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PROGRAMA DE MAESTRÍA Y DOCTORADO EN CIENCIAS MATEMÁTICAS Y DE LA ESPECIALIZACIÓN EN ESTADÍSTICA APLICADA

MODULUS OF A FAMILY OF CURVES, UPPER GRADIENTS AND POTENTIAL THEORY

TESIS
QUE PARA OPTAR POR EL GRADO DE:
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To my beloved mother

There is no tomorrow
Apollo Creed

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Introduction

The classical theory of Sobolev spaces is defined under the concept of weak derivative. However, to define this, it is necessary to consider an open set of $\mathbb{R}^d$, because the weak derivative is defined within compactly supported smooth functions. For this reason, for weak derivative, we cannot generalize immediately the concept of Sobolev space on the metric measure spaces setting. On the other hand, there are many results in practice that suggest how to generalize the notion of derivative, for example Corollary 1.5.23. This lead us to the notion of upper gradient(Definition 3.1.1). Nevertheless, upper gradients are not compatible with limit(Counterexample 3.1.7). For this reason, we need to make adjust to upper gradient definition. In this thesis, we discuss what are those adjustments to the $p-$weak upper gradient definition(Definition 3.1.1 second part).

The relevance of upper gradients lies in the the fact that this definition allow us to define a Sobolev space in the metric measure setting.

Main results

The main result is theorem 3.3.16 where we prove that the potential $f_{E,F}$ is weakly differentiable using the theory of module of a family of curves, so $\nabla f_{E,F}$ exists. Furthermore, the module of a family of curves that join two closed nonempty sets can be computed with the $|\nabla f_{E,F}|$ Item 4 of Theorem 3.3.16. By the module of a family of curves, we mean to a notion of a size of this family.

Structure of this thesis

Chapter 1: We discuss about metric measure spaces, and how the properties of a metric measure space generalize the Lebesgue-Radon-Nikodym theorem for curves in metric measure spaces. Later, in Section 1.6 we introduce the ACL property, this property is fundamental to provide a characterization of $W^{1,p}(\Omega)$ with general notions defined for metric measure spaces.

Chapter 2: We define the modulus of a family of curves, which is a notion of size of a family of curves. In Section 2.1, we define the $p-$modulus of aa family of curves. The most important result in this section is that this is an outer measure in the family of all families of curves. In Section 2.2, we characterize the families with $p-$modulus zero. In Section 2.4 we provide some fundamental examples of modulus, also in this section we will relate ACL property and $p-$modulus. In Section 2.5, we introduce Fuglede lemma (theorem 2.5.1) which will be used to prove the main theorem of Chapter 2 which is Lemma 2.6.1. This chapter culminate in Section 2.6 where we will prove that the Sobolev space $W^{1,p}(\Omega)$ can be characterized with ACL property.

Chapter 3: We will introduce the notion of upper gradients and we will discuss why we need to relax the hypothesis to weak upper gradients. In Section 3.1, we prove the general properties of upper gradients. Later in Section 3.2, we prove that considering $p-$integrable $p-$weak upper gradients, we have automatically properties of absolute continuity on curves. We will finish with Section 3.3, where we use all the properties of $p-$integrable $p-$weak upper gradients in Theorem 3.3.16 to prove that a potential is a Sobolev function.

Some considerations

For this thesis, we assume that the reader has the background of analysis and topology of the level of [Fol99] and [Dug66] respectively. For this work, we state all the basic result and we refer to the reader to the previous references for proofs. If the reader is also familiarized with the radon measures and the properties of the curves in the metric measure spaces setting; the reader we can start in Section 1.6.
To read directly the main results of this thesis, it is desirable that the reader is familiarized with a course on geometric measure theory. In this case the reader can start in Section 1.3 to see the generalization of the properties of Borel regular measures and Radon for metric measure spaces.

Future work

One of the most important application of Sobolev spaces in metric measure setting is the study of potential theory and the posses of PDE in the metric measure setting, see [GM13; MS16]. Another product of my thesis is the paper on the existence and uniqueness of the solutions of the $p-$Poisson equation in metric measure spaces with Takala, [CT25], in my visit to Aalto University, we will send this work on this summer.

Contents

Chapter 1

Preliminaries

1.1 Metric spaces

We start by giving the formal concepts of distance.

Definition 1.1.1: Metric spaces

A **metric** for a set X is a function $d : X \times X \to \mathbb{R}$ such that:

1. **Non-negative:** $d(x, y) \geq 0 \quad \forall x, y \in X$.
2. **Symmetric:** $d(x, y) = d(y, x) \quad \forall x, y \in X$.
3. **Identity of indiscernibles:** $d(x, y) = 0 \Leftrightarrow x = y$.
4. **Triangle inequality:** $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$.

A **metric space** is a pair (X, d) where X is a set and d is a metric on X. If the metric d is clear, we refer to the metric space simply as X.

Now we will provide some canonical examples of metric spaces.
The following example provides the most trivial metric space. However, this is useful for provide counter examples of metric spaces.

Example 1.1.2: Discrete metric

Let X a set we define $d_{\text{dis}} : X \times X \to \mathbb{R}$ given by:

$$\boldsymbol{d_{\text{dis}}}(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Is easy to see that d_{dis} is a metric, it is called the **discrete metric** and the metric space (X, d_{dis}) is called a **discrete space**.

Next we define some sets induced by metric:

Definition 1.1.3: Ball and spheres

Let (X, d) a metric space and $x \in X, r \geq 0$. We define the following sets:

1. **Open ball** of center x and radius r is te set:

$$\boldsymbol{B(x, r)} = \{y \in X \mid d(y, x) < r\}$$

2. **Closed ball** of center x and radius r is te set:

$$\boldsymbol{B[x, r]} = \{y \in X \mid d(y, x) \leq r\}$$

To provide a better notation we refer to closed ball as $\overline{\boldsymbol{B}}(\boldsymbol{x},\boldsymbol{r})$. This is just notation, because the closure of the open ball and the closed ball does not hold in infinite dimensional spaces.

3. **Sphere** of center x and radius r is the set:

$$\boldsymbol{S}(\boldsymbol{x},\boldsymbol{r}) = \{y \in X \mid d(y,x) = r\}$$

Sometimes we need to consider centerless balls, for these reason we provide te next definitions:

4. **Centerless open ball** of center x and radius r is te set:

$$\boldsymbol{B}^*(\boldsymbol{x},\boldsymbol{r}) = B(x,r) \setminus \{x\}\,.$$

5. **Centerless closed ball** of center x and radius r is te set:

$$\boldsymbol{B}^*[\boldsymbol{x},\boldsymbol{r}] = B[x,r] \setminus \{x\}\,.$$

We will use the same notation as the closed ball, that is $\overline{\boldsymbol{B}}^*(\boldsymbol{x},\boldsymbol{r})$

In a discrete space we have the following characterization of balls.

Proposition 1.1.4 (Balls in a discrete space): Let X a discrete space, let $x \in X$. Then

$$B(x,r) = \begin{cases} \{x\} & \text{if } r \leq 1, \\ X & \text{if } r > . \end{cases}$$
$$B[x,r] = \begin{cases} \{x\} & \text{if } r < 1, \\ X & \text{if } r \geq . \end{cases}$$
$$S(x,r) = \begin{cases} \{x\} & \text{if } r = 1, \\ \varnothing & \text{if } r \neq 1. \end{cases}$$
$$B^*(x,r) = \begin{cases} \varnothing & \text{if } r \leq 1, \\ X \setminus \{x\} & \text{if } r > . \end{cases}$$
$$B^*[x,r] = \begin{cases} \varnothing & \text{if } r < 1, \\ X \setminus \{x\} & \text{if } r \geq . \end{cases}$$

Disclaimer 1.1.5: Balls and spheres cannot be characterized by center and radius This is because proposition 1.1.4 shows that the same set could be shown as different balls. Therefore, the center and radius are not sufficient parameters to characterize the balls and spheres.

We use the following notation

Definition 1.1.6: $G_\delta, F_\sigma, G_{\delta\sigma}, F_{\sigma\delta}$ sets

Let (X,τ) be a topological space. We say that $A \subset X$ is a set of type:

1. $\boldsymbol{G_\delta}$ if A is a countable intersection of open sets.
2. $\boldsymbol{F_\sigma}$ if A is a countable union of closed sets.
3. $\boldsymbol{G_{\delta\sigma}}$ if A is a countable union of G_δ sets.
4. $\boldsymbol{F_{\sigma\delta}}$ if A is a countable intersection of F_σ sets.

1.1.1 Totally bounded sets

We introduce the following concepts

Definition: $\varepsilon-$net

Let (X,d) be a metric space. A subset $A \subset X$ is an **$\varepsilon-$net** or an **$\varepsilon-$dense set** if

$$X = \bigcup_{a\in A} B\left(a,\varepsilon\right).$$

A related definition for $\varepsilon-$net is the following:

Definition: $\varepsilon-$separated set

Let (X,d) be a metric space and $\varepsilon > 0$. A subset $A \subset X$ is **$\varepsilon-$separated** if each two distinct points has distance at least ε.

From the above definitions follows immediately the next:

Proposition 1.1.7: Let (X,d) be a metric space, $A \subset X$ and $\varepsilon > 0$. If $N \subset A$ is an $\varepsilon-$net on A and $S \subset A$ is a $2\varepsilon-$separated set, then $|S| \leq |N|$.

Proof: Let $s \in S$. Since $S \subset A$ and N is an $\varepsilon-$net on A, there exists $a_s \in N$ such that $s \in B\left(a_s,\varepsilon\right)$. Now, let us define $F\colon S \to N$ given by $F\left(s\right) = a_s$. Let us prove that F is injective. Indeed, suppose that

$$a_{s_1} = F(s_1) = F(s_2) = a_{s_2},$$

from this equality and by definition of F, follow $s_i \in B\left(a_{s_1},\varepsilon\right)$, thus $d\left(s_1,s_2\right) < 2\varepsilon$ and since S is $\varepsilon-$separated, it follows that $s_1 = s_2$. This proves that F is injective. Therefore, $|S| \leq |N|$.

■

From the above definitions follows immediately the next:

Proposition 1.1.8: Let (X,d) be a metric space, $A \subset X$ and $\varepsilon > 0$. Let $S \subset A$ be a $\varepsilon-$separated set. Then, $S \cup \{a\}$ is an $\varepsilon-$separated set for all $a \in A \setminus \bigcup_{s\in S} B\left(s,\varepsilon\right)$.

Proof: Since S is an $\varepsilon-$separated set, we only need to verify the distance condition for a. Since $a \in A \setminus \bigcup_{s\in S} B\left(s,\varepsilon\right)$, clearly a is distinct from all element of S. Furthermore, we have

$$d(a,s) \geq \varepsilon$$

Therefore, $S \cup \{a\}$ is an $\varepsilon-$separated set.

■

With the definition of $\varepsilon-$net we define the next notion of boundedness.

Definition: Totally bounded set

Let (X,d) be a metric space. A subset $A \subset X$ is **totally bounded** if for every $\varepsilon > 0$ there exist $A' \subset A$ a finite $\varepsilon-$net in A.

Now, we will prove the following result

Theorem 1.1.9: Totally bounded characterization

Let (X,d) be a metric space. For a subset $A \subset X$ the following statements are equivalent:

1. A is totally bounded.

2. Every $\varepsilon-$separated set in A is finite for each $\varepsilon > 0$.

3. For each $\varepsilon > 0$, there exists a finite $\varepsilon-$separated set in A.

Proof:

(1) $\Rightarrow$ (2) Assume that A is totally bounded. Then, there exists a finite $\frac{\varepsilon}{2}-$net in A, from Proposition 1.1.7 it follows that every $\varepsilon-$separated set is finite.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) Suppose that Item 2 holds. Let $a_1 \in A$, if $A \subset B(a_1, \varepsilon)$, we have finished, if not we choose $a_2 \in A$ such that $A \setminus B(a_2, \varepsilon)$ in this way $\{a_1, a_2\}$ is an $\varepsilon-$separated set. Inductively if there exists $\{a_k\}_{k=1}^n \subset A$ such that $A = \bigcup_{k=1}^n B(a_k, \varepsilon)$ we have finished, if not we may assume with loss of generality $\{a_k\}_{k=1}^n$ is an $\varepsilon-$separated set and choose $a_{n+1} \in A \setminus \bigcup_{k=1}^n B(a_k, \varepsilon)$. Since Item 2 holds, this procedure is finite, then there exists $N \in \mathbb{N}$ such that $\bigcup_{k=1}^N B(a_k, \varepsilon) = A$, that is $\{a_k\}_{k=1}^N$ is a $\varepsilon-$net. Therefore, A is totally bounded.

■

The following is a well a know result in analysis.

Theorem 1.1.10: Characterization of compact spaces

Let (X, d) be a metric space. The following statements are equivalent:

1. X is compact.
2. Every infinite subset has an accumulation point.
3. X is complete and totally bounded.

For the proof see [Fol99].

1.1.2 Lipschitz functions

A useful concept on metric spaces is the following:

Definition: Lipschitz function

Let $f : (X, d_X) \to (Y, d_Y)$ a function between metric spaces and $C \geq 0$.

1. We say that f is $\boldsymbol{C}$**−Lipschitz** if
$$\forall x, y \in X \qquad d_Y(f(x), f(y)) \leq C d_X(x, y).$$
We say that f is **Lipschitz** if there exists $C_0 \geq 0$ such that f is C_0-Lipschitz.
2. **The (global) Lipschitz constant** of f, is
$$\mathscr{LIP}(\boldsymbol{f}) := \inf \{C \geq 0 : f \text{ is } C - \text{Lipschitz}\}.$$
3. The set of real valued Lipschitz functions, $\{f : (X, d) \to \mathbb{R} : f \text{ is Lipschitz}\}$, will be denoted by $\mathbf{LIP}(\boldsymbol{X}, \boldsymbol{d})$. When the metric is clear we simply write $\mathbf{LIP}(\boldsymbol{X})$.

Now, we introduce an intuitive notion of *distance* between sets

Definition 1.1.11: Distance between sets

Let (X, d) metric space and $A, B \subset X$ nonempty sets. We define the "**distance" between $\boldsymbol{A}$ and $\boldsymbol{B}$** as:
$$\boldsymbol{d(A, B)} = \inf \{d(a, b) \mid a \in A, b \in B\}. \tag{1.1.12}$$
Sometimes, we will denote (1.1.12) as $\mathbf{dist}(\boldsymbol{A}, \boldsymbol{B})$.

Special case

The **distance between $x \in X$ a point** and a **set $A \subset X$, $d(x, A)$**, is defined as follows:

$$\boldsymbol{d(x, A)} = d(\{x\}, A).$$

With the distance to a set we define a useful set.

Definition 1.1.13

Let (X, d) be a metric space, $A \subset X$ and $\varepsilon > 0$.

$$\boldsymbol{A_\varepsilon} = \{x \in A \mid d(x, \partial A) > \varepsilon\},$$

where ∂A denotes the topological boundary of A.

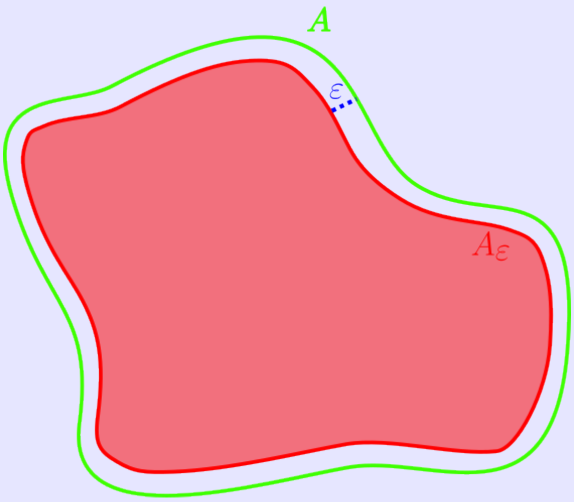

Figure 1.1.14: Set A_ε

An immediate result for the set Ω_ε is that it is a kind of $\varepsilon-$neighborhood. We state that result.

Theorem 1.1.15

Let $\Omega \subset \mathbb{R}^d$ be an open set and $\varepsilon > 0$. Then, $B(x, \varepsilon) \subset \Omega$ for all $x \in \Omega_\varepsilon$.

The proof is a usual proof in analysis.

1.2 Measure theory

In this section we recall the concepts of measure theory. To do this we deal with the power set of X, this set we denote by $\boldsymbol{\mathcal{P}(X)}$

1.2.1 Measurable spaces

First, we define a desirable class of sets to define measures.

Definition: $\sigma-$algebra

Let X be a set and $\Sigma \subset \mathcal{P}(X)$. We say that Σ is a **$\sigma-$algebra** if:

1. **$\varnothing, X \in \Sigma$.**
2. **It is closed under complements:** If $A \in \Sigma$ then $X \setminus A \in \Sigma$.
3. **It is closed under countable unions:** If $(A_n)_{n\in\mathbb{N}}$ then $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma$.

The elements of the $\sigma-$algebra Σ are called **measurable sets**, and a **measurable space** is a pair (X, Σ) where Σ is a $\sigma-$algebra on X.

Motivated by the definition of continuous function, we define a measurable function as follows:

Definition: Measurable function

Let $f:(X,\Sigma_X)\to(Y,\Sigma_Y)$ be a **function** between measurable spaces. We say that f is **measurable** if $f^{-1}[A]\in\Sigma_X$ for every $A\in\Sigma_Y$.

Now, we will state a version of Gluing lemma for measurable functions.

Theorem: Gluing lemma

Let $\{A_i\}_{i\in\Gamma}$ be a covering of a measurable space X such that A_i is measurable for each $i\in\Gamma$. For any countable family of measurable functions $\{f_n\colon A_n\to B\}_{n\in\mathbb{N}}$ such that:

$$f_n|_{A_n\cap A_m}=f_m|_{A_n\cap A_m}\qquad \forall n,m\in\mathbb{N}, \tag{1.2.1}$$

there exists a unique measurable extension for all f_n.

Proof: We define $F\colon X\to B$ as follow, let $x\in X$ then there exists $j\in\mathbb{N}$ such that $x\in A_j$, we define $F(x)=f_j(x)$, from (1.2.1) it follows that F is well defined. Let $M\subset B$ a measurable set. We have that

$$F^{-1}[M]=\bigcup_{n\in\mathbb{N}}f_n^{-1}[M].$$

since M is measurable and f_n is measurable, we have that $f_n^{-1}[M]$ is measurable, then $F^{-1}[M]$ is measurable. Therefore, F is measurable.

■

An application of the previous result is the following:

Lemma 1.2.2: Extension of $f\colon\Omega\subset X\to\mathbb{R}$

Let Σ be a $\sigma-$algebra in X and $\Omega\subset X$ be a measurable set. If $f\colon\Omega\subset X\to\mathbb{R}$ is a measurable function, then, the function $g\colon X\to\mathbb{R}$ defined as

$$g=\begin{cases} f(x) & \text{if } x\in\Omega,\\ 0 & \text{if } x\notin\Omega.\end{cases}$$

is measurable.

Remark 1.2.3 The most useful form of Lemma 1.2.2 arises when $X=\mathbb{R}^d$ is endowed with the Borel $\sigma-$algebra(which we define below). In this case, Ω can be taken to be open. We use this for of Lemma 1.2.2 without further mention.

1.2.2 Measures and outer measures

The measures and outer measures gives a notion of size. We will prove that essentially both notions are the same.

Definition: Measure

Let Σ be a $\sigma-$algebra over a set X. A **measure** over X is a function $\mu\colon\Sigma\to[0,\infty]$ such that:

1. $\boldsymbol{\mu(\varnothing)=0}$.
2. **$\boldsymbol{\sigma}-$additive:** If $(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoints sets then $\mu\left(\bigcup_{n\in\mathbb{N}}A_n\right)=\sum_{n\in\mathbb{N}}\mu(A_n)$.

A **measure space** is a triple (X,Σ,μ) where Σ is a $\sigma-$algebra over X

One of the well known examples of measures is the Lebesgue measure, in this thesis we we denote $\boldsymbol{m_d}$ for the Lebesgue measure.

Definition: Outer measure

Outer measure An **outer measure** over a set X is a function $\mu^*: \mathcal{P}(X) \to [0,\infty]$ such that:

1. $\boldsymbol{\mu^*(\varnothing) = 0}$.
2. **Increasing:** If $A \subset B$ then $\mu^*(A) \subset \mu^*(B)$.
3. $\boldsymbol{\sigma}$**−subadditive:** $\mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n)$

The definition of outer measure provides a general notion of size. Nevertheless, we are looking for more desirable properties to this notion of size, the following definition is motivated by geometry:

Definition: Measurable sets with respect to an outer measure

Let $\mu^*: \mathcal{P}(X) \to [0,\infty]$ be an outer measure and $A \subset X$. We say that A is $\boldsymbol{\mu^*}$**−measurable** if:

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \setminus A) \qquad \forall B \subset X.$$

Lemma 1.2.4 and theorems 1.2.5 and 1.2.6 are standard results on geometric measure theory, the proofs of these results can be found in [Mag12].
The following result optimizes the effort to prove that some set is μ^*−measurable.

Lemma 1.2.4: Characterization of μ^*−measurable sets

Let $\mu^*: \mathcal{P}(X) \to [0,\infty]$ be an outer measure and $A \subset X$. Then, A is μ^*−measurable if and only if

$$\mu^*(B \cap A) + \mu^*(B \setminus A) \leq \mu^*(B) \qquad \forall B \subset X \text{ with } \mu^*(B) < \infty.$$

Now, there is a natural restriction of an outer measure to obtain a measure.

Theorem 1.2.5: Restriction of an outer measure to measurable sets is a complete measure

Let $\mu^*: \mathcal{P}(X) \to [0,\infty]$ an outer measure on a set X. Then:

1. The family of the μ^*−measurable sets, $\mathcal{M}(\mu)$ is a σ−algebra.
2. $\mu^*|_{\mathcal{M}(\mu^*)}$ is a complete measure, that is every subset of a measure zero set is measurable.

On the other hand we can extend a measure to an outer measure as follows:

Theorem 1.2.6: Extension of a measure to an outer measure

If $\mu: \Sigma \to [0,\infty]$ is a measure over X. Then the function $\mu^*: \mathcal{P}(X) \to [0,\infty]$ given by:

$$\boldsymbol{\mu^*(A)} = \inf\{\mu(M) \mid A \subset M \in \Sigma\} \tag{1.2.7}$$

is an other measure that extends to μ and $\Sigma \subset \mathcal{M}(\mu)$.

Definition 1.2.8

If $\mu: \Sigma \to [0,\infty]$ is a measure over X. Then the function $\mu^*: \mathcal{P}(X) \to [0,\infty]$ given by (1.2.7) is called the **extension of the measure $\boldsymbol{\mu}$**.

From Theorems 1.2.5 and 1.2.6 we obtain the following:

Theorem 1.2.9: Compatibility of the restrictions and extensions of measures and outer measures

Let X be a set. The following statements holds:

1. Let $\mu: \Sigma \to [0,\infty]$ be a measure in X. Then, $\mu^*|_\Sigma = \mu$.
2. Let $\mu^*: \mathcal{P}(X) \to [0,\infty]$ be an outer measure. Then, the extension , satisfies $\left(\mu^*|_{\mathcal{M}(\mu^*)}\right)^* = \mu^*$

Proof:

1. From Theorem 1.2.6, we have that μ^* is an extension of μ, that is $\mu^*|_\Sigma = \mu$.

2. It is clear that $\nu = \mu^* \mid_{\mathcal{M}(\mu^*)}$ satisfies $\mu^* \leq \nu^*$. For the other inequality we use the infimum property. Let $A \subset X$ and $\varepsilon > 0$, then there exists $M \in M(\mu)$ such that $A \subset M$ and

$$\mu^*(A) \leq \mu^*(A) < \nu^*(A) + \varepsilon.$$

this proves $\mu^*(A) \leq \nu^*(A)$. Therefore we conclude $\nu^* = \mu^*$.

■

Remark 1.2.10: Measures and outer measures are the same considering restrictions and extensions Theorem 1.2.9 shows that considering the appropriate adjustment with extensions and restrictions measures and outer measures we can take as the same thing. For this reason we put a convenient name for a measurable sets. When we define a property for a measure, we will extend this notion for outer measure considering the extension.

From Remark 1.2.10 follows that we have some examples of measures, one of the most important of (outer) measures is the generalization of the extensions given with the canonical definition in the induced $\sigma-$algebra $\Sigma \cap A$.

Proposition 1.2.11 (Restrictions of measure): Let $\mu \colon \Sigma \to [0, \infty]$ be a measure on a set X and $Y \subset X$. Then, the following statement hold:

1. The restriction of μ to the $\sigma-$algebra $\Sigma \cap A$, $\boldsymbol{\mu_Y} = \mu \mid_{\Sigma \cap Y}$ is again a measure. That is $\mu_Y = \mu|_{\Sigma \cap Y}$ is a measure if μ is a measure, and $\mu_Y = \mu \mid_{\mathcal{P}(Y)}$ is an outer measure if μ is an outer measure.

2. The restriction of μ, $\boldsymbol{\mu_{\llcorner Y}} \colon \Sigma \to [0, \infty]$ given by:

$$\boldsymbol{\mu_{\llcorner Y}}(M) = \mu(Y \cap M) \qquad \forall M \in \Sigma.$$

Notice that the definitions of Proposition 1.2.11 are defined by outer measures but this generalization is simpler than the definitions for $\sigma-$additive measures.

Disclaimer Even within the considerations of Remark 1.2.10 the properties of measures do not inherit immediately to outer measures and viceversa, for example the continuity from below of measure is not inherited to outer measures, an example of this incompatibility is proved in Counterexample 2.5.9. To extend properties of measure to outer measure we need to consider the appropriate extension or restriction as in Remark 1.2.10.

1.2.3 Regular measures

First, we will define the kind of measures to work with metric spaces:

Motivation 1.2.12: Measures in a metric space

Let (X, τ) be a topological space. We aim to make compatible the notions of measure theory with the topology, To do this we have the following options.

1. **Compatibility with measures** To define a measure μ for X compatible with the topology, we require a $\sigma-$algebra to define μ. For any give $\sigma-$algebra Σ we can define a measure $\mu_\Sigma \colon \Sigma \to [0, \infty]$. However, the measure μ_Σ and the topology of X no need to have common elements. A way to do this is

considering the Borel sets contained in the topology of X, that is $\mathbb{B}(X) \subset \Sigma$.

The disadvantage of this definition is that we have a two non-topological notions. These are measure as a function as well its domain.

2. **Compatibility with outer measures** Notice that if we consider an outer measure $\mu\colon \mathcal{P}(X) \to [0, \infty]$ the domain is provided. Considering Remark 1.2.10 and the previous requirement a compatible outer measure with the topology is such that

$$\mathbb{B}(X) \subset \mathcal{M}(\mu). \tag{1.2.13}$$

The advantage of (1.2.13) is that there is essentially a unique different object of the topological notions.

Conclusions

The essence in the proposed definition in Items 1 and 2 are that the Borel sets are measurable. The importance on this example lies in (1.2.13). We will use the definition provided in Item 2. Also this example shown the convenience of consider outer measures and measure as in Remark 1.2.10.

The measures that satisfies the conditions of Motivation 1.2.12 have an special name.

Definition 1.2.14: Borel measure

Let X a topological space. A **Borel measure** is a measure such that every Borel set is measurable.

Notice that Definition 1.2.14 can be stated for either a measure or an outer measure.
Now, we will define concepts about regularity:

Motivation 1.2.15: Regularity for measure

The idea of regularity of a measure is to have approximation properties. The intuitive idea is given by the following:

Definition 1.2.16: Regular measure [Fol99]

Let μ a Borel measure in a topological space X and $E \subset X$ measurable, we say that μ is:

1. **Outer regular on E** if

$$\mu(E) = \inf \{\mu(U) \mid U \supset E \text{ with } U \text{ open}\}.$$

2. **Inner regular(by compact sets) on E** if

$$\mu(E) = \sup \{\mu(K) \mid K \subset E \text{ with } K \text{ compact}\}.$$

3. **Regular on E** if μ is inner an outer regular on E.

4. **Regular** if μ is regular on every measurable set.

However this definition requires to have a topological space, and for that reason, we need make compatible the space with this topological notions.

Generalization

Suppose that μ is outer regular. Let $E \subset X$ be a measurable set, then there exists a sequence $\{U_n\}_{n\in\mathbb{N}}$ of open sets each containing E such that $\lim_{n\to\infty} \mu(U_n) = \mu(E)$. Now, by the monotonicity of μ, we also have that

$$\mu(E) \le \mu\left(\bigcap_{j=1}^{n} U_j\right) \le \mu(U_n),$$

and so we can replace U_n with $V_n = \bigcap_{j=1}^{n} U_j$ and see that

$$E \subset V_{n+1} \subset V_n \text{ and } \lim_{n\to\infty} \mu(V_n) = \mu(E).$$

Now take the Borel set $W = \bigcap_{n\in\mathbb{N}} V_n$ and note that $E \subset W$ and

$$\mu(E) \leq \mu(W) \leq \mu(V_k)$$

for each k, and letting $k \to \infty$ we get that $\mu(E) = \mu(W)$. This provides the notion of regularity in terms of only measurable sets not necessarily open. Then, the notion of regularity can be generalized for any measurable spaces. We can do a same proceeding to extend the notion of inner regularity. However, we use this notion provided by outer regularity it is compatible with the outer measure.

Conclusions

Definition 1.2.16 is in terms of Borel sets but his could be in terms of any measurable set this lead us to the general definition of regular measure, Definition 1.2.18, this is the most usual definition of regularity. For geometrical analysis we focus on Borel regular measures and we prove that in this context we prove that Definitions 1.2.16 and 1.2.18 are equivalent.

There are two definitions for inner regularity because the approximation could be by closed sets. Later, we will prove that for metric measure spaces there is no distinction between two definitions under certain conditions. For either definitions of inner regularity follows immediately the next:

Proposition 1.2.17: Let μ be Borel measure on a topological space X. If μ is a inner regular measure by compact(closed) sets on a measurable set $E \subset X$. Then, there exists an increasing sequence of compact(closed) sets that approximate $\mu(E)$.

Proof: By definition of inner regularity by compact(closed) set, there exist a sequence of compact(closed) sets $\{K_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \mu(K_n) = \mu(E)$ and this limit converges increasingly. Thus, the finite unions of $\{K_n\}_{n\in\mathbb{N}}$ converges to $\mu(E)$.

■

From the discussion of the Motivation 1.2.15, we state the following:

Definition 1.2.18: Regular measure

Let μ be a **measure** on a set X and $\mathcal{C} \subset \mathcal{P}(X)$ a given class. We say that μ is **$\mathcal{C}-$regular** if every set is contained in a set in $\mathcal{C}$ of equal measure.

Borel regular measure

Let μ be a Borel measure in a topological space X is **Borel regular** if every set is contained in a set of $\mathcal{C}$ of equal measure.

Remark 1.2.19: Importance of the regularity

1. The notion of $\mathcal{C}$-regularity allows us to focus on the behavior of the measure within a smaller and known class of sets. Often this class $\mathcal{C}$ the sets are measurable, we require this condition to do arithmetic with μ.

2. To prove properties that only depends on μ, we only need to prove it on the class $\mathcal{C}$ and by regularity we have automatically the extension. **Notice that this statement holds for the notions of inner and outer regularity**.

Now, we will prove some regularity results for measure, those results are stated for Borel regular measures.

However, we can generalize for any regular measure. In the next result we prove that a Borel regular measure can be approximated from below.

Proposition 1.2.20: Let μ is a Borel regular measure on X. If $A \subset X$ is a measurable set with finite measure then there are B_1, B_2 Borel sets such that $B_1 \subset A \subset B_2$ such that $\mu(B_1) = \mu(A) = \mu(B_2)$.

Proof: Indeed, by the definition of Borel regular measure there exist A_1 Borel set such that $A \subset A_1$ with $\mu(A) = \mu(A_1)$; since A has finite measure follows that $\mu(A_1 \setminus A) = \mu(A_1) - \mu(A) = 0$. Again, by the regularity of μ there exist A_2 Borel measurable set such that $A_1 \setminus A \subset A_2$ and $\mu(A_2) = \mu(A_1 \setminus A) = 0$; then $A_1 \setminus A_2$ is a Borel set such that $A_1 \setminus A_2 \subset A$ and $\mu(A_1 \setminus A_2) = \mu(A_1) - \cancelto{0}{\mu(A_2)} = \mu(A)$. Taking $B_1 = A_1 \setminus A_2, B_2 = A_1$ we have the result.

■

Remark: A must be measurable in Proposition 1.2.20 Remember we are considering measures and outer measures as the same, then to do arithmetic used in the proof Proposition 1.2.20 is necessary require A measurable.

Now, we will prove some immediate results about Borel regular measures.

Theorem 1.2.21: Borel regular measures properties

Let μ be a $\sigma-$finite Borel regular measure on X. Then:

1. μ is inner regular by closed sets on measurable sets.
2. μ is outer regular.

Specific cases of regularity

3. Every measurable set $A \subset X$ contains F a F_σ set such that $\mu(F) = \mu(A)$.
4. Every set $E \subset X$ is contained in G a G_δ set such that $\mu(E) = \mu(G)$.
5. For every $A \subset X$ measurable and for each $\varepsilon > 0$ there exists $C \subset X$ and $O \subset X$ closed and open sets respectively such that $C \subset A \subset O$ and satisfy

$$\mu(O \setminus C) < \varepsilon.$$

Proof:

1. First, we prove the result when $\mu(X)$ is finite. Since $\mu(X)$ is finite and A is measurable, we can use Proposition 1.2.20 thus there exist $B' \subset A$ a Borel set such that $\mu(B') = \mu(A)$. Then, we can replace A for B' if necessary. Hence, we need to prove the result only for Borel sets.

   Define $\mathcal{F}$ as the family of subsets of X for which μ is inner regular by compact sets. Thus $\mathcal{F}$ contains the closed sets, hence also contains the open sets, since the open sets are F_σ by separability. Notice that the measure of a countable union of closed sets can be approximated by the measures of finite unions of these closed sets, that is $\mathcal{F}$ contains all the F_σ sets.

   Let $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ and $\varepsilon > 0$. Then for any $n \in \mathbb{N}$ there exist $C_n \subset F_n$ a closed set such that

$$\mu(F_n) - \frac{\varepsilon}{2^n} < \mu(C_n)$$
$$\mu(F_n \setminus C_n) < \frac{\varepsilon}{2^{n+1}}$$

Since all the sets are measurable and $\mu(X) < \infty$.

Then by the series argument, we have

$$\mu\left(\bigcup_{n\in\mathbb{N}} F_n \setminus \bigcup_{n\in\mathbb{N}} C_n\right) \leq \mu\left(\bigcup_{n\in\mathbb{N}} (F_n \setminus C_n)\right) \leq \sum_{n\in\mathbb{N}} \mu\left(F_n \setminus C_n\right) \leq \frac{\varepsilon}{2}. \tag{1.2.22}$$

and similarly as in Theorem 1.2.21, we prove that

Now, let us prove that $\mathcal{F}$ has the following properties:

$\mathcal{F}$ is closed under countable unions.

From the continuity from below of measure, we have

$$\begin{aligned}\lim_{N\to\infty} \mu\left(\bigcup_{n\in\mathbb{N}} F_n \setminus \bigcup_{n=1}^{N} C_n\right) &= \mu\left(\bigcup_{n\in\mathbb{N}} F_n \setminus \bigcup_{n\in\mathbb{N}} C_n\right) \\ &\leq \frac{\varepsilon}{2}. \qquad \text{From (1.2.22).}\end{aligned}$$

Then, there exist $M \in \mathbb{N}$ large enough such that

$$\mu\left(\bigcup_{n\in\mathbb{N}} F_n \setminus \bigcup_{n=1}^{M} C_n\right) < \varepsilon \tag{1.2.23}$$

and clearly $\bigcup_{n=1}^{M} C_n$ is closed and $\bigcup_{n=1}^{M} C_n \subset \bigcup_{n\in\mathbb{N}} F_n$. Therefore $\bigcup_{n\in\mathbb{N}} F_n \in \mathcal{F}$.

$\mathcal{F}$ is closed under countable intersections.

Since $\bigcap_{n\in\mathbb{N}} C_n \subset C_m$ for all $m \in \mathbb{N}$, we have

$$\begin{aligned}\mu\left(\bigcap_{n\in\mathbb{N}} F_n \setminus \bigcap_{n\in\mathbb{N}} C_n\right) &\leq \mu\left(\bigcup_{n\in\mathbb{N}} (F_n \setminus C_n)\right) \\ &\leq \sum_{n\in\mathbb{N}} v\left(F_n \setminus C_n\right) \qquad \text{From (1.2.22).} \\ &\leq \frac{\varepsilon}{2}\end{aligned}$$

and clearly $\bigcap C_n \subset \bigcap_{n\in\mathbb{N}} A_n$ is a closed set. Thus $\bigcap_{n\in\mathbb{N}} F_n \in \mathcal{F}$.

Hence, the family

$$\mathcal{G} := \{A \in \mathcal{F} \colon X \setminus A \in \mathcal{F}\}$$

is a σ-algebra containing all closed sets. Hence, $\mathcal{G}$ must contain all Borel sets, and (1.3.3) follows. We extend the result by $\sigma-$finiteness.

2. First, we prove the result when $\mu\left(X\right)$ is finite. Since μ is Borel regular there exist a Borel set E_0 with $E \subset E_0$ and $\mu(E) = \mu\left(E_0\right)$. Let $\epsilon > 0$ from (1.3.3), we may choose a closed set $C \subset X \setminus E_0$ such that

$$\begin{aligned}\mu\left(X \setminus E_0\right) - \epsilon &< \mu(C) \\ \mu(X \setminus C) &< \mu\left(E_0\right) + \epsilon \\ &= \mu(E) + \epsilon.\end{aligned}$$

   Since $X \setminus C$ is open this proves (1.3.4). We extend the result by $\sigma-$finiteness.

