasi-hereditary algebras and Tilting Theory, through a tilting module that he calls characteristic.

In (24), Ortiz introduces the notion of a quasi-hereditary category, which will be the main concept upon which we will develop this chapter. Therefore, in this section, we aim to present some of the main definitions and results outlined in (24). With these tools, we will be able to extend the main results obtained by Zhu in (30) for quasi-hereditary algebras to the category $Mod(\mathcal{C})$.

In (30, Theorem 3.1), B. Zhu finds certain conditions for the triangular matrix algebra to be quasi-hereditary. Following these ideas, we generalize this result to the context of functor categories. Specifically, as seen in (23), we prove that if $\mathcal{U}$ and $\mathcal{T}$ are quasi-hereditary Hom-finite and Krull-Schmidt categories with respect to the filtrations $\{\mathcal{U}_j\}_{0\leq j\leq n}$ and $\{\mathcal{T}_j\}_{j\geq 0}$ of $\mathcal{U}$ and $\mathcal{T}$, respectively, which consist of additively closed subcategories and $M_T \in \mathcal{F}({}_{\mathcal{U}}\Delta)$ for every $T \in \mathcal{T}$, then it follows that $\Lambda = \begin{pmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{pmatrix}$ is quasi-hereditary with respect to a certain filtration $\{\Lambda_j\}j \geq 0$, which we construct from the filtrations $\{\mathcal{U}_j\}_{0\leq j\leq n}$ and $\{\mathcal{T}_j\}_{j\geq 0}$. Furthermore, we obtain a characterization of the category of filtered Λ-modules ${}_{\Lambda}\Delta$.

3.1 Quasi-hereditary Category respect to a Chain of Two-sided Ideals

Assume $\mathcal{C}$ is a Hom-finite Krull-Schmidt K-category. In order to generalize the notion of quasi-hereditary algebra to K-categories, the notion of heredity ideal and heredity chain is introduced in (24).

Remark 3.1.1. We note that definition of quasi-hereditary category in (24, Definition 3.4) is given for contravariant functors; however, by considering the opposite category $\mathcal{C}^{op}$ we have that contravariant functors over $\mathcal{C}^{op}$ coincide with covariant functors over $\mathcal{C}$. So, we can translate all the results in (24) to the setting of covariant functors.

To begin, we need to provide a series of important definitions, starting with the notion of **Heredity Ideals** in $\mathcal{C}$.

Definition 3.1.2. A two-sided ideal I in $\mathcal{C}$ is called (left) **heredity** if the following conditions hold:

(i) $I^2 = I$, i.e, I is an idempotent ideal;

(ii) $I\mathrm{rad}\,\mathcal{C}(-,?)I = 0$, and

(iii) $I(X,-)$ is a projective finitely generated $\mathcal{C}-$module for all $X \in \mathcal{C}$.

Definition 3.1.3. A chain of two-sided ideals

$$0 = I_0 \subset I_1 \subset \cdots \subset \mathcal{C}(-,?),$$

is **exhaustive** if $\bigcup\limits_{j\in J} I_j = \mathcal{C}(-,?)$. The category $\mathcal{C}$ is called **quasi-hereditary** if there exists an exhaustive chain $\{I_j\}_{j\in J}$, where J is at most countable, of two-sided ideals

$$0 = I_0 \subset I_1 \subset \cdots \subset \mathcal{C}(-,?),$$

such that $\frac{I_j}{I_{j-1}}$ is heredity in the quotient category $\frac{\mathcal{C}}{I_{j-1}}$. Such a chain is called a **heredity chain**.

Remark 3.1.4. (a) We note that we consider exhaustive chain of ideals because as in the classical case, we need to reach $\mathcal{C}(-,?)$ in some way and if the set J is infinite we can do that by requiring the equality $\bigcup_{j\in J} I_j = \mathcal{C}(-,?)$.

(b) In the spirit of the Remark 3.1.1, we should have called left heredity ideal and left quasi-hereditary category to the notions given above for covariant functors and the notions given for contravariant functors in (24) should have been called right heredity ideal and right quasi-hereditary category. However, in order to avoid overloading the notation we will not write the left adjective in all those notions.

(c) By Remark 3.1.1 we have that $\mathcal{C}$ is quasi-hereditary in the sense given above if and only if $\mathcal{C}^{op}$ is quasi-hereditary in the sense of (24).

Recall that, if $\mathcal{B}$ is a full additive subcategory of $\mathcal{C}$, and given $C, C' \in \mathcal{C}$, we denote by $I_{\mathcal{B}}(C, C')$ the subset of $\mathcal{C}(C, C')$ consisting of morphisms which factor through some object in $\mathcal{B}$. This allows us to define the two-sided ideal $I_{\mathcal{B}}(-,?)$ which is an idempotent ideal in $\mathcal{C}$.

Moreover, we denote by $\mathrm{Tr}_{\{\mathcal{C}(E,-)\}_{E\in\mathcal{B}}}\mathcal{C}(X,-)$ with $X \in \mathcal{C}$, the **trace** of $\{\mathcal{C}(E,-)\}_{E\in\mathcal{B}}$ in $\mathcal{C}(X,-)$, that is,

$$\mathrm{Tr}_{\{\mathcal{C}(E,-)\}_{E\in\mathcal{B}}}\mathcal{C}(X,-)=\sum\Big\{\mathrm{Im}(\psi):\psi\in\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}\Big(\mathcal{C}(E,-),\mathcal{C}(X,-)\Big),\ E\in\mathcal{B}\Big\}.$$

By the covariant version of 2.5.1, we have that:

Lemma 3.1.5. *Let $\mathcal{B}$ be a full additive subcategory of $\mathcal{C}$. Then for $X \in \mathcal{C}$ we have that:*

$$\mathrm{Tr}_{\{\mathcal{C}(E,-)\}_{E\in\mathcal{B}}}\mathcal{C}(X,-) = I_{\mathcal{B}}(X,-).$$

3.2 Quasi-hereditary Category respect to an Exhaustive Filtration

Previously, we gave a definition of quasi-hereditary category respect to a chain $\{I_j\}_{j\in J}$ of two-sided ideals. Now, in order to produce a chain of two sided ideals of $\mathcal{C}$ we will need a filtration of the category $\mathcal{C}$. So, assume we have an exhaustive filtration

$$\{0\} = \mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \cdots \subseteq \mathcal{C}$$

of $\mathcal{C}$ into additive full subcategories (that is, $\cup_{j\geq 0}\mathcal{B}_j = \mathcal{C}$). We then have an exhaustive chain of two-sided idempotent ideals:

$$\{0\} = I_{\mathcal{B}_0} \subset I_{\mathcal{B}_1} \subset \cdots I_{\mathcal{B}_{j-1}} \subset I_{\mathcal{B}_j} \subset \cdots \subset \mathcal{C}(-,?).$$

Note that $\frac{I_{\mathcal{B}_j}}{I_{\mathcal{B}_{j-1}}}$ is an idempotent ideal in the quotient category $\frac{\mathcal{C}}{I_{\mathcal{B}_{j-1}}}$ since $I_{\mathcal{B}_j}$ and $I_{\mathcal{B}_{j-1}}$ are idempotents in $\mathcal{C}$ and

$$\left(\frac{I_{\mathcal{B}_j}}{I_{\mathcal{B}_{j-1}}}\right)^2 = \frac{I_{\mathcal{B}_j}}{I_{\mathcal{B}_{j-1}}}\frac{I_{\mathcal{B}_j}}{I_{\mathcal{B}_{j-1}}} = \frac{I^2_{\mathcal{B}_j} + I_{\mathcal{B}_{j-1}}}{I_{\mathcal{B}_{j-1}}} = \frac{I_{\mathcal{B}_j}}{I_{\mathcal{B}_{j-1}}}.$$

The above motivates us to introduce the definition of quasi-hereditary category respect to an exhaustive $\{\mathcal{B}_j\}_{j\geq 0}$ filtration of $\mathcal{C}$.

Definition 3.2.1. ([24], Definition 3.4 (b)) Let $\mathcal{C}$ be a Hom-finite Krull-Schmidt K-category. Assume that $\{\mathcal{B}_j\}_{j\geq 0}$ is an exhaustive filtration of $\mathcal{C}$ into full additive subcategories. We say that $\mathcal{C}$ is quasi-hereditary with respect to $\{\mathcal{B}_j\}_{j\geq 0}$ if

$$\{0\} = I_{\mathcal{B}_0} \subset I_{\mathcal{B}_1} \subset \cdots I_{\mathcal{B}_{j-1}} \subset I_{\mathcal{B}_j} \subset \cdots \subset \mathcal{C}(-,?)$$

is a heredity chain.

Remark *3.2.2.* (a) We note that the advantage of using the Definition 3.2.1 instead of the more general definition of quasi-hereditary category given previously, is that exhaustive filtrations are easier to compute than idempotent ideals in general, and exhaustive filtrations induce an exhaustive chain of idempotent ideals.

(b) Now, by ([24], Lemma 3.9) we can see that if $\mathrm{rad}^{\infty}(\mathcal{C}) = 0$ then all idempotent ideals of $\mathcal{C}$ are of the form $I_{\mathcal{B}}$ for some additive subcategory of $\mathcal{C}$; and hence all the exhaustive chains $\{I_i\}_{i\geq 0}$ of idempotent ideals are constructed in this way. Unfortunately, there are categories where all the idempotent ideals are not of the form $\mathcal{I}_{\mathcal{B}}$ for some $\mathcal{B}$.

For example, if A is a wild algebra over an algebraically closed field, then the transfinite radical rad^{*} is not zero (see ([29], Proposition 4)) and rad^{*} is an idempotent ideal of mod(A) which does not contain any identity morphisms (see (29, Lemma 1)); and hence it is not of the form $\mathcal{I}_B$.

Let $\mathcal{B}$ be an additive subcategory of $\mathcal{C}$, we denote by add$(\mathcal{B})$ the full subcategory of $\mathcal{C}$ whose objects are the direct summands of finite coproducts of objects in $\mathcal{B}$.

A subcategory $\mathcal{B}$ of $\mathcal{C}$ is **closed under direct summands** if add$(\mathcal{B}) = \mathcal{B}$. Let $\mathcal{B}$ be an additive subcategory of $\mathcal{C}$, we denote by **Ind**$(\mathcal{B})$ the class of all the indecomposable objects belonging to $\mathcal{B}$.

Let's consider the following result that we will need later in 3.2.1.

Lemma 3.2.3. *(24, Lema 3.5) Consider a pair of subcategories $\mathcal{B} \subset \mathcal{B}' \subset \mathcal{C}$, closed under direct summands and isomorphisms. Then $\frac{I_{\mathcal{B}'}(-,X)}{I_{\mathcal{B}}(-,X)}$ is a projective $\frac{\mathcal{C}}{I_{\mathcal{B}}}-$module if and only if it is isomorphic to $\frac{\mathcal{C}(-,E)}{I_{\mathcal{B}}(-,E)}$, for some $\mathcal{B}'$.*

Proof:

Assume that $\frac{I_{\mathcal{B}'}(-,X)}{I_{\mathcal{B}}(-,X)}$ is a projective $\frac{\mathcal{C}}{I_{\mathcal{B}}}-$module. Then there exists an isomorphism of $\frac{\mathcal{C}}{I_{\mathcal{B}}}-$modules

$$\varphi : \frac{\mathcal{C}(-,E)}{I_{\mathcal{B}}(-,E)} \longrightarrow \frac{I_{\mathcal{B}'}(-,X)}{I_{\mathcal{B}}(-,X)}.$$

It remains to show that $E \in \mathcal{B}'$.

Set

$$\varphi_E(1_E + I_{\mathcal{B}}(E,E)) = \left(E \xrightarrow{f} B' \xrightarrow{g} X \right) + I_{\mathcal{B}}(E,X) \in \frac{I_{\mathcal{B}'}(E,X)}{I_{\mathcal{B}}(E,X)},$$

with $B' \in \mathcal{B}'$. Let's prove that $1_E \in I_{\mathcal{B}'}(E,E)$.

Indeed, we have the following commutative diagram:

$$\begin{tikzcd}
\frac{\mathcal{C}(B',E)}{I_{\mathcal{B}}(B',E)} \arrow[r, "\varphi_{B'}"] \arrow[d, "\frac{\mathcal{C}(f,E)}{I_{\mathcal{B}}(f,E)}"'] & \frac{I_{\mathcal{B}'}(B',X)}{I_{\mathcal{B}}(B',X)} \arrow[d, "\frac{I_{\mathcal{B}'}(f,X)}{I_{\mathcal{B}}(f,X)}"] \\
\frac{\mathcal{C}(E,E)}{I_{\mathcal{B}}(E,E)} \arrow[r, "\varphi_E"] & \frac{I_{\mathcal{B}'}(E,X)}{I_{\mathcal{B}}(E,X)} .
\end{tikzcd}$$

Thus,

$$\frac{I_{\mathcal{B}'}(f,E)}{I_{\mathcal{B}}(f,E)}(g + I_{\mathcal{B}}(B',X)) = \varphi_E(1_E + I_{\mathcal{B}}(E,E)).$$

Let $h \in \mathcal{C}(B',E)$ such that $\varphi_{B'}(h + I_{\mathcal{B}}(B',E)) = (g + I_{\mathcal{B}}(B',X))$.

By the commutativity of the diagram, we have that

$$\frac{\mathcal{C}(f,E)}{I_{\mathcal{B}}(f,E)}(h+I_{\mathcal{B}}(B',E)) = hf + I_{\mathcal{B}}(E,E) = 1_B + I_{\mathcal{B}}(E,E).$$

Then, since $\mathcal{B} \subset \mathcal{B}'$ it follows that $1_E - hf \in I_{\mathcal{B}}(E,E) \subset I_{\mathcal{B}'}(E,E)$. Thus, $1_E \in I_{\mathcal{B}'}(E,E)$ since $hf \in I_{\mathcal{B}'}(E,E)$. Therefore, there exist morphisms $v \in \mathcal{C}(B'',E)$ and $u \in \mathcal{C}(E,B'')$ with $B'' \in \mathcal{B}'$ such that $1_E = vu$. Thus, u is a split monomorphism and E is a direct summand of B'', i.e. $E \in \mathcal{B}'$.

The other implication is immediate since $\frac{\mathcal{C}(-,E)}{I_{\mathcal{B}}(-,E)}$ is a projective $\frac{\mathcal{C}}{I_{\mathcal{B}}}$−module for every $E \in \mathcal{B}'$. ■

The following result given in (24) will be useful in the remainder of this work.

Theorem 3.2.1. (24, Theorem 3.6) *Let $\mathcal{C}$ be a* Hom-*finite Krull-Schmidt K-category and let $\{\mathcal{B}_j\}_{j\geq 0}$ be a family of closed under direct summands additive subcategories of $\mathcal{C}$. Suppose that $\{\mathcal{B}_j\}_{j\geq 0}$ is an exhaustive filtration of $\mathcal{C}$. Then $\mathcal{C}$ is quasi-hereditary with respect to $\{\mathcal{B}_j\}_{j\geq 0}$ if and only if the following conditions hold:*

(i) $\mathrm{rad}_{\mathcal{C}}(E,E') = I_{\mathcal{B}_{j-1}}(E,E')$, *for all* $E,E' \in \mathrm{Ind}\ \mathcal{B}_j - \mathrm{Ind}\ \mathcal{B}_{j-1}$ *;*

(ii) and for all $X \in \mathcal{C}$ and $j \geq 1$, there exists an exact sequence

$$\mathcal{C}(-,E_{j-1}) \longrightarrow \mathcal{C}(-,E_j) \longrightarrow I_{\mathcal{B}_j}(-,X) \longrightarrow 0\ ,$$

with $E_j \in \mathcal{B}_j$ and $E_{j-1} \in \mathcal{B}_{j-1}$.

Proof:
Given the filtration $\{\mathcal{B}_j\}_{j\geq 0}$, let's see that $\frac{I_{\mathcal{B}_j}}{I_{\mathcal{B}_{j-1}}}$ is hereditary in $\frac{\mathcal{C}}{I_{\mathcal{B}_{j-1}}}$ if and only if conditions (i) and (ii) are satisfied.

