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Determinantal Surfaces and the Noether-Lefschetz Locus

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Chapter 1

Introduction

This thesis is an expanded version of [LLV24] which, together with [LLR24] and [LLV25], makes up the research I carried out during my PhD studies. The present work offers a more accessible and detailed exposition of the main results obtained in [LLV24], which are stated informally inside this introduction as Theorems A, B and C.

We study the interplay between two topics: a special class of algebraic surfaces, called *determinantal surfaces*; and the subarea of algebraic geometry known as *Noether-Lefschetz theory*.

The main geometrical objects studied are algebraic surfaces inside the complex projective 3-space $\mathbb{P}^3$. An algebraic surface $X \subset \mathbb{P}^3$ is defined as the zero locus of a homogeneous polynomial F in the homogeneous coordinate ring of $\mathbb{P}^3$, which is isomorphic to $\mathbb{C}[x, y, z, w]$. For example, the polynomial

$$F = x^4 + y^4 + z^4 + w^4$$

defines a smooth surface known as the *Fermat surface* of degree 4.

There is a remarkable fact about degree 3 surfaces, noted in [Gra55]: any degree 3 homogeneous polynomial $F \in \mathbb{C}[x, y, z, w]$ can be expressed as the determinant of a matrix

$$A = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \tag{1.1}$$

where each $m_{ij} \in \mathbb{C}[x, y, z, w]$ is a homogeneous polynomial of degree 1.

Let us take a closer look at the case of degree 3 surfaces. Given a point $p \in \mathbb{P}^3$, the evaluation $A(p)$ of the matrix (1.1) at p is well-defined up to a scalar multiple; the surface X defined by $F = det(A)$ coincides with the locus of points $p \in \mathbb{P}^3$ for which $A(p)$ has rank strictly smaller than 3. Furthermore, if X is smooth, then for any point $p \in X$ the rank of $A(p)$ is exactly 2. In this case, we can associate to each $p \in X$ the point in $\mathbb{P}^2$ representing the kernel of the linear transformation $A(p) : \mathbb{C}^3 \to \mathbb{C}^3$. This construction yields a morphism $\pi : X \to \mathbb{P}^2$.

It turns out that π exhibits X as the blow-up of $\mathbb{P}^2$ at six points. This is a powerful device, which allows one to describe the Picard group of X; recover the famous Cayley-Salmon result that any smooth cubic surface contains 27 lines; among many other things.

The moral of the story is that, although expressing F as the determinant $det(A)$ is a purely algebraic affair, doing so reveals deep information about the geometric nature of X.

The equivalent statement for degree 1 and 2 homogeneous polynomials is elementary: every homogeneous polynomial of degree ≤ 2 in $\mathbb{C}[x, y, z, w]$ can be written as the determinant of a matrix whose entries are linear forms. For $d \geq 4$, the polynomials with this property are special. One way to understand this is through the Noether-Lefschetz theorem (Theorem 4.1), which claims that most surfaces of degree $d \geq 4$ have Picard group isomorphic to $\mathbb{Z}$.

Indeed, consider a degree 4 surface X defined as the zero locus of a polynomial $F = det(A)$, where

$$A = \left(\begin{array}{ccc|c} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{array}\right) = (A'|A'') \tag{1.2}$$

and each m_{ij} is a homogeneous polynomial of degree 1. The maximal minors of the 4×3 submatrix A' define a curve C of degree 6 which is contained in X. Since X has degree 4, C cannot be obtained as the intersection of X with another surface in $\mathbb{P}^3$, as the degree of such an intersection must be a multiple of 4.

Therefore, X does not satisfy the conclusion of the Noether-Lefschetz theorem, which implies that most degree 4 surfaces cannot be defined by the determinant of a matrix (1.2).

The smooth surfaces for which the Noether-Lefschetz theorem fails are parametrized by a subset $NL(d) \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$, called the *Noether-Lefschetz locus*. The set $NL(d)$ is a countable union of proper algebraic sets (Theorem 4.12), called *Noether-Lefschetz components*. The study of the Noether-Lefschetz components, known as *Noether-Lefschetz theory*, is a rich area of research which has seen many developments since the 80's.

In this thesis we study *determinantal surfaces*, which are surfaces whose defining polynomial is the determinant of a matrix (see Definitions 2.1 and 2.3). As in the previous examples, the definition of a surface being determinantal is purely algebraic, but this property has crucial geometric implications. For example, Proposition 2.7 shows that determinantal surfaces are characterized by containing a curve constructed in a similar fashion to the degree 6 curve C discussed above. This curve also shows up as a generator of the Picard group of X which, for determinantal surfaces, is typically isomorphic to $\mathbb{Z}^2$. We prove this in Corollary 2.15.

The determinantal surfaces are naturally organized in families, which sit inside the Noether-Lefschetz locus $NL(d)$. We discovered that each of these families forms a Noether-Lefschetz component. This is, in the opinion of the author, the main contribution of this thesis and of [LLV24].