3. The particular case follows immediately from the first part and analogously as in Motivation 1.2.15.

4. Follows immediately the inner regularity by closed sets on measurable sets and the outer regularity of μ.

5. First we prove the case when $\mu(X) < \infty$. From (1.3.3) and (1.3.4) there exists $C \subset X$ and $O \subset X$ open and closed sets respectively such that $C \subset A \subset O$ and satisfy

$$\mu(A) - \frac{\varepsilon}{2} < \mu(C)$$
$$\mu(O) < \mu(A) + \frac{\varepsilon}{2}$$

   Since $\mu(X) < \infty$, it follows that

$$\mu(O \setminus C) = \mu(O) - \mu(C) < \mu(A) + \frac{\varepsilon}{2} - \mu(A) + \frac{\varepsilon}{2} = \varepsilon$$

   The general case is by $\sigma-$finiteness.

■

1.2.4 Radon measure

Now, we will define a kind of measure with many desirable regularity properties

Definition: Radon measure

Let μ be a Borel regular measure in X a topological space. We say that μ is a **Radon measure** if:

1. **μ is finite on compact sets.**
2. **Outer regular.** That is for every subset $E \subset X$, we have

$$\mu(E) = \inf\{\mu(O) \mid O \supset E \text{ with } O \text{ open}\}.$$

3. **Inner regular by compact sets on open sets.** That is

$$\mu(O) = \sup\{\mu(K) \mid K \subset O \text{ with } K \text{ compact}\} \tag{1.2.24}$$

   for all $O \subset X$ open.

Remark 1.2.25: Computations for a Radon measures From the outer regularity, to compute μ, we only need to compute μ for open sets, thus by the inner regularity on open sets, we only need to compute μ on the compact sets.

Theorem 1.2.26: $\sigma-$finite Radon measures are inner regular(approximated by compacts)

Let μ be a $\sigma-$finite Radon measure on a topological space X. Then, μ is inner regular by compact sets on every measurable set.

<u>**Proof:**</u> By $\sigma-$finiteness, we may assume that $\mu(X) < \infty$. Let $A \subset X$ be a measurable set, then, there exists $C \subset X$ and $O \subset X$ closed and open sets respectively such that $C \subset A \subset O$ and satisfy

$$\mu(O \setminus C) < \epsilon.$$

Furthermore by the definition of Radon measure, there exists a compact set $K \subset O$ such that

$$\mu(K) > \mu(O) - \epsilon.$$

Then the compact set $K \cap C \subset A$ satisfies

$$\mu(A) \geq \mu(K \cap C) = \mu(K) - \mu(K \setminus C) > \mu(O) - 2\epsilon \geq \mu(A) - 2\epsilon.$$

■

1.2.5 Hausdorff measures

In this section we will introduce Hausdorff measure. In our work we often use the properties of the measure $\mathcal{H}_1$. We start with the following:

Definition: $\delta-$cover

Let X a metric space, $E \subset X$ and $\delta > 0$. A sequence of subsets of X, $(E_n)_{n\in\mathbb{N}} \subset X$ is a $\boldsymbol{\delta}$**−cover** for E if it sequence is a cover for E such that $\operatorname{diam}(E_n) \leq \delta$ for each $n \in \mathbb{N}$. The set of all $\delta-$covers for E is denoted by $\boldsymbol{C(E,\delta)}$.

From the definition follows immediately the next:

Proposition 1.2.27 ($\delta-$covers are increasing): Let X a metric space, $E \subset X$. If $0 < \delta_1 < \delta_2$, then $C(E,\delta_1) \subset C(E,\delta_2)$.

Proof: Let $\{E_n\}_{n\in\mathbb{N}} \in C(E,\delta_1)$, then $\{E_n\}_{n\in\mathbb{N}}$ is a numerable cover of E and

$$\operatorname{diam} E_n \leq \delta_1 < \delta_2$$

then $\{E_n\}_{n\in\mathbb{N}} \in C(E,\delta_2)$.

■

Using $\delta-$covers we define a sort of notion of volume in a metric space as follows:

Definition: $\alpha-$Hausdorff measure of step δ

Let X a metric space, $\alpha \geq 0$ fixed. Let $\delta > 0$ the $\boldsymbol{\alpha}$**−Hausdorff measure of step** $\boldsymbol{\delta}$ is the function $\mathcal{H}_{\alpha,\delta} \colon \mathcal{P}(X) \to [0,\infty]$ defined by

$$\boldsymbol{\mathcal{H}_{\alpha,\delta}(E)} = \inf\left\{ \sum_{n\in\mathbb{N}} v(\alpha)(\operatorname{diam} E_n)^{\alpha} \,\middle|\, (E_n)_{n\in\mathbb{N}} \in C(E,\delta) \right\}$$

where

$$v(\alpha) = \frac{2^{-\alpha}\pi^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}+1\right)}.$$

From the definition of $\mathcal{H}_{\alpha,\delta}$ follows immediately the next:

Lemma 1.2.28: Basic properties of $\mathcal{H}_{\alpha,\delta}$

Let X a metric space, $\alpha \geq 0$ fixed. For each $E \subset X$ and $\delta > 0$,

1. The function $\mathcal{H}_{\alpha,\delta} \colon \mathcal{P}(X) \to [0,\infty]$ is increasing.
2. The family of functions $\{\mathcal{H}_{\alpha,\delta}\}_{\delta>0}$ is decreasing.

Proof:

1. Let $A \subset B$, then $C(B,\delta) \subset C(A,\delta)$ and since the infimum is decreasing with respect to the contention we have $\mathcal{H}_{\alpha,\delta}(A) \leq \mathcal{H}_{\alpha,\delta}(B)$.
2. Let $E \subset X$ and $0 < \delta_1 < \delta_2$. From Proposition 1.2.27 and since the infimum is decreasing with respect to the contention, it follows that

$$\mathcal{H}_{\alpha,\delta_2}(E) \leq \mathcal{H}_{\alpha,\delta_1}(E)$$

■

Definition: $\alpha-$Hausdorff measure

Let X a metric space, $\alpha \geq 0$ fixed. The $\boldsymbol{\alpha}$**−Hausdorff measure** is defined by

$$\mathcal{H}_\alpha(E) = \lim_{\delta \downarrow 0} \mathcal{H}_{\alpha,\delta}(E).$$

The proof of the fact that $\mathcal{H}_\alpha$ is a measure is standard in geometrical measure theory, see [Mag12]. In the next result, we prove the compatibility of the Hausdorff measure with Lipschitz functions.

Lemma 1.2.29: Hausdorff measure and Lipschitz functions

Let $f : X \to Y$ a $L-$Lipschitz function and $E \subset X$. Then:

1. If $\mathcal{F}$ be a $\delta-$cover of E, then $f[\mathcal{F}]$ is a $L\delta-$cover of $f[E]$. That is $f[C(E,\delta)] \subset C[f[E], L\delta]$
2. $$\mathcal{H}_{\alpha,L\delta}(f[E]) \leq L^\alpha \mathcal{H}_{\alpha,\delta}(E). \tag{1.2.30}$$
3. $$\mathcal{H}_\alpha(f[E]) \leq L^\alpha \mathcal{H}_\alpha(E).$$

Proof:

1. Since $\mathcal{F}$ is a $\delta-$cover of E, it follows that $f[\mathcal{F}]$ is a numerable cover of $f[E]$. Since f is $L-$Lipschitz, for all $F \in \mathcal{F}$, we have
$$\begin{aligned} \operatorname{diam} f[F] &\leq L \operatorname{diam} F \\ &< L\delta \end{aligned} \quad \textit{$\mathcal{F}$ is a $\delta-$cover.}$$
This proves that $f[\mathcal{F}]$ is a $L\delta-$cover of $f[E]$.
2. From Item 1, the definition of $\mathcal{H}_{\alpha,\delta}$ and since the infimum is decreasing with respect the contention, it follows that
$$\mathcal{H}_{\alpha,L\delta}(f[E]) \leq \inf \left\{ \sum_{n\in\mathbb{N}} v(\alpha) (\operatorname{diam} f[E_n])^\alpha \,\middle|\, (E_n)_{n\in\mathbb{N}} \in C(E,\delta) \right\}$$
3. Since $\delta > 0$ in (1.2.30) is arbitrary, we can take the limit when $\delta \to 0$, then
$$\begin{aligned} \mathcal{H}_{\alpha,L\delta}(f[E]) &\leq L^\alpha \mathcal{H}_{\alpha,\delta}(E) \\ \mathcal{H}_\alpha(f[E]) &\leq L^\alpha \mathcal{H}_\alpha(E). \end{aligned} \quad \textit{From Item 2 of Lemma 1.2.28 and the definition of $\mathcal{H}_\delta$.}$$

■

The Item 3 of Lemma 1.2.29 is the most useful result of this theorem. Thus, when we invoke Lemma 1.2.29 we refer to Item 3.

Motivation: $\mathcal{H}_1$ and the notion of length

For the specif case of the 1−Hausdorff measure notice that from the properties of function Γ, we have that

$$\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\pi^{\frac{1}{2}}$$
$$v(1) = \frac{2^{-1}\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}+1\right)} = 1$$

Thus, in this case we have that

$$\mathcal{H}_{1,\delta} = \inf \left\{ \sum_{n\in\mathbb{N}} \operatorname{diam} E_n \,\middle|\, (E_n)_{n\in\mathbb{N}} \in C(E,\delta) \right\}$$

This identity give us a glimpse of the relationship with the notion of a length of a curve(see Definition 1.5). But it is better deal with the 1−Hausdorff measure from the point of view of measure theory.

For the present work the previous results are enough. For a further lecture on Hausdorff measures see [Rog70].

1.3 Metric measure spaces

Now, we define the spaces with which we are dealing in this thesis.

Definition: Metric measure space

A **metric measure space** is a triple (X, d, μ) where (X, d) is a separable metric space and μ is nontrivial locally finite Borel regular, that is for every $x \in X$ there is $r > 0$ such that $\mu(B(x, r)) < \infty$.

From the definition of metric measure space, we have the following properties:

Theorem 1.3.1: Metric measure spaces are $\sigma-$finite

1. Metric measure spaces are Lindelöf.

2. Metric measure spaces can be written as a countably union of balls with finite measure. Particularly, every metric measure space is $\sigma-$finite. For this reason in metric measure spaces, we can use Fubini Theorem immediately, because $\sigma-$finitenes is the only necessary condition see [Fol99].

3. Compact subsets of a metric measure space have finite measure.

Proof: Let (X, d, μ) be a metric measure space.

1. Follows from the separability of the metric measure space.

2. Let $x \in X$ since μ is locally finite there exists $r_x > 0$ such that $\mu(B(x, r_x)) < \infty$. Clearly $\{B(x, r_x)\}_{x \in X}$ is a cover of X. From the Lindelöf property it follows that there exists a countable subcover of $\{B(x, r_x)\}_{x \in X}$. Therefore, X is $\sigma-$finite.

3. Similarly as in Item 2, since μ is locally finite, it follows that a compact set can be covered by finitely many balls, each of which has finite measure. Therefore, every compact subset of X has finite measure.

■

We will use the above properties without further mention. By definition of metric measure space and their $\sigma-$finiteness together with Theorem 1.2.21 it follows that immediately the next:

Theorem 1.3.2

Let (X, d, μ) be a metric measure space. Then:

1. μ is inner regular(by closed sets) on measurable sets. That is

$$\mu(A) = \sup\{\mu(C) \,|\, C \subset A \text{ with } C \text{ closed}\}. \tag{1.3.3}$$

   for every $A \subset X$ measurable.

2. μ is outer regular. That is

$$\mu(E) = \inf\{\mu(U) \mid U \supset E \text{ with } U \text{ open}\}. \tag{1.3.4}$$

Specific cases of regularity

3. Every measurable set $A \subset X$ contains F a F_σ set such that $\mu(F) = \mu(A)$.

4. Every set $E \subset X$ is contained in G a G_δ set such that $\mu(E) = \mu(G)$.

5. For every $A \subset X$ measurable and for each $\varepsilon > 0$ there exists $C \subset X$ and $O \subset X$ closed and open sets respectively such that $C \subset A \subset O$ and satisfy

$$\mu(O \setminus C) < \varepsilon.$$

1.3.1 Radon measures on metric measure spaces

By $\sigma-$finiteness of metric measure space and Theorem 1.2.26, we have the next

Theorem 1.3.5: Radon measures in metric measure spaces are inner regular(approximated by compacts)

Let (X, d, μ) be a metric measure space with μ be a Radon measure. Then, every measurable set $A \subset X$ is inner regular by compacts sets

Theorem 1.3.6: Decomposition theorem for metric measure space with a Radon measure

Let (X, d, μ) be a metric measure space with μ be a Radon measure. Then, X can be expressed as a countable union of compact sets plus a set of measure zero.

Proof: From Item 2 of Theorem 1.3.1, we can cover X by countably many balls $\{B_n\}_{n\in\mathbb{N}}$, each of finite measure. Because μ is Radon, for each $j \in \mathbb{N}$, we can find a compact set $K_{n,m} \subset B_n$ such that

$$\mu(B_n) - \frac{\varepsilon}{2^m} < \mu(K_{n,m})$$

Sincel B_n and $K_{n,m}$ are measurable

$$\mu(B_n \setminus K_{n,m}) < \frac{\varepsilon}{2^m}.$$

Using the same argument of Theorem 1.2.21 and the continuity from below of measure, we have

$$\mu\left(B_n \setminus \bigcup_{m\in\mathbb{N}} K_{n,m}\right) = 0,$$

by $\sigma-$additivity, we have

$$\mu\left(X \setminus \bigcup_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} K_{n,m}\right) = 0.$$

Thus, $\{K_{n,m}\}_{n,m\in\mathbb{M}}$ is a countable decomposition of X by compact sets such that

$$\mu\left(X \setminus \bigcup_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} K_{n,m}\right) = 0.$$

■

Remark This decomposition is used to extend the properties of measure.

Theorem 1.3.7: Sufficient conditions for Radon measures on a metric measure space

Let (X, d, μ) a metric measure space. If either

1. μ is inner regular by compacts on closed sets.
2. (X, d) complete.

then μ is a Radon measure. Furthermore, in this case μ is inner regular by compact sets on measurable sets.

Proof: Since μ is a Borel regular measure, we have that μ is outer regular, and from Item 3 of Theorem 1.3.1, we have that the compact sets have measure finite. Then, to prove that μ is Radon, we only need to prove the inner regularity by compact sets on open sets. Let $O \subset X$ a measurable set and $\varepsilon > 0$.

1. Suppose that Item 1 holds. Since μ is Borel regular, it follows that μ is inner regular by closed sets on O, then there exist a sequence of closed subsets $\{C_n\}_{n\in\mathbb{N}}$ of O such that $\mu(C_n) \uparrow \mu(O)$ from Proposition 1.2.17, we have that $\mu\left(\bigcup_{n\in\mathbb{N}} C_n\right) = \mu(O)$. Let $n \in \mathbb{N}$, since Item 1 holds we have that there exists

$K_n \subset C_n$ such that

$$\mu(C_n) - \frac{\varepsilon}{2} < \mu(K_n)$$
$$\mu(C_n \setminus K_n) < \frac{\varepsilon}{2^{n+1}}$$

Using the same argument of Theorem 1.2.21, we obtain a similar inequality as (1.2.23), this together with the fact μ is finite on compact sets

$$\mu(O) - \mu\left(\bigcup_{n=1}^{M} K_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} C_n\right) - \mu\left(\bigcup_{n=1}^{M} K_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} C_n \setminus \bigcup_{n=1}^{M} K_n\right) < \varepsilon$$

Since $\bigcup_{n=1}^{M} K_n \subset O$ is compact, then (1.2.24) holds. Therefore, μ is inner regular by compact sets on measurable sets.

2. Suppose that Item 2 holds. From Item 1, we only need to prove that μ is inner regular by compacts on closed sets, furthermore, by the $\sigma-$finiteness of metric measure spaces, we may assume that $\mu(A) < \infty$.

   Let $A \subset X$ be a closed subset with finite measure and let $n \in \mathbb{N}$. Since A is closed it follows that A is Lindelöf for all $n \in \mathbb{N}$ there exists a countable collection of closed balls $\{\overline{B}_{n_k}\}_{k\in\mathbb{N}}$ with centers in A, each with radius $\frac{1}{n}$, such that

$$A \subset \bigcup_{k=1}^{\infty} \overline{B}_{n_k}.$$

   Thus, $\{A \cap \bigcup_{k=1}^{n} \overline{B}_{n_k}\}_{n\in\mathbb{N}}$ is an increasing sequence of closed sets such that converges to A, and since $\mu(A)$ is finite it follows that there exists $N_n \in \mathbb{N}$

$$\mu(A) - \frac{\varepsilon}{2^n} < \mu\left(A \cap \bigcup_{k=1}^{N_n} \overline{B}_{n_k}\right)$$
$$\mu\left(A \setminus \bigcup_{k=1}^{N_n} \overline{B}_{n_k}\right) < \frac{\varepsilon}{2^n} \tag{1.3.8}$$

   Let us define

$$C_n = \bigcup_{k=1}^{N_n} \overline{B}_{n_k},$$
$$K = \bigcap_{n=1}^{\infty} C_n.$$

   Now, we prove the following properties of K:

   $\boldsymbol{\mu(K)}$ **approximates** $\mu(A)$**.**

   From the subadditivity of μ, we have

$$\begin{aligned}\mu(A) &\leq \mu\left(A \cap \bigcap_{k=1}^{m} C_k\right) + \sum_{n=1}^{m} \mu(A \setminus C_n) \\ &\leq \mu\left(A \cap \bigcap_{k=1}^{m} C_k\right) + \epsilon.\end{aligned} \qquad \textit{From (1.3.8).} \tag{1.3.9}$$

   Since A as finite measure, we can use the continuity from above of measure, then

$$\mu(A \cap K) = \lim_{m\to\infty} \mu\left(A \cap \bigcap_{k=1}^{m} C_k\right),$$

Using this limit in (1.3.9), we obtain that

$$\mu(A \cap K) \geq \mu(A) - \epsilon.$$

This proves that $\mu(A)$ approximates $\mu(K)$.

K is compact.

Let us prove that K is totally bounded, to do this we use the characterization from Theorem 1.1.9. Let $S \subset K$ be a $\varepsilon-$separated set, from the definition of K, we have that $S \subset C_n$ for all $n \in \mathbb{N}$. Considering n large enough such that $\frac{2}{n} < \delta$, from Proposition 1.1.7, it follows that S is finite.

By definition K is closed, hence is complete, and we have already proved that K is totally bounded. Therefore K is compact.

This proves that μ is inner regular by compacts on closed sets. Therefore, the conclusions of Theorem 1.3.7 hold.

■

There is more results about the regularity of metric measure spaces, the above results are sufficient to this thesis. For a further lecture about the properties of metric measure spaces see [Hei+15].

1.3.2 Differentiation of measures on metric measure spaces

We present the extension of the classical results of differentiation of measures for a metric measure space. For the classical results see [Rud87, Chapter 7].

First, we introduce the notion of metric derivative in metric measure spaces. It is a well known fact in the metric measure setting that this generalization require for an extra condition on the measures

Definition 1.3.10: Doubling Space, Doubling Measure

1. For a metric measure space (X, d, μ), we say that μ is a ***C*−doubling measure** if μ is non-trivial and $\exists C > 0$ such that $\mu(B(x, r)) \leq C\mu\left(B\left(x, \frac{r}{2}\right)\right)$ $\forall x \in X, \forall r \in \mathbb{R}^+$.
2. For a metric space (X, d), we say that it is ***C*−doubling**, with $C \in \mathbb{Z}^+$, if any ball of radius r can be covered by at most C balls of radius $\frac{r}{2}$.

Considering the previous restriction, we can define.

Definition: Metric derivative

Let (X, d, μ) be a metric measure space and ν be a Borel regular locally finite measure. If at least one of the measures is locally doubling, we define the **derivative of $\boldsymbol{\nu}$ respect to $\boldsymbol{\mu}$ at $\boldsymbol{x}$** as the limit

$$\frac{\boldsymbol{d\nu}}{\boldsymbol{d\mu}}\boldsymbol{(x)} = \lim_{r \downarrow 0} \frac{\nu(B[x, r])}{\mu(B[x, r])}.$$

The reason to ask for the doubling condition is to make the above definition compatible with the Radon-Nikodym derivative.

First, we extend the definition of variation.

Definition: Variation of a function of one variable

Let $I \subset \mathbb{R}$ be an interval, $f\colon I \subset \mathbb{R} \to (X,d)$ a function. We define the **variation** of f in $[a,b] \subset I$ as follows:

$$\begin{aligned}\mathbf{Var}(\boldsymbol{f},[\boldsymbol{a},\boldsymbol{b}]) &= \boldsymbol{V}(\boldsymbol{f},[\boldsymbol{a},\boldsymbol{b}]) \\ &= \sup\left\{ \sum_{k=1}^{n} d(f(t_k), f(t_{k-1})) \,\middle|\, \{t_k\}_{k=0}^{n} \text{ is a partition of } [a,b] \right\}.\end{aligned}$$

where a partition of the interval $[a,b]$ is an ordered finite collection of points in $[a,b]$, we denote this as follows

$$a = t_0 < \ldots < a_k < \ldots < t_n = b.$$

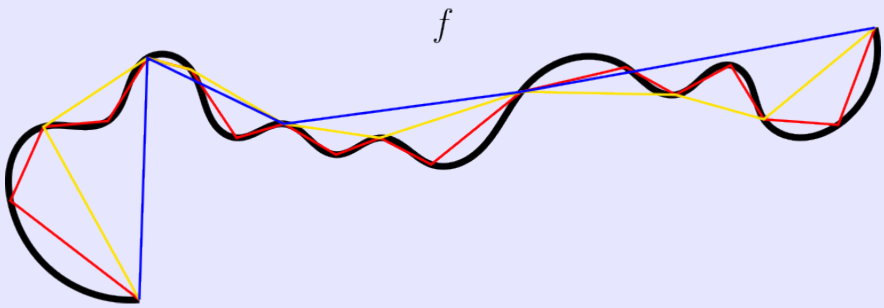

Figure 1.3.11: Variation of a function

Remark Considering extended values the variation always exist.

From the definition of variation follows immediately the next:

Proposition (Monotonicity of the variation): Let $I \subset \mathbb{R}$ be an interval $f\colon I \subset \mathbb{R} \to (X,d)$ a function. If $[a_2,b_2] \subset [a_1,b_1] \subset I$, then

$$V(\gamma,[a_2,b_2]) \leq V(\gamma,[a_1,b_1]).$$

Now, we will prove the following property of the variation:

Lemma: Additive property of the variation

Let X be a metric space and $f\colon I \subset \mathbb{R} \to X$ be a function where $I \subset \mathbb{R}$ is an interval. For all $[a,b] \subset I$, the following identity holds.

$$V(\gamma,[a,b]) = V(\gamma,[a,c]) + V(\gamma,[c,b]) \qquad \forall c \in (a,b). \tag{1.3.12}$$

Proof: We will prove that the following inequalities holds:

$\leq$

Let $\{x_k\}_{k=0}^{n} \subset [a,b]$ be a partition of $[a,b]$. Let us define $\{y_k\}_{k=0}^{l}$ be the partition given by $\{x_k\}_{k=0}^{n} \cup \{c\}$. Thus, we can separate $\{y_k\}_{k=0}^{l}$ as the union of
Therefore:

$$V(\gamma,[a,b]) \leq V(\gamma,[a,c]) + V(\gamma,[c,b]) \qquad \forall c \in (a,b).$$

$\geq$

Without loss of generality, we can assume that $V(\gamma,[a,b])$ is finite. Then, from Proposition 1.3.2, it follows that $V(\gamma,[a,c])$ and $V(\gamma,[c,b])$ are finite. Let $\{x_k\}_{k=0}^{n}, \{y_k\}_{k=0}^{m}$ be arbitrary partitions of $[a,c],[c,b]$ respectively. Thus $\{x_k\}_{k=0}^{n} \cup \{y_k\}_{k=0}^{m}$ is a partition of $[a,b]$.

Therefore:

$$\sum_{k=0}^{n} d(f(x_k), f(x_{k-1})) + \sum_{k=0}^{m} d(f(y_k), f(y_{k-1})) \leq V(\gamma, [a,b]) .$$

Fixing one and varying the another we obtain the result.

$$V(\gamma, [a,c]) + V(\gamma, [c,b]) \leq V(\gamma, [a,b]) .$$

Therefore, (1.3.12) holds.

■

Now, we will extend the classical differentiation for measure theorems in for a curve in a metric measure space. The proof of this result is lengthy and highly technical, this is Theorem 4.4.8 of [Hei+15]. The idea is that we can think a curve in a metric measure space as $\mathbb{R}$ and we use this idea with another techniques together with the classical Lebesgue-Radon-Nikodym to extend it to a metric measure spaces.

Theorem 1.3.13: Metric derivative of a curve

For each continuous function $\gamma : [a,b] \to X$ of bounded variation. We can associate γ with a unique Radon measure ν_γ on $[a,b]$ such that

$$\nu_\gamma(O) = V(\gamma, O) \tag{1.3.14}$$

for every open $O \subset [a,b]$ and

$$\frac{d\nu_\gamma}{dm_1}(t) = \lim_{\substack{u \to t \\ u \neq t}} \frac{d(\gamma(t), \gamma(u))}{|t-u|} := |\gamma'(t)|$$

for m_1- almost every $t \in [a,b]$.

Notice that Theorem 1.3.13, considering curves, gives a notion of differentiability in a metric measure spaces. In the following chapters, we develop the theory to define Sobolev Spaces on metric measure spaces. From now on, we will prove an useful form of Theorem 1.3.13.

Corollary 1.3.15: Let $\gamma\colon [a,b] \to X$ be a curve of bounded variation such that

$$V(\gamma, [t,u]) = t - u \qquad \forall a \leq t \leq u \leq b. \tag{1.3.16}$$

Then the following statements hold:

1. γ is $1-$Lipschitz. Hence γ is absolutely continuous.
2. The limit
$$|\gamma'(t)| = \lim_{\substack{u \to t \\ u \neq t}} \frac{d(\gamma(t), \gamma(u))}{|t-u|} = 1.$$
holds for m_1- almost every $t \in [a,b]$.
3. If $E \subset [a,b]$ satisfies $\mathcal{H}_1(\gamma[E]) > 0$, then $\mathcal{H}_1(E) > 0$. That is γ maps $\mathcal{H}_1$-null sets in $\mathcal{H}_1$-null sets.

Proof:

1. The fact that γ is $1-$Lipschitz follows immediately from (1.3.16), and the second fact is a well known result of real analysis.

2. From Theorem 1.3.13, we have that

$$
\begin{aligned}
\lim_{\substack{u\to t\\ u\neq t}} \frac{d\left(\gamma(t),\gamma(u)\right)}{|t-u|} &= \frac{d\nu_\gamma}{dm_1}(t) \\
&= \lim_{r\downarrow 0} \frac{\nu_\gamma(B[t,r])}{m_1(B[t,r])} && \textit{Definition of metric derivative.} \\
&= \lim_{r\downarrow 0} \frac{V([t-r,t+r])}{m_1([t-r,t+r])} && \textit{From (1.3.14).} \\
&= \lim_{r\downarrow 0} \frac{2r}{2r} && \textit{From (1.3.16).} \\
&= 1.
\end{aligned}
$$

3. From Item 1 we have that γ is $1-$Lipschitz, the we can use Lemma 1.2.29. Then

$$\mathcal{H}_1(\gamma[E]) \leq \mathcal{H}_1(E)$$

which proves the result.

■

We can extend the notions absolute continuity for curves, within this notion, we extend the results of differentiability of absolute continuous functions. We discuss all these details in Section 1.5.3.

1.4 Dini derivatives

In this section, we introduce the notion of derivative for Borel functions in an Euclidean set. The derivative in the Euclidean setting is defined as the limit of the Newtonian quotient, the issue within this limit is its existence. To fix this issue with the existence, we can consider the superior and inferior limits, since both are limit notions which always exists. This lead us to the following definition.

Definition: Dini derivatives

Let $f\colon \Omega\subset\mathbb{R}^d\to\mathbb{R}$ a function on an open set and $i\in\{1,\dots,d\}$. The **$i-$th directional Dini derivatives at $x\in\Omega$** are defined as:

$$
\begin{aligned}
\boldsymbol{\partial_i^+ f(x)} &= \limsup_{r\to 0^+} \frac{f(x+re_i)-f(x)}{r}, \\
\boldsymbol{\partial_{i+} f(x)} &= \liminf_{r\to 0^+} \frac{f(x+re_i)-f(x)}{r}, \\
\boldsymbol{\partial_i^- f(x)} &= \limsup_{r\to 0^-} \frac{f(x+re_i)-f(x)}{r}, \\
\boldsymbol{\partial_{i-} f(x)} &= \limsup_{r\to 0^-} \frac{f(x+re_i)-f(x)}{r}.
\end{aligned}
$$

From the definition of the Dini derivatives, it follows immediately the next

Proposition 1.4.1: Let $f\colon \Omega\subset\mathbb{R}^d\to\mathbb{R}$ a function on an open set and $i\in\{1,\dots,d\}$. Then, the partial $i-$th partial derivative of f at x, $\partial_i f(x)$ exists if and only if all the $i-$th directional Dini derivatives are finite and coincide.

Now, we will prove that the Dini derivatives are Borel.

Theorem 1.4.2: Dini derivatives are Borel functions

Let $f\colon \Omega\subset\mathbb{R}^d\to\mathbb{R}$ a measurable function. Then the Dini derivatives $\partial_i^+ f, \partial_{i+} f, \partial_i^- f, \partial_{i-} f$ are measurable functions.

Proof: We will prove the result for the Dini directional derivative $\partial_i^+ f$. The proof for the rest of the Dini derivatives is the same. Consider $g\colon \mathbb{R}^d \to \mathbb{R}$ be the function defined as

$$g(x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

From Theorem 1.2.1, it follows that g is measurable. For each $n \in \mathbb{N}$ define $g_n\colon \mathbb{R}^d \to \mathbb{R}$

$$g_n(x) = \frac{g\left(x + \frac{1}{n}e_i\right) - g\left(x\right)}{\frac{1}{n}},$$

then g_n is measurable. Then

$$\limsup_{n\to\infty} g_n(x) = \limsup_{r\to 0^+} \frac{g\left(x + re_i\right) - g\left(x\right)}{r}$$

is measurable. Therefore, the $i-$th directional Dini derivative $\partial_i^+ g\colon \mathbb{R}^d \to \mathbb{R}$ is measurable, then $\partial_i^+ f = \partial_i^+ g\big|_\Omega$ is measurable.

■

From the above theorem it follows immediately the next:

Theorem 1.4.3: The set where ∂_i exists is measurable

Let $f\colon \Omega \subset \mathbb{R}^d \to \mathbb{R}$ a measurable function. For $i \in \{1, \ldots, d\}$ define A_i as set where the partial derivative $\partial_i f$ exists is measurable. Then A_i and $\Omega \setminus A_i$ are measurable.

Proof: Let $i \in \{1, \ldots, d\}$, define

$$A_i = \left\{x \in \Omega \mid \partial_i^+ f(x) = \partial_{i+} f(x) = \partial_i^- f(x) = \partial_{i-} f(x) < \infty\right\}.$$

From Theorem 1.4.2 $\partial_i^+ f, \partial_{i+} f, \partial_i^- f, \partial_{i-} f$ are measurable, then A_i is measurable. Thus, $\Omega \backslash A_i$ is also measurable.

■

1.5 Curves in metric measure spaces spaces

We start defining concepts relatives to curves in a metric space:

Definition: Curves

Let X be a metric or a topological space.

1. A **curve** on X is a continuous function $\gamma : I \to X$ where I is an interval in $\mathbb{R}$. From the properties of the interval we say that the curve is **open**, **closed**, **half-open** or **compact**. The interval I is called the **parameter interval**. The **endpoints of the curve γ** are the image of the endpoints of the interval.
2. We say that γ is a **constant curve** if $\gamma[I]$ is a single point. For each $x \in X$, we define $c_x\colon \widetilde{I} \to X$ as the constant curve which is identically x, where x is a customary interval.
3. A **subcurve** of a curve $\gamma : I \subset \mathbb{R} \to X$ is any restriction of γ to some subinterval of I.

Remark: Considerations for the parameter interval

1. **Considering the same type of interval, we can take I bounded** Since there is a canonical homeomorphism between a non bounded interval and a bounded one, we can change the parameter if necessary.

2. The definition of **open, closed, half-open and compact curves indicate the characteristics of the parameter interval but not the properties of the curve on our metric space**. For example there are open curves that can be compact and therefore not open in some cases.

Now, let us define the geometrical notion of length:

Definition: Rectifiable curve

Let γ be a curve in X. We define

1. The **length of the curve** as follow: If the curve is compact we define the length as the total variation of γ. If the curve γ is not compact we define the length as the supremum of the lengths of the compact subcurves of γ. We denote the length by $\mathbf{length}(\gamma)$.

Thus, the length of the curve $\gamma\colon [a,b] \to X$ is defined as:

$$\mathbf{length}(\gamma) = \sup \left\{ \sum_{k=0}^{n} d(\gamma(t_k), \gamma(t_{k-1})) \,\middle|\, \{t_k\}_{k=0}^{n} \text{ is a partition of } [a,b] \right\}.$$

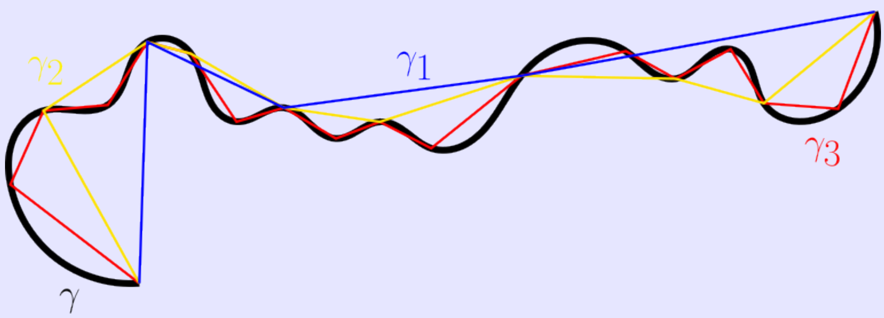

Figure 1.5.1: Length of a curve

2. We say that γ is a **rectifiable curve** if has finite length and **locally rectifiable** if all of its compact subcurves are rectifiable. If a curve is not rectifiable we say that is **nonrectifiable**.

From the definition of rectifiable curves we have Differentiation of measures on metric measure spaces, and follows immediately the following equivalences:

Theorem 1.5.2: Equivalence of rectifiability on compact curves

Let γ a compact curve in a metric space. Then, the following statements are equivalent:

1. γ is rectifiable.

2. γ is locally rectifiable.

3. γ is a function of bounded variation.

Proof:

(1)$\Leftrightarrow$(2) The implication (1)$\Rightarrow$(2) follows immediately from the properties of the variation. On the other hand, since γ is a compact curve follows that γ is a compact subcurve of itself. Thus, the converse implication, (1)$\Leftarrow$(2) holds.

(1)$\Leftrightarrow$(3) Follows immediately from the definition of length.

The previous implications prove that the statements of Theorem 1.5.2 are equivalent.

∎

Remark In this work we only considering compact curves Then we can use Theorem 1.5.2 as a characterization of rectifiability.

The next result give us an useful example of rectifiable curves

Proposition 1.5.3: A Lipschitz curve is locally rectifiable.

Proof: Let γ a $L-$Lipschitz curve and $\gamma\mid_J$ a compact subcurve of γ, then we have

$$\begin{aligned} \text{length}(\gamma\mid_J) &= V(\gamma\mid_J) && \textit{Definition of variation.} \\ &\leq L\,\text{length}(J) < +\infty. && \textit{f is } L\textit{-Lipschitz.} \end{aligned}$$

This proves that each compact subcurve of γ is rectifiable therefore γ is locally rectifiable.

■

The natural generalization of Proposition 1.5.3 is the next:

Theorem 1.5.4: Composition of curves by a Lipschitz function is a Lipschitz curve

Let $f : X \to Y$ a $L-$Lipschitz function and $\gamma \colon I \to X$ be a curve then $f \circ \gamma$ is a curve and

$$\text{length}(f \circ \gamma) \leq L\,\text{length}(\gamma). \tag{1.5.5}$$

Furthermore, if γ is rectifiable then $f \circ \gamma$ is rectifiable.

Proof: The fact that $f \circ \gamma$ is a curve follows from the continuity of f. First, we will prove the result for $I = [a, b]$, let $\{t_k\}_{k=0}^n$ be a partition of $[a, b]$, since f is Lipschitz it follows that

$$\begin{aligned} \sum_{k=1}^n d\left(f\circ\gamma\left(t_k\right), f\circ\gamma\left(t_{k-1}\right)\right) &\leq \sum_{k=1}^n Ld\left(\gamma\left(t_k\right), \gamma\left(t_{k-1}\right)\right) && \textit{Definition of } \gamma. \\ &\leq L\,\text{length}\left(\gamma\right) \end{aligned}$$

From the definition of length we extend the inequality (1.5.5) for any curve.

■

Now, we will introduce the classical operations with curves.