First, let's see that $\frac{I_{\mathcal{B}_j}}{I_{\mathcal{B}_{j-1}}} rad\left(\frac{\mathcal{C}(-,?)}{I_{\mathcal{B}_{j-1}}}\right)\frac{I_{\mathcal{B}_j}}{I_{\mathcal{B}_{j-1}}} = 0$ if and only if condition (i) holds. To do this, note that

$$\begin{aligned}\frac{I_{\mathcal{B}_j}}{I_{\mathcal{B}_{j-1}}} rad\left(\frac{\mathcal{C}(-,?)}{I_{\mathcal{B}_{j-1}}}\right)\frac{I_{\mathcal{B}_j}}{I_{\mathcal{B}_{j-1}}} &= \frac{I_{\mathcal{B}_j}}{I_{\mathcal{B}_{j-1}}}\left(\frac{rad\mathcal{C}(-,?)+I_{\mathcal{B}_{j-1}}}{I_{\mathcal{B}_{j-1}}}\right)\frac{I_{\mathcal{B}_j}}{I_{\mathcal{B}_{j-1}}} \\ &= \frac{I_{\mathcal{B}_j}(rad\mathcal{C}(-,?)+I_{\mathcal{B}_{j-1}})I_{\mathcal{B}_j}+I_{\mathcal{B}_{j-1}}}{I_{\mathcal{B}_{j-1}}}\end{aligned}$$

$$= \frac{I_{\mathcal{B}_j} rad\mathcal{C}(-,?)I_{\mathcal{B}_j} + I_{\mathcal{B}_{j-1}} I_{\mathcal{B}_j} + I_{\mathcal{B}_{j-1}}}{I_{\mathcal{B}_{j-1}}}.$$

Hence, $\frac{I_{\mathcal{B}_j}}{I_{\mathcal{B}_{j-1}}} rad\left(\frac{\mathcal{C}(-,?)}{I_{\mathcal{B}_{j-1}}}\right)\frac{I_{\mathcal{B}_j}}{I_{\mathcal{B}_{j-1}}} = 0$ if and only if

$$I_{\mathcal{B}_j} rad\mathcal{C}(-,?)I_{\mathcal{B}_j} + I_{\mathcal{B}_{j-1}} I_{\mathcal{B}_j} + I_{\mathcal{B}_{j-1}} = I_{\mathcal{B}_{j-1}}. \tag{3.1}$$

On the other hand, $I_{\mathcal{B}_{j-1}} = I^2_{\mathcal{B}_{j-1}} \subset I_{\mathcal{B}_{j-1}} I_{\mathcal{B}_j} \subset I_{\mathcal{B}_{j-1}}$, meaning that $I_{\mathcal{B}_{j-1}} = I_{\mathcal{B}_{j-1}} I_{\mathcal{B}_j}$. From this, we have that condition (3.1) implies that for every j it holds

$$I_{\mathcal{B}_j} rad\mathcal{C}(-,?)I_{\mathcal{B}_j} \subseteq I_{\mathcal{B}_{j-1}}. \tag{3.2}$$

Now let's see that $I_{\mathcal{B}_j} rad\mathcal{C}(-,?)I_{\mathcal{B}_j} \subseteq I_{\mathcal{B}_{j-1}}$ if and only if $rad\mathcal{C}(E,E') = I_{\mathcal{B}_{j-1}}(E,E')$ for all $E, E' \in Ind\ \mathcal{B}_j$ such that non of them is a direct summand of any object in $\mathcal{B}_{j-1}$.

Assume that $I_{\mathcal{B}_j} rad\mathcal{C}(-,?)I_{\mathcal{B}_j} \subseteq I_{\mathcal{B}_{j-1}}$. Let $t \in rad\mathcal{C}(E,E')$, then $t = 1_{E'} t 1_E \in I_{\mathcal{B}_j}(E,E) rad\mathcal{C}(E,E') I_{\mathcal{B}_j}(E',E') \subseteq I_{\mathcal{B}_{j-1}}(E,E')$. Let $f \in I_{\mathcal{B}_{j-1}}(E,E')$. Then, there exists $B \in \mathcal{B}_{j-1}$ for which $f = vu$ for some morphisms $E \xrightarrow{u} B \xrightarrow{v} E'$. We assert that f is not an isomorphism.

Otherwise, there exists $g : E' \longrightarrow E$ such that $fg = 1_{E'}$. Therefore $vug = 1_{E'}$, i.e. E' is a direct summand of B; hence $E' \in \mathcal{B}_{j-1}$, which is a contradiction. This contradiction implies that $I_{\mathcal{B}_{j-1}}(E,E') \subset rad\mathcal{C}(E,E')$.

Conversely, consider $rad\mathcal{C}(E,E') = I_{\mathcal{B}_{j-1}}(E,E')$ and let's see that $I_{\mathcal{B}_j} rad\mathcal{C}(-,?)I_{\mathcal{B}_j} \subseteq I_{\mathcal{B}_{j-1}}$. Let $f \in I_{\mathcal{B}_j}(X,Y) rad\mathcal{C}(Y,Z) I_{\mathcal{B}_j}(Z,W)$. We can assume that $f = str$ with $r \in I_{\mathcal{B}_j}(X,Y)$, $t \in rad\mathcal{C}(Y,Z)$ and $s \in I_{\mathcal{B}_j}(Z,W)$. Note that $r : X \longrightarrow \coprod B'_i \longrightarrow Y$ and $s : Z \longrightarrow \coprod B''_i \longrightarrow W$, where the intermediate terms are finite direct sums of indecomposable objects in $\mathcal{B}_j$. Therefore, the induced maps

$$B'_i \longrightarrow Y \xrightarrow{t} Z \longrightarrow B''_i$$

are all in $rad\mathcal{C}(B'_i, B''_i) = I_{\mathcal{B}_{j-1}}(B'_i, B''_i)$, and hence $f \in I_{\mathcal{B}_{j-1}}(X,W)$.

We prove that $\frac{I_{\mathcal{B}_j}(-,X)}{I_{\mathcal{B}_{j-1}}(-,X)}$ is a projective $\frac{\mathcal{C}}{I_{\mathcal{B}_{j-1}}}$−module if and only if the condition (ii) holds.

(⇒) We will proceed by induction on j. The case $j = 1$ is given by Lemma 2.5.1 because $I_{\mathcal{B}_0} = 0$.

Assume that condition (ii) holds for $j-1$ and that $\frac{I_{\mathcal{B}_j}(-,X)}{I_{\mathcal{B}_{j-1}}(-,X)}$ is a projective $\frac{\mathcal{C}}{I_{\mathcal{B}_{j-1}}}$−module. By Lemma 3.2.3 we have an isomorphism

$$\varphi : \frac{\mathcal{C}(-,E_j)}{I_{\mathcal{B}_{j-1}}(-,E_j)} \cong \frac{I_{\mathcal{B}_j}(-,X)}{I_{\mathcal{B}_{j-1}}(-,X)},$$

with $E_j \in \mathcal{B}_i$. Let's consider the following short exact sequences

$$0 \longrightarrow I_{\mathcal{B}_{j-1}}(-,E_j) \longrightarrow \mathcal{C}(-,E_j) \xrightarrow{u} \frac{\mathcal{C}(-,E_j)}{I_{\mathcal{B}_{j-1}}(-,E_j)} \longrightarrow 0\ ,$$

$$0 \longrightarrow I_{\mathcal{B}_{j-1}}(-,X) \longrightarrow I_{\mathcal{B}_j}(-,X) \xrightarrow{v} \frac{I_{\mathcal{B}_j}(-,X)}{I_{\mathcal{B}_{j-1}}(-,X)} \longrightarrow 0\ .$$

Then the morphism $\phi u : \mathcal{C}(-,E_j) \longrightarrow \frac{I_{\mathcal{B}_j}(-,X)}{I_{\mathcal{B}_{j-1}}(-,X)}$ lifts to v, , since $\mathcal{C}(-,E_j)$ is projective. Thus, we have the following pushout diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
I_{\mathcal{B}_{j-1}}(-,E_j) \arrow[r] \arrow[d] & I_{\mathcal{B}_{j-1}}(-,X) \arrow[d] \\
\mathcal{C}(-,E_j) \arrow[r] \arrow[d, "\phi u"] & I_{\mathcal{B}_j}(-,X) \arrow[d, "v"] \\
\frac{I_{\mathcal{B}_j}(-,X)}{I_{\mathcal{B}_{j-1}}(-,X)} \arrow[r, equal] \arrow[d] & \frac{I_{\mathcal{B}_j}(-,X)}{I_{\mathcal{B}_{j-1}}(-,X)} \arrow[d] \\
0 & 0
\end{tikzcd}$$

which induces the following exact sequence,

$$I_{\mathcal{B}_{j-1}}(-,E_j) \longrightarrow \mathcal{C}(-,E_j) \coprod I_{\mathcal{B}_{j-1}}(-,X) \xrightarrow{\pi} I_{\mathcal{B}_j}(-,X) \longrightarrow 0\ .$$

On the other hand, by the induction hypothesis, there exist exact sequences:

$$\mathcal{C}(-,E'_{j-2}) \longrightarrow \mathcal{C}(-,E'_{j-1}) \xrightarrow{p} I_{\mathcal{B}_{j-1}}(-,X) \longrightarrow 0\ ,$$

$$\mathcal{C}(-,E''_{j-2}) \longrightarrow \mathcal{C}(-,E''_{j-1}) \xrightarrow{q} I_{\mathcal{B}_{j-1}}(-,E_j) \longrightarrow 0\ ,$$

with $E'_{j-1}, E''_{j-1} \in \mathcal{B}_{j-1}$ and $E'_{j-2}, E''_{j-2} \in \mathcal{B}_{j-2}$. Thus, we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Ker\,(1\coprod p) \arrow[r, equal] \arrow[d] & Ker\,(1\coprod p) \arrow[d] & & \\
0 \arrow[r] & Ker\,(\pi\,(1\coprod p)) \arrow[r] \arrow[d] & \mathcal{C}(-,E_j)\coprod\mathcal{C}(-,E'_{j-1}) \arrow[r, "\pi(1\coprod p)"] \arrow[d, "1\coprod p"] & I_{\mathcal{B}_j}(-,X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Ker(\pi) \arrow[r] \arrow[d] & \mathcal{C}(-,E_j)\coprod I_{\mathcal{B}_{j-1}}(-,X) \arrow[r, "\pi"] \arrow[d] & I_{\mathcal{B}_j}(-,X) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

From the epimorphisms

$$\mathcal{C}(-,E''_{j-1}) \xrightarrow{q} I_{B_{i-1}}(-,E_j) \longrightarrow Ker(\pi) \longrightarrow 0\ ,$$

$$\mathcal{C}(-,E'_{j-2}) \longrightarrow Ker(p) = Ker\,(1\coprod p) \longrightarrow 0\ ,$$

it follows, from the exactness of the left column in the previous diagram and by the horseshoe lemma, that there exists an epimorphism

$$\mathcal{C}(-,E'_{j-2})\coprod\mathcal{C}(-,E''_{j-1}) \longrightarrow Ker\,(\pi\,(1\coprod p)) \longrightarrow 0\ .$$

Thus, we have an exact sequence

$$\mathcal{C}(-,E'_{j-2})\coprod\mathcal{C}(-,E''_{j-1}) \longrightarrow \mathcal{C}(-,E_j)\coprod\mathcal{C}(-,E'_{j-1}) \xrightarrow{\pi(1\coprod p)} I_{\mathcal{B}_j}(-,X) \longrightarrow 0\ .$$

($\Leftarrow$) First, let's observe that for a given $\mathcal{C}-$module M and a $\mathcal{C}\times\mathcal{C}^{op}-$module $N(?,-)$ a $\mathcal{C}-$module $F:\mathcal{C}\longrightarrow Ab$ can be defined as follows:

$$F(C) = M\otimes N(C,-),$$

for every $C\in\mathcal{C}$. We will denote this by $M\otimes N(?,-)$.

Assume that we have an exact sequence

$$\mathcal{C}(-,E_{j-1}) \longrightarrow \mathcal{C}(-,E_j) \longrightarrow I_{\mathcal{B}_j}(-,X) \longrightarrow 0\ . \tag{3.3}$$

Then, after applying $-\otimes\frac{\mathcal{C}(C,-)}{I_{\mathcal{B}_{j-1}}(C,-)}$ to the exact sequence (3.3), we obtain the following exact sequence

$$\frac{\mathcal{C}(C,E_{j-1})}{I_{\mathcal{B}_{j-1}}(C,E_{j-1})} \longrightarrow \frac{\mathcal{C}(C,E_j)}{I_{\mathcal{B}_{j-1}}(C,E_j)} \longrightarrow I_{\mathcal{B}_j}(-,X)\otimes\frac{\mathcal{C}(C,-)}{I_{\mathcal{B}_{j-1}}(C,-)} \longrightarrow 0\ .$$

Since E_{j-1} lies in $\mathcal{B}_{j-1}$, we have $\mathcal{C}(C, E_{j-1}) = I_{\mathcal{B}_{j-1}}(C, E_{j-1})$. Therefore,

$$\frac{\mathcal{C}(C, E_{j-1})}{I_{\mathcal{B}_{j-1}}(C, E_{j-1})} \cong 0$$

.

Then, there exists an isomorphism of $\mathcal{C}-$modules

$$\frac{\mathcal{C}(C, E_i)}{I_{\mathcal{B}_{j-1}}(C, E_i)} \cong I_{\mathcal{B}_j}(-, X) \otimes \frac{\mathcal{C}(C, -)}{I_{\mathcal{B}_{j-1}}(C, -)}.$$

Finally, we assert that

$$I_{\mathcal{B}_j}(-, X) \otimes \frac{\mathcal{C}(?, -)}{I_{\mathcal{B}_{j-1}}(?, -)} \cong \frac{I_{\mathcal{B}_j}(?, X)}{I_{\mathcal{B}_{j-1}}(?, X)}.$$

Indeed, let $C \in \mathcal{C}$. Then, after applying $I_{\mathcal{B}_j}(-, X) \otimes -$ to the exact sequence of $\mathcal{C}^{op}-$modules

$$\begin{tikzcd} 0 \arrow[r] & I_{\mathcal{B}_{j-1}}(C, -) \arrow[r, "q"] & \mathcal{C}(C, -) \arrow[r] & \frac{\mathcal{C}(C,-)}{I_{\mathcal{B}_{j-1}}(C,-)} \arrow[r] & 0 \end{tikzcd},$$

we obtain the exact sequence of abelian groups

$$\begin{tikzcd} I_{\mathcal{B}_j}(-, X) \otimes I_{\mathcal{B}_{j-1}}(C, -) \arrow[r, "q*"] & I_{\mathcal{B}_j}(C, X) \arrow[r] & I_{\mathcal{B}_j}(-, X) \otimes \frac{\mathcal{C}(C,-)}{I_{\mathcal{B}_{j-1}}(C,-)} \arrow[r] & 0 \end{tikzcd}. \quad (3.4)$$

First, let's prove that $Im(q*) = Im(I_{\mathcal{B}_j}(-, X) \otimes q) = I_{\mathcal{B}_{j-1}}(C, X)$. Indeed, we have a morphism of $\mathcal{C}^{op}-$modules

$$\left(\varphi^{(E',C)}\right)_{E' \in \mathcal{B}_{j-1}} : \coprod_{E' \in \mathcal{B}_{j-1}} \mathcal{C}(E', -)^{k_{E'}(C)} \to \mathcal{C}(C, -),$$

where $k_{E'}(C) := dim_K \mathcal{C}(C, E')$ and the morphism $\varphi^{(E',C)}$ is defined as

$$\varphi_C^{(E',C)} : \mathcal{C}(E', X)^{k'_E(C)} \to \mathcal{C}(C, X),$$

$$(g_1, \cdots, g_{k_{E'}(C)}) \mapsto \sum_{i=1}^{k_{E'}(C)} g_i f_i \ ,$$

for a $K-$basis, $\{g_1, \cdots, g_{k_E(X)}\}$, of $\mathcal{C}(C, E')$.