Noether-Lefschetz theory can be understood from a Hodge-theoretical point of view, as explained in Chapter 4. In this thesis, we study surfaces and curves contained in them, which fit into the broader context of Hodge cycles of codimension 1. In these terms, our main result states that each family of determinantal surfaces can be constructed as a Hodge locus, defined by the cohomology class of an *arithmetically Cohen-Macaulay* curve (see Section 2.2).

The study of higher codimension Hodge cycles is an active area of research, and we would like to explore its relation to determinantal hypersurfaces in the future. One difficulty that arises as we consider higher codimension Hodge cycles is that we run into the Hodge conjecture, one of the seven *Millennium Prize Problems*. To this date, the Hodge conjecture remains an open problem.

1.1 Results and organization of the thesis

The organization of the thesis is as follows. Chapter 2 covers the preliminary definitions and results about determinantal surfaces used throughout. Chapter 3 proves the following:

Theorem A (Theorem 3.1)**.** There is an explicit formula for the dimension of each family of determinantal surfaces.

This formula is given in terms of the numerical invariants of each family of determinantal surfaces. After reviewing some key results in the Noether-Lefschetz theory, Chapter 4 proves the following:

Theorem B (Theorem 4.18)**.** Each family of determinantal surfaces forms a Noether-Lefschetz component.

Chapter 5 focuses on determinantal surfaces of degree 4. There are five families in this case, each having codimension 1 inside $|\mathcal{O}_{\mathbb{P}^3}(4)| \cong \mathbb{P}^{34}$. The main result in this chapter is the following:

Theorem C (Theorem 5.1)**.** There are five families of degree 4 determinantal surfaces, each of them having degree 320112, 136512, 38475, 2508 and 320 respectively.

At the end of this thesis, Appendix A contains computer code, written in `Macaulay2`, which allows to make explicit computations based on the results presented.

Chapter 2

Determinantal surfaces

This chapter collects basic definitions and results about determinantal surfaces in $\mathbb{P}^3$ that will be used in the following chapters. Although elementary, we include proofs of non-standard results for completeness.

2.1 Definition of determinantal surfaces

An algebraic surface $X \subset \mathbb{P}^3$ is defined by a homogeneous polynomial $F = F(x, y, z, w)$ of degree $d = deg(X)$. We say that X is *determinantal* if $F = det(S)$ is the determinant of a matrix $S = (m_{ij})$, whose entries are homogeneous polynomials of positive degree. For this definition to work, we need to ensure that $F = det(S)$ is in fact a homogeneous polynomial. For example

$$det\begin{pmatrix} x & y^2 \\ z^2 & w^3 \end{pmatrix} = xw^3 - y^2z^2$$

is homogeneous of degree 4, while

$$det\begin{pmatrix} x & y^3 \\ z^2 & w^2 \end{pmatrix} = xw^2 - y^3z^2$$

is not. The following definition makes sure that the determinants we consider are homogeneous, while keeping track of the size of S and the degrees of its entries.

Definition 2.1. [LLV24, Definition 1] An *admissible pair* of degree d and length $t \geq 2$ is a pair (a, b) of integer arrays $a = (a_1, \dots, a_t)$ and $b = (b_1, \dots, b_t)$ such that

(i) $a_1 \leq a_2 \leq \ldots \leq a_t < b_1 \leq b_2 \leq \cdots \leq b_t$,

(ii) $d = \sum_{i=1}^{t} b_i - a_i$.

Two admissible pairs (a, b) and (a', b') of the same length are *equivalent* if there exists an integer $k \in \mathbb{Z}$ such that $a'_i = a_i + k$ and $b'_i = b_i + k$ for all $1 \leq i \leq t$. We write $(a', b') = (a + k, b + k)$ in this case.

The next lemma explains the usefulness of Definition 2.1.

Lemma 2.2. *Let $D = (d_{ij})$ be a $t \times t$ matrix whose entries are positive integers, and consider the vector space $Mat(D)$ of matrices $S = (m_{ij})$ whose entries are homogeneous polynomials of degree $deg(m_{ij}) = d_{ij}$. Then the following are equivalent:*

(i) Every determinant $det(S)$ with $S \in Mat(D)$ is homogeneous.

(ii) $d_{ij} + d_{kl} = d_{il} + d_{kj}$ for any $1 \le i, j, k, l \le t$.

(iii) There exists an admissible pair (a, b) of length t such that $d_{ij} = b_j - a_i$.

Furthermore, if these properties are satisfied, (a, b) is unique up to equivalence, and the degree of (a, b) is equal to the degree of $det(S)$ for each $S \in Mat(D)$.

Proof.