Definition: Concatenation

Let X be a metric space. For $i = 1, 2$ we consider $\gamma_i \colon [0, a_i] \to X$ compact curves in X such that $\gamma_1(a_1) = \gamma_2(0)$. The **concatenation** is the function $\boldsymbol{\gamma_1 * \gamma_2} \colon [0, a_1 + a_2] \to X$ given by

$$\boldsymbol{\gamma_1 * \gamma_2(t)} = \begin{cases} \gamma_1(t) & \text{if } t \in [0, a_1], \\ \gamma_1(t - a_1) & \text{if } t \in [a_1, a_1 + a_2]. \end{cases}$$

From the definition of concatenation and by the additive property of the variation of a curve, follows immediately the next:

Lemma 1.5.6: Concatenation preserve rectifiability

Let X be a metric space and $\gamma_i \colon [0, a_i] \to X$ with $i = 1, 2$ two compact curves in X such that $\gamma_1(a_1) = \gamma_2(0)$. Then

$$\text{length}\left(\gamma_1 * \gamma_2\right) \leq \text{length}\left(\gamma_1\right) + \text{length}\left(\gamma_2\right).$$

Particularly, if the curves γ_1, γ_2 are rectifiable, then the concatenation $\gamma_1 * \gamma_2$ is rectifiable.

Proof: Let $\{t_k\}_{k=0}^n$ a partition of $[0, a_1 + a_2]$, we can assume that a_1 is an element of the partition $\{t_k\}_{k=0}^n$. Thus, we can decompose the partition $\{t_k\}_{k=0}^n$ as $\{r_k\}_{k=0}^{n_1}, \{s_k\}_{k=0}^{n_2}$ partitions of $[0, a_1], [a_1, a_1 + a_2]$ respectively. Considering the previous decompositions, we have

$$\begin{aligned}
\sum_{k=1}^{n} d\left((\gamma_1 * \gamma_2)(t_k), (\gamma_1 * \gamma_2)(t_{k-1})\right) &= \sum_{k=1}^{n} d\left((\gamma_1 * \gamma_2)(r_k), (\gamma_1 * \gamma_2)(r_{k-1})\right) \\
&+ \sum_{k=1}^{n} d\left((\gamma_1 * \gamma_2)(s_k), (\gamma_1 * \gamma_2)(s_{k-1})\right) && \textit{Definition of } \gamma_1 * \gamma_2. \\
&= \sum_{k=1}^{m} d\left(\gamma_1(r_k), \gamma_1(r_{k-1})\right) \\
&+ \sum_{k=1}^{m} d\left(\gamma_2(s_k - a_1), \gamma_2(s_{k-1} - a_1)\right) && \{s_k - a_1\}_{k=0}^{n_2} \textit{ is a partition of } [0, a_2]. \\
&\leq \text{length}(\gamma_1) + \text{length}(\gamma_2)
\end{aligned}$$

■

After continue with our analysis is convenient give the next definition, this is motivated of the topology definitions:

Definition: Rectifiable component of a point

Let X be a metric space and $x \in X$. The **rectifiable component** of x is the set of points $y \in X$ such that there exists a rectifiable path from x to y.

From Lemma 1.5.6, follows immediately the next:

Proposition (The rectifiable components of a point defines a partition): Let X be a metric space and define the relation $x \sim y$ if and only if there is a rectifiable curve with endpoints x and y. Then, the relation $x \sim y$ is an equivalence relation.

Counterexample: Rectifiable component can be strictly contained in the path component

The Collapsed topologist's sine curve is **path connected but not rectifiable connected**. Furthermore, has **three rectifiable components**. That is because any nonconstant curve such that contains to $(0, 0)$ is not rectifiable.

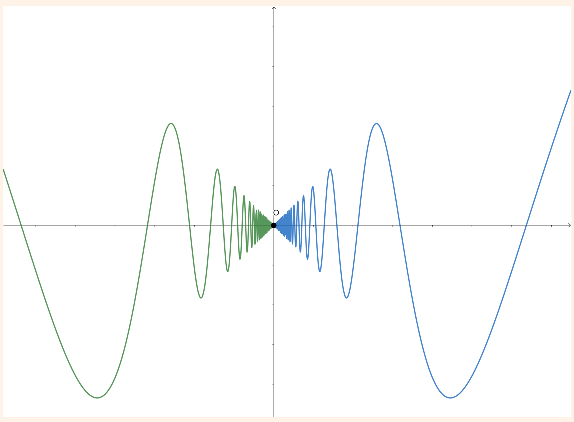

Figure 1.5.7: Topologist's sin curve

1.5.1 The length function s_γ and its right-inverse

Now, let us the introduce the following function:

Definition: Length function

Let $\gamma\colon [a,b] \to X$ a rectifiable curve. We define **length function** $s_\gamma\colon [a,b] \to [0, \text{length}(\gamma)]$ given by:

$$s_\gamma(t) = \text{length}\left(\gamma\mid_{[a,t]}\right).$$

We will use the arc length to define the canonical parametrization of a curve. The reason for using this specific parametrization lies in its desirable properties. The following result is about the arithmetic properties of the arc length function

Proposition (Arithmetic properties of s_γ): Let $\gamma\colon [a,b] \to X$ a rectifiable curve.

$$s_{\gamma|_{[t_1,t_0]}}(t_1) = s_\gamma(t_0) - s_\gamma(t_1) \qquad \forall\, a \le t_0 \le t_1 \le b \tag{1.5.8}$$

Proof: From the additive property of the variation, it follows that

$$\begin{aligned}\text{length}\left(\gamma|_{[a,t_0]}\right) + \text{length}\left(\gamma|_{[t_0,t_1]}\right) &= \text{length}\left(\gamma|_{[a,t_1]}\right) \\ s_{\gamma|_{[t_0,t_1]}}(t_1) &= s_\gamma(t_1) - s_\gamma(t_0)\end{aligned} \quad \textit{Definition of } s_\gamma.$$

■

The following result is a usual technique to prove results when the length is involved.

Proposition 1.5.9: Let $\gamma\colon [a,b] \to X$ a rectifiable curve. If

$$\text{length}(\gamma) > \delta > 0 \tag{1.5.10}$$

then there exists a collection of points $\{t_k\}_{k=0}^n$ such that $a = t_0 < \ldots < t_n < b$ and

$$\sum_{k=1}^{n} d\left(\gamma(a_k), \gamma(a_{k-1})\right) > \delta \tag{1.5.11}$$

Proof: We proceed by contradiction. ※Suppose that the conclusion of Proposition 1.5.9 does not hold, that is every finite set of points $\{t_k\}_{k=0}^n$ such that $a = t_0 < \ldots < t_n < b$ satisfies

$$\sum_{k=1}^{n} d\left(\gamma(t_k), \gamma(t_{k-1})\right) \le \delta \tag{1.5.12}$$

Let $\{s_k\}_{k=0}^m$ be a partition of $[a,b]$, for this partition, we consider $\{t_k\}_{k=0}^n$ defined as follows

$$t_k = \begin{cases} s_k & \text{if } k < m, \\ \text{any number of } (s_{m-1}, s_m) & \text{if } k = m. \end{cases}$$

Then, (1.5.12) holds and since t_m is any number in (s_{m-1}, s_m), we can take the limit in (1.5.12) when $s_n \to t_n^-$; from the continuity of γ and the metric together with the definition of t_k, it follows that

$$\sum_{k=1}^{m} d\left(\gamma(s_k), \gamma(s_{k-1})\right) \le \delta$$

Since the partition $\{s_k\}_{k=0}^m$ is arbitrary, we have

$$\text{length}(\gamma) \le \delta$$

this contradicts (1.5.10). ※Therefore, the desired collection of points exists.

■

Remark The intuitive idea of Proposition 1.5.9 is that to obtain partition such that the estimation (1.5.11) holds, we do not need the whole curve γ. We will use this idea to prove the continuity of the arc length function s_γ.

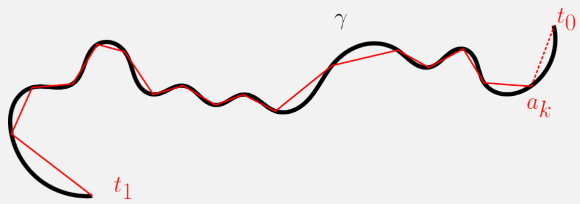

Figure 1.5.13: "Incomplete" partition

Considering the above results, we state the result about the properties of the arc length function.

Lemma 1.5.14: Length function is increasing continuous

The length of a compact rectifiable curve $\gamma\colon [a,b] \to X$, $s_\gamma\colon [a,b] \to [0, \text{length}(\gamma)]$, is increasing continuous.

Proof: The fact that s_γ is an increasing function is clear, now we prove the continuity. Let $t_0 \in [a,b]$ fixed. Since s_γ is increasing, it follows that the one-sided limits

$$\boldsymbol{s(t_0)^-} = \lim_{t \to t_0^-} s(t),$$
$$\boldsymbol{s(t_0)^+} = \lim_{t \to t_0^+} s(t)$$

exists. Now, we will prove that the following equations hold:

$\boldsymbol{s(t_0)^- = s(t_0)}$

We proceed by contradiction ※Suppose that $s_\gamma(t_0) > s_\gamma(t_0)^-$, then, there exists $s_\gamma(t_0) - s_\gamma(t_0)^- > \delta > 0$, thus $t_0 > a$. Let $a < t_1 < t_0$, from (1.5.8) it follows that

$$\begin{aligned} s_{\gamma|_{[t_1,t_0]}}(t_0) &= s_\gamma(t_0) - s_\gamma(t_1) \\ \text{length}\left(\gamma|_{[t_1,t_0]}\right) &\geq s_\gamma(t_0) - s_\gamma^-(t_0) \\ &> \delta \end{aligned} \tag{1.5.15}$$

$s_\gamma(t_1) \leq s_\gamma^-(t_0)$ since s_γ is increasing.

Since the above inequality holds, we can use Proposition 1.5.9, then, there exists $t_1 = a_0 < \cdots < a_k < t_0$ that satisfies:

$$\sum_{j=1}^{k} d\left(\gamma(a_j), \gamma(a_{j-1})\right) > \delta. \tag{1.5.16}$$

Define $t_2 = a_k$. From the above inequality, it follows that length $\left(\gamma|_{[t_1,t_2]}\right) > \delta$ this inequality is essentially the same as (1.5.15), then we can repeat the procedure to obtain a sequence of values $t_1 < t_2 < \cdots < t_i < \cdots < t_0$ such that length length $\left(\gamma|_{[t_i,t_{i+1}]}\right) > \delta$. Then, for every $i \geq 2$, we have

$$\begin{aligned} \text{length}\left(\gamma|_{[t_1,t_0]}\right) &\geq \text{length}\left(\gamma|_{[t_1,t_i]}\right) \\ &= \sum_{k=2}^{i} \text{length}\left(\gamma|_{[t_{k-1},t_k]}\right) \\ &> (i-1)\delta \end{aligned}$$

length $\left(\gamma|_{[t_i,t_{i+1}]}\right) > \delta.$

for every $i = 2, 3, \ldots$, contradicting the rectifiability of γ. ※Therefore, $s_\gamma^-(t_0) = s_\gamma(t_0)$.

$\boldsymbol{s(t_0) = s(t_0)^+}$

This equation is proved analogously by contradiction as the previous one.

This proves that $s(t_0)^- = s(t_0) = s(t_0)^+$.
Therefore, s_γ is continuous.

■

Counterexample: s_γ is not necessarily strictly increasing

Consider the curve $\gamma\colon [-1,1] \to \mathbb{R}$ given by

$$\gamma(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0, \\ x & \text{if} \quad 0 < x \leq 1. \end{cases}$$

Clearly $s_\gamma = \gamma$ which is not strictly increasing function.

Now, we prove some properties of s_γ.

Theorem: Invertibility of s_γ

Let $\gamma\colon [a,b] \to X$ be a curve. The function $s_\gamma^{-1}\colon [0, \operatorname{length}(\gamma)] \to [a,b]$ defined as

$$\boldsymbol{s_\gamma^{-1}(t)} = \sup\{ s \in [a,b] |\, s_\gamma(s) = t\}$$

has the following properties:

1. $$s_\gamma^{-1}(t) = \max\{ s \in [a,b] |\, s_\gamma(s) = t\}. \tag{1.5.17}$$

2. s_γ is a right-inverse of s_γ.

3. s_γ is increasing right-continuous. Furthermore, if $\lim_{t\to t_0^-} s_\gamma^{-1}(t) = s_0 < s_\gamma^{-1}(t_0)$ then s_γ^{-1} is constant on $\left[s_0, s_\gamma^{-1}(t_0)\right]$.

<u>Proof:</u>

1. Notice that $\gamma^{-1}[\{t\}] \subset [a,b]$ is a compact set, thus $\gamma^{-1}[\{t\}]$ has a maximum and clearly $\gamma^{-1}(t) = \sup \gamma^{-1}[\{t\}]$. Therefore $\gamma^{-1}(t) \in \gamma^{-1}[\{t\}]$

2. From (1.5.17) and the definition of s_γ^{-1}, we have $s_\gamma\left(s_\gamma^{-1}(t)\right) = t$. This proves $s_\gamma\left(s_\gamma^{-1}\right) = \operatorname{Id}_{[a,b]}$. Therefore, s_γ^{-1} is a right-inverse of s_γ.

3. From Lemma 1.5.14 we have that s_γ is increasing right-continuous, then the inverse also is.

■

1.5.2 Arc length parametrization

With the arc length, we define the canonical parametrization for a curve.

Definition: Arc length parametrization

Let $\gamma : [a,b] \to X$ a rectifiable curve. The **arc length parametrization** of γ is the curve $\boldsymbol{\gamma_s}\colon [0, \operatorname{length}(\gamma)] \to X$ defined by

$$\boldsymbol{\gamma_s(t)} = \gamma\left(s_\gamma^{-1}(t)\right)$$

where s_γ^{-1} is the one-sided inverse of s_γ.

From the definition of arc length parametrization, we have immediately the next:

Theorem: Properties of arc length parametrization

Let $\gamma\colon [a,b] \to X$ a rectifiable curve. The arc length parametrization has the following properties:

1. γ_s is the unique curve that satisfies:
$$\gamma = \gamma_s \circ s_\gamma. \tag{1.5.18}$$
2. $$\operatorname{length}\left(\gamma_s|_{[t_0,t_1]}\right) = t_1 - t_0 \qquad \forall\, 0 \le t_0 \le t_1 \le \operatorname{length}(\gamma).$$

Proof:

1. By definition of arc length parametrization, it follows that $\gamma_s = \gamma \circ s_\gamma^{-1}$ where s_γ^{-1} is the right-inverse of s_γ. Then, the identity (1.5.18) follows immediately.
2. From the arithmetic properties of the arc length function (1.5.8), we have
$$s_{\gamma|_{\left[s_\gamma^{-1}(t_0),\, s_\gamma^{-1}(t_1)\right]}}\left(s_\gamma^{-1}(t_1)\right) = s_\gamma\left(s_\gamma^{-1}(t_1)\right) - s_\gamma\left(s_\gamma^{-1}(t_0)\right)$$
s_γ^{-1} is the right-inverse of s_γ. $\left[s_\gamma^{-1}(t_0), s_\gamma^{-1}(t_1)\right] = s_\gamma^{-1}[[t_0,t_1]]$.
$$\operatorname{length}\left(\gamma|_{s_\gamma^{-1}[[t_0,t_1]]}\right) = t_1 - t_0$$
$\gamma|_{s_\gamma^{-1}[[t_0,t_1]]} = \gamma \circ s_\gamma^{-1}\big|_{[t_0,t_1]} = \gamma_s|_{[t_0,t_1]}$.
$$\operatorname{length}\left(\gamma_s|_{[t_0,t_1]}\right) = t_1 - t_0$$

■

Notice that γ_s satisfies the conditions of Corollary 1.3.15. Then, we have automatically the following:

Theorem 1.5.19: Metric derivative of γ

The arc length parametrization γ_s of a compact rectifiable curve in a metric measure space satisfies the following:

1. γ_s is $1-$Lipschitz. Hence γ_s is absolutely continuous.
2. $$|\gamma_s'(t)| = \lim_{\substack{u\to t\\ u\neq t}} \frac{d\left(\gamma_s(t), \gamma_s(u)\right)}{|t-u|} = 1.$$
3. If $E \subset [a,b]$ satisfies $\mathcal{H}_1\left(\gamma_s[E]\right) > 0$, then $\mathcal{H}_1(E) > 0$.

1.5.3 Absolute continuity on curves

From Theorem 1.5.2, it follows that the theory of differentiation of functions of bounded variation can be applied to rectifiable curves. Since we will deal with curves, it is convenient to adapt some terms of the theory of functions of bounded variation for rectifiable curves.
We extend the concept of absolutely continuity for curves.

Definition: Absolute continuity on γ

Let $\gamma : [a,b] \to X$ a continuous map of bounded variation. We say that γ is **absolutely continuous** if: For each $\varepsilon > 0$ there exists $\delta > 0$ such that:
$$\sum_{i=1}^{k} d(\gamma(b_i), \gamma(a_i)) < \varepsilon$$
whenever $\{(a_i,b_i)\}_{i=1}^{k}$ are nonoverlapping subintervals of $[a,b]$ with
$$\sum_{i=1}^{k} (b_i - a_i) < \delta.$$

From the above definition, it follows immediately the next:

Theorem 1.5.20: Characterization of absolute continuity of a curve

Let $\gamma\colon [a,b] \to X$ be a rectifiable curve in a metric space X. Then, γ is absolutely continuous if and only if its length function $s_\gamma\colon [a,b] \to [0, \operatorname{length}\gamma]$ is absolutely continuous.

Proof:

$\Rightarrow$ Assume that γ is absolutely continuous. Let $\varepsilon > 0$, and $\delta > 0$ as in the definition of absolutely continuity for γ. Let $\{[a_i, b_i]\}_{i=1}^k$ be nonoverlapping subintervals of $[a,b]$ with $\sum_{i=1}^k |b_i - a_i| < \delta$.
For $i \in \{1, \ldots, k\}$, we have that $s_\gamma(b_i) - s_\gamma(a_i) = \operatorname{length}\gamma\,|_{[a_i,b_i]} < \infty$. Thus, we can divide $[a_i, b_i]$ in k_i intervals $\{[a_{i,j}, b_{i,j}]\}_{j=1}^{k_i}$, such that

$$\begin{aligned}
s_\gamma(b_i) - s_\gamma(a_i) - \frac{\varepsilon}{k} &< \sum_{j=1}^{k_i} d(\gamma(b_{i,j}), \gamma(a_{i,j})) \\
s_\gamma(b_i) - s_\gamma(a_i) &< \frac{\varepsilon}{k} + \sum_{j=1}^{k_i} d(\gamma(b_{i,j}), \gamma(a_{i,j})) \qquad \textit{Adding these } i \textit{ inequalities.} \\
\sum_{i=1}^{k} (s_\gamma(b_i) - s_\gamma(a_i)) &< \sum_{i=1}^{k} \left(\frac{\varepsilon}{k} + \sum_{j=1}^{k_i} d(\gamma(b_{i,j}), \gamma(a_{i,j})) \right) \\
\sum_{i=1}^{k} (s_\gamma(b_i) - s_\gamma(a_i)) &< \varepsilon + \sum_{i=1}^{k} \sum_{j=1}^{k_i} d(\gamma(b_{i,j}), \gamma(a_{i,j})).
\end{aligned} \tag{1.5.21}$$

From the definition of $\{[a_{i,j}, b_{i,j}]\}_{j=1}^{k_i}$, it follows immediately that $\sum_{i=1}^k \sum_{j=1}^{k_i} |b_{i,j} - a_{i,j}| = \sum_{i=1}^k |b_i - a_i| < \delta$. Thus, $\sum_{i=1}^k \sum_{j=1}^{k_i} d(\gamma(b_{i,j}), \gamma(a_{i,j})) < \varepsilon$. From, this inequality and (1.5.21), it follows that.

$$\sum_{i=1}^{k} |s_\gamma(b_i) - s_\gamma(a_i)| < 2\varepsilon$$

Therefore, s_γ is absolutely continuous.

$\Leftarrow$ Conversely, assume that s_γ is absolutely continuous. Let $\varepsilon > 0$, and $\delta > 0$ as in the definition of absolutely continuity for s_γ. Let $\{[a_i, b_i]\}_{i=1}^k$ be nonoverlapping intervals of $[0, \operatorname{length}(\gamma)]$ with $\sum_{i=1}^k |b_i - a_i| < \delta$. Then:

$$\begin{aligned}
\varepsilon &> \sum_{i=1}^{k} |s_\gamma(b_i) - s_\gamma(a_i)| \qquad s_\gamma(b_i) - s_\gamma(a_i) = \operatorname{length}\gamma\,|_{[a_i,b_i]}. \\
&= \sum_{i=1}^{k} \operatorname{length}\gamma\,|_{[a_i,b_i]} \qquad \textit{Properties of the length.} \\
&\geq \sum_{i=1}^{k} d\left(\gamma\left(b_i\right), \gamma\left(a_i\right)\right).
\end{aligned}$$

Therefore, γ is absolutely continuous.

■

Remark: Underlying notion of differentiability on curves that are absolutely continuous
Theorem 1.5.20 gives an equivalence between the absolutely continuity of curves and absolutely continuity of functions. The advantage of absolutely continuity of s_γ is that this is a sufficient condition to have a derivative. Thus, considering appropriate families of curves, we can define notions of partial derivatives.

Now we prove some technical results about Absolute continuity on curves.
Theorem 1.5.20 suggests that the natural definition for a function be absolutely continuous in a rectifiable curve is:

Definition: Absolute continuity of a function on a curve

Let $f\colon X \to \mathbb{R}$ be a function and γ any rectifiable curve on X. We say that f is **absolutely continuous on γ** if $f \circ \gamma_s$ is absolutely continuous on $[0, \text{length}(\gamma)]$.

As in the classical case, we obtain the following

Theorem 1.5.22: Absolute continuity of functions and measures

Let $\gamma\colon [a,b] \to X$ a continuous function of bounded variation. Then γ is absolutely continuous if and only if the associated Radon measure ν_γ is absolutely continuous with respect to the Lebesgue measure m_1.

For the proof see [Rud87]. The following result of Theorem 1.5.22 gives the intuitive idea of the concept of upper gradient.

Corollary 1.5.23 (Glimpse of upper gradients): Let $\gamma : [a,b] \to X$ a continuous function of bounded variation. If γ is absolutely continuous we have:

$$d(\gamma(a), \gamma(b)) \le \int_a^b |\gamma'(t)| \ dt.$$

Proof: From Theorem 1.3.13 follows that:

$$\begin{aligned}
\int_a^b |\gamma'(t)|\, dt &= \int_a^b \frac{d\nu_\gamma}{dm_1}(t)\, dm_1(t) && \textit{Lebesgue-Radon-Nikodym theorem and Theorem 1.5.22.}\\
&= \int_a^b d\nu_\gamma(t) \\
&= \nu_\gamma((a,b)) && \textit{Definition of } \nu_\gamma. \\
&= V(\gamma, (a,b)) && \textit{By the definition of variation.} \\
&\ge d(\gamma(a), \gamma(b)).
\end{aligned}$$

■

Notice that in Corollary 1.5.23, essentially we use concepts that can be defined in a metric measure space. This inequality give us an idea of the size of the derivative along a curve. This is glimpse to the concept of an upper gradient, which generalizes the notion of the derivative in a larger class of spaces.

1.5.4 Line integral

Once we have defined the arc length parametrization, we define the

Definition: Line integration

Line integration

1. Given $\gamma : [a,b] \to X$ a **rectifiable curve** and a nonnegative Borel function $\rho\colon X \to [0,\infty]$ we define the **line integral of ρ over γ** as:

$$\int_\gamma \boldsymbol{\rho\, ds} := \int_0^{\text{length}(\gamma)} \rho(\gamma_s(t))\, dt.$$

2. Given $\gamma : [a,b] \to X$ a **locally rectifiable curve** and a nonnegative Borel function $\rho\colon X \to [0,\infty]$ we define the **line integral of ρ over γ** as:

$$\int_\gamma \boldsymbol{\rho\, ds} := \sup\left\{ \int_\eta \rho\, ds \ \middle|\ \eta \text{ is compact subcurve of } \gamma \right\}.$$

To extend this definition for any Borel function we consider the canonical decomposition $\rho = \rho^+ - \rho^-$ and previous definitions.

Special cases

For upper gradients, we use exclusively nonnegative Borel functions. In this case, we can think the line integral $\int_\gamma \rho$ as the weight of ρ over γ. Therefore, we refer to $\int_\gamma \rho$ as $\boldsymbol{\rho}$**−weight**; hence in this case we refer to ρ as a **density**.

Remark

1. **Integrability on a curve.** By the definition of line integral we have that it admits values in $[0, \infty]$. Therefore the nonintegrability of ρ on γ is equivalent to $\int_\gamma \rho \, ds = \infty$. We use this equivalence without further mention.

2. **The line integral is a pure metric concept.** That is because the line integral is defined within Riemann integral in $\mathbb{R}$, therefore we not need a measure on our space.

To define this, we will introduce the following:

Definition: Integrable curves for a Borel function

Let X be a metric measure space and $\rho\colon X \to [0, \infty]$ be a Borel function. We will denote $\boldsymbol{\Gamma(\rho)}$ as the set of curves γ in X for which ρ is integrable over γ. That is the set of curves for which

$$\int_\gamma \rho \, ds < \infty.$$

The curves whose are not elements of $\Gamma(\rho)$ are called ρ−**heavy**. That is the curves γ such that $\int_\gamma \rho = \infty$.

From the definition of $\Gamma(\rho)$, it follows immediately the next result

Proposition 1.5.24: Let X be a metric measure space and $\rho\colon X \to [0, \infty]$ be a Borel function. Then, $\Gamma(\rho)$ contains all the compact curves.

To calculate line integral, we need change to arc length parametrization. However, this change could be not explicit. Since all theory of functions of bounded variation holds for rectifiable curves, we have the following:

Theorem 1.5.25: Compatibility of the line integral with metric derivative

If $\gamma\colon I \to X$ is an absolutely continue locally rectifiable curve, then:

$$\int_\gamma f \, ds = \int_I f(\gamma(t)) \left|\gamma'(t)\right| \, dt. \tag{1.5.26}$$

<u>**Proof:**</u> From Theorem 1.5.19, we have that s_γ is absolutely continue, and from Theorem 1.3.13, we have

$$\begin{aligned} |\gamma'(t)| &= \frac{d\nu_\gamma}{dm_1}(t) \\ &= s'_\gamma(t). \end{aligned}$$

From the above identity and $\gamma = \gamma_s \circ s_\gamma$, we have

$$\begin{aligned} \int_I f(\gamma(t)) \left|\gamma'(t)\right| \, dt &= \int_I f(\gamma_s \circ s_\gamma(t)) s'_\gamma(t) \, dt \\ &= \int_I f(\gamma_s(s)) \, ds. \end{aligned}$$

Since s_γ is absolutely continuous.

■

Theorem 1.5.25 makes easier the computations of the line integrals. Furthermore, since we focus on the compactly rectifiable curves we can take (1.5.26) as the definition of the line integral.

1.6 The class $\mathrm{ACL}(\Omega)$

We will study the properties of absolutely continuity on curves for the Euclidean space. To do this first, we introduce the following

Definition: Decomposition in parallel lines

For all $i \in \{1, \ldots, d\}$, consider the $\boldsymbol{i}$**–th canonical hyperplane** H_i in $\mathbb{R}^d$.

$$\boldsymbol{H_i} = \left\{ x \in \mathbb{R}^d \,\middle|\, x_i = 0 \right\}.$$

Thus:

$$\mathbb{R}^d = H_i \oplus \mathbb{R}e_i.$$

We call to this decomposition **decomposition in parallel lines to the** i**–th coordinated axis** or **decomposition in parallel hyperplanes to the** H_i **canonical hyperplane** .

We will prove that the functions of our interest are determined for its behavior in a specific family of curves. To do this, we will focus on the definition of Sections of a set for a set in $\mathbb{R}^n$

Definition: Sections of Ω parallel to the i–th axis

Let $i \in \{1, \ldots, d\}$. Consider the decomposition of $\mathbb{R}^d$ in parallel lines to the i–th coordinated axis

$$\mathbb{R}^d = H_i \oplus \mathbb{R}e_i. \tag{1.6.1}$$

Within this decomposition *any set* $\Omega \subset \mathbb{R}^n$ can be considered as a subset of the product as $\Omega \subset \mathbb{R}^{d-1} \times \mathbb{R}$. The **sections of $\boldsymbol{\Omega}$ (associated with the decomposition** (1.6.1)**)** are:

1. For each $x \in \mathbb{R}^{d-1} \cong H_i$, the $\boldsymbol{x}$**–section of $\boldsymbol{\Omega}$ parallel to the $\boldsymbol{i}$–th axis** is:

$$\boldsymbol{\Omega_x} = \{ t \in \mathbb{R} \mid (x, t) \in \Omega \}.$$

   Sometimes, we need to consider the lines parallel to the i–th axis, this denoted by $\boldsymbol{\Gamma_i(\Omega)}$. If the set Ω is clear, we only denote this with $\boldsymbol{\Gamma_i}$. Clearly, this class of lines curves are all the curve in the sect

2. For each $t \in \mathbb{R}$, the $\boldsymbol{t}$**–section of $\boldsymbol{\Omega}$ parallel to $\boldsymbol{H_i}$ canonical hyperplane** is:

$$\boldsymbol{\Omega^t} = \left\{ x \in \mathbb{R}^{d-1} \,\middle|\, (x, t) \in \Omega \right\}.$$

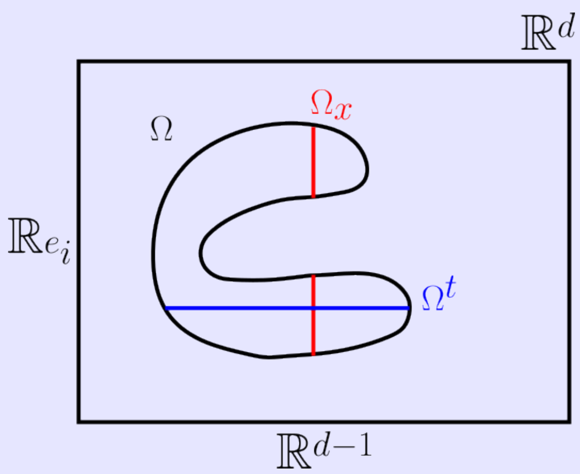

Figure 1.6.2:

Comments about the definition

The sections depend on the decomposition of $\mathbb{R}^d$ given in (1.6.1). Sometimes, we need to consider all the possible decompositions for $i \in \{1, \dots, n\}$. However, to keep simply the notation, we omit the dependence on the decomposition in (1.6.1) as the word parallel in the terminology, only remarking this dependence when the situation merits it.

Suitable representation

Considering this identification, we can characterize the elements of Ω_x as follows:

$$t \in \Omega_x \Leftrightarrow x + te_i \in \Omega.$$

We will use both characterizations indistinctly.

Definition: Absolute continuity on lines (ACL)

Let $\Omega \subset \mathbb{R}^d$ be an open set. A function $f \colon \Omega \subset \mathbb{R}^d \to \mathbb{R}$ is:

1. **Absolutely continuous on almost every line parallel to the i−th axis** if for m_{d-1}-almost every $x \in \mathbb{R}^{d-1}$ we have that f is absolutely continuous on each compact line segment contained in the section Ω_x. The set of all these functions is denoted by $\boldsymbol{ACL_i(\Omega)}$.

2. **Absolutely continuous on lines** (abbreviate **ACL**) if f absolutely continuous on parallel lines to the $\boldsymbol{i}$−th axis for all $i \in \{1, \dots, d\}$. The set of all these functions is denoted by $\boldsymbol{ACL(\Omega)}$.

Remark: Let $f \colon \Omega \subset \mathbb{R}^d \to \mathbb{R}$ be a function. For $i \in \{1, \dots, d\}$ we define F_i as the set as the points in $H_i \cong \mathbb{R}^{d-1}$ such that f is not absolutely continuous on each compact line segment contained in the section Ω_x. The condition $f \in \mathrm{ACL}_i$ F_i is equivalent to F_i is a set of measure zero.

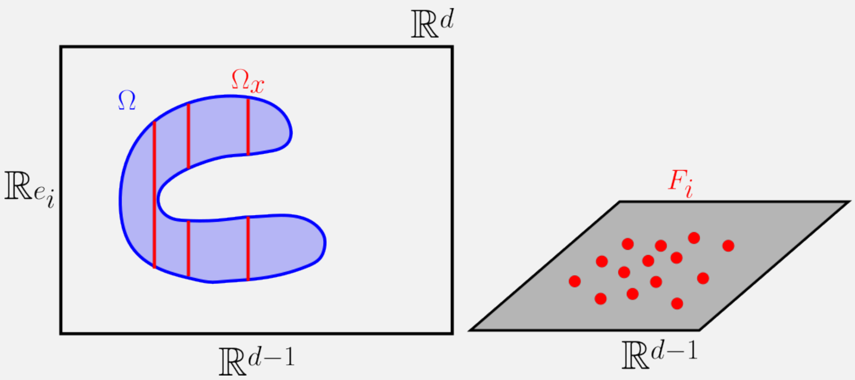

Figure 1.6.3: Sections of Ω parallel to i−th axis where f is not absolutely continuous.

Interpretation

From the theory of absolutely continuous curves it follows that the curves the absolutely continuous on lines are differentiable for almost every direction parallel to the axis. Later, in Lemma 2.6.10, we will prove that the absolute continuity on lines together integrability conditions are sufficient to prove that those functions are differentiable almost everywhere. Also, we will prove by modulus of curves theory that the Sobolev space $W^{1,p}(\Omega)$ can be characterized by absolutely continuity on lines.

Lemma 1.6.4: ACL(Ω) **and classical partial derivates**

Let $\Omega \subset \mathbb{R}^d$ an open subset. If $u \in ACL(\Omega)$, then u has classical partial derivatives m_d−a.e. on Ω. Furthermore, if u is measurable in Ω, then the partial derivatives also are.

Proof: Let L be the set where the partial derivative ∂u_i does not exists. From Theorem 1.4.3, it follows that L and $\Omega \setminus L$ are measurable in Ω.
Now, let us consider the decomposition in parallel lines to the $i-$th coordinated axis:

$$\mathbb{R}^d = H_i \oplus \mathbb{R}e_i.$$

and the sections of parallel to the $i-$th axis Ω, $\{\Omega_x\}_{x\in H_i}$. Since u is absolutely continuous on parallel lines to the $i-$th axis, $f(x,\cdot) : \Omega_x \to \mathbb{R}$ is absolutely continuous for $m_{d-1}-$almost every $x \in \mathbb{R}^{d-1}$. Consider $N \subset \mathbb{R}^{d-1}$ as the set where this property does not hold, then $m_{d-1}(N) = 0$. Let $x \in \mathbb{R}^{d-1} \setminus N$, by the definition of N, $f(x,\cdot) : \Omega_x \to \mathbb{R}$ is differentiable almost everywhere, thus

$$m_1(L_x) = 0 \qquad \forall x \in \mathbb{R}^{d-1} \setminus N.$$

From the above identity and since $m_{d-1}(N) = 0$, from Fubini theorem, we have

$$(N \times \mathbb{R}) \cup \bigcup_{x\in\mathbb{R}^{d-1}\setminus N} L_x$$

and clearly

$$L \subset (N \times \mathbb{R}) \cup \bigcup_{x\in\mathbb{R}^{d-1}\setminus N} L_x$$

Therefore $m_d(L) = 0$.

■

Remark 1.6.5: About the the requirement Ω open in ACL definition Notice that in Definition 1.6, we do not need the requirement of Ω open. We require this condition for existence of Dini derivatives and to establish Lemma 1.6.4. Thus, we can define all the notions for Definition 1.6 for Ω not necessarily open. Considering the zero extension, we can obtain the same results for that extension.

Lemma 1.6.4 motivates the following:

Definition: $\mathrm{ACL}^p(\Omega)$

Let $\Omega \subset \mathbb{R}^d$ an open subset. The class $\mathbf{ACL}^{\boldsymbol{p}}(\boldsymbol{\Omega})$ is defined as:

$$\mathbf{ACL}^{\boldsymbol{p}}(\boldsymbol{\Omega}) = \{u \in \mathrm{ACL}(\Omega) \mid \partial_i u \in L^p(\Omega)\ \forall i \in \{1,\ldots,d\}\}.$$

The following example shows that the absolute continuity is not preserved under changes of sets of measure zero.

Counterexample: Absolute continuity is not a measure theoretical notion

Consider the decomposition of $\mathbb{R}^d$ in parallel lines to the $i-$th coordinated axis

$$\mathbb{R}^d = H_i \oplus \mathbb{R}e_i.$$

Let $\Omega \subset \mathbb{R}^d$ an open set such that contains a not $m_{d-1}-$null set of H_i in some section. Consider the indicator function of the hyperplane H_i on Ω, $\chi_{H_i} : \Omega \to \mathbb{R}$. Now, let us prove that $\chi_{H_i} \notin \mathrm{ACL}_i(\Omega)$. Indeed, let $h \in H_i$. Then, $\chi_{H_i}(h,\cdot) : \Omega_h \subset \mathbb{R} \to \mathbb{R}$ coincide with $\chi_{\{0\}} : \Omega_h \subset \mathbb{R} \to \mathbb{R}$ which is a non-continuous function thus $\chi_{H_i}(h,\cdot)$ is not absolutely continuous. This proves that $\chi_{H_i} \notin \mathrm{ACL}(\mathbb{R}^d)$. Therefore, $\chi_{H_i} \notin \mathrm{ACL}^p(\Omega)$. Notice that $\chi_{H_i} = 0$ in $L^1_{\mathrm{loc}}(\Omega)$. However $0 \in \mathrm{ACL}^p(\Omega)$. This shows that **absolute continuity is not a measure theoretical notion**.

The issue of Counterexample 1.6 lies in that H_i is a large set. We fix this issue below. Before to do this, it is convenient remember the following property of the sections of a set.