Note that by Lemma 2.5.1, we have $Im\left(\left(\varphi^{(E',C)}\right)_{E' \in \mathcal{B}_{j-1}}\right) = I_{\mathcal{B}_{j-1}}(C, -)$. Therefore, we have the following factorization of the morphism $\left(\varphi^{(E',C)}\right)_{E' \in \mathcal{B}_{j-1}}$,

$$\left(\varphi^{(E',C)}\right)_{E' \in \mathcal{B}_{j-1}} : \begin{tikzcd} \coprod_{E' \in \mathcal{B}_{j-1}} \mathcal{C}(E', -)^{k_{E'}(C)} \arrow[r, "p"] & I_{\mathcal{B}_{j-1}}(C, -) \arrow[r, "q"] & \mathcal{C}(C, -) \end{tikzcd}. \quad (3.5)$$

Now, let's observe that for all $E' \in \mathcal{B}_{j-1}$ we have

$$I_{\mathcal{B}_{j-1}}(E',-) = I_{\mathcal{B}_j}(E',-) = \mathcal{C}(E',-),$$

since $\mathcal{B}_{j-1} \subset \mathcal{B}_j$. Therefore, after applying $- \otimes I_{\mathcal{B}_j}(-,X)$ to the exact sequence (3.5), we obtain the following commutative diagram

$$\begin{tikzcd}
\coprod_{E'\in\mathcal{B}_{j-1}} \mathcal{C}(E',-)^{k_{E'}(C)} \arrow[r, "p*"] \arrow[dr, "\left(\varphi^{(E',C)}\right)^{*}_{E'\in\mathcal{B}_{j-1}}"'] & I_{\mathcal{B}_j}(-,X) \otimes I_{\mathcal{B}_{j-1}}(C,-) \arrow[d, "q*"] \\
 & I_{\mathcal{B}_j}(C,X)\ ,
\end{tikzcd}$$

where $p*$ is an epimorphism. Therefore,

$$\begin{aligned}
Im(q*) &= Im\left(\left(\varphi^{(E',C)}\right)_{E'\in\mathcal{B}_{j-1}}\right) \bigcap I_{\mathcal{B}_j}(C,X) \\
&= I_{\mathcal{B}_{j-1}}(C,X) \bigcap I_{\mathcal{B}_j}(C,X) \\
&= I_{\mathcal{B}_{j-1}}(C,X).
\end{aligned}$$

Finally, from the sequence (3.4), we obtain the following exact sequence

$$\begin{tikzcd}
0 \arrow[r] & I_{\mathcal{B}_{j-1}}(?,X) \arrow[r] \arrow[d, equal] & I_{\mathcal{B}_j}(C,X) \arrow[r] \arrow[d, equal] & I_{\mathcal{B}_j}(-,X) \otimes \frac{\mathcal{C}(C,-)}{I_{\mathcal{B}_{j-1}}(C,-)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Im(q*) \arrow[r] & I_{\mathcal{B}_j}(C,X) \arrow[r] & \frac{I_{\mathcal{B}_j}(C,X)}{Im(q*)} \arrow[r] & 0,
\end{tikzcd}$$

from which we conclude the desired isomorphism. ■

By Remark 3.1.1, we will use the covariant version of this result.

Remark 3.2.4. (a). Let us see why the Definition 3.2.1 is a generalization of the classical notion of quasi-hereditary algebra. Let A be a finite dimensional K-algebra. In this case we consider the K-category $\mathcal{C} := \text{proj}(A)$, that is, the full subcategory of $\text{mod}(A)$ whose objects are the finitely generated projective A-modules. Since A is semiperfect we have that $\mathcal{C} = \text{proj}(A)$ is a Krull-Schmidt category (see (13, Proposition 4.1)).

Suppose that $\text{proj}(A)$ is a quasi-hereditary category in the sense of Definition 3.2.1, with respect to the exhaustive filtration

$$\{0\} = \mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \cdots \subseteq \mathcal{B}_{m-1} \subseteq \mathcal{B}_m = \mathcal{C} = \text{proj}(A)$$

of $\operatorname{proj}(A)$ into additive full subcategories and closed under direct summands. Then we have an exhaustive chain of two-sided idempotent ideals of $\mathcal{C}(-,?)$:

$$\{0\} = I_{\mathcal{B}_0} \subset I_{\mathcal{B}_1} \subset \cdots I_{\mathcal{B}_{m-1}} \subset I_{\mathcal{B}_m} = \mathcal{C}(-,?).$$

By (14, Lemma 4.5), we have the corresponding chain of idempotent ideals of A^{op}

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_{m-1} \subseteq J_m \simeq A^{op}$$

where $J_i := I_{\mathcal{B}_i}(A, A)$ for all i. In this case $J_i = Ae_iA$ for some idempotent $e_i \in A$ and by (14, Lemma 4.5), we have that $\mathcal{B}_i = \operatorname{add}(Ae_i)$ for all $i \geq 0$.

Now, let us see that J_1 is a heredity ideal of A^{op}.

(i). By Theorem 3.2.1(i) we have that $\operatorname{rad}_{\mathcal{C}}(E, E') = I_{\mathcal{B}_0}(E, E') = 0$, for all pair of objects $E, E' \in \operatorname{Ind} \mathcal{B}_1$, since $\mathcal{B}_0 = \{0\}$. Hence $\operatorname{rad}(\mathcal{B}_1) = 0$ and thus $e_1\operatorname{rad}(A)e_1 = 0$. We conclude that $J_1 A^{op} J_1 = 0$.
(ii) By Theorem 3.2.1(ii) we have an exact sequence

$$\begin{tikzcd} \mathcal{C}(Z,-) \arrow[r] & \mathcal{C}(Y,-) \arrow[r] & I_{\mathcal{B}_1}(A,-) \arrow[r] & 0 \end{tikzcd},$$

with $Y \in \mathcal{B}_1 = \operatorname{add}(Ae_1)$ and $Z \in \mathcal{B}_0 = \{0\}$.

Hence, $J_1 = I_{\mathcal{B}_1}(A, A) \simeq \mathcal{C}(Y, A) = \operatorname{proj}(Y, A) = \operatorname{Hom}_A(Y, A)$ which is projective over $A^{op} \simeq \operatorname{proj}(A, A)$.

Hence, J_1 is a heredity ideal of A^{op}. We can proceed inductively, and conclude that J_i/J_{i-1} is a heredity ideal of A^{op}/J_{i-1} for all $i \geq 1$. Therefore, A^{op} is quasi-hereditary and hence by (11, Statement 9) we conclude that A is quasi-hereditary. Similarly, by using (14, Lemma 4.5), it can be proved that if A is a quasi-hereditary algebra then $\operatorname{proj}(A)$ is a quasi-hereditary category.

(b). Suppose that A is quasi-hereditary, then there exists a chain of idempotent ideals

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_{m-1} \subseteq J_m = A$$

such that J_t/J_{t-1} is a heredity ideal of A/J_{t-1} for all t.

By (4, Proposition 6.1), there exists projective A-modules $P_1, \cdots, P_m$ such that $J_i = \mathrm{Tr}_{P_1 \oplus \cdots \oplus P_i}(A)$ for $i = 1, \cdots, m$. Moreover, if $\mathcal{B}_i = \mathrm{add}(P_1 \oplus \cdots \oplus P_i)$, then for each $i = 1, \cdots, m$ there exists an exact sequence in $\mathrm{mod}(A)$

$$\begin{tikzcd} P_{i,1} \arrow[r] & P_{i,0} \arrow[r] & J_i \arrow[r] & 0 \end{tikzcd}$$

such that $P_{i,0} \in \mathcal{B}_i$ and $P_{i,1} \in \mathcal{B}_{i-1}$.

We note that these exact sequences are the analogous of the exact sequences given in the Theorem 3.2.1(ii):

$$\begin{tikzcd} \mathcal{C}(P_{i,1}, -) \arrow[r] & \mathcal{C}(P_{i,0}, -) \arrow[r] & I_{\mathcal{B}_j}(X, -) \arrow[r] & 0 \end{tikzcd},$$

with $P_{i,0} \in \mathcal{B}_i$, $P_{i,1} \in \mathcal{B}_{i-1}$ and $X \in \mathcal{C}$.

It is well-known that a semiprimary ring A is quasi-hereditary if and only if A^{op} is quasi-hereditary (see (11, Statement 9) in p. 288). We have somehow a similar result for the context of quasi-hereditary categories.

We recall the following notions. Let $\mathcal{A}$ be an arbitrary category and $\mathcal{B}$ a full subcategory of $\mathcal{A}$. The full subcategory $\mathcal{B}$ is **contravariantly finite** if for every $A \in \mathcal{A}$ there exists a morphism $f_A : B \longrightarrow A$ with $B \in \mathcal{B}$ such that if $f' : B' \longrightarrow A$ is other morphism with $B' \in \mathcal{B}$, then there exist a morphism $g : B' \longrightarrow B$ such that $f' = f_A g$. The morphism f_A is called a **right $\mathcal{B}$-approximation** of A. A right $\mathcal{B}$-approximation $f_A : B \longrightarrow A$ of A is **minimal** if whenever $g : B \longrightarrow B$ is a morphism such that $g f_A = f_A$ then g is an isomorphism. Dually, is defined the notion of **covariantly finite** and **left minimal $\mathcal{B}$-approximation**. We say that $\mathcal{B}$ is **functorially finite** if $\mathcal{B}$ is contravariantly finite and covariantly finite.

Proposition 3.2.5. *Let $\mathcal{C}$ be a Hom-finite and Krull-Schmidt category and let $\{\mathcal{B}_j\}_{j \geq 0}$ be a family of closed under direct summands additive subcategories of $\mathcal{C}$; and suppose* *that each $\mathcal{B}_j$ is covariantly finite. If $\mathcal{C}$ is quasi-hereditary in the sense of (24, Definition 3.4), then $\mathcal{C}^{op}$ is quasi-hereditary in the sense of (24, Definition 3.4).*

Proof:
Suppose that $\mathcal{C}$ is quasi-hereditary with respect to $\{\mathcal{B}_j\}_{j\geq 0}$. By 3.2.1, there exist exact sequences for all $X\in\mathcal{C}$ and $j\geq 1$:

$$\mathcal{C}(-,E_{j-1})\longrightarrow \mathcal{C}(-,E_j)\longrightarrow I_{\mathcal{B}_j}(-,X)\longrightarrow 0\ ,$$

with $E_j\in\mathcal{B}_j$ and $E_{j-1}\in\mathcal{B}_{j-1}$. Now, since B_j is covariantly finite, there exists an epimorphism $\mathcal{C}(X,-)\longrightarrow I_{\mathcal{B}_j}(C,-)$ for every $C\in\mathcal{C}$ by using the proof of (27, Proposition 4.12). Then, this implies that B_j is functorially finite by (27, Proposition 4.12).

We assert that for all $X\in\mathcal{C}$ there exists a monic right $\mathcal{B}_1$-approximation of X. Indeed, by taking, $j=1$, from the above exact sequence we get that $\mathcal{C}(-,E_1)\simeq I_{\mathcal{B}_1}(-,X)$. By Yoneda's Lemma we get a morphism $\gamma: E_1\longrightarrow X$ and this morphism is a right $\mathcal{B}_1$-approximation of X. Now, let $\alpha: Y\longrightarrow E_1$ such that $\gamma\alpha=0$. Since $\mathcal{C}(-,\gamma):\mathcal{C}(-,E_1)\longrightarrow I_{\mathcal{B}_1}(-,X)$ is an isomorphism, we conclude that $\alpha=0$, this implies that γ is a monomorphism.

Now, since $\mathcal{B}_1$ is covariantly finite, $\mathrm{add}(\mathcal{B}_1)=\mathcal{B}_1$ and $\mathcal{C}$ is a Krull-Schmidt category, every object of $\mathcal{C}$ has a left minimal $\mathcal{B}_1$-approximation. So, let $\theta: X\longrightarrow E_1'$ be a left minimal $\mathcal{B}_1$-approximation of X.

We assert that θ is an epimorphism. Indeed, consider $\beta: E_1'\longrightarrow Y$ a morphism such that $\beta\theta=0$. By the first assertion above, there exists $\lambda: E\longrightarrow Y$ a monic right $\mathcal{B}_1$-approximation of Y; and hence $\beta=\lambda\delta$ for some $\delta: E_1'\longrightarrow E$. Since, $0=\beta\theta=\lambda\delta\theta$ and λ is a monomorphism we get that $\delta\theta=0$.

Now, for all $g: E\longrightarrow E_1'$ we have that $(1_{E_1'}-g\delta)\theta=\theta$. Since θ is minimal we get that $(1_{E_1'}-g\delta)$ is an isomorphism; and thus we conclude that $\delta\in\mathrm{rad}_{\mathcal{C}}(E_1',E)=\mathrm{rad}_{\mathcal{B}_1}(E_1',E)=0$ (by Theorem 3.2.1(i)). Hence $\delta=0$ and we obtain that $\beta=\lambda\delta=0$. This proves that θ is an epimorphism. Now, $\mathcal{C}(\theta,-)$ is an epimorphism as established in the proof of (27, Proposition 4.12). Therefore, we get an isomorphism

$$\mathcal{C}(\theta,-):\mathcal{C}(E_1',-)\longrightarrow I_{\mathcal{B}_1}(X,-).$$

We note that $I_{\mathcal{B}_j}^{op}=I_{\mathcal{B}_j^{op}}$ is a bilateral ideal of $\mathcal{C}^{op}$ for all j. Thus, we get that $I_{\mathcal{B}_1^{op}}(-,X)=I_{\mathcal{B}_1}(X,-)\simeq\mathcal{C}(E_1',-)\simeq\mathcal{C}^{op}(-,E_1')$ is a projective $\mathcal{C}^{op}$-module. Since $\mathrm{rad}\mathcal{C}(-,-)$ is a bilateral ideal we have that $I_{\mathcal{B}_1}\mathrm{rad}\ \mathcal{C}(-,?)I_{\mathcal{B}_1}=0$ implies that $I_{\mathcal{B}_1^{op}}\mathrm{rad}\mathcal{C}^{op}(-,?)I_{\mathcal{B}_1^{op}}=0$. Hence, we obtain that $I_{\mathcal{B}_1^{op}}(-,X)$ is a heredity ideal of $\mathcal{C}^{op}$. Proceeding inductively, we conclude that

$$\{0\}=I_{\mathcal{B}_0^{op}}\subset I_{\mathcal{B}_1^{op}}\subset\cdots\subset I_{\mathcal{B}_{j-1}^{op}}\subset I_{\mathcal{B}_j^{op}}\subset\cdots\subset\mathcal{C}^{op}(-,?)$$

is a heredity chain. Hence $\mathcal{C}^{op}$ is quasi-hereditary with respect to $\{\mathcal{B}_j^{op}\}_{j\geq 0}$ in the sense of *(24, Definition 3.4).* ■

Corollary 3.2.6. *Let $\mathcal{C}$ be a Hom-finite and Krull-Schmidt category and let $\{\mathcal{B}_j\}_{j\geq 0}$ be a family of closed under direct summands additive subcategories of $\mathcal{C}$ and suppose that each $\mathcal{B}_j$ is functorially finite. Then $\mathcal{C}$ is quasi-hereditary in the sense of (24, Definition 3.4) if and only if $\mathcal{C}^{op}$ is quasi-hereditary in the sense of (24, Definition 3.4).*

Proof:

It follows from Proposition 3.2.5 and its dual. ■

By the Corollary above and Remark 3.1.4(c) we conclude the following.

Corollary 3.2.7. *Let $\mathcal{C}$ be a Hom-finite and Krull-Schmidt category and let $\{\mathcal{B}_j\}_{j\geq 0}$ be a family of closed under direct summands additive subcategories of $\mathcal{C}$ and suppose that each $\mathcal{B}_j$ is functorially finite. Then $\mathcal{C}$ is quasi-hereditary in the sense of (24, Definition 3.4) if and only if $\mathcal{C}$ is quasi-hereditary in the sense of Definition 3.2.1.*

3.3 Standard $\mathcal{C}$-modules and Filtered $\mathcal{C}$-modules

M. Ortiz in (24), introduce the concept of standard subcategories of $\mathcal{C}$-modules with respect to the given filtration $\{\mathcal{B}_j\}_{j\geq 0}$. The standard subcategories generalize the usual standard modules over finite-dimensional algebras, the reader can see (11) and (12). Let $\mathcal{C}$ be a quasi-hereditary category with respect to a family of additively closed subcategories $\{\mathcal{B}_j\}$.

Each module

$${}_{\mathcal{C}}\Delta_E(j) := \frac{\mathcal{C}(E,-)}{I_{\mathcal{B}_{j-1}}(E,-)}$$

with $E \in \mathrm{Ind}\mathcal{B}_j - \mathrm{Ind}\mathcal{B}_{j-1}$ is called **standard**, and ${}_{\mathcal{C}}\Delta(j)$ denotes the category consisting of the standard $\mathcal{C}$-modules ${}_{\mathcal{C}}\Delta_E(j)$. In addition, ${}_{\mathcal{C}}\Delta$ denotes the full subcategory consisting of the standard $\mathcal{C}$-modules.

Let $\mathcal{A}$ be an abelian category, and $\mathcal{X} \subseteq \mathcal{A}$. We denote by $\mathcal{X}^{\mathrm{II}}$ the class of objects of $\mathcal{A}$, which are a finite direct sum of objects in $\mathcal{X}$. We say that $M \in \mathcal{A}$ is **$\mathcal{X}$-filtered** if there exists a chain $\{M_j\}_{j\geq 0}$ of subobjects of M such that $\frac{M_{j+1}}{M_j} \in \mathcal{X}^{\mathrm{II}}$ for $j \geq 0$.

In case $M = M_n$ for some $n \in \mathbb{N}$, we say that M has a finite $\mathcal{X}$-filtration of length n. We denote by $\mathcal{F}(\mathcal{X})$ the class of objects that are $\mathcal{X}$-filtered and by $\mathcal{F}_f(\mathcal{X})$ the class of objects that have a finite filtration. For $M \in \mathcal{F}_f(\mathcal{X})$, the $\mathcal{X}$-length of M can be defined as follows $l_{\mathcal{X}}(M) := \min\{n \in \mathbb{N} : M \text{ has an } \mathcal{X}\text{-filtration of length } n\}$.