Lemma 1.6.6: Properties of sections

Let $\Omega \subset \mathbb{R}^d$ and $F \subset \mathbb{R}^d$. Then:

$$\{x\} \times \Omega_x \subset \Omega \setminus F \qquad \forall x \in \mathbb{R}^{d-1} \setminus \pi\,[F]\,, \tag{1.6.7}$$

$$(\Omega \setminus F)_x = \Omega_x \qquad \forall x \in \mathbb{R}^{d-1} \setminus \pi\,[F]\,, \tag{1.6.8}$$

where $\pi \colon \mathbb{R}^d \to H_i$ is the canonical projection to the $i-$th hyperplane.

Proof:

(1.6.7). Let $(x, y) \in \{x\} \times \Omega_x$ with $x \notin \pi(F)$, then $(x, y) \in \Omega$. To prove that $(x, y) \notin F$, we proceed by contradiction ※Suppose that $x \in F$. Then

$$\pi(F) \ni \pi(x, y) = x \in F$$

which contradicts $x \notin \pi(F)$. ※

(1.6.8).

Let $y \in \Omega_x$, by (1.6.7) and the definition of Ω_x, we have $(x, y) \in \Omega \setminus F$. The other contention holds by monotonicity. Therefore, (1.6.8) holds.

■

Now, let us prove a result that avoids cases as Counterexample 1.6.

Theorem 1.6.9: Removability for functions that are absolutely continuous on almost every line

Let $\Omega \subset \mathbb{R}^d$ be an open set and $F \subset \Omega$ a closed. The following affirmations hold:

Removability for ACL_i

If F is such that the $i-$th projection of F in $\mathbb{R}^{d-1}$ has measure zero. Then the following statement holds.

1. $u \in \mathrm{ACL}_i(\Omega)$ if and only if $u \in \mathrm{ACL}_i(\Omega \setminus F)$. This equivalence is considering appropriate restrictions and any extension.

Removability for ACL and ACL^p

If F is such that each projection of F in $\mathbb{R}^{d-1}$ has measure zero. Then the following statements hold.

2. $u \in \mathrm{ACL}(\Omega)$ if and only if $u \in \mathrm{ACL}(\Omega \setminus F)$. This equivalence is considering appropriate restrictions and any extension.

3. For all $p \geq 1$, $u \in \mathrm{ACL}^p(\Omega)$ if and only if $u \in \mathrm{ACL}^p(\Omega \setminus F)$. This equivalence is considering appropriate restrictions and any extension.

Proof: First, notice that $\Omega \setminus F$ is an open set since F is closed, then it makes sense to consider the spaces in the statement. Consider the decomposition in parallel lines to the $i-$th coordinated axis:

$$\mathbb{R}^n = H_i \oplus \mathbb{R}e_i.$$

and the sections of parallel to the $i-$th axis Ω, $\{\Omega_x\}_{x \in H_i}$. Consider F_i as the $i-$th projection of F. First, we prove Item 1. By hypothesis, we have that $m_{d-1}(F_i) = 0$.

$\Rightarrow$ Assume that $u \in \mathrm{ACL}(\Omega)$. Then for almost every $x \in H_i$ we have that u is absolutely continuous on each compact subset in Ω_x, let I_i the set where this property holds. Since $m_{d-1}(F_i) = 0$, it follows that almost every point in $\mathbb{R}^{d-1}$ belongs to $I_i \setminus F_i$. From Lemma 1.6.6

$$\bigcup_{x \in I_i \setminus F_i} \{x\} \times \Omega_x \subset \Omega \setminus F$$

and all of the above sections are the sections of $\Omega \setminus F$ and by constructions in all of these sections the absolute continuity property holds. Therefore, $u|_{\Omega\setminus F} \in \mathrm{ACL}(\Omega \setminus F)$.

$\Leftarrow$ Suppose $u \in \mathrm{ACL}(\Omega \setminus F)$. Let $v\colon \Omega \to \mathbb{R}$ any extension of u. By definition for almost every $x \in H_i$, u is absolutely continuous on every compact segment contained in the section $(\Omega \setminus F)_x$, and by hypothesis, we have $m_{d-1}(F_i) = 0$, then for almost every $x \in H_i$ satisfies $x \in H_i \setminus F_i$. From 1.6.8 of Lemma 1.6.6, we have that

$$\bigcup_{x\in H_i\setminus F_i} \{x\} \times \Omega_x \subset \Omega \setminus F \tag{1.6.10}$$

and all the previous sections coincide with the sections of Ω and in all of these sections v is absolutely continuous on lines. Furthermore, by (1.6.10) and since v is an extension of u, it follows that v is absolutely continuous on lines. Therefore, $u \in \mathrm{ACL}(\Omega)$.

This proves Item 1. Considering the construction in the proof of Item 1 for each axis we obtain Item 2. Notice that the argument for the proof of Item 2 does not affect $p-$integrability, thus, this holds for Item 3.

■

Remark 1.6.11: Changes of for functions that are absolutely continuous on lines Notice that the set F in Theorem 1.6.9 has measure zero, this follows from Fubini theorem. Theorem 1.6.9 shows that we can change(in the sense of measure theory) the representative of a function in $\mathrm{ACL}(\Omega)$. However Counterexample 1.6 shows that this change cannot be in the usual measure theoretic sense. The set where the function change is fundamental.

Remark: The hypothesis of Item 1 of Theorem 1.6.9 can be relaxed Considering the restrictions of Remark 1.6.5, it follows that Theorem 1.6.9 the conditions Ω open and F closed can be omitted.

1.7 Mollifiers

Mollifiers are smooth functions that regularize and approximate functions. This concept is useful in the Sobolev space theory. The objective of this section is to prove Theorem 1.7.9 which is the most useful result to operate with mollifications.

Definition: Mollifier

Let $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ be a real valued function. We say that φ is a **mollifier** if satisfies:

$$\varphi \in C_0^\infty\left(\mathbb{R}^d\right), \tag{1.7.1}$$

$$\operatorname{spt}\varphi \subset B[0,1], \tag{1.7.2}$$

$$\int_{\mathbb{R}^d} \varphi = 1. \tag{1.7.3}$$

We also require that φ to be a even function, the reason why symmetry is require is discussed in Remark 1.7. However, we will show that the most essential without the symmetry.

Remark: Essential part of the integral of mollification Since $\operatorname{spt}\varphi \subset B[0,1]$, it follows immediately:

$$\int_{\mathbb{R}^d} \varphi = \int_{B[0,1]} \varphi = 1.$$

We use this fact without further mention.

As an abstract definition, mollifications looks pretty useful. However, it is not obvious the existence of this type of functions. The canonical example is as follows:

Example: Canonical mollification

The function $\eta\colon \mathbb{R}^d \to \mathbb{R}$ defined as follow

$$\boldsymbol{\eta(x)} = \begin{cases} ce^{\frac{1}{\|x\|^2-1}} & \text{if } \|x\| < 1, \\ 0 & \text{if } \|x\| \geq 1. \end{cases}$$

is know as the **Friedrichs' mollifier** or the **standard mollifier**. It is a well known result that η satisfies all the conditions of Definition 1.7.

From the definition of mollifier, it follows the next useful property:

Proposition (Mollifiers are in L^p): Let φ be a mollification. For every $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^d$ measurable set, we have that $\varphi|_\Omega \in L^p(\Omega)$. Furthermore, if Ω is that $B[0,1] \subset \Omega$, then:

$$\left\| \varphi^{\frac{1}{p}} \right\|_{p,\Omega} = 1$$

<u>Proof:</u> Since φ is continuous on $B[0,1]$, such is a compact set, and vanishes outside of $B[0,1]$, it follows that $\varphi \in L^p(\mathbb{R}^d)$, then $\varphi|_\Omega \in L^p(\Omega)$. Now, let suppose that $B[0,1] \subset \Omega$. Hence:

$$\begin{aligned} \left\| \varphi^{\frac{1}{p}} \right\|_{p,\Omega} &= \left(\int_\Omega \varphi \right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{R}^d} \varphi \right)^{\frac{1}{p}} \\ &= 1 \end{aligned}$$

■

From the definition of mollification, it follows immediately that we can extend the essential properties of mollification to larger sets of $\mathbb{R}^d$ considering dilations and translations as the following.

Definition: Mollifiers of step ε

Let φ be a mollification and $x \in \mathbb{R}^d, \varepsilon > 0$. The **mollifier of step $\boldsymbol{\varepsilon > 0}$ in $\boldsymbol{x}$** is the function $\boldsymbol{\varphi_{\varepsilon,x}}\colon \mathbb{R}^d \to \mathbb{R}$ given by:

$$\boldsymbol{\varphi_{\varepsilon,x}(y)} = \frac{1}{\varepsilon^d} \varphi\left(\frac{y-x}{\varepsilon} \right).$$

When x is 0, we simply write $\boldsymbol{\varphi_\varepsilon}$.

From Definition 1.7 follows immediately the next:

Proposition (Mollifiers properties): Let φ a mollification and $x \in \mathbb{R}^d, \varepsilon > 0$. Then the function $\varphi_{\varepsilon,x}\colon \mathbb{R}^d \to \mathbb{R}$ given by:

$$\varphi_{\varepsilon,x}(y) = \frac{1}{\varepsilon^d} \varphi\left(\frac{y-x}{\varepsilon} \right)$$

has te following properties:

$$\varphi_{\varepsilon,x} \in C_0^\infty\left(\mathbb{R}^d\right), \tag{1.7.4}$$

$$\operatorname{spt} \varphi_{\varepsilon,x} \subset B[x,\varepsilon], \tag{1.7.5}$$

$$\int_{B[0,\varepsilon]} \varphi_{\varepsilon,x} = \int_{\mathbb{R}^d} \varphi_{\varepsilon,x} = 1. \tag{1.7.6}$$

Notice that the properties (1.7.4), (1.7.5), (1.7.6) are a slight adjustments of the conditions (1.7.1),(1.7.2), (1.7.3). The functions $\varphi_{\varepsilon,x}$ will be used to regularize nonsmooth functions.
Theorem 1.1.15 specifies a suitable domain for defining mollifiers. The reason why this domain is appropriate is discussed in Remark 1.7. With the domain established, the definition of a mollifier is as follows:

Definition: Mollification of f

Let $\Omega \subset \mathbb{R}^d$ be an open set and $f \in L^1_{\text{loc}}(\Omega)$ (real or complex valuated). We define the **$\varepsilon-$mollification of f**, denoted by $\boldsymbol{f_\varepsilon}$, as the function: $f_\varepsilon \colon \Omega_\varepsilon \to \mathbb{K}$ given by:

$$\boldsymbol{f_\varepsilon(x)} = \int_\Omega \varphi_{\varepsilon,x} f$$

here $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, depending if f is real or complex valuated.

Remark: Mollifiers as convolution Notice that the definition of mollifier and $\operatorname{spt} \varphi_{\varepsilon,x} \subset B[0,\varepsilon]$, it follows that:

$$\begin{aligned}
\boldsymbol{f_\varepsilon(x)} &= \int_{B(x,\varepsilon)} \varphi_{\varepsilon,x} f \\
&= \int_{B(x,\varepsilon)} \varphi_\varepsilon(y-x) f(y)\, dy && (1.7.7) \\
&= \int_{B(0,\varepsilon)} \varphi_\varepsilon(-z) f(x-z)\, dz. && (1.7.8)
\end{aligned}$$

$z = x - y.$

The equation (1.7.7) shows that we must consider $x \in \Omega_\varepsilon$. To avoid the unnecessary sign in z on (1.7.8) some authors ask mollifiers be pair functions. The standard mollifier η is a pair function. Considering symmetry in the mollification we have that

$$f_\varepsilon = \varphi_\varepsilon * f.$$

We will use these facts without further mention.

The following results proves the mollifiers has the desired properties:

Theorem 1.7.9: Properties of mollifications

Let and $\Omega \subset \mathbb{R}^d$ be an open set, $u \in L^1_{\text{loc}}(\Omega)$ and $\varepsilon > 0$. Then, the mollifications of u, $\{u_\varepsilon\}_{\varepsilon>0}$ have the following properties:

Regularity

1. **Directional derivatives:** For any $\omega \in \mathbb{S}^{n-1}$ be an unit vector. The following identity holds:
$$\partial_\omega(u_\varepsilon) = \partial_\omega(u * \varphi_\varepsilon) = u * \partial_\omega \varphi_\varepsilon = u * (\nabla\varphi_\varepsilon \cdot \omega) \tag{1.7.10}$$
2. **Smoothness:** u_ε is a C^∞ function. Furthermore, the following identities holds:
$$D^\alpha(u_\varepsilon) = D^\alpha(\varphi_\varepsilon * u) = D^\alpha\varphi_\varepsilon * u \qquad \forall \alpha \in \mathbb{N}_0^n. \tag{1.7.11}$$

All the previous convolutions can be considered as integrals in Ω.

Approximation

3. **Pointwise approximation:** Let $u \in L^1_{\text{loc}}(\Omega)$. If x a Lebesgue point, then $u_\varepsilon(x) \to u(x)$. Furthermore, if u is continuous, then u_ε converges uniformly on compact sets.
4. **Approximation in L^p:** Let $1 \leq p < \infty$. If $u \in L^p_{\text{loc}}(\Omega)$. Then, for each $\Omega' \subset\subset \Omega$ there exists $\varepsilon_0 = \varepsilon_0(\Omega') > 0$ such that:
$$u_\varepsilon \in L^p(\Omega') \qquad \forall\, 0 < \varepsilon < \varepsilon_0.$$

then $u_\varepsilon \in L^p_{\text{loc}}(\Omega')$ for all $\varepsilon > 0$. Furthermore, $u_\varepsilon \to u$ in $L^p_{\text{loc}}(\Omega)$.

Proof:

1. Let $x \in \Omega_\varepsilon$ fixed and $i \in \{1, \dots, n\}$. Clearly Ω_ε is an open set, then, there exists $h_0 > 0$ such that $\{x + h\omega \mid |h| < h_0\} \subset \Omega_\varepsilon$. From now on, we consider $h \in \mathbb{R}$ such that $|h| < h_0$. Notice that:

$$\begin{aligned} u_\varepsilon(x + h\omega) - u_\varepsilon(x) &= \int_\Omega (\varphi_{\varepsilon,x+h\omega} - \varphi_{\varepsilon,x}) f \\ &= \int_\Omega (\varphi_\varepsilon(x + h\omega - y) - \varphi_\varepsilon(x - y)) f(y)\, dy. \end{aligned} \tag{1.7.12}$$

Recall, φ_ε is compactly supported, then we can find a compact set $K \subset$ independent of h such that:

$$\begin{aligned} \varphi_\varepsilon(x + h\omega - y) - \varphi_\varepsilon(x - y) = 0 &\qquad \forall y \notin K, \\ \partial_\omega \varphi_\varepsilon(x - y) = 0 &\qquad \forall y \notin K, \end{aligned}$$

With this considerations and with (1.7.12), it follows that:

$$\begin{aligned} u_\varepsilon(x + h\omega) - u_\varepsilon(x) &= \int_\Omega (\varphi_\varepsilon(x + h\omega - y) - \varphi_\varepsilon(x - y)) f(y)\, dy \\ &= \int_K (\varphi_\varepsilon(x + h\omega - y) - \varphi_\varepsilon(x - y)) f(y)\, dy \end{aligned} \tag{1.7.13}$$

Considering $h \neq 0$, we can divide by h in (1.7.13). Thus, we have:

$$\frac{u_\varepsilon(x + h\omega) - u_\varepsilon(x)}{h} = \int_K \frac{\varphi_\varepsilon(x + h\omega - y) - \varphi_\varepsilon(x - y)}{h} f(y)\, dy. \tag{1.7.14}$$

Since K is compact and φ_ε is C^∞, it follows that $\frac{\varphi_\varepsilon(x+h\omega-y)-\varphi_\varepsilon(x-y)}{h} \to \partial_\omega \varphi_\varepsilon(x - y)$ when $|h| \to 0$ uniformly for all $y \in K$. Therefore, from (1.7.14); it follows that

$$\begin{aligned} \partial_\omega u_\varepsilon(x) &= \int_K \partial_\omega \varphi_\varepsilon(x - y) f(y)\, dy \\ &= \int_\Omega \partial_\omega \varphi_\varepsilon(x - y) f(y)\, dy \\ &= \partial_\omega \varphi_\varepsilon * f. \end{aligned} \tag{1.7.15}$$

Since K is choosen appropriately.

From this identity, the other identities of (1.7.10) immediately holds.

Another way to prove this is the following: Applying the Fundamental theorem of calculus in (1.7.13) for φ_ε, we have:

$$\begin{aligned} u_\varepsilon(x + h\omega) - u_\varepsilon(x) &= \int_\Omega \left(\int_0^h \partial_\omega \varphi_\varepsilon(x - y + t\omega)\, dt \right) f(y)\, dy \\ &= \int_\Omega \int_0^h \partial_\omega \varphi_\varepsilon(x - y + t\omega) f(y)\, dt\, dy \\ &= \int_0^h \int_\Omega \partial_\omega \varphi_\varepsilon(x - y + t\omega) f(y)\, dy\, dt. \end{aligned} \tag{1.7.16}$$

Considering that $\partial_\omega \varphi_\varepsilon$ is uniformly continuous, we have that

$$\mathbb{R} \ni s \mapsto \int_\Omega \partial_\omega \varphi_\varepsilon(x - y + s\omega) f(y)\, dy$$

is continuous. Thus we can use Fundamental theorem of calculus for the previous function in (1.7.16) to obtain:

$$\begin{aligned} \lim_{h \to 0} \frac{u_\varepsilon(x + h\omega) - u_\varepsilon(x)}{h} &= \lim_{h \to 0} \frac{1}{h} \int_0^h \int_\Omega \partial_\omega \varphi_\varepsilon(x - y + te_i) f(y)\, dy\, dt \\ \partial_\omega u_\varepsilon(x) &= \int_\Omega \partial_\omega \varphi_\varepsilon(x - y) f(y)\, dy\, dt \\ \partial_\omega u_\varepsilon(x) &= \partial_\omega \varphi_\varepsilon * f(x). \end{aligned}$$

Therefore, we have proved that (1.7.10) holds.

2. Now, we will prove the identity (1.7.11) by induction for $|\alpha|$. The basis of the induction is given by (1.7.10) for the specific cases when ω is a vector of the canonical basis. To prove the inductive step, we notice that we can compute the partial derivative of $D_i u_\varepsilon$ applying all the previous steps to (1.7.15) with ω a vector of the canonical basis. This proves that (1.7.11) holds for every multi-index. Therefore, u_ε is a C^∞ function.

3. Notice that:

$$
\begin{aligned}
|u_\varepsilon(x) - u(x)| &= \left| \int_{B(x,\varepsilon)} \varphi_\varepsilon(x-y)\,(f(y) - f(x))\ dy \right| \\
&\leq \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \left| \varphi\left(\frac{x-y}{\varepsilon}\right) \right| |f(y) - f(x)|\ dy
\end{aligned}
$$

φ_ε definition. Absolute value and integral inequality

Since φ is a test function, and for the properties of Lebesgue measure, it follows that there exists a suitable constant $C > 0$ such that

$$
\begin{aligned}
|u_\varepsilon(x) - u(x)| &\leq \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \left| \varphi\left(\frac{x-y}{\varepsilon}\right) \right| |f(y) - f(x)|\ dy \\
&= C \fint_{B(x,\varepsilon)} \left| \varphi\left(\frac{x-y}{\varepsilon}\right) \right| |f(y) - f(x)|\ dy
\end{aligned}
$$

Since we are considering x as a Lebesgue point from the previous estimation we obtain $u_\varepsilon(x) \to u(x)$.

Now, let us suppose that u is continuous. Then, u is uniformly continuous on compacts. Therefore, the convergence $u_\varepsilon(x) \to u(x)$ is uniform in compacts.

4. Since $\mathbb{R}^d$ is a locally compact T_2 space and $\Omega' \subset\subset \Omega$, then there exists an open subset Ω'' and $\varepsilon > 0$ such that:

$$
\Omega' \subset\subset \Omega'' \subset\subset \Omega,
$$

$$
B(x,\varepsilon) \subset \Omega''.
$$

Let q the Hölder conjugated of p and $x \in \Omega_\varepsilon$. From Holder inequality, it follows that:

$$
\begin{aligned}
\int_{B(0,\varepsilon)} \varphi_\varepsilon(z)\,|f(x-z)|\ dz &\leq \int_{B(0,\varepsilon)} \varphi_\varepsilon(z) f(x-z)\ dz \\
\int_{B(x,\varepsilon)} \varphi_{\varepsilon,x}^{\frac{1}{q}+\frac{1}{p}} |f| &\leq \left(\int_{B(x,\varepsilon)} \varphi_{\varepsilon,x}^{\frac{q}{q}} \right)^{\frac{1}{q}} \left(\int_{B(x,\varepsilon)} \varphi_{\varepsilon,x}^{\frac{p}{p}} |f|^p \right)^{\frac{1}{p}} \\
\left| \int_{B(x,\varepsilon)} \varphi_{\varepsilon,x} f \right| &\leq 1^{\frac{1}{q}} \cdot \left(\int_{B(x,\varepsilon)} \varphi_{\varepsilon,x} |f|^p \right)^{\frac{1}{p}} \\
|u_\varepsilon(x)| &\leq \left(\int_{B(x,\varepsilon)} \varphi_{\varepsilon,x} |f|^p \right)^{\frac{1}{p}} \\
|u_\varepsilon(x)|^p &\leq \int_{B(x,\varepsilon)} \varphi_{\varepsilon,x} |f|^p
\end{aligned}
$$

u_ε definition.

The above estimation holds for every $x \in \Omega_\varepsilon$, it follows that:

$$
\begin{aligned}
\int_{\Omega'} |u_\varepsilon(x)|^p\ dx &\leq \int_{\Omega'} \int_{\Omega} \varphi_{\varepsilon,x} |f(y)|^p\ dydx \\
&= \int_{\Omega} \int_{\Omega'} \varphi_{\varepsilon,x} |f(y)|^p\ dxdy \\
&= \int_{\Omega} |f(y)|^p \int_{\Omega'} \varphi_{\varepsilon,x}\ dxdy \\
&\leq \int_{\Omega''} |f(y)|^p\ dy
\end{aligned}
\tag{1.7.17}
$$

$\boldsymbol{u_\varepsilon \to u}$ **in** $\boldsymbol{L^p_{\text{loc}}(\Omega)}$**.**

Let $\delta > 0$, since $f \in L^p(\Omega'')$ and $C_c(\Omega)$ is dense in $L^p(\Omega)$, then there exists $g \in C(\Omega')$ such that $\|f - g\|_{L^p(\Omega)} < \delta$. From (1.7.17), it follows that

$$\|f_\varepsilon - g_\varepsilon\|_{L^p(\Omega)} < \delta$$

Therefore:

$$\|f - f_\varepsilon\|_{L^p(\Omega)} \leq \delta + \|f_\varepsilon - g_\varepsilon\|_{L^p(\Omega)} + \delta < 3\delta.$$

■

Remark

1. **Mollifiers regularize functions** As Item 2 demonstrates, the mollifier is smooth. For this reason, mollifiers are also called as regularization.
2. **Mollifiers commutes with the derivative** This result will be extended to functions in appropriate Sobolev Spaces.

1.8 The Sobolev spaces $L^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$

Sobolev spaces generalize the ideas of derivatives to solve partial differential equations. In this work, we will provide an approach to Sobolev spaces with upper gradients. Remember the classical Sobolev Spaces are defined considering the following:

Definition 1.8.1: Weak derivate, weak gradient

Let $\Omega \subset \mathbb{R}^d$ be an open set and $u \in L^1_{\text{loc}}(\Omega)$, then:

1. For $i \in \{1, \dots, n\}$ a function $v_i \in L^1_{\text{loc}}(\Omega)$ is the $\boldsymbol{i}$**−th weak derivate** of u if

$$\int_\Omega v_i \varphi = -\int_\Omega u \cdot \partial_i \varphi \qquad \forall \varphi \in \mathcal{C}^\infty_c(\Omega).$$

2. We say that u is **weakly differentiable** if all weak derivates exists, v_i , the function $\nabla u : \Omega \to \mathbb{R}^d$ such that

$$\nabla u = (v_1, \dots, v_n)$$

is called the **weak gradient** of u.

Remark 1.8.2: Elements of Definition 1.8.1 From the classical theory of Sobolev spaces, see [EG15; AF03], we know that the properties of the weak derivative came from the properties of the the space of test functions $\mathcal{C}^\infty_c(\Omega)$. We know that the properties of $\mathcal{C}^\infty_c(\Omega)$ related with the weak derivative came form **the differentiable structure of** $\boldsymbol{\Omega}$ **which is in terms of the vectorial structure of** $\mathbb{R}^{\boldsymbol{d}}$**.**

Now, using the weak gradient we define spaces in terms of its regularity.

Definition: The Dirichlet space $\boldsymbol{L^{1,p}(\Omega)}$

Let $\Omega \subset \mathbb{R}^d$ be an open set and $1 \leq p \leq \infty$. We denote the set of functions in $L^1_{\text{loc}}(\Omega)$ with weak derivate in

$L^p(\Omega)$ as:

$$\boldsymbol{L^{1,p}(\Omega)} = \left\{ u \in L^1_{\text{loc}}(\Omega) \,\middle|\, \nabla u \in L^p(\Omega;\mathbb{R}^d) \right\}.$$

$p-$energy

Let $\Omega \subset \mathbb{R}^d$ be an open set and $1 \leq p < \infty$. For the functions in $u \in L^{1,p}(\Omega)$, we will define **$p-$energy of u** as $\boldsymbol{E_p(u)} = \|\nabla u\|_p^p$. That is:

$$\boldsymbol{E_p(u)} = \int_\Omega |\nabla u|_p^p.$$

where $|x|_p$ is the $p-$norm in $\mathbb{R}^d$ of x. That is

$$|(x_1,\ldots,x_d)|_p = \left(\sum_{k=1}^d x_k^p \right)^{\frac{1}{p}}.$$

The reason of the name energy came from the case $p=2$. In this case

$$\boldsymbol{E_2(u)} = \int_\Omega |\nabla u|^2$$

which is a multiple of the kinetic energy.
We always endow $L^{1,p}(\Omega)$ with the seminorm given by the energy, unless we state otherwise. This space is called **Dirichlet space**.

As in in the classical case, we introduce a local version of $L^{1,p}(\Omega)$.

Definition: The Dirichlet space $L^{1,p}_{\text{loc}}(\Omega)$

Let $\Omega \subset \mathbb{R}^d$ be an open set. The **local Dirichlet space** is the set

$$\boldsymbol{L^{1,p}_{\text{loc}}(\Omega)} = \left\{ u \in L^1_{\text{loc}}(\Omega) \,\middle|\, u \in L^{1,p}(\Omega') \quad \forall\, \Omega' \subset\subset \Omega \right\}.$$

endowed with the following notion of

Convergence in $L^{1,p}_{\text{loc}}(\Omega)$

Given a sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^{1,p}_{\text{loc}}(\Omega)$ and $f \in L^{1,p}_{\text{loc}}(\Omega)$ we say that $\boldsymbol{f_n \to f}$ **in** $\boldsymbol{L^{1,p}_{\text{loc}}(\Omega)}$ if $\nabla f_n \to \nabla f$ in $L^p(\Omega')$ for all $\Omega' \subset\subset \Omega$.

Now, we will prove approximations in $L^p_{\text{loc}}(\Omega)$.

Theorem 1.8.3: Approximations by mollifiers in $L^{1,p}_{\text{loc}}(\Omega)$

Let $\Omega \subset \mathbb{R}^d$ be an open set and $1 \leq p < \infty$. If $u \in L^1_{\text{loc}}(\Omega)$ has $i-$th weak derivative, then

$$\partial_{x_i} u_\varepsilon = (\partial_{x_i} u)_\varepsilon = \varphi_\varepsilon * u_{x_i}. \tag{1.8.4}$$

Convergence

Let $\Omega \subset \mathbb{R}^d$ be an open set and $1 \leq p < \infty$. If $u \in L^1_{\text{loc}}(\Omega)$ has $i-$th weak derivative in $L^p_{\text{loc}}(\Omega)$, then $\partial_i u_\varepsilon \to \partial_i u$ when $\varepsilon \to 0$ in $L^p_{\text{loc}}(\Omega)$.

Particular case

Let $\Omega \subset \mathbb{R}^d$ be an open set and $1 \leq p < \infty$. If $u \in L^{1,p}_{\text{loc}}(\Omega)$, then the mollifications u_ε satisfies that $u_\varepsilon \to u$ when $\varepsilon \to 0$ in $L^{1,p}_{\text{loc}}(\Omega)$.

Proof: From the regularity of the mollifications, check Theorem 1.7.9, we have that:

$$\begin{aligned}\partial_{x_i} u_\varepsilon(x) &= (\partial_{x_i}\varphi_\varepsilon) * f \\ &= \int_\Omega (\partial_{x_i}\varphi_\varepsilon(x-y)) \cdot f(y)\, dy && \text{From Theorem 1.7.9.} \\ &= \int_\Omega -\partial_{y_i}(\varphi_\varepsilon(x-y)) \cdot f(y)\, dy \\ &= \int_\Omega \varphi_\varepsilon(x-y) \cdot \partial_{y_i} f(y)\, dy && i\text{–th weak parial derivative definition.} \\ &= \varphi_\varepsilon * u_{x_i} \\ &= (\partial_i u)_\varepsilon .\end{aligned}$$

From the convergence properties of the mollifications, check Theorem 1.7.9, we have that $(\partial_i u)_\varepsilon \to \partial_{x_i} u$ when $\varepsilon \to 0$ in $L^p_{\text{loc}}(\Omega)$. From this convergence and (1.8.4), we conclude that $\partial_{x_i} u_\varepsilon \to \partial_{x_i} u$ when $\varepsilon \to 0$ in $L^p_{\text{loc}}(\Omega)$.

Since $u \in L^{1,p}_{\text{loc}}(\Omega)$, we have that all the partial derivatives are in $L^p_{\text{loc}}(\Omega)$. Thus, we can apply the part of convergence in this result to obtain $\partial_{x_i} u_\varepsilon \to \partial_{x_i} u$ when $\varepsilon \to 0$ in $L^{1,p}_{\text{loc}}(\Omega)$ for all $i \in \{1, \dots, d\}$. Hence, $\nabla u_\varepsilon \to \nabla u$ when $\varepsilon \to 0$ in $L^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^d)$. Therefore, $u_\varepsilon \to u$ when $\varepsilon \to 0$ in $L^{1,p}_{\text{loc}}(\Omega)$.

■

For completeness, we define the classical Sobolev Space $W^{1,p}(\Omega)$.

Definition: $W^{1,p}(\Omega)$

Let $\Omega \subset \mathbb{R}^d$ be an open set and $1 \leq p \leq \infty$. We define the **Sobolev space $W^{1,p}(\Omega)$** as follows:

$$\begin{aligned}\boldsymbol{W^{1,p}(\Omega)} &= L^p(\Omega) \cap L^{1,p}(\Omega) \\ &= \left\{ u \in L^1_{\text{loc}}(\Omega) \,\middle|\, u, \partial_i u \in L^p(\Omega) \quad \forall i \in \{1, \dots, d\} \right\}\end{aligned}$$

Norms

In $W^{1,p}(\Omega)$, we define the norms:

$$\|\boldsymbol{u}\|_{\boldsymbol{1,p,\Omega}} = \|u\|_p + \|\nabla u\|_p$$

We always endow $W^{1,p}(\Omega)$ with the norm $\|\cdot\|_{1,p,\Omega}$, unless we state otherwise.

Local version

Let $\Omega \subset \mathbb{R}^d$ be an open set.

$$\boldsymbol{W^{1,p}_{\text{loc}}(\Omega)} = \left\{ u \in L^1_{\text{loc}}(\Omega) \,\middle|\, u \in W^{1,p}(\Omega') \qquad \forall \Omega' \subset\subset \Omega \right\}.$$

Convergence in $W^{1,p}_{\text{loc}}(\Omega)$

Given a sequence $\{f_n\}_{n \in \mathbb{N}} \subset W^{1,p}_{\text{loc}}(\Omega)$ and $f \in L^{1,p}_{\text{loc}}(\Omega)$ we say that $\boldsymbol{f_n \to f}$ in $\boldsymbol{W^{1,p}_{\text{loc}}(\Omega)}$ if $f_n \to f$ in $W^{1,p}(\Omega')$ for all $\Omega' \subset\subset \Omega$.

In the following chapters, that $W^{1,p}(\Omega)$ is characterized by the absolute continuity on curves and we use this result to extend the notion of Sobolev spaces for metric measure spaces.

Chapter 2

Modulus of a family of curves

The essence of modulus is the size of a family of curves. This concept is motivated by Motivation 2.1.1. The first appearance of the notion of modulus is as *extremal length* in [AB50]. The most important result for this work is Fuglede lemma due to [Fug57], this result is about convergence in L^p for curves. In the next chapter, we will show that Fuglede lemma is fundamental for the theory of upper gradients.

2.1 Modulus of a family of curves

The modulus of a family of curves is defined as the $p-$energy of certain kind of functions. In the first subsection we define those functions and later we define the concept of $p-$modulus.

2.1.1 Admissible densities

Motivation: Admissible density

Let $\gamma\colon I \to X$ be a locally rectifiable curve in a metric measure space. The $\rho-$weight on γ is $\int_\gamma \rho\, ds$. The minimal weight is reached with $\rho \equiv 0$. However, this does not provide us with more information. Thus, we need to restrict the functions we are considering. Even if we only ask for positive weights, we have that $\int_\gamma \rho\, ds > 0$, but considering the functions $\varepsilon\rho$, we have that the best would be 0 again. One way to avoid this pathological case is to only consider weights such that

$$\int_\gamma \rho\, ds \geq x_0 > 0$$

for some x_0 fixed. We can suppose without loss of generality that $x_0 = 1$.

The previous example motivates the following:

Definition: Admissible density

Let Γ be a family of curves in a metric measure space X. A nonnegative Borel function $\rho\colon X \to [0, \infty]$ is an **admissible density for $\boldsymbol{\Gamma}$** if

$$\int_\gamma \rho\, ds \geq 1$$

for all locally rectifiable $\gamma \in \Gamma$. The set of all admissible densities for Γ is denoted by $\boldsymbol{D_\Gamma}$. Sometimes, we will refer an admissible density for a single curve γ, this means admissible density for $\{\gamma\}$, we will denote $\boldsymbol{D_\gamma} = D_{\{\gamma\}}$.

The following result provide examples of admissible density and shows its compatibility with the derivative.

Theorem 2.1.1: Admissible densities for a curve in $\mathbb{R}$

Let $\gamma\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a rectifiable curve. Let $f\colon [a,b] \to \mathbb{R}$ be an absolutely continuous function such that and $f(b) = 0 f(a) \geq 1$. Then, $|f'|$ is an admissible density for γ.

Proof: Since $f\colon I \to \mathbb{R}$ is absolutely continuous function such that $f(0) = 1$ and $\operatorname{spt} f \subset (0, L)$. From Corollary 1.5.23, we have that:

$$\int_\gamma |f'| = \int_a^b |f'| \geq |f(b) - f(a)| \geq 1.$$

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The above result will be discussed for a larger family of curves in theorem 2.4.5.

We present some basic properties of admissible densities.

Proposition 2.1.2 (Properties of admissible densities): Let $\Gamma, \Gamma_1, \Gamma_2$ families of curves in X.

1. **Decreasing.** If $\Gamma_1 \subset \Gamma_2$, then $D_{\Gamma_2} \subset D_{\Gamma_1}$.
2. If γ is a constant curve then $D_\gamma = \varnothing$.
3. Γ has a constant curve then $D_\Gamma = \varnothing$.
4. $D_\varnothing$ is the set of all nonnegative Borel functions.
5. **Convex combination of admissible densities is an admissible density**.

Proof:

1. Let $\rho \in D_{\Gamma_2}$, then
$$1 \leq \int_\gamma \rho \, ds \qquad \forall \gamma \in \Gamma_2. \tag{2.1.3}$$
Since $\Gamma_1 \subset \Gamma_2$, we have that (2.1.3) holds for all $\gamma \in \Gamma_1$. Therefore $\rho \in D_{\Gamma_1}$.
2. Since γ is constant then $\int_\gamma f \; ds = 0$ for all Borel function, then there is no admissible densities for γ. Therefore, $D_\gamma = \varnothing$.
3. Let γ_0 a constant curve of Γ. From Items 1 and 2 follows that $D_\Gamma \subset D_{\gamma_0} = \varnothing$.
4. Clearly $D_\varnothing$ is subset of the set of all nonnegative Borel functions.
5. Let $\{\rho_k\}_{k=1}^n \subset D_\Gamma$ be a family of admissible densities for Γ and let
$$\rho = \sum_{k=1}^n \lambda_k \rho_k$$
be a convex combination of $\{\rho_k\}_{k=1}^n$. Let $\gamma \in \Gamma$, then
$$\int_\gamma \rho \; ds = \int_\gamma \sum_{k=1}^n \lambda_k \rho_k \; ds = \sum_{k=1}^n \lambda_k \int_\gamma \rho_k \; ds \geq \sum_{k=1}^n \lambda_k = 1$$
Therefore, any convex combination of admissible densities is an admissible density

■

Notice that there is no contradiction between Items 2 and 4 because the conditions $D_\Gamma = \varnothing, \Gamma = \varnothing$ are different. Now, we will prove more general properties of admissible densities. First notice the upward grown property for admissible densities:

Lemma: Admissible densities are increasing

Let Γ be a family of curves in a metric space X. If $\rho \in D_\Gamma$ and $\sigma\colon X \to [0,\infty]$ is Borel such that $\rho \leq \sigma$ then $\sigma \in D_\Gamma$.