By using the notion of $\mathcal{X}$-length and induction, the following useful remark can be proven.

Remark 3.3.1. Let $\mathcal{X}$ be a class of objects in an abelian category A. Then, the class $\mathcal{F}_f(\mathcal{X})$ is closed under extensions.

Given F a $\mathcal{C}-$module, its trace filtration with respect to $\{\mathcal{B}_j\}_{j\geq 0}$ is given by

$$\{0\} = F^{[0]} \subset F^{[1]} \subset F^{[2]} \subset \cdots \subset F^{[j-1]} \subset F^{[j]} \subset \cdots ,$$

where $F^{[j]} := \mathrm{Tr}_{\{\mathcal{C}(E,-)\}_{E\in\mathcal{B}_j}} F$ and $F = \bigcup\limits_{j\geq 0} F^{[j]}$.

It is of interest to study the $\mathcal{C}$-modules F that possesses a trace filtration that satisfies $\frac{F^{[j]}}{F^{[j-1]}} \in {}_{\mathcal{C}}\Delta(j)^{\amalg}$ for all $j \geq 1$. It then follows that these $\mathcal{C}$-modules are Δ-filtered. We denote the full subcategory of the Δ-filtered modules by $\mathcal{F}({}_{\mathcal{C}}\Delta)$.

The following results will be very useful, and the item (b) of Lemma 3.3.3 is a generalization of a result of Dlab (see (9, Proposition A.3.2)).

Lemma 3.3.2. (24, Lemma 3.17) *Let* $F \in \mathcal{F}(\Delta)$.

(a) For all $j \geq 0$, $F^{[j]}$ *has a presentation*

$$\mathcal{C}(-, E_{j-1}) \to \mathcal{C}(-, E_j) \to F^{[j]} \to 0,\ \ E_{j-1} \in \mathcal{B}_{j-1}, E_j \in \mathcal{B}_j. \tag{3.6}$$

(b) $F^{[j]} \cong \mathcal{C} \otimes_{\mathcal{B}_j} (F|_{\mathcal{B}_j})$.

Proof:

(a) Let's prove it by induction on the index j.

Since $I_{\mathcal{B}_0}(-, E) = 0$ for all $E \in \mathcal{B}_1$, we have $\Delta(1) = \{\mathcal{C}(-, E) \mid E \in \mathcal{B}_1\}$. Let $F \in \mathcal{F}(\Delta)$. Then $F^{(0)} = 0$ and $\frac{F^{(1)}}{F^{(0)}} = F^{(1)}$ is a finite direct sum of objects from $\Delta(1)$.

Assume that there exists an exact sequence for $F^{(j-1)}$,

$$\begin{tikzcd} \mathcal{C}(-,E_{j-2}) \arrow[r] & \mathcal{C}(-,E_{j-1}) \arrow[r] & F^{(j-1)} \arrow[r] & 0 . \end{tikzcd} \tag{3.7}$$

Note that for every $\mathcal{C}-$module F, we have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & F^{(j-1)} \arrow[r] & F^{(j)} \arrow[r] & \frac{F^{(j)}}{F^{(j-1)}} \arrow[r] & 0 . \end{tikzcd} \tag{3.8}$$

As $F \in \mathcal{F}(\Delta)$, it follows that $\frac{F^{(j)}}{F^{(j-1)}}$ is a finite direct sum of objects ${}_{\mathcal{C}}\Delta_E(j)$, $E \in \mathcal{B}_j$. We then write ${}_{\mathcal{C}}\Delta_E(j) = \frac{\mathcal{C}(-,E)}{I_{\mathcal{B}_{j-1}}(-,E)}$. Thus, we obtain the following short exact sequence for every $E \in \mathcal{B}_j$,

$$\begin{tikzcd} 0 \arrow[r] & I_{\mathcal{B}_{j-1}}(-,E) \arrow[r] & \mathcal{C}(-,E) \arrow[r] & \frac{\mathcal{C}(-,E)}{I_{\mathcal{B}_{j-1}}(-,E)} \arrow[r] & 0 . \end{tikzcd}$$

Moreover, by Theorem 3.2.1, there exists an exact sequence

$$\begin{tikzcd} \mathcal{C}(-,E'_{j-2}) \arrow[r] & \mathcal{C}(-,E'_{j-1}) \arrow[r] & I_{\mathcal{B}_{j-1}}(-,E) \arrow[r] & 0 , \end{tikzcd}$$

with $E'_{j-2} \in \mathcal{B}_{j-2}$ and $E'_{j-1} \in \mathcal{B}_{j-1}$.

We then have an exact sequence,

$$\begin{tikzcd} \mathcal{C}(-,E'_{j-1}) \arrow[r] & \mathcal{C}(-,E) \arrow[r] & {}_{\mathcal{C}}\Delta_E(j) \arrow[r] & 0 , \end{tikzcd} \tag{3.9}$$

for every $E \in \mathcal{B}_j$. Finally, from the exact sequences (3.7), (3.8) and (3.9), the desired resolution follows.

(b) Note that $\left(\frac{F}{F^{(j)}}\right)^{(j)} = 0$. Then $\left((-,E), \frac{F}{F^{(j)}}\right) = 0$ for every $E \in \mathcal{B}_j$, and thus for all $E \in \mathcal{B}_j$ we have,

$$\frac{F}{F^{(j)}}(E) = \frac{F}{F^{(j)}}\Big|_{\mathcal{B}_j}(E) = \left((-,E), \frac{F}{F^{(j)}}\right) = 0,$$

i.e., $\frac{F}{F^{(j)}}\Big|_{\mathcal{B}_j} = 0$.

Applying the exact functor $\big|_{\mathcal{B}_j}$ to the exact sequence $\begin{tikzcd} 0 \arrow[r] & F^{(j)} \arrow[r] & F \arrow[r] & \frac{F}{F^{(j)}} \arrow[r] & 0 \end{tikzcd}$, we obtain

$$F^{[j]}\Big|_{\mathcal{B}_j} \cong F\big|_{\mathcal{B}_j}. \tag{3.10}$$

On the other hand, from equation (3.6) we have that $F^{(j)}$ is projectively presented over $\mathcal{B}_j$.

Then, by Proposition 2.3.2 we have $\mathcal{C} \otimes_{\mathcal{B}_j} (F^{[j]}\big|_{\mathcal{B}_j}) \cong F^{[j]}$. Therefore, from (3.10), we have that

$$F^{[j]} \cong \mathcal{C} \otimes_{\mathcal{B}_j} \left(F^{[j]}\Big|_{\mathcal{B}_j} \right) \cong \mathcal{C} \otimes_{\mathcal{B}_j} (F|_{\mathcal{B}_j}).$$

■

Lemma 3.3.3. (24, Lemma 3.18) *Let $F \in \mathcal{F}(\Delta)$.*

(a) F is locally finite, that is, $dim_K F(B) < \infty$ for all $B \in \mathcal{C}$.

(b) F is finitely presented if and only if $F \cong F^{[j]}$ for some $i \in \mathbb{N}$.

(c) If the filtration $0 = \mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \cdots \subseteq \mathcal{B}_n = \mathcal{C}$ is finite, then $\mathcal{F}(\Delta)$ consists of finitely presented functors.

Proof:

(a) Let $B \in \mathcal{C}$, then $B \in \mathcal{B}_j$ for some $j \in \mathbb{N}$. By Lemma 3.3.2, there exists an exact sequence (3.6), $\mathcal{C}(-, E_{j-1}) \to \mathcal{C}(-, E_j) \to F^{[j]} \to 0$. From this, it follows that $dim_K F^{[j]}(B) < \infty$.

Since $F|_{\mathcal{B}_j} \cong F^{[j]}|_{\mathcal{B}_j}$, by (3.10), we have $F(B) = F|_{\mathcal{B}_j}(B) = F^{[j]}(B)$, and therefore $dim_K F(B) < \infty$.

(b) Assume that there exists an exact sequence

$$\mathcal{C}(-, C) \to \mathcal{C}(-, C') \to F \to 0,$$

with $C, C' \in \mathcal{C}$. Then, there exists some $j \geq 0$ such that $C, C' \in \mathcal{B}_j$. Thus, F is projectively presented over $\mathcal{B}_j$ and $C \otimes_{\mathcal{B}_j} F|_{\mathcal{B}_j} \cong F$, but, by (b) of Lemma 3.3.2, we have $F^{[j]} \cong \mathcal{C} \otimes_{\mathcal{B}_j} (F|_{\mathcal{B}_j})$. The other implication follows from part (a) of Lemma 3.3.2.

(c) If the filtration is finite, we have $F^{[n]} = F$, and the assertion follows from the previous part.

■

By Remark 3.1.1, we will use the covariant versions of this results.

3.4 Triangular Matrix Categories

In this section we will prove the main result of this thesis. First, we recall by 2.4.2, that if $\mathcal{U}$ and $\mathcal{T}$ have finite coproducts, then $\Lambda = \begin{pmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{pmatrix}$ has finite coproducts. For the convenience of the reader we will give an idea of how construct finite coproducts in Λ. Let $\begin{pmatrix} T_1 & 0 \\ M & U_1 \end{pmatrix}, \begin{pmatrix} T_2 & 0 \\ M & U_2 \end{pmatrix}$ two objects in Λ. Consider the coproduct $\{u_i : U_i \longrightarrow U_1 \oplus U_2\}_{i=1}^2$ of the family $\{U_i\}_{i=1}^2$ in $\mathcal{U}$ and the coproduct $\{v_i : T_i \longrightarrow T_1 \oplus T_2\}_{i=1}^2$ of the family $\{T_i\}_{i=1}^2$ in $\mathcal{T}$. Then we can construct the following morphisms in Λ

$$\left\{ \begin{pmatrix} u_i & 0 \\ 0 & v_i \end{pmatrix} : \begin{pmatrix} U_i & 0 \\ 0 & T_i \end{pmatrix} \longrightarrow \begin{pmatrix} U_1 \oplus U_2 & 0 \\ 0 & T_1 \oplus T_2 \end{pmatrix} \right\}_{i=1}^2,$$

and it is straightforward to see that this family is a coproduct for the family of objects $\left\{ \begin{pmatrix} T_i & 0 \\ M & U_i \end{pmatrix} \right\}_{i=1}^2$. Now, let $\mathcal{U}$ and $\mathcal{T}$ be Hom-finite Krull-Schmidt quasi-hereditary K-categories with respect to filtrations $\{\mathcal{U}_j\}_{0 \leq j \leq n}$ and $\{\mathcal{T}_j\}_{j \geq 0}$, respectively, consisting of full additive subcategories which are closed under direct summands. Assume that the $\mathcal{U}$-module $M_T = M(-, T)$ lies in $\mathcal{F}({}_{\mathcal{U}}\Delta)$ for all $T \in \mathcal{T}$; therefore M_T is finitely presented since the filtration of $\mathcal{U}$ is finite (see Lemma 3.3.3 (c)). Thus, by Proposition 2.4.9, we have that $\Lambda = \begin{pmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{pmatrix}$ is a Hom-finite Krull-Schmidt K-category.

Consider the filtration of Λ into subcategories $\{\Lambda_j\}_{j \geq 0}$ given by

$$\begin{aligned} \Lambda_0 &= \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}; \\ \Lambda_j &= \begin{pmatrix} 0 & 0 \\ M & \mathcal{U}_j \end{pmatrix} := \left\{ \begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix} : U \in \mathcal{U}_j \right\}, \text{ if } 1 \leq j \leq n; \\ \Lambda_{n+j} &= \begin{pmatrix} \mathcal{T}_j & 0 \\ M & \mathcal{U} \end{pmatrix} = \left\{ \begin{pmatrix} T & 0 \\ M & U \end{pmatrix} : T \in \mathcal{T}_j, U \in \mathcal{U} \right\}, \text{ if } j \geq 1. \end{aligned} \tag{3.11}$$

It is clear that $\Lambda_j \subseteq \Lambda$ is an additive full subcategory for all $j \geq 0$. For and object $\begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix} \in \Lambda$ with $U \in \operatorname{Ind} \mathcal{U}_j$ we have that

$$\operatorname{Hom}_\Lambda\left(\begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{U}(U, U) \end{pmatrix} \simeq \mathcal{U}(U, U)$$

is local. Hence we get that $\begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix}$ is an indecomposable object in Λ_j. Similarly, for an object $\begin{pmatrix} T & 0 \\ M & 0 \end{pmatrix} \in \Lambda$ with $T \in \operatorname{Ind} \mathcal{T}_j$ we have that $\begin{pmatrix} T & 0 \\ M & 0 \end{pmatrix}$ is an indecomposable object in Λ_{n+j}. It follows that

$$\begin{aligned}
\text{Ind } \Lambda_j &= \left\{ \begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix} : U \in \text{Ind } \mathcal{U}_j \right\}, \text{ if } 1 \leq j \leq n, \text{ and;} \\
\text{Ind } \Lambda_{n+j} &= \left\{ \begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix} : U \in \text{Ind } \mathcal{U} \right\} \cup \left\{ \begin{pmatrix} T & 0 \\ M & 0 \end{pmatrix} : T \in \text{Ind } \mathcal{T}_j \right\}, \text{ if } j \geq 1.
\end{aligned}$$

In this way, we have that Λ_j , for $j \geq 0$, is closed under direct summands. Moreover,

$$\text{Ind } \Lambda_j - \text{Ind } \Lambda_{j-1} = \begin{cases} \left\{ \begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix} : U \in \text{Ind } \mathcal{U}_j - \text{Ind } \mathcal{U}_{j-1} \right\}, \text{ if } 1 \leq j \leq n, \\ \\ \left\{ \begin{pmatrix} T & 0 \\ M & 0 \end{pmatrix} : T \in \text{Ind } \mathcal{T}_{j-n} - \text{Ind } \mathcal{T}_{j-n-1} \right\}, \text{ if } j > n. \end{cases}$$

One of the main results of this Section is a generalization of the Theorem 4.1.12, due to B. Zhu in (30, Theorem 3.1). The proof of such Theorem 3.4.1 will be a consequence of a series of results that are presented below.

Lemma 3.4.1. *Let $\mathcal{U}$ be a quasi-hereditary category with respect to a filtration $\{\mathcal{U}_j\}_{j\geq 0}$. Let M be a $\mathcal{U}$-module, and set*

$$M^{[j]} := \text{Tr}_{\{\mathcal{U}(U,-)\}_{U\in\mathcal{U}_j}}(M).$$

Assume that $M \in \mathcal{F}({}_{\mathcal{U}}\Delta)$. Then for all $U' \in \mathcal{U}$ we have that

$$M^{[j]}(U') = \Big\{ m : m = M(s)(m') \textit{ for } s \in \mathcal{U}(U'', U'), U'' \in \mathcal{U}_j \textit{ and } m' \in M(U'') \Big\}.$$

Proof:
($\subseteq$). By Yoneda's isomorphism $Y^{U'} : \text{Nat}\big((U',-), M^{[j]}\big) \cong M^{[j]}(U')$, $\eta \mapsto \eta_{U'}(1_{U'})$.