Proof: Let $\gamma \in \Gamma$, then

$$1 \leq \int_\gamma \rho \, ds \leq \int_\gamma \sigma \, ds$$

■

This result is useful in practice, we use it without further mention.

2.1.2 Modulus of a family of curves

Theorem 2.1.1 suggest that admissible densities can be considered as derivatives. Using this identification, a natural notion of size is through the $p-$energy. Notice that the definition of admissible density is defined for curves, however, we aim to define a concept of measure of a family of curves in the whole space, then, the $p-$energy must be considered in the whole space. Then, a natural notion of size for a family of curves is as follows:

Definition: Modulus of a family of curves

Let Γ be a family of curves in a metric measure space X and $1 \leq p < \infty$. The $\boldsymbol{p-}$**modulus of** $\boldsymbol{\Gamma}$ is defined as:

$$\mathbf{Mod}_{\boldsymbol{p}}(\boldsymbol{\Gamma}) = \inf_{\rho \in D_\Gamma} \int_X \rho^p.$$

From the definition of admissible density and Mod_p follows immediately the next:

Lemma 2.1.4: Usual considerations for the modulus

Let Γ a family of curves in a metric space X. Then we can consider the following conditions for Γ:

Only consider families of locally rectifiable curves

1. If $\widetilde{\Gamma}$ is the family of locally rectifiable curves in Γ. Then $\mathrm{Mod}_p(\Gamma) = \mathrm{Mod}_p(\widetilde{\Gamma})$.

Only consider nonconstant curves

2. If Γ has a constant curve, then $\mathrm{Mod}_p(\Gamma) = \infty$.

Proof:

1. The definition of admissible density is only considered for locally rectifiable curves, then $D_\Gamma = D_{\widetilde{\Gamma}}$. Thus, $\mathrm{Mod}_p(\Gamma) = \mathrm{Mod}_p(\widetilde{\Gamma})$.

2. Since Γ has a constant curve, from Proposition 2.1.2, it follows that $D_\Gamma = \varnothing$. By definition, we have $\mathrm{Mod}_p(\Gamma) = \inf \varnothing$ and we are considering $\mathrm{Mod}_p(\Gamma) \in [0,\infty]$. Therefore, $\mathrm{Mod}_p(\Gamma) = \infty$.

■

We develop the modulus for a family of curves to support the upper gradients theory. Then, it is useful to have theorems to compute modulus. One of our objectives of this section is proved approximation for $p-$modulus by special type of functions. From now, we prove the local behavior of the $p-$modulus. To do this, we introduce the following:

Definition: Refinement Borel no negative functions

Let X be a metric measure space, and let D_1, D_2 be two sets of Borel no negative functions on X. Let $1 \leq p < \infty$, we say that D_1 **$p-$refine** D_2, $\boldsymbol{D_1 \leq_p D_2}$, if $D_1 \subset D_2$ and satisfy for every $\rho \in D_2$ there exists $\rho_1 \in D_1$ such that

$$\int_X \rho_1^p \leq \int_X \rho^p.$$

With this notion, we will prove the following:

Lemma 2.1.5: Reduction of densities

Let Γ a family of curves in a metric measure space X. If D is a set of Borel no negative functions on X such that $D \leq_p D_\Gamma$, then

$$\mathrm{Mod}_p(\Gamma) = \inf_{\rho \in D} \int_X \rho^p.$$

Proof: We will prove the following inequalities:

$\boldsymbol{\mathrm{Mod}_p(\Gamma) \leq \inf_{\rho\in D} \int_X \rho^p}$ Since $D \subset D_\Gamma$ follows that

$$\left\{ \int_X \rho^p \,\middle|\, \rho \in D \right\} \subset \left\{ \int_X \rho^p \,\middle|\, \rho \in D_\Gamma \right\}$$

$$\mathrm{Mod}_p(\Gamma) \leq \inf_{\rho \in D} \int_X \rho^p.$$

Taking infimums is decreasing.

$\boldsymbol{\inf_{\rho\in D} \int_X \rho^p \leq \mathrm{Mod}_p(\Gamma)}$ Let $\rho \in D_\Gamma$, since $D \leq_p D_\Gamma$, then there exists $\rho_0 \in D$ such that

$$\inf_{\rho \in D} \int_X \rho^p \leq \int_X \rho_0^p \leq \int_X \rho^p$$

Since $\rho \in D_\Gamma$ is arbitrary, it follows that $\inf_{\rho \in D} \int_X \rho^p \leq \mathrm{Mod}_p(\Gamma)$.

■

Lemma 2.1.5 is useful since it provides two useful techniques to compute $p-$modulus.

Corollary 2.1.6 (Local behavior of the $p-$module): Let Γ a family of curves in a metric space X and $A \subset X$ a Borel set such that every curve of Γ lies in A. Then, the set

$$D = \{\rho \chi_A \mid \rho \in D_\Gamma\}$$

satisfies $D \leq_p D_\Gamma$ for all $p \geq 1$. Furthermore, the following equality holds:

$$\mathrm{Mod}_p(\Gamma) = \inf_{\rho \in D_\Gamma} \int_A \rho^p.$$

Proof: Clearly $\rho\chi_A \in D_\Gamma$, then $D \subset D_\Gamma$. Furthermore, $\rho\chi_A \leq \rho$. This proves that $D \leq_p D_\Gamma$, from Lemma 2.1.5 follows that

$$\begin{aligned} \mathrm{Mod}_p(\Gamma) &= \inf_{\rho \in D} \int_X \rho^p \\ &= \inf_{\rho \in D_\Gamma} \int_A \rho^p. \end{aligned}$$

Definition of D.
$\int_X (\rho\chi_A)^p = \int_A \rho^p$.

■

Corollary 2.1.7: Let Γ a family of curves in a metric measure space X. If D is a set of Borel no negative functions on X such that for all $\rho \in D_\Gamma$ there exists $F_\rho \in D$ such that $F_\rho \leq \rho$ almost everywhere. Then

$D \leq_p D_\Gamma$, and in consequence

$$\operatorname{Mod}_p(\Gamma) = \inf_{\rho \in D} \int_X F_\rho^p.$$

Proof: Since $F_\rho \leq \rho$ almost everywhere, it follows that

$$\int_X F_\rho^p \leq \int_X \rho^p.$$

Then, $D \leq_p D_\Gamma$ and the computation for $p-$modulus follows immediately from Lemma 2.1.5.

■

Through all this work, we show that Corollary 2.1.7 is an standard technique to compute $p-$modulus. From Lemma 2.1.4 and corollary 2.1.6, when we dealing with modulus of a family of curves we can suppose the following:

Remark: Usual considerations for modulus

1. **The family only has nonconstant locally rectifiable curves.**
2. **We only analyze the behavior in an appropriate set that contains all the curves.**

We use both considerations without further mention.

Now, we will prove that modulus is a measure. Now we prove basic properties of the modulus of a family of curves:

Lemma 2.1.8: Basic properties of $p-$modulus

Let Γ_1, Γ_2 families of curves in a metric space X. Then, the following statements holds:

1. $\operatorname{Mod}_p(\varnothing) = 0$.
2. If $\Gamma_1 \subset \Gamma_2$, then $\operatorname{Mod}_p(\Gamma_1) \leq \operatorname{Mod}_p(\Gamma_2)$.

Proof:

1. If $\Gamma = \varnothing$, by emptiness, we have that $\rho \equiv 0$ is an admissible density.
2. From Proposition 2.1.2, $D_{\Gamma_2} \subset D_{\Gamma_1}$, and since the infimum is decreasing, we conclude $\operatorname{Mod}_p(\Gamma_1) \leq \operatorname{Mod}_p(\Gamma_2)$.

■

Remember our objective is to prove that the modulus is a measure, we almost done this only remains to prove that the modulus is $\sigma-$subadditive.

Lemma 2.1.9: Mod_p is $\sigma-$subadditive

Let $\{\Gamma_n\}_{n\in\mathbb{N}}$ families of curves in a metric measure space X. Then:

$$\operatorname{Mod}_p\left(\bigcup_{n\in\mathbb{N}} \Gamma_n\right) \leq \sum_{n\in\mathbb{N}} \operatorname{Mod}_p(\Gamma_n). \tag{2.1.10}$$

Proof: Without loss of generality we can assume that the right-hand side of the inequality (2.1.10) is finite, hence all the terms in the sum are finite. Let $\varepsilon > 0$ fixed and $n \in \mathbb{N}$, by the definition of $\operatorname{Mod}_p(\Gamma_n)$, there is an admissible density $\rho_n \in D_{\Gamma_n}$ such that:

$$\int_X \rho_n^p \leq \operatorname{Mod}_p(\Gamma_n) + \frac{\varepsilon}{2^n} \tag{2.1.11}$$

Now, define:

$$\rho = \left(\sum_{n\in\mathbb{N}} \rho_n^p \right)^{\frac{1}{p}}$$

Now, we prove that $\rho \in D_\Gamma$. Indeed, clearly ρ is Borel measurable and nonnegative. Let $\gamma \in \bigcup_{n\in\mathbb{N}} \Gamma_n$, then there exists $m \in \mathbb{N}$ such that $\gamma \in \Gamma_m$, since $\rho_m \leq \rho$, we have

$$\int_\gamma \rho \geq \int_\gamma \rho_m \geq 1.$$

This proves that $\rho \in D_\Gamma$.
On the other hand, from the definition of ρ and (2.1.11), we have:

$$\mathrm{Mod}_p \left(\bigcup_{n\in\mathbb{N}} \Gamma_n \right) \leq \int_X \rho^p = \int_X \sum_{n\in\mathbb{N}} \rho_n^p = \sum_{n\in\mathbb{N}} \int_X \rho_n^p \leq \sum_{n\in\mathbb{N}} \mathrm{Mod}_p(\Gamma_n) + \varepsilon.$$

Since the above inequality holds for any $\varepsilon > 0$, we conclude that (2.1.10) holds.

■

We introduce the following:

Notation

Let X a metric space. We denote $\boldsymbol{\Gamma(X)}$ as the set of all families of curves in X.

From Lemmas 2.1.8 and 2.1.9 Then we have that the modulus defines a measure on the set of all families of curves in X, for completeness we write the result:

Theorem: $p-$modulus is a measure

Let X a metric space and let $\boldsymbol{\Gamma(X)}$ be the set of all families of curves in X and $1 \leq p < \infty$. Then $\mathrm{Mod}_p \colon \Gamma(X) \to [0, \infty]$ defines an outer measure on $\Gamma(X)$.

Now that we have already proved that $p-$modulus is a measure, we will give the usual notions of measure theory:

Definition: $p-$exceptional, $p-$a.e. curve

Let X be a metric space and $1 \leq p < \infty$. We say that:

1. A **family of curves** Γ is **$p-$exceptional** if $\mathrm{Mod}_p(\Gamma) = 0$.
2. A property holds for **$p-$almost every ($p-$a.e.) curve** if the collection of curves that not satisfy the property is $p-$exceptional.

The $p-$modulus is a measure that have a well behaved characterization of $p-$exceptionality, which we analyze in Section 2.2.

2.1.3 Subcurves, majorization and modulus

In this section, we introduce the notion of majorization which is motivated to make compatible the $p-$modulus of a family of curves Γ with the family of all subcurves of Γ. We need to introduce the majorization relationship because $p-$modulus of a family of curves Γ with the family of all subcurves of Γ are not immediately compatible. First, we introduce the following:

Notation: Family of subcurves

Let X be a metric space and Γ a family of curves in X. We will denote the family of all nonconstant subcurves Γ by $\mathbf{Sub}\,(\boldsymbol{\Gamma})$.

From the definition of $\mathrm{Sub}(\Gamma)$, it follows immediately the next

Proposition 2.1.12:

1. Let X be a metric space, then $\Gamma \subset \mathrm{Sub}(\Gamma)$. So

$$\mathrm{Mod}_p(\Gamma) \leq \mathrm{Mod}_p(\mathrm{Sub}(\Gamma)). \tag{2.1.13}$$

2. Let X be a metric space, then for all $x \in \bigcup \Gamma$, we have that the constant curve c_x is in $\mathrm{Sub}(\Gamma)$.

We will use the above facts without further mention. From Proposition 2.1.12, we have the following:

Remark From Item 2, we have that $\mathrm{Mod}_p(\mathrm{Sub}(\Gamma))$ is trivial. However, this is not a problem since the computations those we will use in this work consider $\Gamma(X) \setminus \mathrm{Sub}(\Gamma)$. See Lemma 3.2.3 for further clarification.

The estimation (2.1.13) follows from the decreasing property of $p-$modulus. Alternatively, (2.1.13) can be obtained as a consequence of the following basic fact:

Proposition 2.1.14: An admissible density for a subcurve of γ is and admissible density for γ.

The above result motivates the following:

Definition: Majorization of families of curves

Majorization of families of curves Let Γ, Γ_0 two families of curves in X. We say that Γ **majorizes** $\mathbf{\Gamma_0}$, $\mathbf{\Gamma \geq \Gamma_0}$, if for each curve $\gamma \in \Gamma$ there is γ_0 subcurve of γ such that $\gamma_0 \in \Gamma_0$.

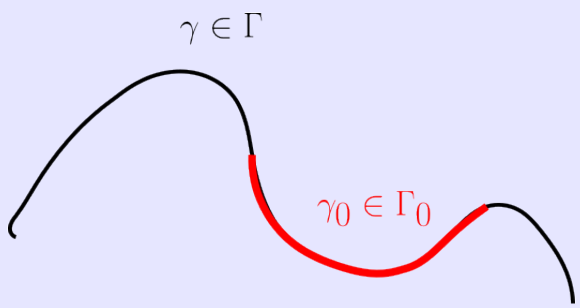

Figure 2.1.15: Subcurve

Using this notion, we will generalize the compatibility of admissible densities with subcurves as follows:

Proposition 2.1.16 (Admissible densities preserve majorization): Let Γ, Γ_0 two families of curves in X such that $\Gamma_0 \leq \Gamma$. Then, $D_{\Gamma_0} \subset D_\Gamma$.

Proof: Let $\rho \in D_{\Gamma_0}$. Now, let $\gamma \in \Gamma$. Then, there exist $\gamma_0 \in \Gamma_0$ subcurve of γ hence

$$\begin{aligned} \int_\gamma \rho\, ds &\geq \int_{\gamma_0} \rho\, ds \\ &\geq 1. \end{aligned} \quad \text{Since } \rho_0 \in D_{\Gamma_0}.$$

Therefore, $\rho \in D_\Gamma$. This proves that $D_{\Gamma_0} \subset D_\Gamma$.

■

Now we will relate the majorization with modulus

Lemma 2.1.17: Majorization and modulus

Let Γ, Γ_0 two families of curves in a metric space X such that $\Gamma_0 \leq \Gamma$. Then

$$\mathrm{Mod}_p(\Gamma) \leq \mathrm{Mod}_p(\Gamma_0).$$

Proof: From Proposition 2.1.16, we have $D_{\Gamma_0} \subset D_\Gamma$, then

$$\left\{ \int_X \rho^p \,\middle|\, \rho \in D_{\Gamma_0} \right\} \subset \left\{ \int_X \rho^p \,\middle|\, \rho \in D_\Gamma \right\},$$

since the infimum is decreasing, from the above inequality, we conclude

$$\mathrm{Mod}_p\left(\Gamma_0\right) = \inf \left\{ \int_X \rho^p \,\middle|\, \rho \in D_{\Gamma_0} \right\} \geq \inf \left\{ \int_X \rho^p \,\middle|\, \rho \in D_\Gamma \right\} = \mathrm{Mod}_p\left(\Gamma\right).$$

■

The most trivial example of majorization is stated in the next:

Proposition 2.1.18 (Properties of $\mathrm{Sub}\,\Gamma$): Let X be a metric space and Γ be a family of curves in X. Then, $\mathrm{Sub}\,\Gamma \leq \Gamma$.

Proposition 2.1.18 provides an alternative proof of (2.1.13). From definition of $\Gamma(\rho)$, we have immediately the next:

Lemma 2.1.19

Let X be a measure space and $\rho\colon X \to [0, \infty]$ be a $p-$integrable nonnegative Borel function. Then

$$\Gamma(\rho) = \mathrm{Sub}\left(\Gamma(\rho)\right). \tag{2.1.20}$$

The identity (2.1.20) will be used in the properties for $p-$almost every curve to inherit properties to each subcurve.

2.1.4 Estimations of $\mathrm{Mod}_p(\Gamma)$ by length

Using the Local behavior of the $p-$module we can obtain an estimate for the modulus. This is detailed in the following:

Lemma 2.1.21: Estimation by minimal length

Let X be a metric measure space. Let Γ a family of functions in a Borel set $A \subset X$ such that each $\gamma \in \Gamma$ satisfies $\mathrm{length}(\gamma) \geq L > 0$ then

$$\mathrm{Mod}_p(\Gamma) \leq \mu(A) L^{-p}.$$

Proof: Since A is Borel we have that $\frac{1}{L}\chi_A$ is a nonnegative Borel function. Now, let $\gamma \in \Gamma$ then γ is in the Borel set A hence

$$\begin{aligned}\int_\gamma \frac{1}{L}\chi_A \, ds &= \int_\gamma \frac{1}{L} \, ds \\ &= \frac{1}{L}\mathrm{length}(\gamma) \quad \text{length}(\gamma) \geq L > 0. \\ &\geq 1,\end{aligned}$$

then $L^{-1}\chi_A \in D_\Gamma$. Therefore

$$\mathrm{Mod}_p(\Gamma) \leq \int_X \left(L^{-1}\chi_A\right)^p = \mu(A) L^{-p}.$$

■

Later, in Theorem 2.4.5, we will prove the relationship of $p-$modulus with the *extremal length*. From now, we prove an immediate consequence of Lemma 2.1.21.

Corollary 2.1.22: Let X be a metric measure space. Let Γ a family of nonconstant curves in a null Borel set $A \subset X$. Then $\operatorname{Mod}_p(\Gamma) = 0$.

Proof: For $L > 0$, define $\Gamma_L = \{\gamma \in \Gamma \mid \operatorname{length}\gamma \geq L\}$, from Lemma 2.1.21, we have

$$\operatorname{Mod}_p(\Gamma_L) \leq \cancelto{0}{\mu(A)} L^{-p}.$$

This proves that $\operatorname{Mod}_p(\Gamma_L) = 0$. Since Γ has non constant curves follows that $\Gamma = \bigcup_{L \in \mathbb{R}^+} \Gamma_L$, moreover, by Archimedean property we have that $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_{\frac{1}{n}}$, using this decomposition and the $\sigma-$subadditivity of $p-$modulus we have that

$$\operatorname{Mod}_p(\Gamma) \leq \sum_{n \in \mathbb{N}} \cancelto{0}{\operatorname{Mod}_p(\Gamma_{\frac{1}{n}})}$$

Therefore, $\operatorname{Mod}_p(\Gamma) = 0$.

■

Remark The intuitive idea of Corollary 2.1.22 is a family of curves such that does not fill the space is $p-$null.

With Corollary 2.1.22 we can made many examples of $p-$exceptional families of curves considering all the rectifiable components of a null Borel set.

2.2 $p-$exceptional families of curves

In this work, we study properties that hold for $p-$almost every curve. As in classical measure theory, we employ the following technique to prove properties for $p-$almost every curve.

Lemma 2.2.1: Technique to prove properties $p-$almost every curve

Let X be a metric space and $1 \leq p < \infty$ and let Γ_1, Γ_2 families of curves in X. If $p-$almost every curve is in Γ_1 and $\Gamma_1 \subset \Gamma_2$, then $p-$almost every curve is in Γ_2.

Generalization

Let X be a metric space and $1 \leq p < \infty$ and let $\{\Gamma_n\}_{n \in \mathbb{N}}$ families of curves in X such that $p-$almost every curve is in Γ_n for all $n \in \mathbb{N}$. Let Γ be a family of curves such that

$$\bigcap_{n \in \mathbb{N}} \Gamma_n \subset \Gamma,$$

then $p-$almost every curve is in Γ.

Proof: By hypothesis, we have $X \setminus \Gamma_2 \subset X \setminus \Gamma_1$, so

$$\operatorname{Mod}_p (X \setminus \Gamma_2) \subset \operatorname{Mod}_p (X \setminus \Gamma_1) = 0.$$

Hence $\operatorname{Mod}_p (X \setminus \Gamma_2) = 0$. Therefore, $p-$almost every curve is in Γ_2.

The generalization follows from the first part and the fact that $p-$almost every curve is in $\bigcap_{n \in \mathbb{N}} \Gamma_n$.

■

Now we give a characterization of $p-$exceptionality, which provide a canonical criterion for $p-$exceptionality.

Lemma 2.2.2: $p-$exceptionality criterion

Let Γ a family of locally rectifiable curves in X. Then, Γ is $p-$exceptional if and only if there is a $p-$integrable nonnegative Borel function $\rho\colon X \to [0,\infty]$ such that

$$\int_\gamma \rho \, ds = \infty \qquad \forall \gamma \in \Gamma. \tag{2.2.3}$$

Proof:

$\Rightarrow$ Assume that Γ is $p-$exceptional, that is $\mathrm{Mod}_p(\Gamma) = 0$. Then, by the definition of the modulus for each $n \in \mathbb{N}$ there is an admissible density $\rho_n \in D_\Gamma$ such that

$$\int_X \rho_n^p \leq 2^{-np}. \tag{2.2.4}$$

Now, define $\rho = \sum_{n\in\mathbb{N}} \rho_n$. Notice that ρ satisfies:

ρ is $p-$integrable nonnegative Borel function.

Notice that ρ is Borel. From (2.2.4), we have

$$\|\rho\|_p = \left\| \sum_{n\in\mathbb{N}} \rho_n \right\|_p \leq \sum_{n\in\mathbb{N}} \|\rho_n\|_p \leq \sum_{n\in\mathbb{N}} 2^{-n} < \infty$$

Then, ρ is $p-$integrable.

(2.2.3) holds.

Since $\rho_n \in D_\Gamma$ we have $\int_\gamma \rho_n \, ds \geq 1$ then:

$$\int_\gamma \sum_{n\in\mathbb{N}} \rho_n \, ds \geq \sum_{n\in\mathbb{N}} \int_\gamma \rho_n \, ds = \infty,$$

$\Leftarrow$ Conversely, suppose that there exists ρ such that (2.2.3) holds. Then for every $\varepsilon > 0$ we have that $\varepsilon\rho$ is an admissible density and therefore $\mathrm{Mod}_p(\Gamma) = 0$.

■

From $p-$exceptionality criterion and the definition of $\Gamma(\rho)$ follows immediately the next:

Lemma 2.2.5: $p-$integrable function is integrable over $p-$a.e. curve

Let $p-$integrable nonnegative Borel function $\rho\colon X \to [0,\infty]$. Then, ρ is integrable for $p-$almost every curve.

Proof: Let $\Gamma = \Gamma(X) \setminus \Gamma(\rho)$. By definition of $\Gamma(\rho)$, it follows that ρ is not integrable over any $\gamma \in \Gamma$, then

$$\int_\gamma \rho \, ds = \infty \qquad \forall \gamma \in \Gamma(X) \setminus \Gamma(\rho).$$

Since ρ is $p-$integrable we can use the $p-$exceptionality criterion, then it follows that Γ is $p-$exceptional.

■

Lemma 2.2.5 has the following important remarks.

Remark 2.2.6: Lemma 2.2.5 allow us to change $p-$integrability in the space for $1-$ on curves

1. One of the consequences of this exchanges is Lemma 2.6.1, which is one of the theorems that motivates the notion of upper gradients.

2. For the particular case this can be interpreted as the integrability on almost every x and $y-$sections of Fubini theorem. This details are discussed in Theorem 2.4.11.

Lemma 2.2.2 also can be interpreted as follows

Remark 2.2.7: There is a canonical family to prove $p-$nullity which is $\boldsymbol{\Gamma(X) \setminus \Gamma(\rho)}$. From (2.2.3), we have

$$\Gamma \subset \Gamma(X) \setminus \Gamma(\rho) \tag{2.2.8}$$

and from Lemma 2.2.5, we have $\Gamma(X) \setminus \Gamma(\rho)$ is a $p-$exceptional family of curves. Therefore, Γ is $p-$exceptional. In practice is more useful to consider the equivalent contention of (2.2.8)

$$\Gamma(\rho) \subset \Gamma(X) \setminus \Gamma \tag{2.2.9}$$

In the contention (2.1.20) is the key point to inherit properties to all subcurves. We discuss this in Lemma 2.2.10.

2.2.1 $p-$exceptionality and subcurves

The family $\Gamma(\rho)$ has Lemma 2.2.5 .

Lemma 2.2.10: Properties $p-$almost every curve and subcurves

Let X be a metric measure space. Let $\mathcal{P}$ be a property for curves. Let Γ be a family of curves in X. If $p-$almost every curve of Γ satisfies $\mathcal{P}$, then $p-$almost every curve in Γ satisfies $\mathcal{P}$ as well on every subcurve of Γ.

Proof: Let Γ_1 be the family of curves in Γ that satisfies $\mathcal{P}$. From the definition Γ, from $p-$exceptionality criterion, it follows that there exists $\rho\colon X \to [0,\infty]$ $p-$integrable such that:

$$\int_\gamma \rho \, ds = \infty \qquad \forall \gamma \in \Gamma \setminus \Gamma_1.$$

Let Γ_2 be the family of curves in Γ such that satisfies $\mathcal{P}$ as well on every subcurve of γ. Now, notice that if $\gamma \in \Gamma(\rho) \cap \Gamma$, we have that

$$\int_\gamma \rho \, ds < \infty. \tag{2.2.11}$$

Thus $\gamma \in \Gamma_1$. Furthermore, every subcurve $\widetilde{\gamma}$ of γ satisfies (2.2.11), so $\widetilde{\gamma} \in \Gamma_1$, hence $\widetilde{\gamma}$ satisfies $\mathcal{P}$. Thus, $\gamma \in \Gamma_2$. This proves

$$\Gamma(\rho) \cap \Gamma \subset \Gamma_2$$

Therefore, every subcurve of γ belongs Γ_1. This proves that $\Gamma(\rho) \cap \Gamma \subset \Gamma_2$, that is all the subcurves of $\Gamma(\rho)$ satisfies $\mathcal{P}$. On the other hand, from Lemma 2.2.5, we have that $p-$almost every curve is in $\Gamma(\rho)$.

$$\mathrm{Mod}_p(\Gamma \setminus \Gamma_2) \leq \mathrm{Mod}_p(\Gamma \setminus \Gamma(\rho)) = 0.$$

Thus, we conclude that for $p-$almost every curve in Γ satisfies $\mathcal{P}$ as well on every subcurve of γ.

■

Remark We cannot apply Proposition 2.1.18 in the proof of Lemma 2.2.10 directly. It is true that γ_0 does not satisfy $\mathcal{P}$, however, we cannot assure that γ_0 belong to Γ. For this reason we applied $p-$exceptionality criterion.
To prove Lemma 2.2.10, we used that $\Gamma(\rho) \cap \Gamma \subset \Gamma_2$. However, it is possible that $\Gamma(\rho) \cap \Gamma \subsetneq \Gamma_2$.

2.2.2 Application to extensions of admissible densities

Another useful application of the $p-$exceptionality criterion is the following result that provides a kind of extension of an admissible densities.

Lemma 2.2.12: Extension of admissible densities

Let X be a metric measure space and let Γ be a family of curves. Let Γ_0 be a $p-$null family of curves. For each $\rho \in D_{\Gamma \setminus \Gamma_0}$ and for each $\varepsilon > 0$ define

$$\rho_\varepsilon = \varepsilon \rho_0 + \rho$$

where ρ_0 is the $p-$integrable function given by the $p-$exceptionality criterion. Then, ρ_ε is an admissible density for Γ such that $\rho_\varepsilon \geq \rho$ and $\rho_\varepsilon \to \rho$ pointwise.

Proof: The properties $\rho_\varepsilon \geq \rho$ and $\rho_\varepsilon \to \rho$ pointwise follow immediately from the definition of ρ_ε. Then, we only need to prove that ρ_ε is an admissible density. Indeed, since ρ_0 is the function given by the $p-$exceptionality criterion, we have

$$\int_\gamma \rho_0 \, ds = \infty \qquad \forall \gamma \in \Gamma_0.$$

Then

$$\int_\gamma \rho_\varepsilon \, ds = \infty \qquad \forall \gamma \in \Gamma_0.$$

On the other hand, if $\gamma \in \Gamma \setminus \Gamma_0$, we have that

$$\int_\gamma \rho_\varepsilon \, ds \geq \int_\gamma \rho \, ds \geq 1.$$

■

This technique will be used in the principal theorem of this thesis(Theorem 3.3.16).

2.3 Curves on a set of measure zero

We introduce the notion of the length of a curve in a certain set.

Definition: Length in a set

Let $E \subset X$ be any set in a metric space X and γ a rectifiable curve in X. We define the **length in E**, denoted by $\mathbf{length}(\gamma \cap E)$, as the number

$$\mathbf{length}(\gamma \cap E) = m_1(\gamma_s^{-1}[E]).$$

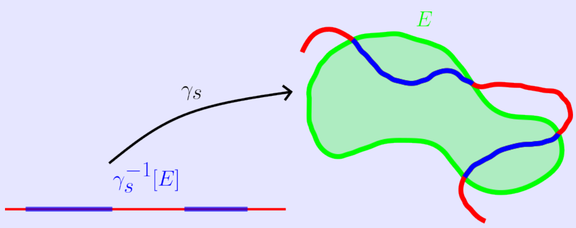

Figure 2.3.1: Length in E.

From the definition of length in a set, it follows the next results:

Lemma 2.3.2: Length of a curve in a set of measure zero

Let X be a metric measure space. If E is a set of measure zero. Then for $p-$a.e. curve γ in X, the length of γ in E is zero.

Particular case

If E is a set of measure zero, then for $p-$a.e. curve γ in X, we have $\mathcal{H}_1(\gamma \cap E) = 0$.

Proof: Let Γ the family of all locally rectifiable curves such that the length of γ in E is positive. Let E_1 a Borel set containing E with zero measure, then $\rho = \infty \cdot \chi_{E_1}$ is $L^p(X)$. Moreover, ρ is an admissible density. Indeed, let $\gamma \in \Gamma$, since $E \subset E_1$ and the definition of ρ we have:

$$\begin{aligned}\int_\gamma \rho \, ds &\geq \int_\gamma \infty \cdot \chi_E \, ds \\ &= \infty \cdot \text{length}(\gamma \cap E) \\ &= \infty \qquad \textit{Since } \text{length}(\gamma \cap E) > 0. \\ &\geq 1\end{aligned}$$

therefore $\rho \in D_\Gamma$. By definition of ρ we have $\int_X \rho = 0$. Therefore, $\text{Mod}_p(\Gamma) = 0$.

■

We will use the above result to prove Lemma 2.6.1.

2.4 Fundamental examples

In this section we give theorems to compute the fundamental examples of $p-$modulus of certain families of curves. The above theorems focus on the computation itself together with a suitable approximation.

2.4.1 Modulus of a single curve

We start computing the modulus of a family with a single compact rectifiable curve γ, since γ is compact follows that $\gamma[I]$ is a Borel set. Now, if $\gamma[I]$ is null we finished. Then, the important case is when $\gamma[I]$ has positive measure. Thus, for understand this case we will analyze this case in $\mathbb{R}$.

Theorem 2.4.1: $\text{Mod}_p(\{\gamma\})$

Let $[a, b]$ be an interval in $\mathbb{R}$ of finite positive length L, and consider γ as the curve given by the canonical injection $[a, b] \hookrightarrow \mathbb{R}$. Then

$$\text{Mod}_p(\{\gamma\}) = \frac{1}{L^{p-1}} = \frac{L}{L^p}.$$

Reduction of densities

For each admissible density $\rho \in D_\gamma$ such that ρ is integrable over γ, there exists an admissible density $F\colon \mathbb{R} \to [0, \infty]$ that depends on ρ, $F = F(\rho)$, that satisfies:

1. $\text{spt}\, F \subset [a, b]$.
2. $F(a) = 1, F(b) = 0$.
3. F is absolutely continuous.
4. $|F'| \leq \rho\,|_{[a,b]}$ almost everywhere in $[a, b]$.

Furthermore, considering D as the set of all the functions previously described, we have the following computation for $p-$modulus:

$$\text{Mod}_p(\{\gamma\}) = \inf_{F \in D} \int_a^b |F'|^p$$

Proof: From Local behavior of the $p-$module, we can consider that the space is $[a, b]$. Let $p > 1$, and, let $q > 1$ the Holder conjugate of p. From Holder inequality we have:

$$
\begin{aligned}
\int_\gamma 1 \cdot \rho \, ds &\leq \left(\int_\gamma 1^q \, ds \right)^{\frac{1}{q}} \left(\int_\gamma \rho^p \, ds \right)^{\frac{1}{p}} && \textit{γ_x has length l.} \\
1 &\leq L^{1-\frac{1}{p}} \left(\int_\gamma \rho^p \, ds \right)^{\frac{1}{p}} && \textit{Raising to the $p-$th power.} \\
1 &\leq L^{p-1} \int_\gamma \rho^p \, ds && \textit{Definition of line integral.} \\
\frac{1}{L^{p-1}} &\leq \int_a^b \rho^p(t) \, dt && \textit{γ is parametrized by arc length.}
\end{aligned}
$$

Clearly this inequality holds for $p = 1$. Since $\rho \in D_\Gamma$ was arbitrary, it follows that $\frac{m_1([a,b])}{L^p} \leq \operatorname{Mod}_p(\Gamma)$, the converse inequality is by Lemma 2.1.21. Therefore:

$$\operatorname{Mod}_p(\{\gamma\}) = \frac{1}{L^{p-1}} = \frac{L}{L^p}.$$

Given $\rho \in D_\gamma$. Define $f \colon \mathbb{R} \to [0, \infty]$ as

$$f(x) = \int_x^b \rho(t) \, dm_1(t) \qquad \forall x \in [a, b] \, ,$$

and zero otherwise. By definition, we have that f is absolutely continuous. Moreover, in the points in such that exists the derivative in $[a, b]$, we have:

$$
\begin{aligned}
f'(x) &= -\rho(x) && \textit{Considering absolute values} \\
|f'(x)| &= \rho(x).
\end{aligned}
$$

By the definition of f and since ρ is integrable over γ, it follows that:

$$\infty > f(a) = \int_a^b \rho(t) \, dm_1(t) \geq 1$$

Then $\frac{1}{f(a)} \leq 1$. Thus the function $F \colon [a, b] \to [0, \infty]$ defined as $F = \frac{f}{f(a)}$ is absolutely continuous hence satisfies the desired properties. Then, F satisfies the hypothesis of Theorem 2.1.1. Hence F is an admissible density. Thus, $\left\{ \left| F'_\rho \right| \right\}_{\rho \in D_\Gamma}$ satisfies the hypothesis of Corollary 2.1.7. Therefore, the following computation holds:

$$\operatorname{Mod}_p(\Gamma) = \inf_{\rho \in D} \int_X F_\rho^p.$$

∎

The customary way tho write

$$\operatorname{Mod}_p(\{\gamma\}) = \frac{1}{L^{p-1}} = \frac{L}{L^p}.$$

will make sense with Theorem 2.4.5 because this will shown the relationship with the extremal length.
Now, we use Theorem 2.4.1 to compute the $p-$modulus of the family of all the curves such that pass through a point in $\mathbb{R}$. To prove this result, it is convenient to prove first the following:

Theorem 2.4.2: Admissible densities of the family of curves that pass through a point in $\mathbb{R}$

Let $x \in \mathbb{R}$ be any point. Let Γ be either the family of all the curves that pass through x in $\mathbb{R}$ or the family of all curves such that one of its end points is x. Then, every $\rho \colon \mathbb{R} \to [0, \infty]$ admissible density for Γ is not integrable.

Proof: We proceed by contradiction. ※Suppose ρ integrable. For any $n \in \mathbb{N}$, define $\gamma_n \colon \left[x, x+\frac{1}{n}\right] \to \mathbb{R}$ as the canonical inclusion, then $\{\gamma_n\}_{n\in\mathbb{N}} \subset \Gamma$. Since $\rho \in D_\Gamma$, we have that

$$\int_x^{x+\frac{1}{n}} \rho\,(t)\ dt = \int_{\gamma_n} \rho\, ds \geq 1. \tag{2.4.3}$$

Define $\rho_n = \rho|_{\left[x,x+\frac{1}{n}\right]}$. Then, $\{\rho_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of nonnegative functions that converges pointwise to $\rho(x)$, and all the functions are in the subspace $[x, x+1]$ which have finite measure. Then

$$\lim_{n\to\infty} \int \rho_n\, dm_1 = 0.$$

On the other hand, by (2.4.3) we have that

$$\lim_{n\to\infty} \int \rho_n\, dm_1 \geq 1$$

which is a contradiction. ※This contradiction comes from assume ρ integrable. Therefore, ρ is not integrable.

■

Theorem 2.4.4: $p-$modulus of of curves that pass though a point in $\mathbb{R}$

Let Γ be the family of all curves in $\mathbb{R}$ such that one of the endpoints is any fixed point $x \in \mathbb{R}$. Then, $\mathrm{Mod}_p\,(\Gamma) = \infty$.

Proof: Let $x \in \mathbb{R}$ and let Γ be the family of all the curves that pass through x. Now, we will prove that $\mathrm{Mod}_p\,(\Gamma) = \infty$. We separate the proof in the next cases:

$\boldsymbol{p > 1}$

Let $L > 0$ arbitrary and let γ_L be any curve such that pass through x and has longitude L. Then, $\{\gamma_L\} \subset \Gamma$. From Theorem 2.4.1, we have

$$\frac{1}{L^{p-1}} = \mathrm{Mod}_p\,(\{\gamma_L\}) \leq \mathrm{Mod}_p\,(\Gamma)$$

This inequality holds for any $L > 0$. Taking the limit when $L \to 0$ in the above inequality, we obtain $\mathrm{Mod}_p\,(\Gamma) = \infty$.