Let $m \in M^{[j]}(U')$, then there exists $\eta^m : \mathcal{U}(U',-) \to M^{[j]}$ such that $\eta^m_{U'}(1_{U'}) = m$. On the other hand, by Lemma 3.3.2, there exists $p : \mathcal{U}(U'',-) \to M^{[j]} \to 0$, with $U'' \in \mathcal{U}_j$ and since $\mathcal{U}(U',-)$ is a projective $\mathcal{U}$-module, there exists a morphism $s : U'' \to U'$ for which the following diagram is commutative:

$$\begin{tikzcd}
 & \mathcal{U}(U',-) \arrow[dl, dashed, "-\circ s=\mathcal{U}(s,-)"'] \arrow[d, "\eta^m"] & \\
\mathcal{U}(U'',-) \arrow[r, "p"] & M^{[j]} \arrow[r] & 0 .
\end{tikzcd}$$

Again by Yoneda's lemma we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Nat}(\mathcal{U}(U'',-),M^{[j]}) \arrow[r, "\mathrm{Nat}(\mathcal{U}(s,-),M^{[j]})"] \arrow[d, "Y^{U''}"] & \mathrm{Nat}(\mathcal{U}(U',-),M^{[j]}) \arrow[d, "Y^{U'}"] \\
M^{[j]}(U'') \arrow[r, "M^{[j]}(s)"'] & M^{[j]}(U').
\end{tikzcd} \tag{3.12}$$

Let $m' := Y^{U''}(p)$. Since $M^{[j]}$ is a subfunctor of M, we obtain the equality $M^{[j]}(s)(m') = M(s)(m')$ and hence by the above commutative diagram we have that

$$\begin{aligned} M(s)(m') = M^{[j]}(s)(m') = M^{[j]}(Y^{U''}(p)) &= Y^{U'}\Big(\mathrm{Nat}(\mathcal{U}(s,-),M^{[j]})(p)\Big) \\ &= Y^{U'}(p \circ \mathcal{U}(s,-)) \\ &= Y^{U'}(\eta^{m}) = m. \end{aligned}$$

($\supseteq$). Let $m \in M(U')$ and assume there exists $s : U'' \to U'$, with $U'' \in \mathcal{U}_j$, such that $m = M(s)(m')$ for some $m' \in M(U'')$. Then there exist two morphisms $\eta^m : \mathcal{U}(U',-) \to M$ and $p^{m'} : \mathcal{U}(U'',-) \to M$ satisfying the following: $Y^{U'}(\eta^m) = \eta^m_{U'}(1_{U'}) = m$ and $Y^{U''}(p^{m'}) = p^{m'}_{U''}(1_{U''}) = m'$. Thus, by using the diagram (3.12) (with M instead of $M^{[j]}$), we have that

$$\begin{aligned} Y^{U'}(p^{m'} \circ \mathcal{U}(s,-)) = Y^{U'}\Big(\mathrm{Nat}(\mathcal{U}(s,-),M)(p^{m'})\Big) &= M(s)\Big(Y^{U''}(p^{m'})\Big) \\ &= M(s)(m') = m. \end{aligned}$$

Since $Y^{U'}$ is an isomorphism, we conclude that $p^{m'} \circ \mathcal{U}(s,-) = \eta^m$. Note that $\mathrm{Im}(p^{m'})$ is a subfunctor of M and is generated by $\mathcal{U}(U'',-)$. Since $U'' \in \mathcal{U}_j$, $\mathrm{Im}(p^{m'})$ is contained in the largest $\mathcal{U}$-submodule of M generated by $\{\mathcal{U}(U,-) : U \in \mathcal{U}_j\}$, namely $M^{[j]}$, we obtain that $\mathrm{Im}(p^{m'}) \subset M^{[j]}$. It follows that

$$m = \eta^m_{U'}(1_{U'}) = p^{m'}_{U'} \circ \mathcal{U}(s,-)_{U'}(1_{U'}) = p^{m'}_{U'}(s) \in \mathrm{Im}(p^{m'})(U') \subset M^{[j]}(U'),$$

that is, $m \in M^{[j]}(U')$. ■

Let $\mathcal{U}$ and $\mathcal{T}$ be Hom-finite Krull-Schmidt categories and consider exhaustive filtrations $\{\mathcal{U}_j\}_{0 \leq j \leq n}$ and $\{\mathcal{T}_j\}_{j \geq 0}$ of $\mathcal{U}$ and $\mathcal{T}$, consisting of full additive subcategories which are closed under direct summands.

In the remainder of this Section, we will assume that the categories $\mathcal{U}$ and $\mathcal{T}$ are quasi-hereditary categories with respect to the filtrations $\{\mathcal{U}_j\}_{0 \leq j \leq n}$ and $\{\mathcal{T}_j\}_{j \geq 0}$, respectively, and $M_T \in \mathcal{F}({}_{\mathcal{U}}\Delta)$ for all $T \in \mathcal{T}$.

Proposition 3.4.2. *Let* $E = \begin{pmatrix} T & 0 \\ M & U \end{pmatrix}$ *and* $E' = \begin{pmatrix} T' & 0 \\ M & U' \end{pmatrix}$ *in* Λ. *Then,*

$$\begin{aligned} I_{\Lambda_j}(E,E') &= \begin{pmatrix} 0 & 0 \\ M_T^{[j]}(U') & I_{\mathcal{U}_j}(U,U') \end{pmatrix}, \text{ if } 0 \leq j \leq n, \text{ and} \\ I_{\Lambda_j}(E,E') &= \begin{pmatrix} I_{\mathcal{T}_{j-n}}(T,T') & 0 \\ M_T(U') & \mathcal{U}(U,U') \end{pmatrix}, \text{ if } j > n. \end{aligned}$$

Proof:

Let $\begin{pmatrix} f & 0 \\ m & h \end{pmatrix} \in \operatorname{Hom}_\Lambda(E,E')$. Therefore, $f \in \operatorname{Hom}_{\mathcal{T}}(T,T')$, $m \in M(U',T)$ and $h \in \operatorname{Hom}_{\mathcal{U}}(U,U')$.

Consider the case $0 \leq j \leq n$. The morphism $\begin{pmatrix} f & 0 \\ m & h \end{pmatrix}$ lies in $I_{\Lambda_j}(E,E')$ if and only if there is a commutative diagram

$$\begin{tikzcd} \begin{pmatrix} T & 0 \\ M & U \end{pmatrix} \arrow[rr, "\begin{pmatrix} f & 0 \\ m & h \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} 0 & 0 \\ m' & r \end{pmatrix}"'] & & \begin{pmatrix} T' & 0 \\ M & U' \end{pmatrix} \\ & \begin{pmatrix} 0 & 0 \\ M & U'' \end{pmatrix} \arrow[ur, "\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix}"'] & \end{tikzcd}$$

with $\begin{pmatrix} 0 & 0 \\ M & U'' \end{pmatrix} \in \Lambda_j$, $1 \leq j \leq n$. Thus, $U'' \in \mathcal{U}_j$ and $\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ m' & r \end{pmatrix} = \begin{pmatrix} f & 0 \\ m & h \end{pmatrix}$; therefore, $f = 0$, $m = s \bullet m'$ and $h = s \circ r$. It is clear that $h \in I_{\mathcal{U}_j}(U,U')$ because $U'' \in \mathcal{U}_j$.

In this way we conclude that $\begin{pmatrix} f & 0 \\ m & h \end{pmatrix} \in I_{\Lambda_j}(E,E')$ if and only if $h \in I_{\mathcal{U}_j}(U,U')$ and $m = s \bullet m' = M(s \otimes 1_T)(m') = M_T(s)(m')$ (see Definition 2.4.1) and $f = 0$ where $h : U \xrightarrow{r} U'' \xrightarrow{s} U'$ and $U'' \in \mathcal{U}_j$. In other words, $h \in I_{\mathcal{U}_j}(U,U')$ and $m \in M_T^{[j]}(U')$, by Lemma 3.4.1. Thus, $I_{\Lambda_j}(E,E') = \begin{pmatrix} 0 & 0 \\ M_T^{[j]}(U') & I_{\mathcal{U}_j}(U,U') \end{pmatrix}$.

Now, consider the case $j > n$. We have that $\begin{pmatrix} f & 0 \\ m & h \end{pmatrix} \in I_{\Lambda_j}(E,E')$ if and only if there is a commutative diagram

$$\begin{tikzcd}[ampersand replacement=\&]
\begin{pmatrix} T & 0 \\ M & U \end{pmatrix} \arrow[rr, "\begin{pmatrix} f & 0 \\ m & h \end{pmatrix}"] \arrow[dr, "\begin{pmatrix} r & 0 \\ m_1 & h_1 \end{pmatrix}"'] \& \& \begin{pmatrix} T' & 0 \\ M & U' \end{pmatrix} \\
\& \begin{pmatrix} T'' & 0 \\ M & U'' \end{pmatrix} \arrow[ur, "\begin{pmatrix} s & 0 \\ m_2 & h_2 \end{pmatrix}"'] \&
\end{tikzcd}$$

with $\begin{pmatrix} T'' & 0 \\ M & U'' \end{pmatrix} \in \Lambda_j$, for $j = n + (j - n) > n$, that is, $T'' \in \mathcal{T}_{j-n}$ and $U'' \in \mathcal{U}$.

Since $\begin{pmatrix} s & 0 \\ m_2 & h_2 \end{pmatrix} \begin{pmatrix} r & 0 \\ m_1 & h_1 \end{pmatrix} = \begin{pmatrix} f & 0 \\ m & h \end{pmatrix}$, we get that $f = s \circ r$, $m = m_2 \bullet r + h_2 \bullet m_1$ and $h = h_2 \circ h_1$; moreover, $m \in M(U', T)$, $h \in \mathcal{U}(U, U')$, and $f \in I_{\mathcal{T}_{j-n}}(T, T')$ since $T'' \in \mathcal{T}_{j-n}$. Therefore, $\begin{pmatrix} f & 0 \\ m & h \end{pmatrix} \in I_{\Lambda_j}(E, E')$ if and only if $m \in M(U', T)$, $h \in \mathcal{U}(U, U')$, and $f \in I_{\mathcal{T}_{j-n}}(T, T')$.

■

Proposition 3.4.3. *For each pair $E, E' \in \Lambda_j - \Lambda_{j-1}$, we have*

$$\operatorname{rad}_\Lambda(E, E') = I_{\Lambda_{j-1}}(E, E'), \quad \forall j \geq 0.$$

Proof:

The proof is divided in two cases.

Case $1 \leq j \leq n$.

Let $E = \begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix}$ and $E' = \begin{pmatrix} 0 & 0 \\ M & U' \end{pmatrix}$ for which $U, U' \in \operatorname{Ind} \mathcal{U}_j - \operatorname{Ind} \mathcal{U}_{j-1}$. Therefore, by Proposition 2.4.3 we can see that

$$\operatorname{rad}_\Lambda \left(\begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ M & U' \end{pmatrix} \right) = \begin{pmatrix} \operatorname{rad}_{\mathcal{T}}(0,0) & 0 \\ M(U', 0) & \operatorname{rad}_{\mathcal{U}}(U, U') \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \operatorname{rad}_{\mathcal{U}}(U, U') \end{pmatrix}.$$

On the other hand, by Theorem 3.2.1 we have $\operatorname{rad}_{\mathcal{U}}(U, U') = I_{\mathcal{U}_{j-1}}(U, U')$. Therefore, by Proposition 3.4.2 we conclude that

$$\operatorname{rad}_\Lambda \left(\begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ M & U' \end{pmatrix} \right) = I_{\Lambda_{j-1}} \left(\begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ M & U' \end{pmatrix} \right).$$

Case $j > n$.

Let $E = \begin{pmatrix} T & 0 \\ M & 0 \end{pmatrix}$ and $E' = \begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix}$ such that the objects $T, T' \in \operatorname{Ind} \mathcal{T}_{j-n} - \operatorname{Ind} \mathcal{T}_{j-n-1}$. By Proposition 2.4.3 we have

$$\mathrm{rad}_{\Lambda}\left(\begin{pmatrix} T & 0 \\ M & 0 \end{pmatrix}, \begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix}\right) = \begin{pmatrix} \mathrm{rad}_{\mathcal{T}}(T,T') & 0 \\ M(0,T) & \mathrm{rad}_{\mathcal{U}}(0,0) \end{pmatrix} = \begin{pmatrix} \mathrm{rad}_{\mathcal{T}}(T,T') & 0 \\ 0 & 0 \end{pmatrix}.$$

Again by Theorem 3.2.1 we know that $\mathrm{rad}_{\mathcal{T}}(T,T') = I_{\mathcal{T}_{j-n-1}}(T,T')$. It follows from Proposition 3.4.2 that

$$\mathrm{rad}_{\Lambda}\left(\begin{pmatrix} T & 0 \\ M & 0 \end{pmatrix}, \begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix}\right) = I_{\Lambda_{j-1}}\left(\begin{pmatrix} T & 0 \\ M & 0 \end{pmatrix}, \begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix}\right).$$

■

Remark 3.4.4. For $X = \begin{pmatrix} T & 0 \\ M & U \end{pmatrix} \in \Lambda$, we consider $I_{\Lambda_j}(X,-) \in \mathrm{Mod}(\Lambda)$. By Theorem 2.4.1, there exists $I^{(1)}_{\Lambda_j}(X,-) \in \mathrm{Mod}(\mathcal{T})$ and $I^{(2)}_{\Lambda_j}(X,-) \in \mathrm{Mod}(\mathcal{U})$ and a morphism of $\mathcal{T}$-modules $f : I^{(1)}_{\Lambda_j}(X,-) \longrightarrow \mathbb{G}(I^{(2)}_{\Lambda_j}(X,-))$ such that

$$I_{\Lambda_j}(X,-) \simeq \mathfrak{f}\Big(I^{(1)}_{\Lambda_j}(X,-), f, I^{(2)}_{\Lambda_j}(X,-)\Big).$$

As a direct result of the above proposition, we have:

Lemma 3.4.5. *Let* $X = \begin{pmatrix} T & 0 \\ M & U \end{pmatrix} \in \Lambda$. *Let us identify* $I_{\Lambda_j}(X,-)$ *with its corresponding object* $\Big(I^{(1)}_{\Lambda_j}(X,-), f, I^{(2)}_{\Lambda_j}(X,-)\Big)$ *in* $(\mathrm{Mod}(\mathcal{T}), \mathbb{G}(\mathrm{Mod}(\mathcal{U})))$. *Then the following holds:*

(i) $I^{(1)}_{\Lambda_j}(X,-) = 0$ *and* $I^{(2)}_{\Lambda_j}(X,-) \cong M^{[j]}_T \coprod I_{\mathcal{U}_j}(U,-)$, *if* $0 \leq j \leq n$;

(ii) $I^{(1)}_{\Lambda_j}(X,-) \cong I_{\mathcal{T}_{j-n}}(T,-)$ *and* $I^{(2)}_{\Lambda_j}(X,-) \cong M_T \coprod \mathcal{U}(U,-)$, *if* $j > n$.

Proof:
Let $T' \in \mathcal{T}$ and $U' \in \mathcal{U}$. The proof is divided into two cases.

Case $1 \leq j \leq n$.
By Proposition 3.4.2, we have that

$$(1) \quad I^{(1)}_{\Lambda_j}(X,T') = I_{\Lambda_j}\left(X, \begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where the first equality is by Theorem 2.4.1, the and similarly by Proposition 3.4.2 and Theorem 2.4.1 we have that

$$(2)\quad I^{(2)}_{\Lambda_j}(X,U') = I_{\Lambda_j}\left(X, \begin{pmatrix} 0 & 0 \\ M & U' \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ M^{[j]}(U') & I_{\mathcal{U}_j}(U,U') \end{pmatrix}$$
$$\cong M^{[j]}(U') \coprod I_{\mathcal{U}_j}(U,U').$$

Case $j > n$.
By Theorem 2.4.1, and Proposition 3.4.2 we have that

$$(1)\quad I^{(1)}_{\Lambda_j}(X,T') = I_{\Lambda_j}\left(X, \begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix}\right) = \begin{pmatrix} I_{\mathcal{T}_{j-n}}(T,T') & 0 \\ M^{[j]}_T(0) & 0 \end{pmatrix} \cong I_{\mathcal{T}_{j-n}}(T,T'),$$

$$(2)\quad I^{(2)}_{\Lambda_j}(X,U') = I_{\Lambda_j}\left(X, \begin{pmatrix} 0 & 0 \\ M & U' \end{pmatrix}\right) = \begin{pmatrix} I_{\mathcal{T}_{j-n}}(T,0) & 0 \\ M_T(U') & \mathcal{U}(U,U') \end{pmatrix}$$
$$\cong M_T(U') \coprod \mathcal{U}(U,U').$$

■

Proposition 3.4.6. *Let* $X = \begin{pmatrix} T & 0 \\ M & U \end{pmatrix} \in \Lambda$, *and assume* $M_T \in \mathcal{F}({}_{\mathcal{U}}\Delta)$ *for all* $T \in \mathcal{T}$. *Then, for all* $j \geq 1$ *the following exact sequence exists:*

$$\Lambda(E_{j-1},-) \longrightarrow \Lambda(E_j,-) \longrightarrow I_{\Lambda_j}(X,-) \longrightarrow 0\ ,$$

with $E_j \in \Lambda_j$ *and* $E_{j-1} \in \Lambda_{j-1}$.

Proof:
The proof is divided into two cases.

Case $1 \leq j \leq n$.
By Lemma 3.3.2 and Theorem 3.2.1, there exist exact sequences

$$\mathcal{U}(U'_{j-1},-) \longrightarrow \mathcal{U}(U'_j,-) \longrightarrow I_{\mathcal{U}_j}(U,-) \longrightarrow 0$$

$$\mathcal{U}(U''_{j-1},-) \longrightarrow \mathcal{U}(U''_j,-) \longrightarrow M^{[j]}_T(U,-) \longrightarrow 0$$

with $U'_j, U''_j \in \mathcal{U}_j$, $U'_{j-1}, U''_{j-1} \in \mathcal{U}_{j-1}$. Since in functor categories limits and colimits are computed pointwise, we have an exact sequence

$$\mathcal{U}(U_{j-1},-) \longrightarrow \mathcal{U}(U_j,-) \longrightarrow M^{[j]}_T \coprod I_{\mathcal{U}_j}(U,-) \cong I^{(2)}_{\Lambda_j}(X,-) \longrightarrow 0,$$

with $U_{j-1}=U'_{j-1}\coprod U''_{j-1}\in\mathcal{U}_{j-1}$ and $U_j=U'_j\coprod U''_j\in\mathcal{U}_j$.