$\boldsymbol{p = 1}$

Let $\rho \in D_\Gamma$. From Theorem 2.4.2, we have that $\int_{\mathbb{R}} \rho\, dm_1 = \infty$. Therefore, $\mathrm{Mod}_p\,(\Gamma) = \infty$.

■

The modulus of the family of curves such that pass through a point in $\mathbb{R}$ infinite but in more dimensions there are a more suitable characterization given in Theorem 2.5.10.

2.4.2 Curves in a rectangle

The following examples essentially are computations for the $p-$modulus of a family of curves that join two closed sets. For accuracy, we introduce the following:

Notation: Curves that join two sets

Let E, F subsets of a topological space X. We denote by $\boldsymbol{\Gamma(E,F,X)}$ the family of curves that connect E with F. If X is a metric space, we will denote

$$\mathbf{Mod}_{\boldsymbol{p}}\boldsymbol{(E,F,X)} = \mathrm{Mod}_p(\Gamma(E,F,X)).$$

If the space X is clear we only write $\boldsymbol{\Gamma(E,F)}$ and $\mathbf{Mod}_{\boldsymbol{p}}\boldsymbol{(E,F)}$.
Sometimes we need to consider the case when one of the sets is a singleton. In this case we simplify the notation to $\boldsymbol{\Gamma(x,F,X)}$ and $\mathbf{Mod}_{\boldsymbol{p}}\boldsymbol{(x,F,X)}$, if the space is clear we will simply denote $\boldsymbol{\Gamma(x,F)}$ and $\mathbf{Mod}_{\boldsymbol{p}}\boldsymbol{(x,F)}$.

Some considerations

Due to the approach of this notes, we focus previous notation to study $p-$modulus. Therefore we consider the following conditions

1. **Usually, we take E, F be closed, disjoint and nonempty.** In general, if either E, F is empty it follows that $\Gamma(E,F) = \varnothing$. If X is a metric measure space, we have that $\text{Mod}_p(E,F) = 0$. If $E \cap F \neq \varnothing$ we can consider trivial curves, and, in a metric space we have that $\text{Mod}_p(E,F) = \infty$. To avoid these trivial cases we require to E, F to be nonempty and disjoint. To simplify cases we will also require E, F be closed.

2. **The curves in $\Gamma(E,F)$ are compact.** We mean that the endpoints lie in each of the sets. Therefore, we are considering compact curves.

Considering the previous notation we will proceed with the following example.

Theorem 2.4.5: Vertical curves in a rectangle

Let $L, l > 0$. Consider the rectangle $R = [0,L] \times [0,l]$. Define $R_B = [0,L] \times \{0\}$ as the bottom edge of the rectangle R, and $R_T = [0,L] \times \{l\}$ as the top edge of R. Then

$$\text{Mod}_p(R_B, R_T, R) = \frac{m_2(R)}{l^p} = \frac{L}{l^{p-1}}.$$

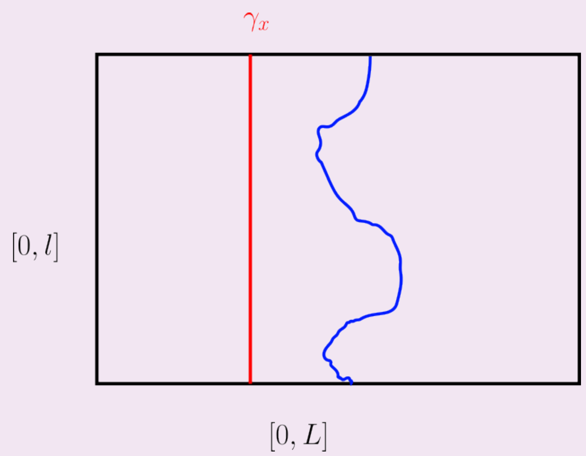

Figure 2.4.6: Curves that connect both edges of the rectangle.

Reduction of densities

For each admissible density $\rho \in D_{\Gamma(R_B,R_T,R)}$ such that ρ is integrable over R, there exists an admissible density $F\colon R \to [0,\infty]$ that depends on ρ, $F = F(\rho)$, that satisfies:

1. $F[R_B] = \{1\}, F[R_T] = \{0\}$.

2. F is absolutely continuous on almost every line parallel to the $y-$axis, that is $F \in \text{ACL}_2(R)$.

3. $|F'| \leq \rho$ almost everywhere in R.

Furthermore, considering D as the set of all the functions previously described, we have the following computation for $p-$modulus:

$$\text{Mod}_p(R_B, R_T, R) = \inf_{F \in D} \int_a^b |F'|^p$$

<u>Proof:</u> First, let us define an auxiliary family of curves. For $x \in [0,L]$ we define $\gamma_x\colon [0,l] \to R$ given by

$\gamma_x(t) = (x, t)$.

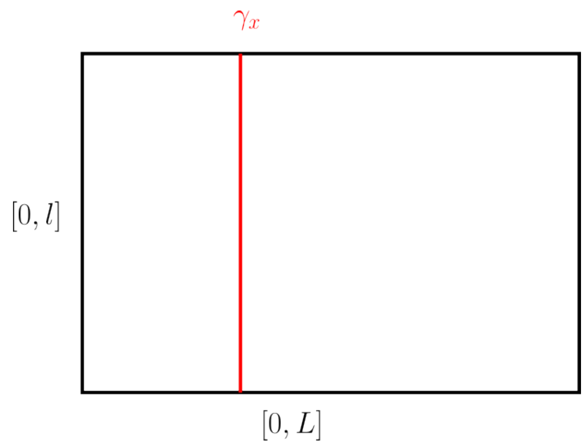

Figure 2.4.7: Vertical lines.

Now, we will calculate the $p-$modulus of the family $\Gamma = \{\gamma_x \mid x \in [0, L]\}$. Let $\rho \in D_\Gamma$. Consider $p > 1$, and, let $q > 1$ the Holder conjugate of p, by the Holder inequality we have:

$$\begin{aligned}
\int_{\gamma_x} 1 \cdot \rho \, ds &\leq \left(\int_{\gamma_x} 1^q \, ds \right)^{\frac{1}{q}} \left(\int_{\gamma_x} \rho^p \, ds \right)^{\frac{1}{p}} && \textit{γ_x has length l.} \\
1 &\leq l^{1-\frac{1}{p}} \left(\int_{\gamma_x} \rho^p \, ds \right)^{\frac{1}{p}} && \textit{Raising to the $p-$th power.} \\
1 &\leq l^{p-1} \int_{\gamma_x} \rho^p \, ds && \textit{Definition of line integral. γ_x is parametrized by arc length.} \\
\frac{1}{l^{p-1}} &\leq \int_0^l \rho^p(x, t) \, dt
\end{aligned} \tag{2.4.8}$$

Notice that the above estimation holds for $p = 1$, as well for each $x \in [0, L]$. Now, integrating in $[0, L]$ and using Fubini's theorem we obtain:

$$\begin{aligned}
\frac{L}{l^{p-1}} &\leq \int_0^L \int_0^l \rho^p(x, t) \, dt dx && \textit{Fubini's theorem. Multiplying and divding by l.} \\
\frac{m_2(R)}{l^p} &\leq \int_R \rho^p.
\end{aligned}$$

Since $\rho \in D_\Gamma$ was arbitrary, it follows that $\frac{m_2(R)}{l^p} \leq \mathrm{Mod}_p(\Gamma)$. Furthermore, since length $\gamma_x = l$ for every $\gamma_x \in \Gamma$, by Lemma 2.1.21 follows that $\mathrm{Mod}_p(\Gamma) \leq \frac{m_2(R)}{l^p}$, hence, this proves that

$$\mathrm{Mod}_p(\Gamma) = \frac{m_2(R)}{l^p}.$$

Now we will extend this result for $\Gamma(R_B, R_T, R)$. Since $\Gamma \subset \Gamma(R_B, R_T, R)$, it follows that (2.4.8) holds, then we can repeat the procedure to obtain:

$$\frac{m_2(R)}{l^p} \leq \mathrm{Mod}_p \left(\Gamma(R_B, R_T, R) \right)$$

The converse inequality follows from this fact about Γ: The curves of $\Gamma(R_B, R_T, R)$ with minimal length are the family of curves Γ(this result is proved below). Together with Lemma 2.1.21.

By Fubini theorem. We have that ρ is integrable on γ_x for almost every $x \in [0, L]$. Let P the set of points in $[0, L]$ where ρ is integrable over γ_x. For all $x \in P$, we can repeat the construction of Theorem 2.4.1 to define a function from the union of all the $x-$sections, $F \colon \bigcup_{x \in P} R_x \to \mathbb{R}$ such that $F \in \mathrm{ACL}_2 \left(\bigcup_{x \in P} R_x \right)$. Furthermore, since

$$R \setminus \bigcup_{x \in P} R_x = \bigcup_{x \in [0,L] \setminus P} R_x$$

and the projection of $\bigcup_{x\in[0,L]\setminus P} R_x$ to $2nd-$canonical hyperplane(the $x-$axis) is $[0, L] \setminus P$ which have m_1 measure zero, we can use Item 1 of Theorem 1.6.9. Then, it follows that the suitable extension with the required properties belongs to $\mathrm{ACL}_2(F)$.

■

Remark Notice that the application of Item 1 of Theorem 1.6.9 in the proof Theorem 2.4.5. We need to adapt the theory of this result for measurable sets because Theorem 1.6.9 requires Ω open and F closed, and in the last part we do not have this hypothesis. For this reason is necessary Remark 1.6.5.

Notice that to complete the proof of Theorem 2.4.5, we need to prove the fact about the minimality of Γ. This is consequence of a more general result proved below. We proceed in this way to generalize Theorem 2.4.5 for a larger class of spaces. To do this, we introduce the next:

Definition: Generalized cylinder

Let X be a metric measure space and let $I \subset \mathbb{R}$ be a nondegenerate interval with height $h > 0$. The product $X \times I$ metric measure space equipped with the product measure $v = \mu \times m_1$ and any distance that restricts to the distances on the factors is called the **generalized cylinder of base X and height h**. The **subcylinders** of $X \times I$ are defined in the canonical way considering a Borel set and a subinterval. For generalized cylinders always, we denote $\boldsymbol{d_X}$ as the metric of X.

Edges of the cylinder

Let $C = X \times I$ be a generalized cylinder with $I = [a, b]$ is a nondegenerate interval. We define $\boldsymbol{C_B} = X \times \{a\}$ and $\boldsymbol{C_T} = X \times \{b\}$ as the **bottom edge** and the **top edge of the cylinder $\boldsymbol{C}$** respectively.

Remark

1. From our courses of analysis, it follows that all the metric that is compatible with the generalized cylinder, d, satisfies
$$d_1 \leq d \leq d_\infty.$$
2. Since X is a metric measure space, we have that X is $\sigma-$finite as well I. Hence we can use Fubini theorem automatically.

We will use these facts without further mention.
Now, we prove the result to complete the proof of Theorem 2.4.5.

Lemma 2.4.9: Curves of minimal length in a generalized cylinder are the vertical curves

Let X be a metric measure space. Let $C = X \times [a, b]$ a generalized cylinder of height $h > 0$. The curves of minimal height in $\Gamma(C_B, C_T, C)$ are the vertical curves. Furthermore, the minimal length is h.

Proof: Let $\gamma \in \Gamma(C_B, C_T, C)$ with end points $(x_1, a), (x_2, b)$. Then
$$\text{length}(\gamma) \geq d((x_1, a), (x_2, b)) \geq d_X(x_1, x_2) + |a - b| \geq |a - b| = h. \tag{2.4.10}$$
Therefore, h is a lower bound for the length of the curves of $\Gamma(C_B, C_T, C)$. Now, we will prove that the image minimal curves are contained in a set of the form $\{x\} \times [a, b]$. Suppose that (x_3, c) is a point of the curve such that $x_3 \neq x_1$. In this case, we have that the inequality (2.4.10) is strict. Hence the image of γ is contained in $\{x_1\} \times [a, b]$. Therefore, the curves of minimal length are the vertical curves.

■

Then, the proof of Theorem 2.4.5 is finally complete. Furthermore, notice that this result can be generalized as follows:

Theorem 2.4.11: Modulus of vertical curves in the generalized rectangle

Let X be a metric measure space. Let $C = X \times [a, b]$ a generalized cylinder. Let $S = E \times J$ be subcylinder of C of height $h > 0$. Then,

$$\mathrm{Mod}_p(\Gamma\left(S_B, S_T, S\right)) = \frac{\mu(E)}{h^{p-1}} \tag{2.4.12}$$

Proof: Notice that the arguments for the proof of Theorem 2.4.5 only depends on the vertical curves and in Lemma 2.1.21 which holds in general. The conclusion follows from the minimal property of the vertical lines proved in Lemma 2.4.9.

■

Theorem 2.4.11 has a lot of consequences. First, we can show that the reciprocal of the modulus is consider as the **extremal length**.

Remark: Comparison of $p-$modulus with extremal length [AB50] For a family of curves Γ in the plane and a nonnegative Borel function $\rho\colon \mathbb{R}^2 \to [0, \infty]$. We define the minimal length as

$$L(\rho) = \inf_{\gamma \in \Gamma} \int_{\gamma} \rho \, ds$$

and the area of ρ as

$$A(\rho) = \int_{\mathbb{R}^2} \rho^2$$

The extremal length is defined as:

$$\lambda(\Gamma) = \sup_{\rho} \frac{L(\rho)^2}{A(\rho)}.$$

considering that inf, sup change each other under decreasing functions, we obtain that

$$\lambda(\Gamma) = \frac{1}{\mathrm{Mod}_2(\Gamma)}.$$

However, to define the minimal length in the above definition, we used the structure of the plane.

Another important result from Theorem 2.4.11 is the relationship of the notion absolute continuity on lines and the $p-$modulus of a family of curves. To prove that result, it is convenient notice the following:

Corollary 2.4.13: Let X be a metric measure space. Let $C = X \times [a, b]$ a generalized cylinder. Let $S = E \times J$ be subcylinder of C of height $h > 0$. Then, the following statements hold:

1. $\Gamma\left(S_B, S_T, S\right)$ is $p-$exceptional if and only if $\mu(E) = 0$.

2. $\Gamma\left(S_B, S_T, S\right)$ is $p-$exceptional for all $p \geq 1$ if and only if $\Gamma\left(S_B, S_T, S\right)$ is p_0-null for some $p_0 \geq 1$.

Proof: Item 1 follows immediately from (2.4.12). To prove Item 2, we only need to prove p_0-exceptionality of $\Gamma\left(S_B, S_T, S\right)$ implies $p-$exceptionality of $\Gamma\left(S_B, S_T, S\right)$ for all $p \geq 1$. Indeed, if $\Gamma\left(S_B, S_T, S\right)$ is p_0-exceptional, from Item 1, we have $\mu\left(E\right) = 0$. Notice that Item 1 holds for any $p \geq 1$. Therefore, $\Gamma\left(S_B, S_T, S\right)$ is $p-$exceptional.

■

We use the above result to prove the next:

Theorem 2.4.14: Absolutely continuity on lines and $p-$modulus

Let $f\colon \Omega \subset \mathbb{R}^d \to \mathbb{R}$ a function. Then, the following statements are equivalent:

1. f is absolutely continuous on lines.
2. For **all** $1 \leq p < \infty$ we have that: f is absolutely continuous on $p-$a.e. line segment parallel to the coordinated axis contained in Ω.
3. There is **some** $1 \leq p < \infty$ we have that: f is absolutely continuous on $p-$a.e. line segment parallel to the coordinated axis contained in Ω.

Proof: The equivalence between absolute continuity and the $p-$modulus of the family of all vertical lines follows from the Item 1 of Corollary 2.4.13 while the immateriality of p follows from the Item 2 of Corollary 2.4.13.

■

Theorem 2.4.14 shows that the notion of almost every curve in the definition of $\mathrm{ACL}(\Omega)$ can be considered in both senses, considering fibers and considering $p-$modulus of all vertical lines. Notice that we can consider any $p-$modulus in this setting.

2.4.3 Curves in annulus

Now we calculate the modulus in an annulus.

Theorem 2.4.15: Modulus of radial curves

Let $0 < r < R < \infty$. Consider the annulus $A = A(x, r, R)$ in $\mathbb{R}^d$ with $d \geq 2$, and their two connected components $B[x,r], \mathbb{R}^d \setminus B(x,R)$. Then

$$\mathrm{Mod}_p(\Gamma\left(B[x,r], \mathbb{R}^d \setminus B(x,R), \mathbb{R}^d\right)) = \frac{\omega_{d-1}}{C_{r,R}}$$

where

$$\boldsymbol{C_{r,R}} = \begin{cases} \left(\frac{p-1}{p-d}\, t^{\frac{p-d}{p-1}}\Big|_r^R\right)^{p-1} & \text{if } p \neq 1, d \\ \left(\ln\left(\frac{R}{r}\right)\right)^{p-1} & \text{if } p = d, \\ r^{1-d} & \text{if } p = 1. \end{cases}$$

Proof: First, let us define an auxiliary family of curves. For $\omega \in \mathbb{S}^{d-1}$ we define $\gamma_\omega\colon [r,R] \to \mathbb{R}^d$ given by $\gamma_\omega(t) = t\omega + x$.

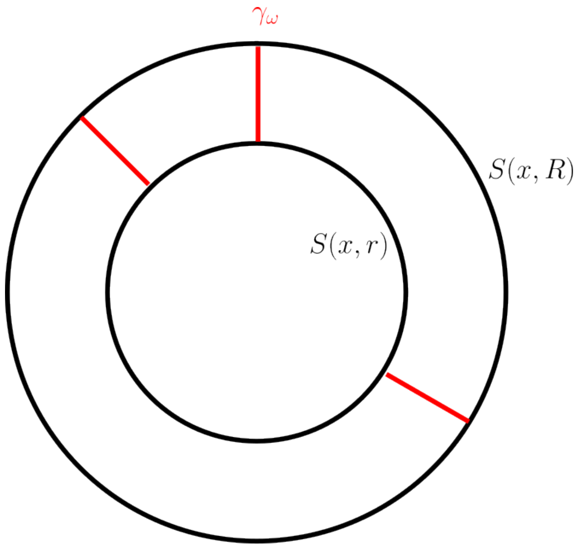

Figure 2.4.16: Radial curves.

We will calculate the $p-$modulus of the family $\Gamma = \left\{\gamma_\omega \mid \omega \in \mathbb{S}^{d-1}\right\}$. Let $\rho \in D_\Gamma$. We will analyze the following cases:

If $p > 1$.

Let $q > 1$ be the Holder conjugate of p, by the Holder inequality we have:

$$\begin{aligned} \int_r^R t^{-\frac{d-1}{p}} \cdot \rho(t\omega + x) t^{\frac{d-1}{p}}\, dt &\le \left(\int_r^R t^{-\frac{(d-1)q}{p}}\, dt\right)^{\frac{1}{q}} \left(\int_r^R \rho^p(t\omega + x) t^{d-1}\, dt\right)^{\frac{1}{p}} \\ 1 &\le \left(\int_r^R t^{-\frac{d-1}{p-1}}\, dt\right)^{1-\frac{1}{p}} \left(\int_r^R \rho^p(t\omega + x) t^{d-1}\, dt\right)^{\frac{1}{p}} \\ 1 &\le \left(\int_r^R t^{-\frac{d-1}{p-1}}\, dt\right)^{p-1} \int_r^R \rho^p(t\omega + x) t^{d-1}\, dt \\ \frac{1}{C_{r,R}} &\le \int_r^R \rho^p(t\omega + x) t^{d-1}\, dt. \end{aligned}$$

$\frac{q}{p} = \frac{1}{p-1}$. $\int_{\gamma_\omega} \rho\, ds = \int_r^R \rho(t\omega + x)\, dt$.

Raising to the $p-$th power.

Definition of $C_{r,R}$.

Since the above inequality holds for every $\omega \in \mathbb{S}^{d-1}$, we consider the integral over $\mathbb{S}^{d-1}$ to obtain

$$\begin{aligned} \frac{\omega_{d-1}}{C_{r,R}} &\le \int_{\mathbb{S}^{d-1}} \int_r^R \rho^p(t\omega + x) t^{d-1}\, dt d\omega. \\ \frac{\omega_{d-1}}{C_{r,R}} &\le \int_A \rho^p. \end{aligned} \tag{2.4.17}$$

Integration in polar coordinates.

Since the above inequality holds for any $\rho \in D_\Gamma$, it follows that $\mathrm{Mod}_p(\Gamma) \ge \frac{\omega_{n-1}}{C_{r,R}}$. Now, we will prove the converse inequality. To do this, we define $\rho_0 \colon A \to [0, \infty]$ as follow:

$$\rho_0(y) = C_{r,R}^{-\frac{1}{p-1}} |y - x|^{-\frac{d-1}{p-1}}.$$

Then, for each $\omega \in \mathbb{S}^{d-1}$ we have:

$$\begin{aligned} \int_{\gamma_\omega} \rho_0\, ds &= \int_r^R \rho_0(t\omega + x)\, dt \\ &= C_{r,R}^{-\frac{1}{p-1}} \int_r^R t^{-\frac{(d-1)}{p-1}}\, dt \\ &= 1. \end{aligned}$$

Definition of ρ_0.

Definition of $C_{r,R}$.

This proves that $\rho_0 \in D_\Gamma$. Again, from the definition of ρ_0 we have:

$$
\begin{aligned}
\int_A \rho_0^p &= C_{r,R}^{-\frac{p}{p-1}} \int_A |y-x|^{-\frac{p(d-1)}{p-1}} \\
&= C_{r,R}^{-\frac{p}{p-1}} \int_r^R \int_{\mathbb{S}^{d-1}} |t\omega + x - x|^{-\frac{p(d-1)}{p-1}} t^{d-1}\, d\omega dt && \text{Integration in polar coordinates.} \\
&= C_{r,R}^{-\frac{p}{p-1}} \int_r^R \int_{\mathbb{S}^{d-1}} |t\omega|^{-\frac{p(d-1)}{p-1}} t^{d-1}\, d\omega dt && \text{t is independient in the surface integral. Grouping appropriately the exponents.} \\
&= \omega_{d-1} C_{r,R}^{-\frac{p}{p-1}} \int_r^R t^{-\frac{d-1}{p-1}}\, dt && \text{Definition of } C_{r,R}. \\
&= \frac{\omega_{d-1}}{C_{r,R}}.
\end{aligned}
\tag{2.4.18}
$$

This proves that ρ_0 achieves the equality in the previous modulus estimation. Therefore $\operatorname{Mod}_p(\Gamma) = \frac{\omega_{d-1}}{C_{r,R}}$.

If $p = 1$.

Using the integration in polar coordinates formula, we have:

$$
\begin{aligned}
\int_A \rho &= \int_{\mathbb{S}^{d-1}} \int_r^R \rho(t\omega + x) t^{d-1}\, dt d\omega && r \le t \le R. \text{ Monoticity and linearity of the integral.} \\
&\ge r^{d-1} \int_{\mathbb{S}^{d-1}} \int_r^R \rho(t\omega + x)\, dt d\omega \\
&= r^{d-1} \int_{\mathbb{S}^{d-1}} \int_{\gamma_\omega} \rho\, ds d\omega && \rho \in D_\Gamma. \\
&\ge r^{d-1} \omega_{d-1}.
\end{aligned}
$$

Since $\rho \in D_\Gamma$ was arbitrary, it follows that $\omega_{d-1} r^{d-1} \le \operatorname{Mod}_1(\Gamma)$. Now, we will prove that the equality is achieved. Indeed, for $j \in \mathbb{N}$ large enough, define $\rho_j \colon A \to [0,\infty]$ as $\rho_j = j\chi_{A(x,r,r+\frac{1}{j})}$. Evidently, $(\rho_j)_{j \ge j_0} \subset D_\Gamma$. Moreover,

$$
\begin{aligned}
\int_A \rho_j &= \int_{A(x,r,r+\frac{1}{j})} j \\
&= \frac{j\omega_{d-1}}{d} \left(\left(r + \frac{1}{j} \right)^d - r^d \right) && \text{Measure of an annulus.} \\
&= \frac{\omega_{d-1}}{d} \frac{\left(r + \frac{1}{j} \right)^d - r^d}{\frac{1}{j}}.
\end{aligned}
$$

From the above equation, follows that:

$$
\begin{aligned}
\lim_{j\to\infty} \int_A \rho_j &= \frac{\omega_{d-1}}{d} \lim_{j\to\infty} \frac{\left(r + \frac{1}{j} \right)^d - r^d}{\frac{1}{j}} && \tfrac{d}{dx} x^d = n x^{d-1} \\
&= \omega_{d-1} r^{d-1}.
\end{aligned}
$$

This proves that $(\rho_j)_{j\in\mathbb{N}}$ reaches the previous modulus estimation. By the infimum properties we conclude that

$$
\operatorname{Mod}_1(\Gamma) = \omega_{d-1} r^{d-1} = \frac{\omega_{d-1}}{C_{r,R}}
$$

Notice that **$\operatorname{Mod}_1(\Gamma)$ only depends on the radius of minor sphere.**

Now, we will extend the previous computation for $\widetilde{\Gamma} = \Gamma\left(B\left[x,r\right], \mathbb{R}^d \setminus B\left(x,R\right), \mathbb{R}^d\right)$.

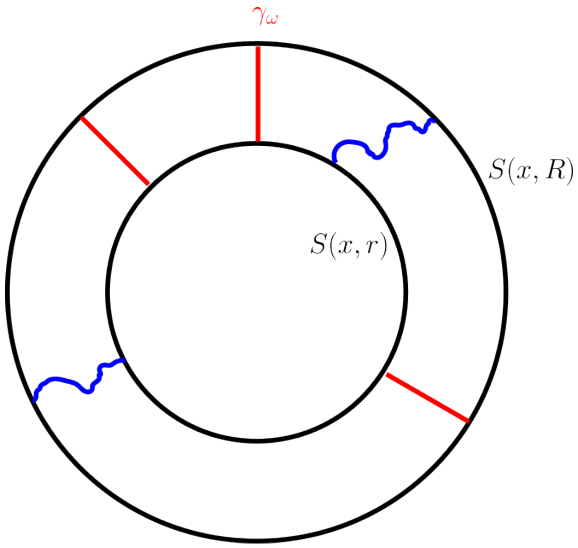

Figure 2.4.19: Curves that connect both circles of the annulus.

Notice that the estimation (2.4.17) holds for any $\rho \in D_{\widetilde{\Gamma}}$ as well for $p \geq 1$, so it follows that $\operatorname{Mod}_p(\widetilde{\Gamma}) \geq \frac{\omega_{n-1}}{C_{r,R}}$. Now, we will prove again that the estimation is reached with $\rho_0 \in D_{\widetilde{\Gamma}}$. Indeed, let $\gamma \in \widetilde{\Gamma}$, consider $\gamma \colon [a,b] \to X$ and $(\widehat{r}, \omega)$ the polar coordinates around of x of γ. We can suppose without loss of generality that $\gamma(a) \in B\left[x,r\right], \gamma(b) \in \mathbb{R}^d \setminus B(x,R)$. Since γ is locally rectifiable, by (1.5.26) we have:

$$
\begin{aligned}
\int_\gamma \rho_0 \, ds &= \int_a^b \rho_0(\gamma) \left|\gamma'\right| \, dt && \text{Definition of } \rho_0. \\
&\geq \int_a^b C_{r,R}^{-\frac{1}{p-1}} \widehat{r}^{-\frac{n-1}{p-1}} \left|\widehat{r}'\right| \, dt && \text{Polar coordinates around of } x. \text{ Change of variables.} \\
&= C_{r,R}^{-\frac{1}{p-1}} \int_{\widehat{r}(a)}^{\widehat{r}(b)} \widehat{r}^{-\frac{n-1}{p-1}} \, d\widehat{r} && \text{The curves are in } \Gamma(S(x,r), S(x,R), A). \\
&= C_{r,R}^{-\frac{1}{p-1}} \int_r^R \widehat{r}^{-\frac{n-1}{p-1}} \, d\widehat{r} && \gamma(a) \in S(x,r), \gamma(b) \in S(x,R). \\
&= 1. && \text{Definition of } C_{r,R}.
\end{aligned}
$$

This proves that $\rho_0 \in D_{\widetilde{\Gamma}}$ and by (2.4.18) we have that $\operatorname{Mod}_p(\widetilde{\Gamma}) = \frac{\omega_{d-1}}{C_{r,R}}$.

■

The idea is to For now, we will compute some limit values of $C_{r,R}$.

Lemma 2.4.20: Limits of $C_{r,R}$

Consider $C_{r,R}$ as in Theorem 2.4.15, with $d \geq 2$. We have that:

$$
\lim_{r \downarrow 0} C_{r,R} = \begin{cases} \infty & \text{if } 1 \leq p \leq d, \\ \left(\frac{p-1}{p-d}\right)^{p-1} R^{p-d} & \text{if } p > d. \end{cases} \tag{2.4.21}
$$

$$
\lim_{R \to \infty} C_{r,R} = \begin{cases} \left(-\frac{p-1}{p-d}\right)^{p-1} r^{p-d} & \text{if } 1 \leq p < d, \\ \infty & \text{if } p \geq d. \end{cases} \tag{2.4.22}
$$

$$
\lim_{R \to r} C_{r,R} = \begin{cases} 0 & \text{if } p \geq 1, \\ R^{1-n} & \text{if } p = 1. \end{cases} \tag{2.4.23}
$$

Proof:

$p > d$.

We have that $\frac{p-d}{p-1}, \frac{p-1}{p-d} > 0$. Then

$$\begin{aligned}
\lim_{r\downarrow 0} C_{r,R} &= \lim_{r\downarrow 0}\left(\frac{p-1}{p-d}\, t^{\frac{p-d}{p-1}}\Big|_r^R\right)^{p-1}\\
&= \left(\frac{p-1}{p-d}R^{\frac{p-d}{p-1}}\right)^{p-1}\\
&= \left(\frac{p-1}{p-d}\right)^{p-1} R^{p-d}.\\
\lim_{R\to\infty} C_{r,R} &= \lim_{R\to\infty}\left(\frac{p-1}{p-d}\, t^{\frac{p-d}{p-1}}\Big|_r^R\right)^{p-1}\\
&= \left(\frac{p-1}{p-d}\left(\infty - r^{\frac{p-d}{p-1}}\right)\right)^{p-1}\\
&= \left(\frac{p-1}{p-d}\infty\right)^{p-1} \qquad p-1,\ \tfrac{p-1}{p-d} > 0.\\
&= \infty.
\end{aligned}$$

$\boldsymbol{p = d.}$

By the continuity of logarithm and $(p-1)-$power, we have that:

$$\lim_{r\downarrow 0}\left(\ln\left(\frac{R}{r}\right)\right)^{p-1} = \infty.$$
$$\lim_{R\to\infty}\left(\ln\left(\frac{R}{r}\right)\right)^{p-1} = \infty.$$

$\boldsymbol{1 < p < d.}$

Notice that $\frac{p-d}{p-1}, \frac{p-1}{p-d} < 0$. Hence

$$\begin{aligned}
\lim_{r\downarrow 0} C_{r,R} &= \lim_{r\downarrow 0}\left(\frac{p-1}{p-d}\, t^{\frac{p-d}{p-1}}\Big|_r^R\right)^{p-1} \qquad \tfrac{p-d}{p-1} < 0.\\
&= \left(\frac{p-1}{p-d}\left(R^{\frac{p-d}{p-1}} - \infty\right)\right)^{p-1}\\
&= \left(-\frac{p-1}{p-d}\infty\right)^{p-1} \qquad -\tfrac{p-1}{p-d} > 0 \quad p > 1.\\
&= \infty.\\
\lim_{R\to\infty} C_{r,R} &= \lim_{R\to\infty}\left(\frac{p-1}{p-d}\, t^{\frac{p-d}{p-1}}\Big|_r^R\right)^{p-1} \qquad \tfrac{p-d}{p-1} < 0.\\
&= \left(\frac{p-1}{p-d}\left(-r^{\frac{p-d}{p-1}}\right)\right)^{p-1}\\
&= \left(-\frac{p-1}{p-d}r^{\frac{p-d}{p-1}}\right)^{p-1} \qquad -\tfrac{p-1}{p-d} > 0 \quad p > 1.\\
&= \left(-\frac{p-1}{p-d}\right)^{p-1} r^{p-d}.
\end{aligned}$$

$\boldsymbol{p = 1.}$

Since $1 - d < 0$, we have:

$$\begin{aligned}
\lim_{r\downarrow 0} C_{r,R} &= \lim_{r\downarrow 0} r^{1-d} = \infty.\\
\lim_{R\to\infty} C_{r,R} &= \lim_{R\to\infty} r^{1-d}\\
&= r^{1-d}\\
&= \cancelto{1}{\left(-\frac{1-1}{1-d}\right)^{1-1}} r^{1-d}.
\end{aligned}$$

From all these equalities, we have proved (2.4.21),(2.4.22). To calculate, (2.4.23) we would do the explicit calculations. However, this is unnecessary. We only need use the definition of $C_{r,R}$

$$\boldsymbol{C_{r,R}} = \begin{cases} \left(\int_r^R t^{-\frac{d-1}{p-1}}\, dt\right)^{p-1} & \text{if } p > 1, \\ r^{1-d} & \text{if } p = 1, \end{cases}$$

and the Lebesgue's dominated convergence theorem to conclude.

■

The computations of the $p-$module of the curves is annulus is really fruitful. Considering appropriate approximations, we can compute the $p-$modulus of the family of all curves with an endpoint in a fixed point $x \in \mathbb{R}^d$. However, to justify this argument we need to prove the $p-$subadditivity of $p-$modulus which is a consequence of Fuglede lemma.

2.5 Fuglede's lemma

In this section we discuss Fuglede lemma which is one of the most important theorems on the modulus theory of curves.

Theorem 2.5.1: Fuglede lemma

Let X be a metric measure space. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of Borel functions such that converges in $L^p(X)$. Then, there is a subsequence $\{g_{n_k}\}_{k\in\mathbb{N}}$ such that: For every Borel representative g of the L^p-limit of $(g_n)_{n\in\mathbb{N}}$, the following limit

$$\lim_{k\to\infty} \int_\gamma |g_{n_k} - g|\; ds = 0 \tag{2.5.2}$$

holds for $p-$a.e. curve γ in X.

Improved version

Let X be a metric measure space. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of Borel functions such that converges in $L^p(X)$. Then, there is a subsequence $\{g_{n_k}\}_{k\in\mathbb{N}}$ such that: For every Borel representative g of the L^p-limit of $(g_n)_{n\in\mathbb{N}}$ the following statements holds for $p-$a.e. curve γ in X:

1. $\lim_{k\to\infty} \int_\gamma |g_{n_k} - g|\; ds = 0$
2. g is integrable on γ.

Proof: Let $j \in \mathbb{N}$. Since $\{g_n\}_{n\in\mathbb{N}}$ converges in $L^p(X)$, it follows that there exists a subsequence $\{g_{n_k}\}_{k\in\mathbb{N}}$ of $\{g_n\}_{n\in\mathbb{N}}$ that converges in $L^p(X)$ and such that

$$\int_X |g_{n_k} - g|^p\; d\mu \le \frac{1}{2^{(p+1)k}} \tag{2.5.3}$$

Note that, the subsequence is independent of the representative (This is because the limits almost everywhere). To improve the readability of the proof, for all $k \in \mathbb{N}$, let us define $\rho_k = |g_{n_k} - g|$.
A continuation, we will define some families of curves. Let Γ be the family of locally rectifiable curves γ in X for which the statement

$$\lim_{k\to\infty} \int_\gamma \rho_k\; ds = 0$$

fails to hold. Considering the characterization of limit of sequences that converges to 0 by superior limit, gives a natural decomposition of Γ. To do this, for all $m \in \mathbb{N}$ define Γ_m be the family of all locally rectifiable curves

γ in X for which

$$\limsup_{k\to\infty}\int_\gamma \rho_k\, ds > \frac{1}{m}.$$

Thus:

$$\Gamma = \bigcup_{m=1}^{\infty}\Gamma_m$$

Now, we will provide an appropriate decomposition of Γ_m. For $l \geq m$, let us define $\Gamma_{m,l}$ as the family of rectifiable curves such that:

$$\int_\gamma \rho_k\, ds > \frac{1}{2^k}. \tag{2.5.4}$$

Since $l \geq m$, it follows that $\frac{1}{m} > \frac{1}{2^l}$. Thus:

$$\Gamma_m \subset \bigcup_{l=M}^{\infty}\Gamma_{m,l} \qquad \forall M \geq m$$

From the inequality (2.5.4), it follows that $2^l\rho_l$ is an admissible density for $\Gamma_{m,l}$. Estimating $\mathrm{Mod}_p\left(\Gamma_{m,l}\right)$ within this admissible density, and from (2.5.3):

$$\mathrm{Mod}_p\left(\Gamma_{m,l}\right) \leq 2^{pl}\int_X \left|g_{n_l} - g\right|^p\, d\mu \leq \frac{1}{2^l}$$

Considering the sum of the previous terms for $l \geq M \geq m$, we obtain:

$$\sum_{l=M}^{\infty}\mathrm{Mod}_p\left(\Gamma_{m,l}\right) \leq \sum_{l=M}^{\infty}\frac{1}{2^l}$$

$$\Gamma_m \subset \bigcup_{l=M}^{\infty}\Gamma_{m,l}$$

$$\mathrm{Mod}_p(\Gamma_m) \leq \frac{1}{2^{M-1}}$$

This inequality holds for every $M \geq m$. Thus, $\mathrm{Mod}_p(\Gamma_m) = 0$; and since $\Gamma = \bigcup_{m=1}^{\infty}\Gamma_m$ we conclude that $\mathrm{Mod}_p(\Gamma) = 0$. Therefore, the limit (2.5.2) holds for $p-$a.e. curve γ in X.
The improved version follows from the main part and lemma 2.2.5

■

Remark: Useful form of Fuglede lemma In practice, we will use the improved version of Fuglede lemma directly. However, it is important remark that in the improved version of Fuglede lemma, the representative is important.