By Remark 3.4.4, we get that $\mathfrak{f}\Big((I^{(1)}_{\Lambda_j}(X,-),f,I^{(2)}_{\Lambda_j}(X,-)\Big)\simeq I_{\Lambda_j}(X,-)$, and by Lemma 3.4.5 we have that $I^{(1)}_{\Lambda_j}(X,-)=0$ and hence $\mathfrak{f}\Big((I^{(1)}_{\Lambda_j}(X,-),0,I^{(2)}_{\Lambda_j}(X,-)\Big)\simeq I_{\Lambda_j}(X,-)$.

It follows by Proposition 2.4.4, that there exists a exact sequence of Λ-modules

$$\Lambda\left(\begin{pmatrix}0&0\\M&U_{j-1}\end{pmatrix},-\right)\to\Lambda\left(\begin{pmatrix}0&0\\M&U_{j}\end{pmatrix},-\right)\to\mathfrak{f}\Big((I^{(1)}_{\Lambda_j}(X,-),0,I^{(2)}_{\Lambda_j}(X,-)\Big)\to 0$$

where $\begin{pmatrix}0&0\\M&U_{j-1}\end{pmatrix}\in\Lambda_{j-1}$ and $\begin{pmatrix}0&0\\M&U_{j}\end{pmatrix}\in\Lambda_{j}$.

Case $j>n$.

Since $M_T\in\mathcal{F}({}_{\mathcal{U}}\Delta)$, M_T is a finitely presented $\mathcal{U}$-module, then $M_T\coprod\mathcal{U}(U,-)\cong I^{(2)}_{\Lambda_j}(X,-)$ is finitely presented. Then, there exists an exact sequence of $\mathcal{U}$-modules

$$\begin{tikzcd}\mathcal{U}(U'',-)\arrow[r]&\mathcal{U}(U',-)\arrow[r]&I^{(2)}_{\Lambda_j}(X,-)\arrow[r]&0.\end{tikzcd}$$

On the other hand, by Theorem 3.2.1, there exists an exact sequence of $\mathcal{T}$-modules

$$\begin{tikzcd}\mathcal{T}(T_{j-n-1},-)\arrow[r]&\mathcal{T}(T_{j-n},-)\arrow[r]&I_{\mathcal{T}_{j-n}}(T,-)\arrow[r]&0\end{tikzcd}$$

with $T_{j-n}\in\mathcal{T}_{j-n}$ and $T_{j-n-1}\in\mathcal{T}_{j-n-1}$. Thus, we get an exact sequence

$$\Lambda\left(\begin{pmatrix}T_{j-n-1}&0\\M&U''\end{pmatrix},-\right)\longrightarrow\Lambda\left(\begin{pmatrix}T_{j-n}&0\\M&U'\end{pmatrix},-\right)\longrightarrow\mathfrak{f}\Big((I^{(1)}_{\Lambda_j},f,I^{(2)}_{\Lambda_j})\Big)\longrightarrow 0,$$

where $\mathfrak{f}((I^{(1)}_{\Lambda_j},f,I^{(2)}_{\Lambda_j}))\cong I_{\Lambda_j}(X,-)$ such that $\begin{pmatrix}T_{j-n-1}&0\\M&U''\end{pmatrix}\in\Lambda_{j-1}$ and $\begin{pmatrix}T_{j-n}&0\\M&U'\end{pmatrix}\in\Lambda_j$. ■

Theorem 3.4.1. *Let $\mathcal{U}$ and $\mathcal{T}$ be* Hom*-finite Krull-Schmidt categories and consider exhaustive filtrations $\{\mathcal{U}_j\}_{0\leq j\leq n}$ and $\{\mathcal{T}_j\}_{j\geq 0}$ of $\mathcal{U}$ and $\mathcal{T}$, consisting of full additive subcategories which are closed under direct summands. Suppose that $\mathcal{U}$ and $\mathcal{T}$ are quasi-hereditary categories with respect to the filtrations $\{\mathcal{U}_j\}_{0\leq j\leq n}$ and $\{\mathcal{T}_j\}_{j\geq 0}$ respectively. Assume that $M_T\in\mathcal{F}({}_{\mathcal{U}}\Delta)$ for all $T\in\mathcal{T}$. Then $\Lambda=\begin{pmatrix}\mathcal{T}&0\\M&\mathcal{U}\end{pmatrix}$ is quasi-hereditary with respect to the filtration $\{\Lambda_j\}_{j\geq 0}$ given in Equation (3.11).*

Proof:

It follows from Theorem 3.2.1; and Propositions 3.4.3 and 3.4.6. ■

Remark *3.4.7.* Now, let us explain why Theorem 3.4.1 is a generalization of the Theorem 4.1.12 given by B. Zhu in (30). Suppose that U and T are quasi-hereditary algebras and that ${}_UM_T$ is a bimodule such that ${}_UM \in \mathcal{F}({}_U\Delta)$. Consider the triangular matrix algebra $A := \begin{pmatrix} T & 0 \\ M & U \end{pmatrix}$.

By Remark 3.2.4(a), we get that $\mathrm{proj}(U)$ and $\mathrm{proj}(T)$ are quasi-hereditary categories in the sense of the Definition 3.2.1. Since ${}_UM \in \mathcal{F}({}_U\Delta)$, we can see that M satisfies the hypothesis in Theorem 3.4.1. Thus, by Theorem 3.4.1, we get that $\begin{pmatrix} \mathrm{proj}(T) & 0 \\ M & \mathrm{proj}(U) \end{pmatrix}$ is a quasi-hereditary category. Now, by using (5, Proposition 2.3) in p. 75, we have that $\mathrm{proj}(A) = \begin{pmatrix} \mathrm{proj}(T) & 0 \\ M & \mathrm{proj}(U) \end{pmatrix}$. Therefore, by Remark 3.2.4(a), we obtain that A is a quasi-hereditary algebra.

3.5 The standard modules in $\mathrm{Mod}(\Lambda)$ and $\mathcal{F}({}_{\Lambda}\Delta)$.

We will proceed from the premise that $\mathcal{U}$ and $\mathcal{T}$ are Hom-finite Krull-Schmidt and quasi-hereditary $K-$categories with respect to filtrations $\{\mathcal{U}_j\}_{0\leq j\leq n}$ and $\{\mathcal{T}_j\}_{j\geq 0}$. If $M : \mathcal{U} \otimes_K \mathcal{T}^{op} \to \mathrm{mod}(K)$ is a functor such that $M_T = M(-,T) : \mathrm{Mod}(\mathcal{U}) \to \mathrm{mod}(K)$ is finitely presented $\mathcal{U}$-module, then by Theorem 3.4.1, the triangular matrix category $\Lambda = \begin{pmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{pmatrix}$ is a quasi-hereditary K-category with respect to some filtration $\{\Lambda_j\}_{j\geq 0}$.

In this part, we study the relation between the full standard subcategories ${}_{\mathcal{U}}\Delta$, ${}_{\mathcal{T}}\Delta$, and ${}_{\Lambda}\Delta$ of $\mathrm{Mod}(\mathcal{U})$, $\mathrm{Mod}(\mathcal{T})$ and $\mathrm{Mod}(\Lambda)$, respectively. More concretely, we will show in Theorem 3.5.1, by using the notation of Theorem 2.4.1, that

$$\mathcal{F}_f({}_{\Lambda}\Delta) = \Big\{(F^{(1)}, g, F^{(2)}) : F^{(1)} \in \mathcal{F}_f({}_{\mathcal{T}}\Delta) \text{ and } F^{(2)} \in \mathcal{F}_f({}_{\mathcal{U}}\Delta)\Big\};$$

see Theorem 4.1.12 given by Zhu in (30).

Recall that given an abelian category $\mathcal{A}$ and $\mathcal{X} \subseteq \mathcal{A}$. We denote by $\mathcal{X}^{\amalg}$ the class of objects of $\mathcal{A}$, which are a finite direct sum of objects in $\mathcal{X}$. We say that $M \in \mathcal{A}$ is $\mathcal{X}$-**filtered** if there exists a chain $\{M_j\}_{j\geq 0}$ of subobjects of M such that $M_{j+1}/M_j \in \mathcal{X}^{\amalg}$ for $j \geq 0$. In case $M = M_n$ for some $n \in \mathbb{N}$, we say that M has a finite $\mathcal{X}$-filtration. We denote by $\mathcal{F}(\mathcal{X})$ the class of objects that are $\mathcal{X}$-filtered and by $\mathcal{F}_f(\mathcal{X})$ the class of objects that have a finite filtration.

Regardless, we need the following result.

Proposition 3.5.1. *The functor* $\mathfrak{f} : (\mathrm{Mod}(\mathcal{T}), \mathbb{G}(\mathrm{Mod}(\mathcal{U}))) \to \mathrm{Mod}(\Lambda)$ *induce equivalences of full subcategories:*

$$(0, 0, {}_{\mathcal{U}}\Delta(j)) \longleftrightarrow {}_{\Lambda}\Delta(j), \quad \text{if } 1 \leq j \leq n, \quad \text{and,}$$

$$({}_{\mathcal{T}}\Delta(j-n), 0, 0) \longleftrightarrow {}_{\Lambda}\Delta(j), \quad \text{if } j > n.$$

Proof:
First, for $E \in \mathrm{Ind}\,\Lambda_j - \mathrm{Ind}\,\Lambda_{j-1}$, consider ${}_{\Lambda}\Delta_E = \frac{\Lambda(E,-)}{I_{\Lambda_{j-1}}(E,-)}$. Then there exists a triple $(\Delta_E^{(1)}, g, \Delta_E^{(2)})$ such that $\mathfrak{f}\Big((\Delta_E^{(1)}, g, \Delta_E^{(2)})\Big) = {}_{\Lambda}\Delta_E$ where $\Delta_E^{(1)} : \mathcal{T} \to \mathbf{Ab}$ and $\Delta_E^{(2)} : \mathcal{U} \to \mathbf{Ab}$. Now, we consider two cases.

Case $1 \leq j \leq n$.
Let $T' \in \mathcal{T}$ and $E = \begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix} \in \mathrm{Ind}\,\Lambda_j - \mathrm{Ind}\,\Lambda_{j-1}$, with $U \in \mathrm{Ind}\,\mathcal{U}_j - \mathrm{Ind}\,\mathcal{U}_{j-1}$. Then

$${}_{\Lambda}\Delta_E^{(1)}(T') = \frac{\Lambda\left(\begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix}, \begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix}\right)}{I_{\Lambda_{j-1}}\left(\begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix}, \begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix}\right)} \cong 0.$$

On the other hand, if $U' \in \mathcal{U}$, we get

$${}_{\Lambda}\Delta_E^{(2)}(U') = \frac{\Lambda\left(\begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ M' & U \end{pmatrix}\right)}{I_{\Lambda_{j-1}}\left(\begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ M & U' \end{pmatrix}\right)} \cong \frac{\mathcal{U}(U, U')}{I_{\mathcal{U}_{j-1}}(U, U')}.$$

In this way, ${}_{\Lambda}\Delta_E^{(1)} \cong 0$ and ${}_{\Lambda}\Delta_E^{(2)} \cong \frac{\mathcal{U}(U,-)}{I_{\mathcal{U}_{j-1}}(U,-)} = {}_{\mathcal{U}}\Delta_U$, where the object U belongs to $\mathrm{Ind}\,\mathcal{U}_j - \mathrm{Ind}\,\mathcal{U}_{j-1}$.

Case $j > n$.
Let $T' \in \mathcal{T}$ and $E = \begin{pmatrix} T & 0 \\ M & 0 \end{pmatrix}$ with $T \in \mathrm{Ind}\,\mathcal{T}_{j-n} - \mathrm{Ind}\,\mathcal{T}_{j-n-1}$. Thus we obtain that:

$${}_{\Lambda}\Delta_E^{(1)}(T') = \frac{\Lambda\left(\begin{pmatrix} T & 0 \\ M & 0 \end{pmatrix}, \begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix}\right)}{I_{\Lambda_{j-1}}\left(\begin{pmatrix} T & 0 \\ M & 0 \end{pmatrix}, \begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix}\right)} \cong \frac{\mathcal{T}(T, T')}{I_{\mathcal{T}_{j-n-1}}(T, T')}.$$

If $U' \in \mathcal{U}$, we obtain

$${}_\Lambda\Delta_E^{(2)}(U') = \frac{\Lambda\left(\begin{pmatrix} T & 0\\ M & 0\end{pmatrix}, \begin{pmatrix} 0 & 0\\ M & U'\end{pmatrix}\right)}{I_{\Lambda_{j-1}}\left(\begin{pmatrix} T & 0\\ M & 0\end{pmatrix}, \begin{pmatrix} 0 & 0\\ M & U'\end{pmatrix}\right)} \cong 0.$$

Therefore, ${}_\Lambda\Delta_E^{(1)} \cong \frac{\mathcal{T}(T,-)}{I_{\mathcal{T}_{j-n-1}}(T,-)} \cong {}_{\mathcal{T}}\Delta_T$, with $T \in \mathrm{Ind}\,\mathcal{T}_{n-j} - \mathrm{Ind}\,\mathcal{T}_{n-j-1}$ and ${}_\Lambda\Delta_E^{(2)} \cong 0$. ■

Theorem 3.5.1. *Let* $F = \mathfrak{f}\big(F^{(1)}, g, F^{(2)}\big) \in \mathrm{mod}(\Lambda)$*, and consider its trace filtration* $\{F^{[j]}\}_{j\geq 0} = \Big\{\Big((F^{[j]})^{(1)}, g^{[j]}, (F^{[j]})^{(2)}\Big)\Big\}_{j\geq 0}$ *with respect to* $\{\Lambda_j\}_{j\geq 0}$*. Then the following conditions hold.*

(i) $(F^{[j]})^{(1)} \cong 0$ *if* $0 \leq j \leq n$*, and* $(F^{[j]})^{(2)} \cong F^{(2)}$ *if* $j > n$*.*

(ii) If $F = \mathfrak{f}\big((F^{(1)}, g, F^{(2)})\big) \in \mathcal{F}({}_\Lambda\Delta)$*, then we have that* $F^{(1)} \in \mathcal{F}({}_{\mathcal{T}}\Delta)$ *and* $F^{(2)} \in \mathcal{F}({}_{\mathcal{U}}\Delta)$*.*

(iii) $\mathcal{F}_f({}_\Lambda\Delta) = \Big\{\mathfrak{f}\big(F^{(1)}, g, F^{(2)}\big) : F^{(1)} \in \mathcal{F}_f({}_{\mathcal{T}}\Delta)$ *and* $F^{(2)} \in \mathcal{F}_f({}_{\mathcal{U}}\Delta)\Big\}$*.*

Proof:

(i) Since $F \in \mathrm{mod}(\Lambda)$, we have an exact sequence of Λ-modules

$$(X', -) \to (X, -) \to F \to 0 \tag{3.13}$$

with $X' = \begin{pmatrix} T' & 0\\ M & U'\end{pmatrix}$ and $X = \begin{pmatrix} T & 0\\ M & U\end{pmatrix}$.

By applying $\mathrm{Tr}_{\{\Lambda(E,-)\}_{E\in\Lambda_j}}(-)$ to the previous exact sequence and using that each $\Lambda(E,-)$ is a projective Λ-module and by Lemma 3.1.5, we get an exact sequence

$$I_{\Lambda_j}(X', -) \to I_{\Lambda_j}(X, -) \to F^{[j]} \to 0. \tag{3.14}$$

By Proposition 2.4.5 and Theorem 2.4.1, we identify the exact sequence in (3.13) with the following exact sequence in the comma category $(\mathrm{Mod}(\mathcal{T}), \mathbb{G}(\mathrm{Mod}(\mathcal{U})))$:

$$\begin{tikzcd} \big(\mathcal{T}(T',-), h', M_{T'} \amalg \mathcal{U}(U',-)\big) \arrow[d] & & \\ \big(\mathcal{T}(T,-), h, M_T \amalg \mathcal{U}(U,-)\big) \arrow[r] & \big(F^{(1)}, g, F^{(2)}\big) \arrow[r] & 0. \end{tikzcd}$$

By Proposition 2.4.6, we have the exact sequence in $\mathrm{Mod}(\mathcal{U})$:

$$M_{T'} \amalg \mathcal{U}(U',-) \to M_T \amalg \mathcal{U}(U,-) \to F^{(2)} \to 0. \tag{3.15}$$

Secondly, by Theorem 2.4.1, we identify the exact sequence in (3.14) with the exact sequence in the comma category $(\mathrm{Mod}(\mathcal{T}), \mathbb{G}(\mathrm{Mod}(\mathcal{U})))$:

$$\begin{tikzcd} \big(I^{(1)}_{\Lambda_j}(X',-), p', I^{(2)}_{\Lambda_j}(X',-)\big) \arrow[d] & & \\ \big(I^{(1)}_{\Lambda_j}(X,-), p, I^{(2)}_{\Lambda_j}(X,-)\big) \arrow[r] & \big((F^{[j]})^{(1)}, g^{[j]}, (F^{[j]})^{(2)}\big) \arrow[r] & 0. \end{tikzcd}$$

By Proposition 2.4.6, we get exact sequences

$$I^{(k)}_{\Lambda_j}(X',-) \to I^{(k)}_{\Lambda_j}(X,-) \to (F^{[j]})^{(k)} \to 0, \text{ for } k = 1, 2,$$

in $\mathrm{Mod}(\mathcal{T})$ and $\mathrm{Mod}(\mathcal{U})$.

By Lemma 3.4.5(i), we have that $I^{(1)}_{\Lambda_j}(X',-) \cong 0$ and $I^{(1)}_{\Lambda_j}(X,-) \cong 0$ if $0 \leq j \leq n$; therefore, $(F^{[j]})^{(1)} \cong 0$ for $0 \leq j \leq n$.

If $j > n$, by Lemma 3.4.5(ii), we have $I^{(2)}_{\Lambda_j}(X',-) \cong M_{T'} \amalg \mathcal{U}(U',-)$ and $I^{(2)}_{\Lambda_j}(X,-) \cong M_T \amalg \mathcal{U}(U,-)$. By the exact sequence (3.15), we get $(F^{[j]})^{(2)} \cong F^{(2)}$ if $j > n$.

(ii). If $F \in \mathcal{F}(\Delta)$, we have that $F^{[j]}/F^{[j-1]}$ is a sum of copies of elements in ${}_{\Lambda}\Delta(j)$ and hence by using item (i) we have that

$$\mathfrak{f}\left(\frac{F^{[j]}}{F^{[j-1]}}\right) \cong \begin{cases} \left(0, 0, \dfrac{(F^{[j]})^{(2)}}{(F^{[j-1]})^{(2)}}\right), & \text{if } 1 \leq j \leq n; \\ \left(\dfrac{(F^{[j]})^{(1)}}{(F^{[j-1]})^{(1)}}, 0, 0\right), & \text{if } j > n. \end{cases}$$

Then (ii) follows by Proposition 3.5.1.

(iii) Assume that $F = \mathfrak{f}\big(F^{(1)}, g, F^{(2)}\big) \in \mathcal{F}_f({}_{\Lambda}\Delta)$. By (ii), it only remains to prove that if $F^{(1)} \in \mathcal{F}_f({}_{\mathcal{T}}\Delta)$ and $F^{(2)} \in \mathcal{F}_f({}_{\mathcal{U}}\Delta)$, then $F \in \mathcal{F}_f({}_{\Lambda}\Delta)$.

In fact, the Λ-modules $\mathfrak{f}\big(F^{(1)}, 0, 0\big)$ and $\mathfrak{f}\Big(0, 0, F^{(2)}\Big)$ are in $\mathcal{F}_f({}_{\Lambda}\Delta)$ by Proposition 3.5.1. It follows that we have a short exact sequence

$$0 \to \mathfrak{f}\big(F^{(1)}, 0, 0\big) \to \mathfrak{f}\big(F^{(1)}, g, F^{(2)}\big) \to \mathfrak{f}\big(0, 0, F^{(2)}\big) \to 0\ .$$

Thus, $\mathfrak{f}\left(F^{(1)}, g, F^{(2)}\right)$ is in $\mathcal{F}_f({}_{\Lambda}\Delta)$ since $\mathcal{F}_f({}_{\Lambda}\Delta)$ is closed under extensions by Remark 3.3.1. ■

3.6 An Important Example

To visualize the results obtained in this chapter, we want to apply them in a final example, hoping that it will help to better understand what has been presented previously.

Example 3.6.1. Consider the following infinite quivers

$$\begin{tikzcd} R: 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\beta_1"] \arrow[r, bend left, "\alpha_2"] & 3 \arrow[l, bend left, "\beta_2"] \arrow[r, bend left, "\alpha_3"] & \cdots \arrow[l, bend left, "\beta_3"] \end{tikzcd} \quad .$$

$$\begin{tikzcd} Q: 1' & 2' \arrow[l, "\gamma_1"'] \arrow[r, "\gamma_2"] & 3' & 4' \arrow[l, "\gamma_3"'] \arrow[r, "\gamma_4"] & 5' & \cdots \end{tikzcd} \quad .$$

Let $\mathcal{U} = K\mathcal{Q}$ and $\mathcal{T} = K\mathcal{R}/\mathcal{J}$ be the path categories of the above quivers, where $\mathcal{J}$ is the ideal in $K\mathcal{R}$ generated by the set of relations

$$\{\beta_1\alpha_1 \text{ and } \alpha_{t+1}\alpha_t, \beta_t\beta_{t+1}, \alpha_t\beta_t - \beta_{t+1}\alpha_{t+1}, t \geq 1\}. \tag{3.16}$$

First, we will see that $\mathcal{T}$ and $\mathcal{U}$ are quasi-hereditary categories.

Set $\mathcal{T}_0 = \{0\}$, and let $\mathcal{T}_j = \text{add}\{t : 1 \leq t \leq j\}$, for $j \geq 1$. Therefore,

$$\{0\} = \mathcal{T}_0 \subset \mathcal{T}_1 \subset T_2 \subseteq \cdots\cdots$$

is a filtration of $\mathcal{T}$ into additively closed subcategories.

(i) It is clear that $\text{rad}_{\mathcal{T}}(1,1) = 0$ because $\beta_1\alpha_1 = 0$. Since we have that $\text{Ind}\mathcal{T}_j - \text{Ind}\mathcal{T}_{j-1} = \{j\}$ for all $j \geq 1$, we conclude the following: $\text{rad}_{\mathcal{T}}(j,j) = (\beta_j\alpha_j) = (\alpha_{j-1}\beta_{j-1}) = I_{\mathcal{T}_{j-1}}(j,j)$.

(ii) $I_{\mathcal{T}_1}(1,-) \cong \mathcal{T}(1,-)$, $I_{\mathcal{T}_1}(2,-) \cong \mathcal{T}(1,-)$ and $I_{\mathcal{T}_1}(j,-) = 0$, if $j \geq 3$.

For $j \geq 2$, we can readily check that there exists an exact sequence

$$0 \to I_{\mathcal{T}_{j-1}}(j,-) \to \mathcal{T}(j,-) \to I_{\mathcal{T}_j}(j+1,-) \to 0$$

and $I_{\mathcal{T}_j}(j+t,-) = 0$ if $t \geq 2$.

$$\begin{tikzcd}
 & & & & & 0 \arrow[d] \\
\cdots & K \arrow[r, bend left, "1"] \arrow[d, "1"] & K \arrow[l, bend left, "1"] \arrow[r, bend left] \arrow[d, "\begin{bmatrix}0\\1\end{bmatrix}"] & 0 \arrow[l, bend left] \arrow[d] & \cdots & I_{\mathcal{T}_{j-1}}(j,-) \arrow[d] \\
\cdots & K \arrow[r, bend left, "\begin{bmatrix}0\\1\end{bmatrix}"] \arrow[d, "0"] & K^2 \arrow[l, bend left, "\begin{bmatrix}1 & 0\end{bmatrix}"] \arrow[r, bend left, "\begin{bmatrix}1 & 0\end{bmatrix}"] \arrow[d, "\begin{bmatrix}1 & 0\end{bmatrix}"] & K \arrow[l, bend left, "\begin{bmatrix}0\\1\end{bmatrix}"] \arrow[d, "1"] & \cdots & \mathcal{T}(j,-) \arrow[d] \\
\cdots & 0 \arrow[r, bend left] & K \arrow[l, bend left] \arrow[r, bend left, "1"] & K \arrow[l, bend left, "0"] & \cdots & I_{\mathcal{T}_j}(j+1,-) \arrow[d] \\
 & j-1 & j & j+1 & & 0
\end{tikzcd}$$

By Theorem 3.2.1, we have that $\mathcal{T}$ is quasi-hereditary. Now, we define the following subcategories $\mathcal{U}_0 = \{0\}$, $\mathcal{U}_1 = \operatorname{add}\{j' \in \mathbb{N} : j \text{ is odd } \}$ and consider $\mathcal{U}_2 = \operatorname{add}\{j' : j \in \mathbb{N}\} = K\mathcal{Q}$. Thus $K\mathcal{Q}$ is quasi-hereditary with respect to the finite filtration

$$\{0\} = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 = K\mathcal{Q}.$$

The condition (i) in Definition 3.2.1, clearly holds since $\operatorname{rad}_{\mathcal{U}}(E, E') = I_{\mathcal{U}_{j-1}}(E, E') = 0$ for all pairs $E, E' \in \operatorname{Ind} \mathcal{U}_j - \operatorname{Ind} \mathcal{U}_{j-1}$ and $j = 1, 2$. On the other hand, $I_{\mathcal{U}_1}(j, -) = \mathcal{U}(j, -)$ if j is odd and $I_{\mathcal{U}_1}(j, -) \cong \mathcal{U}(j-1, -) \oplus \mathcal{U}(j+1, -) = \mathcal{U}((j-1) \oplus (j+1), -)$ if j is even. Then in this case we have that condition (ii) in Theorem 3.2.1 also holds and hence $K\mathcal{Q}$ is quasi-hereditary.

Now, by a result of Z. Leszczyński given in (17, Lemma 1.3), we have that $(K\mathcal{Q}) \otimes_K (K\mathcal{R}/\mathcal{J})^{op} \simeq K(\mathcal{Q} \times \mathcal{R}^{op})/0\square\mathcal{J}$ where $K(\mathcal{Q} \times \mathcal{R}^{op})$ is the product quiver given in p. 145 in (17); and $0\square\mathcal{J}$ is generated by the sets of relations $Q_0 \times \mathcal{J}$ and

$$\Big\{(\gamma, t(\alpha))(s(\gamma), \alpha) - (t(\gamma), \alpha)(\gamma, s(\alpha)), (\gamma, t(\beta))(s(\gamma), \beta) - (t(\gamma), \beta)(\gamma, s(\beta))\Big\},$$

where $\gamma \in Q_1$ and $\alpha, \beta \in R_1$. A functor $M : K\mathcal{Q} \otimes K(\mathcal{R}/\mathcal{J})^{op} \to \operatorname{mod} K$ can be identified with a functor $M : K(\mathcal{Q} \times \mathcal{R}^{op})/0\square\mathcal{J} \to \text{mod } K$.

In this example, we consider M as the following representation on the right below:

$$\begin{tikzcd}
\vdots & \vdots & \\
(1',3) \arrow[d, bend right, "1\otimes\alpha_2^{op}"'] & (2',3) \arrow[l, "\gamma_1\otimes 1"'] \arrow[r, "\gamma_2\otimes 1"] \arrow[d, bend right, "1\otimes\alpha_2^{op}"'] & \cdots \\
(1',2) \arrow[u, bend right, "1\otimes\beta_2^{op}"'] \arrow[d, bend right, "1\otimes\alpha_1^{op}"'] & (2',2) \arrow[l, "\gamma_1\otimes 1"'] \arrow[r, "\gamma_2\otimes 1"] \arrow[u, bend right, "1\otimes\beta_2^{op}"'] \arrow[d, bend right, "1\otimes\alpha_1^{op}"'] & \cdots \\
(1',1) \arrow[u, bend right, "1\otimes\beta_1^{op}"'] & (2',1) \arrow[l, "\gamma_1\otimes 1"'] \arrow[r, "\gamma_2\otimes 1"] \arrow[u, bend right, "1\otimes\beta_1^{op}"'] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
0 \arrow[d, bend right] & 0 \arrow[l] \arrow[r] \arrow[d, bend right] & 0\cdots \arrow[d, bend right] \\
K \arrow[u, bend right] \arrow[d, bend right, "\begin{bmatrix}0\\1\end{bmatrix}"'] & 0 \arrow[l] \arrow[r] \arrow[u, bend right] \arrow[d, bend right] & 0\cdots \arrow[u, bend right] \arrow[d, bend right] \\
K^2 \arrow[u, bend right, "\begin{bmatrix}1 & 0\end{bmatrix}"'] & 0 \arrow[l] \arrow[r] \arrow[u, bend right] & 0\cdots \arrow[u, bend right]
\end{tikzcd}$$

We see that $M(1',1) = K^2$, $M(1',2) = K$. Thus, the functor M satisfies that $M_T : K\mathcal{Q} \to \text{mod } K$ is projective for all T since we have that $M_1 \cong \mathcal{U}(1,-)^2$, $M_2 \cong \mathcal{U}(1,-)$, and $M_t \cong 0$, for all $t > 2$, which are all in $\mathcal{F}({}_{\mathcal{U}}\Delta)$ because $\mathcal{U}$ is quasi-hereditary.

In this way, using results in (17), we can see that the matrix category $\begin{pmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{pmatrix}$ is equivalent to the path category of the quiver Q'' given below, modulo the ideal generated by the set of relations (3.16) and is quasi-hereditary with respect to the filtration

$$\{0\} = \Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \cdots ,$$

where $\Lambda_1 = \mathcal{U}_1$, $\Lambda_2 = \mathcal{U}_2$ and $\Lambda_{j+2} = \text{add}\left(\{j' : j \in \mathbb{N}\} \cup \{t \in \mathbb{N} : 1 \leq t \leq j\}\right)$ and

$$\begin{tikzcd}
Q'' : & \cdots \arrow[r, "\gamma_3"'] & 3' & 2' \arrow[l, "\gamma_2"'] \arrow[r, "\gamma_1"'] & 1' & 1 \arrow[l, "\varphi"'] \arrow[r, bend left, "\alpha_1"] & {} \arrow[l, bend left, "\beta_1"] \arrow[r, bend left, "\alpha_2"] & 3 \arrow[l, bend left, "\beta_2"] \arrow[r, bend left, "\alpha_3"] & \cdots \arrow[l, bend left, "\beta_3"] & .
\end{tikzcd}$$

Chapter 4

Appendices

4.1 Quasi-Hereditary Algebras

The objective of this chapter is to present Theorem 4.1.12, given by B. Zhu in (30, Theorem 3.1), which motivated the development of the main results presented in this thesis in the previous chapter. The results of sections 4.1.1 and 4.1.2 are classical and were developed by C.M. Ringel in (26).

4.1.1 Filtered Modules

Let R be a commutative artin ring and A an artin algebra defined over R. If R is a field K, then A is a finite$-$dimensional algebra over K. The category of all $A-$ modules will be denoted by mod$A$, and all subcategories considered will be full and closed under isomorphisms.

Definition 4.1.1. Given a class Θ of objects in mod$A$, we denote by $\mathcal{F}(\Theta)$ the full subcategory of mod$A$ whose objects are the A-modules that are Θ-filtered; that is, those $M \in \text{mod}A$ for which there exists a Θ-filtration ξ of M, that is, a finite chain

$$\xi : 0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_m = M$$

of submodules of M such that $\forall i = 1, \ldots, m$, $\frac{M_i}{M_{i-1}}$ is isomorphic to a module from Θ. We will call each quotient $\frac{M_i}{M_{i-1}}$ a composition factor of the Θ-filtration ξ of M.

The modules in $\mathcal{F}(\Theta)$ are called $\Theta-$good modules, and the category Θ is called the $\Theta-$good module category. Also, let us observe that $0 \in \mathcal{F}(\Theta)$ with the trivial filtration $0 = M_0$ and, for a class $\Theta = \emptyset$, we define $\mathcal{F}(\Theta) := \{0\}$.

Definition 4.1.2. Let A be an artin algebra and consider $\Theta = \{\theta(1), ..., \theta(n)\} \subset \text{mod}A$ such that $\text{Ext}^1(\theta(j), \theta(i)) = 0$, $\forall j \geq i$. We denote:

i) $\mathcal{F}(\Theta)$ the class of those $M \in \text{mod}A$ for which there exists a chain of submodules with $\frac{M_i}{M_{i-1}} \cong \theta(k) \in \Theta$; that is, M possesses a filtration with quotients in Θ.

ii) $\mathcal{X}(\Theta)$ the subcategory of mod$A$ of the modules that are direct summands of modules in $\mathcal{F}(\Theta)$.

Let's observe that $\mathcal{F}(\Theta)$ is closed under extensions, that is, it is contained in $\mathcal{X}(\Theta)$; however, $\mathcal{F}(\Theta)$ is not necessarily closed under direct summands.