Fuglede lemma has a lot of important applications in the theory of Sobolev Spaces in the metric measure space setting, we discuss one of this application in section 2.6. First of all is to prove a continuity from below of Mod_p using Fuglede lemma.

Theorem 2.5.5: Continuity from below of $p-$modulus

Let $p > 1$ and $\{\Gamma_n\}_{n\in\mathbb{N}}$ an increasing sequence of families of curves

$$\lim_{n\to\infty}\mathrm{Mod}_p\left(\Gamma_n\right) = \mathrm{Mod}_p\left(\bigcup_{n\in\mathbb{N}}\Gamma_n\right).$$

Proof: Define $\Gamma = \bigcup_{n\in\mathbb{N}}\Gamma_n$. We will prove the following inequalities:

$\boldsymbol{\lim_{n\to\infty}\mathrm{Mod}_p\left(\Gamma_n\right) \leq \mathrm{Mod}_p\left(\Gamma\right))}$

By definition, $\Gamma_n \subset \Gamma$, then $\mathrm{Mod}_p(\Gamma_n) \leq \mathrm{Mod}_p(\Gamma)$. Taking the limit when $n \to \infty$ we obtain $\lim_{n\to\infty}\mathrm{Mod}_p\left(\Gamma_n\right) \leq \mathrm{Mod}_p\left(\Gamma\right)$.

$\boxed{\mathbf{Mod}_p\,(\boldsymbol{\Gamma}) \leq \mathbf{lim}_{n\to\infty}\,\mathbf{Mod}_p\,(\boldsymbol{\Gamma}_n))}$

For this inequality we can suppose without loss of generality that $L := \lim_{n\to\infty} \mathrm{Mod}_p\,(\Gamma_n) < \infty$. For each $n \in \mathbb{N}$ there exists an admissible density ρ_n for Γ_n such that

$$\int_X \rho_n^p \, d\mu < L + \frac{1}{n}$$

Then, the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ is bounded in $L^p(X)$ and

$$\lim_{n\to\infty} \|\rho_n\|_p^p = L \tag{2.5.6}$$

Since $p > 1$, we have that $L^p(X)$ is reflexive. Since $\{\rho_n\}_{n\in\mathbb{N}}$ is bounded in $L^p(X)$, then there exists a subsequence $\{\rho_{n_m}\}_{m\in\mathbb{N}}$ of $\{\rho_n\}_{n\in\mathbb{N}}$ converges weakly to some ρ in $L^p(X)$. By the properties of the weak convergence and (2.5.6), we have

$$\|\rho\|_p^p \leq \left(\liminf_{m\to\infty} \|\rho_{n_m}\|_p\right)^p = \lim_{n\to\infty} \|\rho_n\|_p^p = L \tag{2.5.7}$$

By Mazur lemma, there exists a sequence of convex combinations $\{\lambda_m\}_{m\in\mathbb{N}}$ of the functions $\{\rho_{n_m}\}_{m\in\mathbb{N}}$ such that $\lambda_m \to \rho$ in $L^p(X)$. Since $\{\Gamma_n\}_{n\in\mathbb{N}}$ is increasing, and by the Properties of admissible densities, the sequence $\{\lambda_m\}_{m\in\mathbb{N}}$ can be chosen in such a way that λ_m is admissible for Γ_m for each $m \in \mathbb{N}$, then $\|\lambda_m\|_p^p \geq \mathrm{Mod}_p\,(\Gamma_m)$. By this inequality and (2.5.7), we have

$$L = \lim_{n\to\infty} \mathrm{Mod}_p\,(\Gamma_n) \leq \lim_{m\to\infty} \|\lambda_m\|_p^p = \|\rho\|_p^p \leq L. \tag{2.5.8}$$

Since the measure is Borel regular measures, we know that we can modify ρ if necessary to make ρ Borel, and considering Fuglede lemma (the improved version with the integrability of the limit), we may assume that ρ is a Borel function and for $p-$almost every curve in Γ, we have the following:

$$\int_\gamma \rho \, ds = \lim_{k\to\infty} \int_\gamma \lambda_{m_k} \, ds \geq 1$$

for a some subsequence of $\{\lambda_m\}_{m\in\mathbb{N}}$. Then, there exist $\Gamma_0 \subset \Gamma$ a $p-$null family of curves such that ρ is an admissible density for $\Gamma \setminus \Gamma_0$. Then

$$\mathrm{Mod}_p\,(\Gamma \setminus \Gamma_0) \leq \int_X \rho^p \, ds,$$

From the above inequality, the subadditivity of the modulus and by the identity (2.5.8), it follows that

$$\mathrm{Mod}_p(\Gamma) \leq \mathrm{Mod}_p(\Gamma \setminus \Gamma_0) + \cancelto{0}{\mathrm{Mod}_p(\Gamma_0)} \leq \int_X \rho^p \, d\mu = L$$

This proves that $\lim_{n\to\infty} \mathrm{Mod}_p\,(\Gamma_n) = \mathrm{Mod}_p\,(\Gamma)$.

■

The proof of the continuity from below of Mod_p is needed because Mod_p is an outer measure.

Counterexample 2.5.9: $p > 1$ is necessary for Theorem 2.5.5

Let $d \geq 2$, $x \in \mathbb{R}^d$ and $r > 0$. For $n \in \mathbb{N}$, define Γ_n as the family of all curves in $\mathbb{R}^d$ with one end point in the closed ball $B\,[x, r]$, and the other in $\mathbb{R}^d \setminus \left(B\left(x, r + \frac{1}{n}\right)\right)$. So $\{\Gamma_n\}_{n\in\mathbb{N}}$ is an increasing sequence. From Theorem 2.4.15, we have

$$\mathrm{Mod}_1\,(\Gamma_n) = \omega_{d-1} r^{d-1}$$

for every n. Then

$$\lim_{n\to\infty} \mathrm{Mod}_1\,(\Gamma_n) = \omega_{d-1} r^{d-1}.$$

Considering the generalization of the argument of Theorem 2.4.2 to radial lines, it follows that a polar coordinate integration shows that there are no admissible 1-integrable densities for the family $\Gamma = \cup_{i=1}^{\infty}\Gamma_i$ thus, $\text{Mod}_1(\Gamma) = \infty$. Therefore

$$\lim_{n\to\infty} \text{Mod}_1(\Gamma_n) \neq \text{Mod}_1(\Gamma)$$

Now that we have proved the continuity from below of $p-$modulus, we can give a complete proof for the computation $p-$modulus of the family of curves with an endpoint in a fixed point in the Euclidean space.

2.5.1 $p-$modulus of the family of curves with a fixed endpoint in $\mathbb{R}^d$

The $p-$modulus of the family of curves through a point in $\mathbb{R}^d$ was computed for the unidimensional case in Theorem 2.4.4, and it is trivial. Now we use the computations of the annulus and the continuity from below of $p$modulus to compute the $p-$modulus of the family of curves with an end point.

Theorem 2.5.10: Modulus of curves through a point

Let $d \geq 2$. Then, the family of curves such that pass through a point $x \in$ is $p-$null if $1 < p \leq d$ and it is infinite otherwise.

Proof: The computations for the 1-modulus was done. Now, we analyze the other cases. We consider $0 < r < R < \infty$ and consider the annulus $A(x, r, R)$. First, we fix $R > 0$ and consider the approximation $\text{Mod}_p\left(\Gamma\left(B\left[x, \frac{1}{n}\right], \mathbb{R}^d \setminus B(x, R), \mathbb{R}^d\right)\right)$. From the computations of Lemma 2.4.20 and Theorem 2.4.15, it follows that this families are $p-$null therefore $\text{Mod}_p\left(\Gamma\left(\{x\}, \mathbb{R}^d \setminus B(x, R), \mathbb{R}^d\right)\right) = 0$. Considering this computation when $R \to 0$ together with monoticity of $p-$modulus, we conclude.

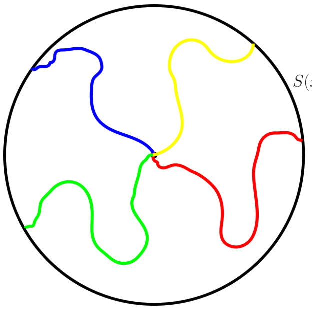

Figure 2.5.11: Curves through a point

■

2.6 The Sobolev space $W^{1,p}(\Omega)$ and the class $\text{ACL}^p(\Omega)$

Now, we will prove that $W^{1,p}(\Omega)$ is characterized by functions that are absolutely continuous on lines.

Lemma 2.6.1: Characterization of the Dirichlet space $L^{1,p}(\Omega)$

Let $\Omega \subset \mathbb{R}^d$ an open subset and $1 \leq p < \infty$. Then:

$$\text{ACL}^p(\Omega) = L^{1,p}(\Omega).$$

Thus, we can choose a representative $u \in L^{1,p}(\Omega)$ such that u has classical derivatives almost everywhere.

Generalization to any curve

Let $\Omega \subset \mathbb{R}^d$ an open subset. Then $u \in L^{1,p}(\Omega)$ if and only if u there is a Lebesgue representative of u that is absolutely continuous on p-a.e. compact curve in Ω and the partial derivatives of u belong to $L^p(\Omega)$.

Thus, we can choose a representative $u \in L^{1,p}(\Omega)$ such that u has classical derivatives almost everywhere.

Proof: We will prove the following contentions holds:

$\boldsymbol{L^{1,p}(\Omega) \subset \mathrm{ACL}^p(\Omega)}$ Let $u \in L^{1,p}(\Omega)$. The family of all compact line segments in Ω can be expressed as a countable union of families of line segments, each contained in a relatively compact subdomain of Ω. Therefore, it is sufficient to consider segments that lie in a fixed relatively compact subdomain Ω' of Ω.

Let $i \in \{1, \ldots, d\}$. First, we will define appropriate families of curves to prove the contention $L^{1,p}(\Omega) \subset \mathrm{ACL}^p(\Omega)$. We will define $\Gamma_i(\Omega')$ as the family of all compact line segments in Ω' parallel to the i-th coordinate axis. Since $u \in L^{1,p}(\Omega)$, we have that $\nabla u \in L^p(\Omega, \mathbb{R}^d)$. From Theorem 1.8.3, it follows that the mollifications u_ϵ have the property that $\partial_i u_\epsilon = (\partial_i u)_\epsilon$ and converges to $\partial_i u$ in $L^p(\Omega')$.

To prove the result, we need to consider certain subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ an special representative of u. We will detail this construction as follows:

First refinement.

Fixing an appropriate representative of the weak derivative $\partial_i u$, from Fuglede lemma,it follows that there is a subsequence $\{\partial_i u_{\epsilon_k}\}_{k\in\mathbb{N}}$ such that: For $p-$almost every curve γ in Ω', the following conditions holds:

$$\lim_{k\to\infty} \int_\gamma |\partial_i u_{\epsilon_k} - \partial_i u| \, \mathrm{d}s = 0 \tag{2.6.2}$$

$$\int_\gamma |\partial_i u| \, \mathrm{d}s < \infty \tag{2.6.3}$$

Thus in particular for $p-$a.e. γ of $\Gamma_i(\Omega')$ satisfies the previous conditions. Considering this conditions; let us define $\widetilde{\Gamma}_1$ as the family of curves that satisfies (2.6.2) and (2.6.3). Thus, from the definition of $\widetilde{\Gamma}_1$, it follows that for $p-$a.e. curve in Ω' belongs to $\widetilde{\Gamma}_1$. Furthermore, let us define Γ_1 as the family of curves that satisfies (2.6.2) and (2.6.3) as well for every subcurve. From the definition of $\widetilde{\Gamma}_1$ and Lemma 2.2.10, it follows that for $p-$a.e. curve in Ω' belongs to Γ_1.

Second refinement.

Since $u \in L^1_{\mathrm{loc}}(\Omega)$, it follows that $u_\epsilon \to u$ in $L^1(\Omega')$. Thus, there exists $\left\{u_{\epsilon_{k_j}}\right\}_{j\in\mathbb{N}}$ a subsequence of $\{u_{\epsilon_k}\}_{k\in\mathbb{N}}$ such that $u_{\epsilon_{k_j}}$ converges almost everywhere in Ω' to u. From now on, we will work with the sequence $u_j = u_{\epsilon_{k_j}}$. We redefine u in such a way that $u(x) = \lim_{j\to\infty} u_j(x)$ whenever this limit exists.

By Fubini theorem, for almost every line segment γ in Ω' that is parallel to the $i-$th axis, we have $u_j \to u$ pointwise outside a set of $\mathcal{H}_1$-measure zero along γ. Let us define Γ_2 as the family of curves $\widetilde{\gamma}$ in $\Gamma_i(\Omega')$ such that $u_j \to u$ pointwise outside a set of $\mathcal{H}_1$-measure zero along $\widetilde{\gamma}$

Now, let us define Δ_i as the family of all compact line segments in Ω' parallel to the i-th coordinate axis, α, such that (2.6.2) and (2.6.3) holds as well for every compact subsegment γ of α, and $u_j \to u$ $\mathcal{H}_1$-almost everywhere on α. Clearly,

$$\Delta_i = \Gamma_1 \cap \Gamma_2$$

and by definition of Γ_j with $j = 1, 2$, we have that $p-$a.e. curve lies in Γ_j. Thus $p-$a.e. curve lies in Δ_i.

Now, within this family of curves, we will prove the contention $L^{1,p}(\Omega) \subset \mathrm{ACL}^p(\Omega)$. From now on, we consider $\alpha \in \Delta_i$, we assume that α as a curve is parametrized by its arc length. Let $s, r \in [0, \mathrm{length}(\alpha)]$ with $r \le s$. Recall, $u_j \to u$ pointwise outside a set of $\mathcal{H}_1$-measure zero along α. Hence, we will analyze the case in that $\alpha(r)$ satisfies this condition. Thus, $u_j(\alpha(r)) \to u(\alpha(r))$. Now, we will prove that s also satisfies this condition. Indeed, since u_j is a mollification, it is a smooth function. Thus, we can apply the Fundamental theorem of calculus:

$$\int_{\alpha|_{[r,s]}} \partial_i u_j \, ds = u_j(\alpha(s)) - u_j(\alpha(r)) \tag{2.6.4}$$

Recall, we redefine u in such way that $u = \lim_{j\to\infty} u_j$; and we take $\alpha \in \Delta_i$, then α satisfies (2.6.2) and (2.6.3). Considering these two conditions and taking the limit when $j \to \infty$ in (2.6.4), we conclude that s satisfies

$u_j(\alpha(s)) \to u(\alpha(s))$. Within an analogous argument we prove that we have the same result for all $0 \le r \le s$. Particularly, the limit holds at the endpoints of α. This proves that:

$$u_j(\alpha(r)) \to u(\alpha(r)) \qquad \forall r \in [0, \text{length}(\alpha)] \, .$$

From this convergence, and applying the Fundamental theorem of calculus to the mollifications u_j, it follows that:

$$\begin{aligned} \lim_{j\to\infty} |u_j(\alpha(r)) - u_j(\alpha(s))| &\le \lim_{j\to\infty} \int_{\alpha|[r,s]} |\partial_i u_j| \ ds \\ |u(\alpha(r)) - u(\alpha(s))| &\le \int_{\alpha|[r,s]} |\partial_i u| \ ds. \end{aligned} \qquad \begin{aligned} &u_j \to u \ \textit{pointwise.} \\ &\textit{From (2.6.2).} \end{aligned} \tag{2.6.5}$$

Since $\alpha \in \Delta_i$, we have that $\int_\alpha |\partial_i u| \ ds < \infty$. Thus, by the absolutely continuity of the integral and the estimation (2.6.5), we conclude that u is absolutely continuous in α, which is an arbitrary curve of the family Δ_i. Then $u \in \mathrm{ACL}^p(\Omega)$. This proves that $L^{1,p}(\Omega) \subset \mathrm{ACL}^p(\Omega)$.

$\mathbf{ACL}^{\boldsymbol{p}}(\boldsymbol{\Omega}) \subset \boldsymbol{L}^{\mathbf{1},\boldsymbol{p}}(\boldsymbol{\Omega})$ Follows immediately from the definition of ACL^p the integration by parts formula for absolutely continuous functions.

Therefore, $L^{1,p}(\Omega) = \mathrm{ACL}^p(\Omega)$. The observation follows immediately from Lemma 1.6.4.

Generalization to any curve.

$\Rightarrow$ Let $u \in L^{1,p}(\Omega)$. We follow the line of argument of the first part. Instead of line segments, we consider general rectifiable curves and replace ∂u_k with ∇u_k. Since we are considering rectifiable curves, it is sufficient to consider rectifiable compact curves that lie in a fixed relatively compact subdomain Ω' of Ω. Since $u \in L^{1,p}(\Omega)$, we have that $\nabla u \in L^p(\Omega, \mathbb{R}^d)$. From Theorem 1.8.3, it follows that the mollifications u_ϵ have the property that $\nabla u_\epsilon = (\nabla u)_\epsilon$ and converge to ∇u in $L^p(\Omega', \mathbb{R}^d)$.

To prove the result, we need to consider the certain subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ an special representative a of u. We will detail this construction as follows:

First refinement.

Fixing an appropriate representative of the weak gradient ∇u, from Fuglede lemma,it follows that there is a subsequence $\{\nabla u_{\epsilon_k}\}_{k\in\mathbb{N}}$ such that: For $p-$almost every curve γ in Ω', the following conditions holds:

$$\lim_{k\to\infty} \int_\gamma |\nabla u_{\epsilon_k} - \nabla u| \ \mathrm{d}s = 0 \tag{2.6.6}$$

$$\int_\gamma |\nabla u| \ \mathrm{d}s < \infty \tag{2.6.7}$$

Considering this conditions; let us define $\widetilde{\Gamma}_1$ as the family of curves that satisfies (2.6.2) and (2.6.3). Thus, from the definition of $\widetilde{\Gamma}_1$, it follows that for $p-$a.e. curve in Ω' belongs to $\widetilde{\Gamma}_1$. Furthermore, let us define Γ_1 as the family of curves that satisfies (2.6.2) and (2.6.3) as well for every subcurve. From the definition of $\widetilde{\Gamma}_1$ and Lemma 2.2.10, it follows that for $p-$a.e. curve in Ω' belongs to Γ_1.

Second refinement.

Since $u \in L^1_{\text{loc}}(\Omega)$, it follows that $u_\epsilon \to u$ in $L^1(\Omega')$. Therefore, there exists $\left\{u_{\epsilon_{k_j}}\right\}_{j\in\mathbb{N}}$, a subsequence of $\{u_{\epsilon_k}\}_{k\in\mathbb{N}}$, such that $u_{\epsilon_{k_j}}$ converges almost everywhere in Ω' to u. We work with the sequence $u_j = u_{\epsilon_{k_j}}$. We redefine u such that $u(x) = \lim_{j\to\infty} u_j(x)$ whenever this limit exists.

Since the pointwise convergence $u_j \to u$ holds almost every where, there exists $N \subset \Omega'$ a null set such that contains all the points were $u_j \not\to u$. Thus, $u_j \to u$ pointwise in $\Omega' \setminus N$. Let us define

$$\Gamma_2 = \{\gamma \in \Gamma(X) \,|\, \mathcal{H}_1(\gamma \cap N) = 0\} \, .$$

That is the curves that the set of points where the convergence can fail has $\mathcal{H}_1$ measure no null. From Lemma 2.3.2, it follows that for p-a.e. curve is in Γ_2. From the definition of Γ_2, we have $\mathcal{H}_1$-almost everywhere convergence along the curve in question.

Now, define Δ as the family of all compact curves on Ω', α, such that every compact subcurve γ of α is in Γ_1 and $\alpha \in \Gamma_2$. Thus, for every $\alpha \in \Delta$ we have that: (2.6.6) and (2.6.7) hold for every compact subsegment γ of α, and $u_j \to u$ $\mathcal{H}_1$-almost everywhere on α. Clearly,

$$\Delta_i = \Gamma_1 \cap \Gamma_2$$

and by definition of Γ_j with $j = 1, 2$, we have that $p-$a.e. curve lies in Γ_j. Thus $p-$a.e. curve lies in Δ_i.

Now, we prove $L^{1,p}(\Omega) \subset \mathrm{ACL}^p(\Omega)$. For $\alpha \in \Delta$, we parametrize α by its arc length. Let $s, r \in [0, \text{length}(\alpha)]$ with $r \leq s$. Recall, $u_k \to u$ pointwise outside a set of $\mathcal{H}_1$-measure zero along α. Hence, analyzing the case in which $\alpha(r)$ satisfies this condition, we have $u_k(\alpha(r)) \to u(\alpha(r))$.

Next, we prove that s also satisfies this condition. Since u_j is a mollification, it is smooth. Applying the Fundamental theorem of calculus with the fact that α is absolutely continuous to obtain:

$$\int_{\alpha|_{[r,s]}} \nabla u_j \; ds = u_j(\alpha(s)) - u_j(\alpha(r)) \tag{2.6.8}$$

Recall, we redefine u such that $u = \lim_{j\to\infty} u_j$; and we take $\alpha \in \Delta$, then α satisfies (2.6.6) and (2.6.7). Taking the limit as $j \to \infty$ in (2.6.8), we conclude that s satisfies $u_j(\alpha(s)) \to u(\alpha(s))$. Applying a similar argument for all $0 \leq r \leq s$, the limit holds at the endpoints of α. This proves:

$$u_j(\alpha(r)) \to u(\alpha(r)) \qquad \forall r \in [0, \text{length}(\alpha)]\,.$$

From this convergence, and applying the Fundamental theorem of calculus to u_j, it follows that:

$$\begin{aligned} \lim_{j\to\infty} |u_j(\alpha(r)) - u_j(\alpha(s))| &\leq \lim_{j\to\infty} \int_{\alpha|[r,s]} |\nabla u_j| \; ds \\ |u(\alpha(r)) - u(\alpha(s))| &\leq \int_{\alpha|[r,s]} |\nabla u| \; ds. \end{aligned} \quad u_j \to u \text{ pointwise, from (2.6.6).} \tag{2.6.9}$$

Since $\alpha \in \Delta$, $\int_\alpha |\nabla u| \; ds < \infty$. By the absolute continuity of the integral and the estimate (2.6.9), u is absolutely continuous in α. This arbitrary curve of the family Δ_i implies that $u \in \mathrm{ACL}^p(\Omega)$.

Therefore, $L^{1,p}(\Omega) \subset \mathrm{ACL}^p(\Omega)$.

$\Leftarrow$ Follows immediately from the first part.

Obviously, the observation about derivatives still holds in the previous construction.

■

Remark: Key points of Lemma 2.6.1

Generalization to metric measure spaces

The generalization of Lemma 2.6.1 only needs the idea of absolutely continuity on curves which is a notion for general metric spaces. When we consider functions that absolutely continuous on curves we have automatically more regularity for our function. This is discussed below.

Special representative

Lemma 2.6.1 told us that we can choose an appropriate representative of $u \in L^{1,p}(\Omega)$. Furthermore, by Lemma 1.6.4, this representative we have the classical notions of differentiation.

The suitable representative of Sobolev function most important difference between the classical theory for Sobolev spaces and the approach with upper gradients because in the classical theory there is no difference considering changes almost everywhere. However, we shown in Remark 1.6.11 that this changes are not admissible with the absolute continuity on lines.

From Lemma 2.6.1, it follows this generalization of Lemma 1.6.4.

Lemma 2.6.10: $\mathrm{ACL}^p(\Omega)$ and classical partial derivates

Let $\Omega \subset \mathbb{R}^d$ an open subset and $1 \leq p < \infty$. If $u \in \mathrm{ACL}^p(\Omega)$, then u has classical directional derivative m_d-a.e. on Ω. Furthermore, u is differentiable in all directions almost everywhere in Ω.

Proof: Let $\omega \in \mathbb{S}^{d-1}$ arbitrary, $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathbb{S}^{d-1}$ a dense set in $\mathbb{S}^{d-1}$; thus, there exists $\{\omega_{n_m}\}_{m\in\mathbb{N}}$ subsequence of $\{\omega_n\}_{n\in\mathbb{N}}$ such that $\omega_{n_m} \to \omega$. Since we are considering rectifiable curves, it is sufficient to consider segments that lie in a fixed relatively compact subdomain Ω' of Ω. Let us restate the construction of the sequence u_j and the families of curves Γ_1, Γ_2 in the generalization of Lemma 2.6.1 for any curve.

Let us define $\widetilde{\Gamma}_1$ as the family of curves that satisfies (2.6.6) and (2.6.7). Thus, from the definition of $\widetilde{\Gamma}_1$, it follows that for $p-$a.e. curve in Ω' belongs to $\widetilde{\Gamma}_1$. Furthermore, let us define Γ_1 as the family of curves that satisfies (2.6.6) and (2.6.7) as well for every subcurve. From the definition of $\widetilde{\Gamma}_1$ and Lemma 2.2.10, it follows that for $p-$a.e. curve in Ω' belongs to Γ_1. On the other hand, from the pointwise convergence $u_j \to u$ holds almost every where, there exists $N \subset \Omega'$ a null set such that contains all the points were $u_j \not\to u$. Thus, $u_j \to u$ pointwise in $\Omega' \setminus N$. Let us define

$$\Gamma_2 = \{\gamma \in \Gamma(X) \mid \mathcal{H}_1(\gamma \cap N) = 0\}.$$

Now, define Δ_ω as the family of all compact segments on Ω', α, such that (2.6.6) and (2.6.7) hold for every compact subsegment γ of α, and $u_j \to u$ $\mathcal{H}_1$-almost everywhere on α. Defining $\Omega'_{\mathbb{S}^{d-1}}$ as the family of all line segments in Ω'. Clearly,

$$\Delta_\omega = \bigcap_{j=1} (\Omega'_{\mathbb{S}^{d-1}} \cap \Gamma_j)$$

and by definition of Γ_j with $j = 1, 2$, it follows that every line segment in $\Omega'_{\mathbb{S}^{d-1}}$ lies in Δ_ω.

Now, let us define

$$\Delta = \bigcap_{n\in\mathbb{N}} \Delta_{\omega_n}.$$

For $\alpha \in \Delta$, we parametrize α by its arc length, let us say, α is defined as:

$$[0, l] \ni t \mapsto x_0 + t\omega_n.$$

Let $s, r \in [0, \mathrm{length}(\alpha)]$ with $r \leq s$. As in the generalization of Lemma 2.6.1 for any curve, we prove that:

$$\int_{\alpha|_{[r,s]}} \nabla u_j \; ds = u_j(\alpha(s)) - u_j(\alpha(r)) \tag{2.6.11}$$

and similarly as in the generalization of Lemma 2.6.1 for any curve we prove the absolutely continuity as well. The identity (2.6.11) holds particularly for the curves:

$$[0, l] \ni t \mapsto x_0 + t\omega_n,$$

it follows that:

$$\int_r^s \nabla u_j \left(x_0 + t\omega_n\right) \; dt = u_j(x_0 + s\omega_n) - u_j(x_0 + r\omega_n)$$

This identity holds in particular for ω_{n_m}, and since u_j is smooth, because this function is a mollification, it follows that:

$$\int_r^s \nabla u_j\left(x_0 + t\omega\right)\, dt = u_j(x_0 + s\omega) - u_j(x_0 + r\omega)$$

$$\begin{aligned} \int_{\alpha|_{[r,s]}} \nabla u_j \, ds &= u_j(\alpha(s)) - u_j(\alpha(r)) \\ & \qquad \textit{Taking the limit when } j \to \infty. \\ \int_{\alpha|_{[r,s]}} \nabla u \, ds &= u(\alpha(s)) - u(\alpha(r)). \end{aligned} \tag{2.6.12}$$

Similarly as in the generalization of Lemma 2.6.1 for any curve we prove the absolutely continuity as well. Using this fact together with (2.6.12). We conclude the proof.

■

Disclaimer We need consider the class $\mathrm{ACL}^p(\Omega)$ not only $\mathrm{ACL}(\Omega)$ because we need to change the direction from e_i to a direction $\omega \in \mathbb{S}^{n-1}$ in Lemma 2.6.1.

From the previous result, it follows immediately the next:

Lemma 2.6.13: $W^{1,p}(\Omega) = L^p(\Omega) \cap L^{1,p}(\Omega)$

Let $\Omega \subset \mathbb{R}^d$ an open subset and $1 \leq p < \infty$. Then

$$W^{1,p}(\Omega) = L^p(\Omega) \cap L^{1,p}(\Omega) = L^p(\Omega) \cap \mathrm{ACL}^p(\Omega). \tag{2.6.14}$$

Notice that the right-hand side of (2.6.14) are notions that can be defined in the metric measure spaces, and the proof of both was done by curves. In the general metric measure setting the control on the right-hand side is done by upper gradients and they prove generalization of the Sobolev Spaces on metric measure spaces.

Chapter 3

Upper gradients and potentials

We defined the weak derivative which is a generalized notion of derivative. We discussed in Remark 1.8.2 that the properties of weak derivative came from the structure of vectorial space of $\mathbb{R}^d$. However, general metric measure spaces there are no structure of vectorial space, there are only notions provided by the metric and measure, so, we focus on the size of the derivative which is the essence of upper gradient.

3.1 Upper gradients

First, recall some results that give a notion of derivative in a metric measure space.

Motivation: Upper gradients

Corollary 1.5.23 and the line integral inequality for conservative fields states that the following inequalities holds:

$$d(\gamma(a), \gamma(b)) \leq \int_a^b |\gamma'(t)| \; dt.$$
$$|f(\gamma(b)) - f(\gamma(a))| \leq \int_\gamma |\nabla f| \; ds.$$

These inequalities suggests that the definition should be:

Definition 3.1.1: Upper gradient

Let be $u : X \to Z$ a map between metric measure spaces and $\rho : X \to [0, \infty]$ a Borel function. We say that ρ is an **upper gradient** of u if for every rectifiable curve $\gamma : [a, b] \to X$ satisfies:

$$d_Z(u(\gamma(a)), u(\gamma(b))) \leq \int_\gamma \rho \, ds. \tag{3.1.2}$$

This inequality is called the **upper gradient inequality for the pair (u, ρ) in γ**. Also this pair is called a **function-upper gradient pair**.
It is possible that u is only defined in a subset A of the space X we define the same concepts considering A as a subspace of X.

Weak upper gradients

For $1 \leq p < \infty$, we say that ρ is a **$p-$weak upper gradient of u**, if the upper gradient inequality for the pair (u, ρ), (3.1.2), holds for $p-$a.e. curve.
As always to encode the information of both concepts we can use a pair (u, ρ) to make the notation as simple as possible if ρ is an upper gradient or a $p-$weak upper gradient for u then we say that (u, ρ) is a **function-upper gradient pair**, of curse if the context is clear; the point of this notation is avoid p if it is clear.
It is possible that u is only defined in a subset A of the space X we define the same concepts considering A

as a subspace of X.

Notice that upper gradients are good enough to extend the notion of derivative because in the definition only appears metric notions. The essence of the left-hand side of the upper gradient inequality is the dependence only of the endpoints in X. This property give us a criterion to discard trivial upper gradient inequalities. Those facts are detailed in the following:

Remark: Properties of the upper gradient inequality

The left-hand side of the upper gradient inequality only depends of the value of u at the endpoints of γ

If $\eta\colon [c,d] \to X$ is a curve with the same endpoints as γ, it follows that

$$d_Z(u(\gamma(a)), u(\gamma(b))) = d_Z(u(\eta(c)), u(\eta(d))), \tag{3.1.3}$$

That is, the value of (3.1.3) does not depend on the chosen curve, it only depends on the endpoints in X. Moreover, from the Identity of indiscernibles, follows immediately that:

$$d_Z(u(\gamma(a)), u(\gamma(b))) = 0, \tag{3.1.4}$$

if and only if $u(\gamma(a)) = u(\gamma(b))$. This means that (3.1.4) is equivalent to **the endpoints of γ lying in the same fiber of u**. When it occurs the upper gradient inequality holds automatically for any nonnegative Borel function ρ, then, **we only need verify the upper gradient inequality for curves that connect different fibers of u**. For this reason, we **discard the constant curves** since these satisfies (3.1.4). Therefore, when dealing with upper gradients, we consider only nonconstant rectifiable curves. **This consideration does not depend on the function ρ.**

Nontrivial upper gradient inequalities

Another trivial case for the upper gradient inequality (u, ρ) in γ occurs when the right-hand side is infinite, *i.e.*, when the curve γ is $\rho-$heavy. **In this case, u does not affect on the upper gradient inequality.**
In summary, given a nonnegative Borel function ρ, to prove ρ is an upper gradient, we only need verify the upper gradient inequality for **non-$\rho-$heavy curves that connect two different fibers of u**. We use this fact without further mention.

Later, in Counterexample 3.1.7, we discuss about the reason to include the notion $p-$almost every curve in the definition of upper gradient.

3.1.1 Non-compatibility between upper gradients and limits

We start with the characterization of $\rho \equiv 0$ being an upper gradient.

Lemma 3.1.5: $\rho \equiv 0$ upper gradient characterization

Let $u : X \to Z$. Then, the function $\rho \equiv 0$ is an upper gradient for u if and only if u is constant of all rectifiable components.

Proof:

$\Rightarrow$ Assume that $\rho \equiv 0$ is an upper gradient for u. Let $\gamma\colon [a,b] \to X$ a rectifiable curve in X. From the upper gradient inequality for $(u, 0)$ in γ follows that $d_Z(u(\gamma(a)), u(\gamma(b))) = 0$, then $u(\gamma(a)) = u(\gamma(b))$; therefore u is constant at the endpoints of γ. Since γ is an arbitrary rectifiable curve, follows that u is constant on any rectifiable component.

$\Leftarrow$ Conversely, assume that u is constant of all rectifiable components.

Then, every rectifiable curve γ are in the same fiber of u. Therefore the upper gradient inequality $(u, 0)$ holds for γ. This proves that $\rho \equiv 0$ is an upper gradient.

■

From Lemma 3.1.5 follows immediately the next:

Corollary 3.1.6: Let $u : X \to Z$. If there are curves that connect different fibers of u then $\rho \equiv 0$ is not an upper gradient for u.

Remark: The interesting cases do not have $\rho \equiv 0$ as an upper gradient As we stated in Remark 3.1, we only need to verify the upper gradient inequality for curves that connect different fibers of u. If these curves does not exists, then we are working on the particular case of Lemma 3.1.5, for which we already have a characterization. Therefore, we may implicitly assume that $\rho \equiv 0$ is not an upper gradient.

Now, we broke the necessary condition on u of Lemma 3.1.5. This is detailed in the next:

Counterexample 3.1.7: Upper gradients are not compatible limit

Consider the space $X = \mathbb{R}^d$, with $d \geq 2$, which has only one rectifiable component. Now, let u the Dirac delta. Since u is not constant in $\mathbb{R}$, it follows form Lemma 3.1.5 that $\rho \equiv 0$ is not an upper gradient for u. Next, we will define another upper gradient, ρ_0, for u that will shown the non-compatibility with the limit. To show the existence of such ρ_0, and to fix this issue in the upper gradient definition we will use the notion of $p-$modulus of a family of curves.
Notice that by the definition of u, it has only two nonempty fibers, which are:

$$u^{-1}[\{c\}] = \begin{cases} \{0\} & \text{if } c = 1, \\ \mathbb{R}^d \setminus \{0\} & \text{if } c = 0. \end{cases}$$

Define Γ as the family of all the locally rectifiable curves that connect two different fibers of u. Clearly, Γ is the family of all locally rectifiable curves in $\mathbb{R}^d$ with an endpoint at the origin. From theorem 2.5.10, we have that Γ is $p-$null for all $1 \leq p \leq d$. Then, from $p-$exceptionality criterion, we have that there exist a $p-$integrable nonnegative Borel function $\rho_0 \colon X \to [0, \infty]$ such that

$$\int_\gamma \rho_0 \, ds = \infty \qquad \forall \gamma \in \Gamma. \tag{3.1.8}$$

The above identity proves the nontrivial upper gradient inequalities, then, ρ_0 is an upper gradient for u. Furthermore, notice that for each $\varepsilon > 0$, $\varepsilon\rho_0$ also satisfies (3.1.8). Thus, $\varepsilon\rho_0$ satisfies the nontrivial upper gradient inequalities. Therefore, $\varepsilon\rho_0$ is an upper gradient for u. Nevertheless, the limit of $\varepsilon\rho_0$ is the function 0, which we have already mentioned is not an upper gradient for u. Therefore, **upper gradients are not compatible with limit**.
We do not have this issue if we discard the family Γ, we can do it without loss of information from the measure theoretical point of view because Γ is $p-$null. Then, the we can generalize upper gradients by $p-$modulus of a family of curves. Later, we will prove that the generalization by the $p-$modulus is compatible with limits.

The following result give us several trivial examples of upper gradients functions.

Proposition: Let $u : X \to Z$ an arbitrary mapping between metric measure spaces then:

1. If X has no nonconstant rectifiable curves then any nonnegative Borel function ρ is an upper gradient of u.

2. If u is constant then any nonnegative Borel function ρ is an upper gradient for u.

Proof:

1. Since X has no nonconstant rectifiable curves we have that the only rectifiable curves are constant therefore both terms of the upper gradient inequality for (u, ρ) over a rectifiable curve γ is zero.