Definition 4.1.3. A full subcategory $\mathcal{X}$ of mod$A$ is called contravariantly finite in mod$A$ if for every module C, there exists a **$\mathcal{X}$-right approximation**, that is, there exists a morphism $f : X \to C$ with $X \in \mathcal{X}$ such that the induced sequence $\text{Hom}_A(X', X) \to \text{Hom}_A(X', C) \to 0$ is exact for all $X' \in \mathcal{X}$. An $\mathcal{X}$-right approximation $f : X \to C$ is said to be a **minimal $\mathcal{X}$-right approximation** if the restriction of f to any non-zero direct summand of X' is non-zero. Dually, the notions of $\mathcal{X}$-left approximation and covariantly finite category are defined. A category that is both covariantly finite and contravariantly finite is called **functorially finite**.

Let $\mathcal{X}$ be a full subcategory of mod$A$. Then we will denote by $\mathcal{Y}$ the full subcategory of mod$A$ of all modules Y that satisfy $Ext^1_A(X, Y) = 0$ for all $X \in \mathcal{X}$.

Lemma 4.1.4. *Let* $0 \to Y \to X \xrightarrow{\gamma} M \to 0$ *be an exact sequence, with* $X \in \mathcal{X}$, $Y \in \mathcal{Y}$. *Then* γ *is a right* $\mathcal{X}$*-approximation.*

Proof:
That γ is a right $\mathcal{X}$-approximation directly follows from applying $Hom_A(X^{'}, -)$ to the sequence for all $X^{'} \in \mathcal{X}$. ■

Lemma 4.1.5. *Assume that* $\mathcal{X}$ *is closed under extensions, that is, if* $0 \to N \to L \to M \to 0$ *is an exact sequence with* $M, N \in \mathcal{X}$, *then* $L \in \mathcal{X}$. *If for every* A*-module* N *there exists an exact sequence* $0 \to N \to Y^N \to X^N \to 0$ *with* $X^N \in \mathcal{X}$ *and* $Y^N \in \mathcal{Y}$. *Then every* A*-module* M *has a right* $\mathcal{X}$*-approximation.*

Proof:

Let M be an A-module. First, assume there exists an epimorphism $\pi : X \to M$ with $X \in \mathcal{X}$; let $K = \ker \pi$. The exact sequence $0 \to K \to Y^K \to X^K \to 0$ results in the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & Y^K \arrow[r] \arrow[d] & X^K \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d, "\pi"] & Z \arrow[r] \arrow[d, "\gamma"] & X^K \arrow[r] & 0 \\
 & M \arrow[r, equal] \arrow[d] & M \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $X, X^K \in \mathcal{X}$ and $\mathcal{X}$ is closed under extensions, then $Z \in \mathcal{X}$. Since $Y^K \in \mathcal{Y}$ using Lemma 4.1.4, we see that in the central column, $\gamma : Z \to M$ is a right $\mathcal{X}$-approximation.

In general, let M' be the submodule of M generated by the images of the morphisms $X' \to M$ with $X' \in \mathcal{X}$. There exists a finite set of such morphisms $\pi_i : X_i \to M$, with $X_i \in \mathcal{X}$ such that their images generate M'. Since $\mathcal{X}$ is closed under direct sums, we form $X = \oplus_i X_i \in \mathcal{X}$, and there exists an epimorphism $\pi : X \to M'$.

The previous considerations give rise to a right $\mathcal{X}$-approximation $\gamma' : Z \to M'$. Let $\mu : M' \to M$ be the inclusion, then the morphism $\gamma' \mu$ is a right $\mathcal{X}$-approximation. ■

Let now $\mathcal{X} = \mathcal{F}(\Theta)$. Then $\mathcal{Y} = Y(\Theta)$ can be alternatively characterized as the full subcategory of mod$A$ of all modules Y that satisfy $Ext^1_A(\Theta(i), Y) = 0$ for $1 \leq i \leq n$.

Lemma 4.1.6. *Let* $1 \leq i \leq n$. *Let* N *be an* A*-module with* $Ext^1_A(\Theta(j), N) = 0$ *for* $j > i$. *Then there exists an exact sequence* $0 \to N \to N' \to Q \to 0$ *with* Q *a direct sum of copies of* $\Theta(i)$ *and* $Ext^1_A(\Theta(j), N') = 0$ *for all* $j \geq i$.

Proof:

Let us take $\epsilon = (0 \to N \to N' \to Q \to 0)$ to be a universal extension of copies of N from above by copies of $\Theta(i)$, this means the following: take the exact sequences $\epsilon_s = (0 \to N \to T_s \to \Theta(i) \to 0)$ such that the corresponding equivalence classes $[\epsilon_1], \ldots, [\epsilon_m]$ generate $Ext^1_A(\Theta(i), N)$ as an $End_A(\Theta(i))$ module, and let ϵ be the exact sequence such that the s-th inclusion of $\Theta(i)$ in $Q = \Theta(i)^m$ induces the sequence ϵ_s. Then, the connecting homomorphism induced by ϵ, $\delta : Hom_A(\Theta(i), Q) \to Ext^1_A(\Theta(i), N)$, is surjective. Consider the exact sequence

$$Hom_A(\Theta(j), Q) \to Ext^1_A(\Theta(j), N) \to Ext^1_A(\Theta(j), N') \to Ext^1_A(\Theta(j), Q).$$

For $j \geq i$, the last term $Ext^1_A(\Theta(j), \Theta(i)^m)$ becomes 0. For $j > i$ we know that $Ext^1_A(\Theta(j), N) = 0$. For $j = i$, the first morphism is δ, which is surjective. Hence, for $j \geq i$ we have $Ext^1_A(\Theta(j), N^{'}) = 0$.
■

Lemma 4.1.7. *Let $1 \leq i \leq n$. Let N be an A-module with $Ext^1_A(\Theta(j), N) = 0$ for $j > i$. Then there exists an exact sequence $0 \to N \to Y \to X \to 0$ with $X \in \mathcal{F}(\{\Theta(1), \dots, \Theta(i)\})$ and $Y \in Y(\Theta)$.*

Proof:
Using the Lemma 4.1.6, we construct monomorphisms

$$N = N_{i+1} \xrightarrow{\mu_i} N_i \xrightarrow{\mu_{i-1}} \dots \xrightarrow{\mu_1} Y$$

with $Q_s = coker\mu_s$ a direct sum of copies of $\Theta(s)$, and $Ext^1_A(\Theta(j), N_i) = 0$ for $j \geq i$. Let $\mu = \mu_1 \dots \mu_i : N \to Y$, and $X = coker\mu$. Then $Y \in Y(\Theta)$, and X has a filtration with factors Q_s. This is because, without loss of generality, we can assume that all μ_s are inclusions; the filtration of $X = Y/N$ is given by the modules N_s/N; for $1 \leq s \leq i+1$, and $(N_s/N)/(N_{s+1}/N) \cong N_s/N_{s+1} \cong Q_s$ for $1 \leq s \leq i$.
■

From the previous results, we can infer that every $A-$module has a right $\mathcal{F}(\Theta)-$approximation, and since the construction of $\mathcal{F}(\Theta)$ is self-dual, we can also obtain left $\mathcal{F}(\Theta)-$approximations using duality. Thus, we have the following result.

Theorem 4.1.8. *The subcategory $\mathcal{F}(\Theta)$ is functorially finite in* mod$A$*, i.e. $\mathcal{F}(\Theta)$ is covariantly finite and contravariantly finite in modA.*

4.1.2 Standard and Costandard Modules

Let $\mathbf{e} = (e_1, e_2, ...e_n)$ be a fixed complete ordered set of primitive and orthogonal idempotents of the $K-$algebra A. Note that considering the sequence $\mathbf{e}$ is equivalent to considering a fixed order of the set of all non-isomorphic simple $A-$modules $\mathcal{S}_i \cong e_i \frac{A}{radA}$ (we know that since A is Artin, it has a finite number). For any $i \in \{1, \cdots, n\}$, let $\mathcal{P}_i \cong e_iA$ be their projective covers of e_i and dually, let $\mathcal{Q}_i \cong D(Ae_i)$ be their injective hulls of e_i.

Definition 4.1.9. *A normal series in an $A-$module M is a sequence of submodules*

$$0 = M_0 \subsetneq M_1 \subsetneq ... \subsetneq M_{\mathcal{T}} = M$$

The number $\mathcal{T}$ *is called the length of the series. The quotients* $\frac{M_{i+1}}{M_i}$ *are called the factors of the series. A composition series is a normal series whose factors are simple modules or, equivalently, a normal series that cannot be refined to another series of greater length.*

If X is an $A-$module, we denote by $[X : \mathcal{S}_i]$ the number of factors isomorphic to $\mathcal{S}_i$ in the composition series of X, that is, the multiplicity of $\mathcal{S}_i$ as a composition factor of X.

Definition 4.1.10. *The standard module* $\Delta(i)$*, for* $1 \leq i \leq n$*, is defined as the maximal factor module of* $\mathcal{P}_i$ *with composition factors of the form* $\mathcal{S}_j$*, with* $j \leq i$*. Dually, the costandard module* $\nabla(i)$*, for* $1 \leq i \leq n$*, is defined as the maximal submodule of* $\mathcal{Q}_i$ *with composition factors of the form* $\mathcal{S}_j$*, with* $j \leq i$*.*

We will denote by Δ *and* ∇ *the full subcategory consisting of these standard and costandard modules, respectively.*

Note that the previous definition implies that $[\Delta(i) : \mathcal{S}_j] = 0$ for $j \leq i$ and that the module $\Delta(n) = \mathcal{P}_n$. Dually, $[\nabla(i) : \mathcal{S}_j] = 0$ for $j \leq i$ and the module $\nabla(n) = \mathcal{Q}_n$. Furthermore, the following results are obtained;

Proposition 4.1.11. *Let* $\Delta = \{\Delta(1), ..., \Delta(n)\}$ *and* $\nabla = \{\nabla(1), ..., \nabla(n)\}$*. The following holds:*

a) $Hom_A(\Delta(i), \Delta(j)) = 0$*, for* $i > j$*.*

b) $Ext^1(\Delta(i), \Delta(j)) = 0$*, for* $i \geq j$*.*

c) $Ext^1(\Delta(i), \nabla(j)) = 0$*, for all* i, j*.*

d) $Hom_A(\mathcal{P}_i, \nabla(j)) = 0$*, for* $i > j$*.*

e) $Hom_A(\Delta(j), \mathcal{Q}_i) = 0$*, for* $i > j$*.*

f) $Hom_A(\Delta(i), \nabla(j)) \neq 0$ *if and only if* $i = j$*.*

As we have seen, the categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ are functorially finite in mod$A$, furthermore, admits the following descriptions, see (26),

$$\mathcal{F}(\Delta) = \{X \in \text{mod}A \mid \text{Ext}^1(X, \nabla) = 0\}, \text{ and}$$
$$\mathcal{F}(\nabla) = \{X \in \text{mod}A \mid \text{Ext}^1(\Delta, X) = 0\}.$$

4.1.3 Triangular Matrix Algebras over Quasi-Hereditary Algebras

We conclude by stating an important result obtained by B. Zhu, where he, in (30, Theorem 3.1), proves the quasi-hereditary of the triangular matrix algebras of quasi-hereditary algebras A and B by a bimodule ${}_AM_B$ under a suitable conditions on the bimodule M. Furthermore, he describe the good module category over this quasi-hereditary triangular matrix algebra. This same results are generalized to the functor categories in the previous chapter, in Theorems 3.4.1 and 3.5.1.

All the algebras we will consider will be artin algebras over a commutative artin ring R. Let $E = (e_1, e_2, ...e_n)$ be a fixed ordered and complete set of primitive and orthogonal idempotents of the R-algebra A. Recall that he pair (A, E), or the algebra A whenever it does not cause confusion, is said to be standardly stratified if ${}_AA$ belongs to $\mathcal{F}(\Delta)$. Furthermore, if $End_A(\Delta(i))$ is a division ring for every $1 \leq i \leq n$, then we say that the algebra is quasi-hereditary.

Let

$$\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix},$$

be the triangular matrix algebra, where M, is an $A-B$-bimodule such that Λ is an artin $R-$algebra. Recall, see (5), that it is possible to identify a $\Lambda-$module with a triple (X, Y, f), where X and Y are an $A-$module and a $B-$module respectively, and $f : M \otimes_B Y \to X$ is a morphism of $A-$modules.

Let $(A, {}_AE)$ and $(B, {}_BE)$ be quasi-hereditary algebras and consider ${}_\Lambda E = ({}_BE, {}_AE)$ the ordering on simple $\Lambda-$modules. In (30), it is shown that $(\Lambda, {}_\Lambda E)$ is a quasi-hereditary algebra with standard modules

$${}_\Lambda\Delta(1) = \begin{pmatrix} 0 & 0 \\ 0 & {}_B\Delta(1) \end{pmatrix},$$

$$\vdots$$

$${}_\Lambda\Delta(m) = \begin{pmatrix} 0 & 0 \\ 0 & {}_B\Delta(m) \end{pmatrix},$$

$${}_\Lambda\Delta(m+1) = \begin{pmatrix} {}_A\Delta(m+1) & 0 \\ 0 & 0 \end{pmatrix},$$

$$\vdots$$

$${}_\Lambda\Delta(m+n) = \begin{pmatrix} {}_A\Delta(m+n) & 0 \\ 0 & 0 \end{pmatrix}.$$

Furthermore, let $\mathcal{T}$ be the subcategory of modΛ formed by all triples (X, Y, f) where X belongs to $\mathcal{F}({}_A\Delta)$ and Y is from $\mathcal{F}({}_B\Delta)$. For any triple (X, Y, f) in $\mathcal{T}$, we can construct the exact sequence

$$0 \to (X,0,0) \to (X,Y,f) \to (0,Y,0) \to 0,$$

where $(X,0,0)$ and $(0,Y,0)$ elements of $\mathcal{F}({}_{\Lambda}\Delta)$. Consequently, since $\mathcal{F}({}_{\Lambda}\Delta)$ is closed under extentions in modΛ, it follows that $(X,Y,f) \in \mathcal{F}({}_{\Lambda}\Delta)$. Thus, we conclude that $\mathcal{T} \subseteq \mathcal{F}({}_{\Lambda}\Delta)$.

In addition, Zhu proves in (30) that $\mathcal{F}({}_{\Lambda}\Delta) \subseteq \mathcal{T}$. Indeed, we have that all standard $\Lambda-$modules ${}_{\Lambda}\Delta(i)$ are in $\mathcal{T}$, where $1 \leq i \leq m+n$. By identifying an $A-$module X with a triple $(X,0,0)$, and a $B-$module Y with a triple $(0,Y,0)$, we can consider both mod$A$ and mod$B$ as subcategories of modΛ. Thus, we identify mod$A$ with the subcategory $(\text{mod}A,0,0)$ and mod$B$ with the subcategory $(0,\text{mod}B,0)$.

Then, since $Ext^1_{\Lambda}(\text{mod}A, \text{mod}B) = 0$, both $\mathcal{F}({}_{A}\Delta)$ and $\mathcal{F}({}_{B}\Delta)$ are closed under extensions in modΛ. Using the notation from (30), we know from (26) that

$$\begin{aligned}\mathcal{T} = \mathcal{F}({}_{B}\Delta)\textstyle\int\mathcal{F}({}_{A}\Delta) := \{N \in \text{mod}\,\Lambda \mid & \text{ there is an exact sequence} \\ & 0 \to X \to N \to Y \to 0, \text{ with } X \in \mathcal{F}({}_{A}\Delta), Y \in \mathcal{F}({}_{B}\Delta)\},\end{aligned}$$

is a subcategory that is closed under extensions in modΛ. Therefore, for any $\Delta-$good $\Lambda-$module N, it follows that $N \in \mathcal{T}$ since N has a ${}_{\Lambda}\Delta-$filtration and all ${}_{\Lambda}\Delta(i) \in \mathcal{T}$.

Therefore,

$$\mathcal{F}({}_{\Lambda}\Delta) = \mathcal{T} = \{(X,Y,f) \mid X \in \mathcal{F}({}_{A}\Delta),\ Y \in \mathcal{F}({}_{B}\Delta)\}.$$

And in this way, Zhu obtains the following result with which we conclude this work.

Theorem 4.1.12. *(30, Theorem 3.1) Let $(A, {}_{A}E)$ and $(B, {}_{B}E)$ be quasi-hereditary algebras and ${}_{\Lambda}E = ({}_{B}E, {}_{A}E)$. If ${}_{A}M \in \mathcal{F}({}_{A}\Delta)$, then $(\Lambda, {}_{\Lambda}E)$ is a quasi-hereditary algebra. Moreover,*

$$\mathcal{F}({}_{\Lambda}\Delta) = \{(X,Y,f) \mid X \in \mathcal{F}({}_{A}\Delta),\ Y \in \mathcal{F}({}_{B}\Delta)\}.$$

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