2. Since u is constant the left-hand side of the upper gradient inequality is zero then does not matter whose is ρ.

■

3.1.2 Basic properties

We must be careful with the concept of upper gradient since by generalizing the notion of derivative we lose several of his properties, including the uniqueness, this is because upper gradients are increasing. This can be seen in the next:

Lemma 3.1.9

If ρ is an upper gradient of $u : X \to Z$ and σ is an nonnegative Borel function such that $\sigma \geq \rho$ then σ is an upper gradient for u.

Proof: First, note that it makes sense for σ to be an upper gradient, because $\sigma \geq \rho \geq 0$. Now, let $\gamma\colon [a, b] \to X$ a rectifiable curve since ρ is an upper gradient we have

$$\begin{aligned} d_Z(u(\gamma(a)), u(\gamma(b))) &\leq \int_\gamma \rho \, ds \\ &\leq \int_\gamma \sigma \, ds, \end{aligned} \qquad \textit{Since } \sigma \geq \rho.$$

then the upper gradient inequality holds for every $\gamma\colon [a, b] \to X$ an arbitrary rectifiable curve we conclude that ρ is an upper gradient.

■

From Lemma 3.1.9 follows immediately the following:

Corollary 3.1.10: If ρ is an upper gradient of $u : X \to Z$ and σ is an nonnegative Borel function then $\rho + \sigma, \max\{\rho, \sigma\}$ are upper gradients for u.

Remark

1. **No uniqueness.** Lemma 3.1.9 shows that there are many upper gradients and suggest that for the uniqueness we must consider the "minimal" upper gradient nevertheless Counterexample 3.1.7 shows that the minimal of a strictly upper gradient might not be an upper gradient again. In Section 3.2.2 we ask for condition to consider minimal upper gradients.

2. **The natural lattice structure for the family of upper gradients.** As we mentioned we are looking for taking minimal upper gradients and Corollary 3.1.10 also shows upper gradients are closed to taking maximums therefore the upper gradients will have a *lattice structure*.

Now we continue proving results that give us examples of upper gradients:

Proposition 3.1.11: If $u : X \to Z$ is $L-$Lipschitz then $\rho \equiv L$ is an upper gradient for u.

Proof: Let $\gamma\colon [a,b] \to X$ a rectifiable curve. Since u is $L-$Lipschitz we have:

$$\begin{aligned} d_Z(u(\gamma(a)), u(\gamma(b))) &\leq L d_X(\gamma(a), \gamma(b)) \\ &\leq L \operatorname{length}\gamma \\ &= \int_\gamma \rho\, ds. \end{aligned}$$

By the definition of length(γ).

Since $\rho \equiv L$.

since $\gamma\colon [a,b] \to X$ an arbitrary rectifiable curve we conclude that ρ is an upper gradient.

■

We can extend Proposition 3.1.11 with the composition and obtain a kind of chain rule for upper gradients:

Proposition 3.1.12: If ρ is an upper gradient for $u : X \to Z$ and $f : Z \to W$ is $L-$Lipschitz then $L\rho$ is an upper gradient $f \circ u\colon X \to W$.

Proof: Let $\gamma\colon [a,b] \to X$ a rectifiable curve. Since f is $L-$Lipschitz we have:

$$\begin{aligned} d_W((f\circ u)(\gamma(a)), (f\circ u)(\gamma(b))) &\leq L d_Z(u(\gamma(a)), u(\gamma(b))) \\ &\leq \int_\gamma L\rho\, ds, \end{aligned}$$

Becasue ρ is an upper gradient for u and $L \geq 0$.

then the upper gradient inequality holds for every $\gamma\colon [a,b] \to X$ an arbitrary rectifiable curve we conclude that $L\rho$ is an upper gradient for $f \circ u$.

■

From Proposition 3.1.12 follows immediately the next:

Corollary 3.1.13: If ρ is an upper gradient for $u : X \to Z$ and $z_0 \in Z$ then ρ is an upper gradient for $d(u, z_0)$.

Proof: We know that $d(\cdot, z_0)$ is $1-$Lipschitz and by hypothesis ρ is a upper gradient for u, by Proposition 3.1.12 we have that $\rho = 1\rho$ is an upper gradient for $d(u, z_0) = d(\cdot, z_0) \circ u$.

■

Another elementary but important fact is the compatibility of upper gradients with restrictions:

Lemma 3.1.14: Compatibility with restrictions

If ρ is an upper gradient for $u : X \to Z$ and $A \subset X$ then $\rho\,|_A$ is an upper gradient for $u\,|_A$.

Now we are looking for extend the properties of usual derivative to upper gradient, in the next result we prove a weaker property than linearity:

Lemma 3.1.15: The family of upper gradients is a convex set

If ρ_1, ρ_2 are upper gradients for $u : X \to Z$ then for every $\lambda \in [0,1]$ we have that $\lambda\rho_1 + (1-\lambda)\rho_2$ is an upper gradient.

Remark 3.1.16

1. **Lemma 3.1.15 cannot be extend to any linear combination of ρ_1, ρ_2** This is because an upper gradient must be nonnegative nevertheless any linear combination of ρ_1, ρ_2 might not satisfy be nonnegative.

2. **Even Lemma 3.1.15 cannot be extended for positive linear combinations** This is because the left-hand side of the upper gradient inequality is a specific nonnegative value. This value is not preserved for general positive linear combinations.

Then we must be careful when we consider linear combinations of upper gradient. Now with the necessary precautions, we will extend Lemma 3.1.15.

Lemma 3.1.17: (Weak) linearity for upper gradients

Let $(Z, |\cdot|)$ is a normed space and for $i = 1,2$ we have that ρ_i is an upper gradient for $u_i : X \to Z$. If $\lambda_1, \lambda_2 \in \mathbb{R}$ then $|\lambda_1|\,\rho_1 + |\lambda_2|\,\rho_2$ is an upper gradient for $\lambda u_1 + u_2$.

Proof: Let $\gamma\colon [a,b] \to X$ a rectifiable curve. By the triangle inequality and the homogeneity of the norm we have:

$$\begin{aligned}
|(\lambda_1 u_1 + \lambda_2 u_2)(\gamma(a)) - (\lambda u_1 + \lambda_2 u_2)(\gamma(b))| &\leq |\lambda_1|\,|u_1(\gamma(a)) - u_1(\gamma(b))| \\
&\quad + |\lambda_2|\,|u_2(\gamma(a)) - u_2(\gamma(b))| \\
&\leq \int_\gamma |\lambda_1|\,\rho_1 + |\lambda_2|\,\rho_2 \, ds.
\end{aligned}$$

Since ρ_i is an upper gradient for u_i.
Linearity of the integral.

then the upper gradient inequality holds for every $\gamma\colon [a,b] \to X$ an arbitrary rectifiable curve we conclude that $|\lambda_1|\,\rho_1 + |\lambda_2|\,\rho_2$ is an upper gradient.

■

Remark: The absolute value is necessary That is because an upper gradient must be nonnegative as we mentioned in Remark 3.1.16. To made the terminology friendly as possible we say that the property of Lemma 3.1.17 is the linearity for upper gradients.

Now, we summarize all these basic results in the next:

Theorem: Basic properties of (weak) upper gradients

1. **.** If ρ is an upper gradient of $u : X \to Z$ and σ is an nonnegative Borel function such that $\sigma \geq \rho$ then σ is an upper gradient for u. Moreover if ρ is an upper gradient of $u : X \to Z$ and σ is an nonnegative Borel function then $\rho + \sigma, \max\{\rho, \sigma\}$ are upper gradients for u.

2. **Upper gradient for Lipschitz functions.** If $u : X \to Z$ is $L-$Lipschitz then $\rho \equiv L$ is an upper gradient for u.

3. **Chain rule.** If ρ is an upper gradient for $u : X \to Z$ and $f : Z \to W$ is $L-$Lipschitz then $L\rho$ is an upper gradient $f \circ u\colon X \to W$. Particularly we have if ρ is an upper gradient for $u : X \to Z$ and $z_0 \in Z$ then ρ is an upper gradient for $d(u, z_0)$.

4. **Compatibility with restrictions.** If ρ is an upper gradient for $u : X \to Z$ and $A \subset X$ then $\rho\,|_A$ is an upper gradient for $u\,|_A$.

5. **The family of upper gradients is a convex set.** If ρ_1, ρ_2 are upper gradients for $u : X \to Z$ then for every $\lambda \in [0,1]$ we have that $\lambda\rho_1 + (1-\lambda)\rho_2$ is an upper gradient.

6. **(Weak) linearity for upper gradients.** Let $(Z, |\cdot|)$ is a normed space and for $i = 1,2$ we have that ρ_i is an upper gradient for $u_i : X \to Z$. If $\lambda \in \mathbb{R}$ then $|\lambda_1|\,\rho_1 + |\lambda_2|\,\rho_2$ is an upper gradient for $\lambda u_1 + u_2$.

Now that we have summarized all the results, we take this opportunity to mention that all these results apply to weak upper gradients.

3.2 Weak upper gradients

In this section we prove some properties $p-$weak upper gradients.

3.2.1 Absolute continuity on curves

We star defining the notion of a function absolutely continuous on a curve.

Definition: Absolute continuity on curves

Let γ be a rectifiable curve on a metric space X. A map $u : X \to Z$ is said to be **absolutely continuous on a curve γ** if $u \circ \gamma_s : [0, \mathrm{length}(\gamma)] \to Z$ is absolutely continuous.

Now we find sufficient conditions to have absolute continuity on curves.

Lemma 3.2.1: Absolute continuity on curves and upper gradient inequality

Let $u : X \to Z$ a map and γ a rectifiable compact curve. If $\rho \colon X \to [0, \infty]$ is a Borel function integrable over γ such upper gradient inequality (u, ρ) holds as well as on each every compact subcurve of γ. Then u is absolutely continuous on γ.

Proof: Since ρ is integrable on γ, it follows that:

$$\int_\gamma \rho \, ds = \int_0^{\mathrm{length}(\gamma)} \rho(\gamma_s(t)) \, dt < \infty. \tag{3.2.2}$$

From the absolutely continuity of the integral, it follows that for ever $\varepsilon > 0$ there exists $\delta > 0$ such that: For all $E \subset [0, \mathrm{length}(\gamma)]$ satisfies the following implication:

$$m_1(E) < \delta \qquad \Rightarrow \qquad \int_E \rho\left(\gamma_s\right) \, dt < \varepsilon$$

Now, consider $\{[a_i, b_i]\}_{i=1}^k$ a family of nonoverlapping subintervals of $[0, \mathrm{length}(\gamma)]$ with $\sum_{i=1}^k |b_i - a_i| < \delta$. Then:

$$m_1\left(\bigcup_{i=1}^k [a_i, b_i]\right) = \sum_{i=1}^k |b_i - a_i| < \delta.$$

Thus, from (3.2.2), it follows that

$$\begin{aligned} \int_{\bigcup_{i=1}^k [a_i, b_i]} \rho\left(\gamma_s\right) \, dt &< \varepsilon && \{[a_i, b_i]\}_{i=1}^k \text{ are nonoverlapping.} \\ \sum_{i=1}^k \int_{[a_i, b_i]} \rho\left(\gamma_s\right) \, dt &= && \text{The upper gradient inequality } (u,\rho) \text{ holds on } \gamma_s|_{[a_i,b_i]}. \\ \sum_{i=1}^k d_Z\left(u\left(\gamma_s(b_i)\right), u\left(\gamma_s(a_i)\right)\right) &\le \end{aligned}$$

This proves that $u \circ \gamma_s$ is absolutely continuous. Therefore, u is absolutely continuous on γ.

■

Remark: In Lemma 3.2.1 it is not required that ρ be an upper gradient of u This is because we only require that the upper gradient inequality holds in γ as well in all subcurve. Thus, the upper gradient inequality could be not fulfilled in all the other curves. Of course if ρ is an upper gradient the conclusions of Lemma 3.2.1 holds.

3.2.2 Maps with $p-$integrable upper gradients

Now, we will prove that when we have an $p-$integrable $p-$weak upper gradient, we have automatically absolute continuity properties.

Lemma 3.2.3: $p-$integrable $p-$weak upper gradients and absolutely continuity

Let $\rho : X \to [0, \infty]$ a $p-$integrable $p-$weak upper gradient of $u : X \to Z$. Then $p-$almost every compact rectifiable curve γ in X have the following property: ρ is integrable in γ and the upper gradient inequality for (u, γ) holds as well as on each every compact subcurve of γ.

Special case

Every map $u : X \to Z$ with a a $p-$integrable $p-$weak upper gradient is absolutely continuous on $p-$a.e. compact curve in X.

Proof: First, define Γ_1 as the family of the compact curves that (u, ρ) does not hold in γ. Since ρ is a p-weak upper gradient of u, it follows that Γ_1 is p-exceptional. Since ρ is p-integrable, from Lemma 2.2.5, we have $\Gamma(X) \setminus \Gamma(\rho)$ is p-exceptional. Therefore, Γ_1, $\Gamma(X) \setminus \Gamma(\rho)$ are p-exceptional. Thus, $\Gamma_0 := \Gamma_1 \cup (\Gamma(X) \setminus \Gamma(\rho))$ is p-exceptional. Finally, defining Γ as the family of all compact curves that have a subcurve in Γ_0, then by definition $\Gamma_0 \leq \Gamma$, hence Γ is p-exceptional.
Obviously, the curves such that does not hold the desired property is contained in Γ. Therefore, the desired property holds $p-$almost every compact rectifiable curve γ in X.

From the first part, we have Lemma 3.2.1 hypothesis for $p-$almost every compact rectifiable curve in X. Then, u is absolutely continuous on $p-$a.e. compact curve in X.

■

The useful version of the above theorems is as follows:

Lemma 3.2.4: Absolute continuity and upper gradient inequality

1. Let $u : X \to Z$ a map and $\gamma : [0, \text{length}(\gamma)] \to X$ a rectifiable curve parametrized by its arc length. If $\rho \colon X \to [0, \infty]$ is Borel function such that is integrable on γ and the upper gradient inequality holds for the pair (u, ρ) on γ as well as on each compact subcurve of γ. Then u is absolutely continuous on γ and the inequality
$$\left|(u \circ \gamma)'(t)\right| \leq (\rho \circ \gamma)(t) \tag{3.2.5}$$
holds for m_1-almost every $t \in [0, \text{length}(\gamma)]$.

2. If $\rho \colon X \to [0, \infty]$ is a $p-$integrable $p-$weak upper gradient of $u : X \to Z$ then the inequality (3.2.5) holds $p-$almost every rectifiable curve $\gamma : [0, \text{length}(\gamma)] \to X$ parametrized by its arc length.

3. If u has a $p-$integrable $p-$weak upper gradient and if ρ is a nonnegative $p-$integrable Borel measurable function such that the inequality (3.2.5) holds for $p-$a.e. absolutely continuous rectifiable curve. Then ρ is a $p-$weak upper gradient.

Proof:

1. The absolute continuity is inherited by the absolutely continuity of the integral. Then, we can invoke the classical comparison with the metric derivative, considering $t \in [0, \text{length}(\gamma))$ and $h \in (0, \text{length}(\gamma) - t)$
$$\begin{aligned} \frac{d_Z((u \circ \gamma)(t), (u \circ \gamma)(t+h))}{h} &\leq \frac{1}{h} \int_t^{t+h} (\rho \circ \gamma)(s)\ ds \\ \lim_{h \downarrow 0} \frac{d_Z((u \circ \gamma)(t), (u \circ \gamma)(t+h))}{h} &\leq \lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} (\rho \circ \gamma)(s)\ ds. \end{aligned} \quad \text{Taking limit.} \tag{3.2.6}$$
From Theorem 1.3.13 and the Lebesgue's differentiation theorem, it follows that for m_1-almost every $t \in [0, \text{length}(\gamma)]$, applying these theorems in the inequality (3.2.6) we conclude that (3.2.5) holds for m_1-almost every $t \in [0, \text{length}(\gamma)]$.

2. Notice that we have the hypothesis of Lemma 3.2.3 then for $p-$almost every rectifiable curve $\gamma\colon [0, \text{length}(\gamma)] \to X$ parametrized by its arc length have the hypothesis of Item 1 therefore this result holds.

3. Notice that we have the conditions of Item 1, then we can use the fact: the inequality

$$d(\gamma(a), \gamma(b)) \leq \int_a^b |\gamma'(t)| \ dt.$$

holds for $p-$a.e. curve conclude that the upper gradient inequality holds; this proves that ρ is $p-$weak upper gradient.

■

In summary, above results shows that $p-$integrable $p-$weak upper gradient have automatically the properties of absolute continuity on curves. This is the principal difference between the classical theory of Sobolev spaces as we have discussed in Remark 2.6 and remark 1.6.11. In the next section we will all this properties to prove that a special ty of functions are in $W^{1,p}(\Omega)$.

3.3 Potentials

We will start by giving the general definition of potential.

Definition: Potential

Let X be a topological space and $E, F \subset X$ be (closed and nonempty) subsets of X. A **potential (E, F)** is a function $f\colon X \to \mathbb{R}$ such that f is 1 in precisely one of the sets and 0 in the other set.

Now, we starting to defining the bases of potential theory. We will start by defining a notion related with $\rho-$weight:

Definition: Best density from a set to a point

Let X be a metric space. For a nonnegative Borel function $\rho\colon X \to [0, \infty]$ and $E \subset X$. We define **$\rho-$pseudo-potential of x with respect E** as:

$$\boldsymbol{p_E(x)} = \inf_{\gamma \in \Gamma(x,E)} \int_\gamma \rho \, ds \tag{3.3.1}$$

and, we define the **$\rho-$pseudo-potential function with respect to E, $p_E\colon X \to [0, \infty]$** as function defined pointwise by (3.3.1).

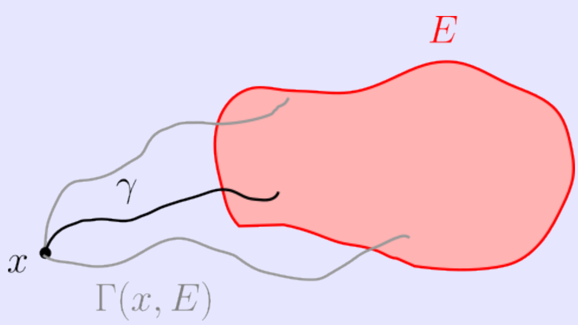

Figure 3.3.2: $\Gamma(x, E)$.

Some considerations

The value of $p_E(x)$ depends on the curves of $\Gamma(x, E)$ and ρ. Therefore, in general the function p_E depends on ρ and on the curves $\Gamma(E, F)$. We will omit this fact in the notation for simplicity. However, you should keep this dependence in mind.

In the following result is clear the reason why this the function p_E is called pseudo-potential.

Proposition 3.3.3: Let X be a metric space $\rho\colon X \to [0,\infty]$ be a nonnegative Borel function. Then, the pseudo-potential with respect to a closed nonempty subset $F \subset X$, p_F, satisfies $p_F = 0$ over F.

Special case

Furthermore, if $E \subset X$ is a closed nonempty subset such that is disjoint with F, and we consider $\rho \in D(E,F)$. Then, $p_F \geq 1$ over E.

Proof: Let $x \in E$. Since $x \in E$, the constant curve c_x satisfies $c_x \in \Gamma(x,E)$, and for this curve, we have $\int_{c_x} \rho\, ds = 0$. Therefore, $p_F(x) = 0$.

Now, we will study what happens in the special case. Let $x \in E$. Then $\Gamma(x,E) \subset \Gamma(E,F)$, since $\rho \in D_{\Gamma(E,F)}$, we have:

$$1 \leq \int_\gamma \rho\, ds. \qquad \gamma \in \Gamma(x,F)$$

Therefore:

$$p_F(x) = \inf_{\gamma\in\Gamma(x,E)} \int_\gamma \rho\, ds \geq 1.$$

■

Of course, the pseudo-potential $p_F(x)$ can be infinite, even in the special case of Proposition 3.3.3. For example if $\rho \equiv \infty$, then, p_E is not necessarily a potential. However, if we bound p_E taking a minimum with the constant 1, this could fix the issue. Considering this, we will prove in Proposition 3.3.6 that this fixes the problem. This motivates the following:

Definition: Potential between sets

Let X be a metric space and $E, F \subset X$ be (closed) disjoint subsets. For any admissible density $\rho \in D_{\Gamma(E,F)}$. We define **ρ−potential of x with respect E** as:

$$\boldsymbol{f_{E,F}(x)} = \min\{p_F(x), 1\} = \min\left\{\inf_{\gamma\in\Gamma(x,F)} \int_\gamma \rho\, ds, 1\right\}, \tag{3.3.4}$$

and, we define the **ρ−potential function of F with respect to E, $f_{E,F}\colon X \to [0,\infty]$** as function defined pointwise by (3.3.4).

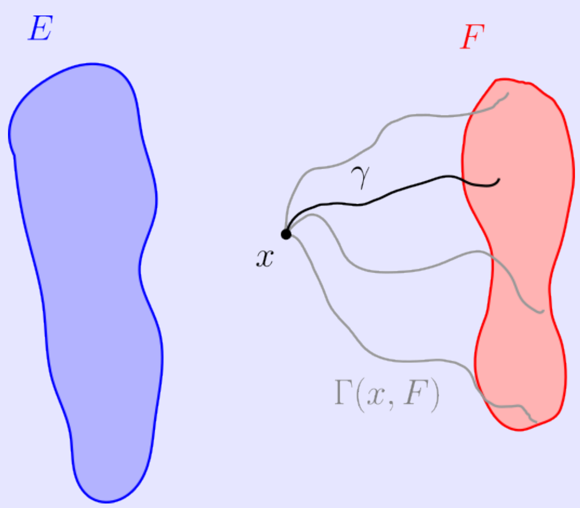

Figure 3.3.5: Elements of $f_{E,F}$.

Proposition 3.3.6 ($f_{E,F}$ is well defined): Let X a metric space and $E, F \subset X$ disjoint sets. If $\rho \in D_{\Gamma(E,F)}$, then $f_{E,F}$ is well defined. That is:

$$\begin{cases} f_{E,F} = 0 & \text{on } F, \\ f_{E,F} = 1 & \text{on } E. \end{cases}$$

Proof: Let $x \in E$. Then $\Gamma(x, E) \subset \Gamma(E, F)$, since $\rho \in D_{\Gamma(E,F)}$, we have:

$$1 \leq \int_\gamma \rho \, ds. \qquad \gamma \in \Gamma(x, F)$$

From this inequality and the definition of $f_{E,F}$ we have $f_{E,F}(x) = 1$. The equality $f_{E,F} = 0$ on F is due to Proposition 3.3.3.

■

Now, we will relate the pseudo-potential f_E with concatenation. Recall this definition is:

Definition: Concatenation

Let X be a metric space. For $i = 1, 2$ we consider $\gamma_i \colon [0, a_i] \to X$ compact curves in X such that $\gamma_1(a_1) = \gamma_2(0)$. The **concatenation** is the function $\boldsymbol{\gamma_1 * \gamma_2} \colon [0, a_1 + a_2] \to X$ given by

$$\boldsymbol{\gamma_1 * \gamma_2(t)} = \begin{cases} \gamma_1(t) & \text{if } t \in [0, a_1], \\ \gamma_2(t - a_1) & \text{if } t \in [a_1, a_1 + a_2]. \end{cases}$$

Clearly, we have the following results:

Proposition: If γ is a curve from y to x in a topological space X, then:

$$\gamma * \Gamma(x, E) \subset \Gamma(y, E). \tag{3.3.7}$$

Furthermore, this contention holds considering special conditions on the curves and space. For example, this contention holds considering rectifiable curves.

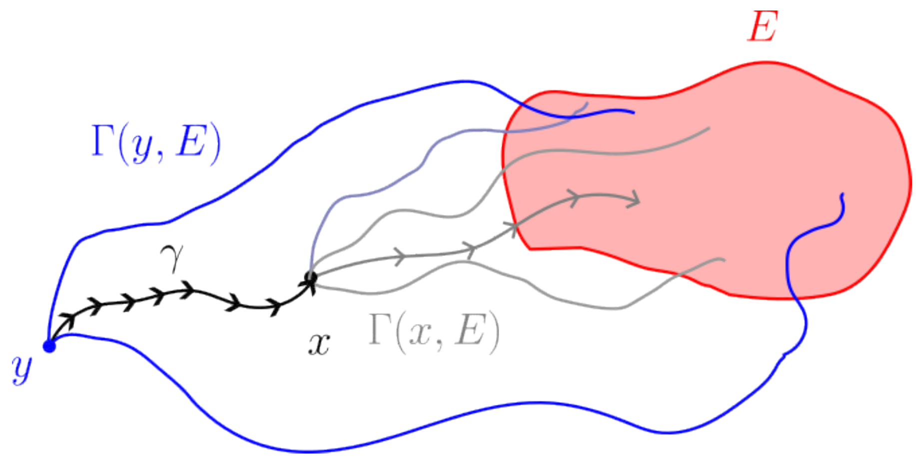

Figure 3.3.8: Concatenation and the family of curves $\Gamma(x, F)$

In general the contention (3.3.7) is strict. As we argument in the following:

Counterexample: The contention (3.3.7) is strict

From the definition of concatenation, all the curves of $\gamma * \Gamma(y, F)$ pass through x. However, not all the curves in $\Gamma(y, F)$ pass through x. Therefore the contention in general is strict.

Proposition 3.3.9: Let $f\colon X \to [-\infty, \infty]$ a measurable function in a metric measure space X. Then

$$\int_{\gamma_1 * \gamma_2} f\, ds = \int_{\gamma_1} f\, ds + \int_{\gamma_2} f\, ds$$

With these previous results we will prove an useful potential identity:

Lemma 3.3.10: Estimates of the pseudo-potential in point

Let X be a metric space and $\rho\colon X \to [0, \infty]$ be a nonnegative Borel function. Consider $\gamma_{\overrightarrow{yx}}$ locally rectifiable curve from y to x. Then

$$p_E(y) \leq \int_{\gamma_{\overrightarrow{yx}}} \rho\, ds + p_E(x) \tag{3.3.11}$$

Proof: Let $\gamma \in \Gamma(x, F)$ arbitrary. From Proposition 3.3.9, we have

$$\begin{aligned}
\int_{\gamma_{\overrightarrow{yx}} * \gamma} \rho\, ds &= \int_{\gamma_{\overrightarrow{yx}}} \rho\, ds + \int_{\gamma} \rho\, ds && \textit{Taking infimums.} \\
\inf_{\gamma \in \Gamma(x,E)} \int_{\gamma_{\overrightarrow{yx}} * \gamma} \rho\, ds &= \inf_{\gamma \in \Gamma(x,E)} \left(\int_{\gamma_{\overrightarrow{yx}}} \rho\, ds + \int_{\gamma} \rho\, ds \right) && \inf(x + A) = x + \inf A. \\
\inf_{\gamma \in \Gamma(x,E)} \int_{\gamma_{\overrightarrow{yx}} * \gamma} \rho\, ds &= \int_{\gamma_{\overrightarrow{yx}}} \rho\, ds + \inf_{\gamma \in \Gamma(x,E)} \int_{\gamma} \rho\, ds. && (3.3.12)
\end{aligned}$$

Recall (3.3.7), we have that $\gamma_{\overrightarrow{yx}*\gamma} * \Gamma(y, F) \subset \Gamma(x, F)$; then, $\left\{ \int_{\gamma_{\overrightarrow{yx}*\gamma}} \rho\, ds \,\middle|\, \gamma \in \Gamma(x, E) \right\} \subset \left\{ \int_{\gamma} \rho\, ds \,\middle|\, \gamma \in \Gamma(x, F) \right\}$. Therefore, $\inf_{\gamma \in \Gamma(x,F)} \int_{\gamma} \rho\, ds \leq \inf_{\gamma \in \Gamma(x,E)} \int_{\gamma_{\overrightarrow{yx}*\gamma}} \rho\, ds$ From this inequality and (3.3.12) follows:

$$\begin{aligned}
\inf_{\gamma \in \Gamma(x,F)} \int_{\gamma} \rho\, ds &\leq \int_{\gamma_{\overrightarrow{yx}}} \rho\, ds + \inf_{\gamma \in \Gamma(x,E)} \int_{\gamma} \rho\, ds && f_E \textit{ definition.} \\
p_E(y) &\leq \int_{\gamma_{\overrightarrow{yx}}} \rho\, ds + p_E(x).
\end{aligned}$$

■

As we have already noted, the pseudo-potential can be infinite. Thus, we must be careful when extending (3.3.11) to $f_{E,F}$. The extension to this function is proven in the following:

Lemma 3.3.13: Estimates of the pseudo-potential in point

Let X be a metric space and $E, F \subset X$ closed disjoint sets. Let $\rho \in D(E, F)$ an admissible density. Consider $\gamma_{\overrightarrow{yx}}$ locally rectifiable curve from y to x. Then

$$f_{E,F}(y) \leq \int_{\gamma_{\overrightarrow{yx}}} \rho\, ds + f_{E,F}(x) \tag{3.3.14}$$

Proof: From (3.3.11), it follows that

$$f_{E,F}(y) \leq \int_{\gamma_{\overrightarrow{yx}}} \rho\, ds + p_F(x).$$

If $p_F(x) \leq 1$, the previous inequality proves (3.3.14). Now, lets suppose that $p_F(x) > 1$, then $f_{E,F}(x) = 1$. Therefore:

$$\begin{aligned}
1 &\leq \int_{\gamma_{\overrightarrow{yx}}} \rho\, ds + 1 && f_{E,F}(y) \leq 1. \\
f_{E,F}(y) &\leq \int_{\gamma_{\overrightarrow{yx}}} \rho\, ds + f_{E,F}(x) && f_{E,F}(x) = 1.
\end{aligned}$$

■

From Lemma 3.3.13 follows immediately the next:

Corollary 3.3.15 (ρ is an upper gradient of $f_{E,F}$): Let X be a metric space and $E, F \subset X$ closed disjoint sets. Let $\rho \in D(E,F)$ an admissible density. Then ρ is an upper gradient of $f_{E,F}$.

Proof: Consider $\gamma\colon [a,b] \to X$ any rectifiable curve, define $x = \gamma(a), y = \gamma(b)$. Applying Lemma 3.3.10 to the curves $\gamma_{\overrightarrow{yx}} = \gamma$ and taking the canonical $\gamma_{\overleftarrow{yx}}$, we obtain two versions of (3.3.11); using both of them, and $\int_{\gamma_{\overrightarrow{yx}}} \rho\, ds = \int_{\gamma_{\overleftarrow{yx}}} \rho\, ds$, we obtain:

$$\begin{aligned} |f_{E,F}(x) - f_{E,F}(y)| &\leq \int_{\gamma_{\overrightarrow{yx}}} \rho\, ds \\ |f_{E,F}(\gamma(a)) - f_{E,F}(\gamma(b))| &\leq \int_{\gamma} \rho\, ds. \end{aligned} \qquad \textit{Definition of } x, y \textit{ and } \gamma_{\overrightarrow{yx}}.$$

This proves that $(f_{E,F}, \rho)$ satisfies the upper gradient inequality on any locally rectifiable curve γ in X.

■

3.3.1 Relationship between modulus and potential

Considering the requirements and the previous definitions, we have the following:

Theorem 3.3.16: $f_{E,F}$ is weakly differentiable

Let $E, F \subset \mathbb{R}^n$ be closed nonempty disjoint sets. Let $\rho \in D_{\Gamma(E,F)}$ an admissible density. If $\rho \in L^p(\Omega)$. Then, the following statements holds:

1. $f_{E,F} \in L^{1,p}(\mathbb{R}^n)$ for all $1 \leq p < \infty$. Therefore, $f_{E,F}$ is weakly differentiable.

2. ρ is a $p-$integrable $p-$weak upper gradient for $f_{E,F}$. Furthermore, the inequality $|\nabla f_{E,F}| \leq \rho$ holds almost everywhere in Ω. Thus, $\nabla f_{E,F}$ is $p-$integrable.

3. $|\nabla f_{E,F}|$ is an admissible density for $\Gamma(E,F) \setminus \Gamma_0$, where Γ_0 is a $p-$null family of curves.

4. Considering that $f_{E,F}$ depends on $\rho \in D_{\Gamma(E,F)}$, $f_{E,F} = f_{E,F}(\rho)$, we have that $\mathrm{Mod}_p(\Gamma(E,F))$ is computed only by the $p-$energy of the gradient of the potential associated with the admissible density. That is

$$\mathrm{Mod}_p(\Gamma(E,F)) = \inf_{\rho \in D_{\Gamma(E,F)}} \int_X |\nabla f_{E,F}(\rho)|^p$$

Proof:

1. First notice that $f_{E,F}$ is bounded by 1, thus $f_{E,F} \in L^1_{\mathrm{loc}}(\Omega)$. From Corollary 3.3.15, we have that ρ is a $p-$integrable upper gradient for $f_{E,F}$. Particularly, ρ is a $p-$integrable $p-$weak upper gradient. Thus, from Lemma 3.2.3, it follows that $f_{E,F}$ is absolutely continuous on $p-$almost every compact curve contained in Ω, particularly it is absolutely continuous the parallel curves to the axis. Therefore, $f_{E,F}$ is ACL^p on lines, from Lemma 2.6.1, we conclude that $f_{E,F}$ is weakly differentiable.

2. Recall from Corollary 3.3.15, we have that ρ is $p-$integrable upper gradient of $f_{E,F}$. Therefore, ρ is a $p-$integrable $p-$weak upper gradient of $f_{E,F}$.

   Let us define the curve $\omega(x,t)\colon [0,t] \to \Omega$ as the line segment from x to $x + t\omega$ parametrized canonically, define Γ as the family of all of those curves. From Item 2 of Lemma 3.2.4, it follows that for $p-$a.e. curve in Γ.

$$\left|\left(f_{E,F} \circ \omega(x,s)\right)'(t)\right| \leq \left(\rho \circ \omega(x,s)\right)(t) \tag{3.3.17}$$

holds for m_1-almost every $t \in [0, t_0]$. From Lemma 2.6.10, we can consider that we are in the set where all the partial directional derivatives exists. Therefore, $|\nabla f(x)|$ exists, thus we can consider the computation, if $\nabla f(x) = 0$ the inequality holds, then, suppose that $|\nabla f(x)| \neq 0$. Then, we can evaluate (3.3.17) in $\omega_0 = \frac{\nabla f(x_0)}{|\nabla f(x_0)|}$, where x_0 is any point of Ω for which ω_0 exists. From the definition of x_0 we can use the classical results of calculus for the directional derivative, then evaluating (3.3.17) in the correspondent t_0 for which $\omega(x, t_0) = x_0$, we obtain:

$$|\nabla f_{E,F}| = \frac{|\nabla f_{E,F}|^2}{|\nabla f_{E,F}|} \leq \rho\left(x_0\right).$$

Therefore, the inequality $|\nabla f_{E,F}| \leq \rho$ holds almost everywhere in Ω.

3. Since ρ is a $p-$integrable $p-$weak upper gradient for $f_{E,F}$, it follows that there exists Γ_0 a $p-$null family of curves such that every $\gamma \in \Gamma\left(X\right) \setminus \Gamma_0$ satisfies:

   (a) The upper gradient inequality holds γ as well on every subcurve.

   (b) The derivative of $f_{E,F}$ exists almost everywhere on γ.

   Then, we can use Corollary 1.5.23 and since $f_{E,F}$ is a potential we have

   $$\int_\gamma |\nabla f_{E,F}| \geq |f_{E,F}\left(\gamma\left(b\right)\right) - f_{E,F}\left(\gamma\left(a\right)\right)| = 1$$

   this proves that $|\nabla f_{E,F}|$ is an admissible density for $\Gamma(E, F) \setminus \Gamma_0$.

4. Let us define M as follows:

   $$M = \inf_{\rho \in D_{\Gamma(E,F)}} \int_X |\nabla f_{E,F}\left(\rho\right)|^p.$$

   Now, we will prove that:

   $\boldsymbol{\mathrm{Mod}_p(\Gamma(E, F) \leq M}$

   Since Γ_0 is $p-$null, from $p-$exceptionality criterion, there exists $\rho_0 \colon X \to [0, \infty]$ a Borel nonnegative $p-$integrable function that satisfies

   $$\int_\gamma \rho_0 \, ds = \infty \qquad \forall \gamma \in \Gamma_0.$$

   For any $\varepsilon > 0$ consider the function

   $$\rho_\varepsilon = \varepsilon \rho_0 + |\nabla f_{E,F}|,$$

   from Lemma 2.2.12, we have that ρ_ε are admissible densities for $\Gamma\left(E, F\right)$, then

   $$\mathrm{Mod}_p(\Gamma(E, F)) \leq \int_X \rho_\varepsilon^p$$

   Taking the limit $\varepsilon \to 0$ and by Lemma 2.2.12, we have $\rho_\varepsilon \to |\nabla f_{E,F}|$ pointwise as $\varepsilon \to 0$.

   $$\mathrm{Mod}_p(\Gamma(E, F)) \leq \int_X |\nabla f_{E,F}|^p$$

   since $f_{E,F}$ is arbitrary, we conclude

   $$\mathrm{Mod}_p(\Gamma(E, F) \leq M.$$

   $\boldsymbol{M \leq \mathrm{Mod}_p(\Gamma(E, F)}$

From the definition of M and Item 2, we have:

$$M \leq \int_X |\nabla f_{E,F}|^p \leq \int_X \rho^p.$$

Since $\rho \in D_{\Gamma(E,F)}$ is arbitrary, we obtain $M \leq \mathrm{Mod}_p(\Gamma(E,F)$.

This proves that the modulus $\mathrm{Mod}_p(\Gamma(E,F)$ only considering the norm of the gradient ot the potentials.

■

The conclusion is that the potentials associated to admissible densities provide a better family of functions to compute $\mathrm{Mod}_p\,\Gamma(E,F)$ in an Euclidean space.

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