(i)⇒(ii) Fix indices i, j, k, l. If $i = k$ or $j = l$, there is nothing to prove. Otherwise, there exists a permutation $\sigma \in S_t$ on the letters $\{1, 2, \dots, t\}$ such that $\sigma(i) = j$ and $\sigma(k) = l$. Consider the matrix $S = (m_{pq})$ given by the rule

$$m_{pq} = \begin{cases} x^{d_{pq}} & \text{if } \sigma(p) = q \\ x^{d_{pq}} & \text{if } (p,q) = (i,l) \text{ or } (k,j) \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $S \in Mat(D)$ and that $det(S) = \pm x^N (x^{d_{ij}+d_{kl}} - x^{d_{il}+d_{kj}})$, where

$$N := \sum_{p \neq i,k} d_{p,\sigma(p)}.$$

Therefore, if $det(S)$ is homogeneous $d_{ij} + d_{kl} = d_{il} + d_{kj}$ must be satisfied.

(ii)⇒(iii) Choose any a_1, and define

$$a_i := d_{11} - d_{i1} + a_1 \qquad b_j := d_{1j} + a_1.$$

Then for any (i, j),

$$\begin{aligned} d_{ij} &= d_{i1} + d_{1j} - d_{11} \\ &= (d_{11} - a_i + a_1) + (b_j - a_1) - d_{11} \\ &= b_j - a_i. \end{aligned}$$

(iii)⇒(i) Suppose that the admissible pair (a, b) such that $d_{ij} = b_j - a_i$ has degree d. For any $S \in Mat(D)$ and any permutation $\sigma \in S_t$, the term

$$(-1)^\sigma \prod_{i=1}^{t} m_{i,\sigma(i)}$$

has degree

$$\sum_{i=1}^{t} d_{i,\sigma(i)} = \sum_{i=1}^{t} b_{\sigma(i)} - a_i = \sum_{i=1}^{t} b_i - a_i = d.$$

Therefore, $det(S)$ is homogeneous of degree d.

To finish the proof, assume that the properties above hold, and that two admissible pairs (a, b) and (a', b') satisfy

$$d_{ij} = b_j - a_i = b'_j - a'_i.$$

If we define $k := b'_1 - b_1$, we see that

$$a'_i = b'_1 - d_{i1} = b'_1 - b_1 + a_i = a_i + k$$

and

$$b'_j = d_{1j} + a'_1 = (b_j - a_1) + (a_1 + k) = b_j + k.$$

Therefore $(a', b') = (a + k, b + k)$. □

Definition 2.3. [LLV24, Definition 2] Let $X \subset \mathbb{P}^3$ be a surface defined by a homogeneous polynomial $F \in \mathbb{C}[x, y, z, w]$, and let (a, b) be an admissible pair of length t. Then X is *determinantal of type* (a, b) if there exists a matrix $S = (m_{ij})$ of size $t \times t$, whose entries are homogeneous polynomials of degree $\deg(m_{ij}) = b_j - a_i$, and such that $F = det(S)$.

The loci of determinantal surfaces, defined next, are the main object of study in this thesis.

Definition 2.4. The closure in $|\mathcal{O}_{\mathbb{P}^3}(d)|$ of the locus of determinantal surfaces of type (a, b) is denoted by $det(a, b)$.

By Lemma 2.2, $det(a, b) = det(a', b')$ for equivalent pairs (a, b) and (a', b').

Proposition 2.5. *[LLV24, Proposition 1.1] Let (a, b) be an admissible pair. Then $det(a, b)$ is irreducible and the general element in $det(a, b)$ is a smooth surface.*

Proof. Consider the vector space $Mat(a, b)$ of matrices S whose entries are homogeneous polynomials of degree $b_j - a_i$. In other words, $Mat(a, b) = Mat(D)$ for $D = (b_j - a_i)$ as in Lemma 2.2. By taking the determinant one gets a rational map

$$det : Mat(a, b) \dashrightarrow |\mathcal{O}_{\mathbb{P}^3}(d)|$$

dominating $det(a, b)$, which proves the irreducibility. The second part follows from the fact that smoothness is an open condition in $|\mathcal{O}_{\mathbb{P}^3}(d)|$. Thus, it is enough to exhibit a matrix realizing a smooth surface. One such matrix is

$$S := \begin{pmatrix} f_1 & 0 & \dots & 0 & g_t \\ g_1 & f_2 & \dots & 0 & 0 \\ 0 & g_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & f_{t-1} & 0 \\ 0 & 0 & \dots & g_{t-1} & f_t \end{pmatrix}$$

where $\prod_i f_i = x^d + y^d$ and $(-1)^t \prod_i g_i = z^d + w^d$. In fact, $det(S) = x^d + y^d + z^d + w^d$ defines the Fermat surface of degree d, which is smooth. □

2.2 Determinantal characterization

This section characterizes the determinantal surfaces of type (a, b) as those containing a curve of special type. The precise statement is Proposition 2.7.

A curve $C \subset \mathbb{P}^3$ is said to be *arithmetically Cohen-Macualay*, or ACM, if its ideal sheaf $\mathcal{I}_C$ admits a minimal free resolution of the form

$$0 \to \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}^3}(-b_j) \xrightarrow{S_+} \bigoplus_{i=0}^{t} \mathcal{O}_{\mathbb{P}^3}(-a_i) \to \mathcal{I}_C \to 0. \tag{2.1}$$

The morphism S_+ is given by a matrix, which we also denote by S_+, of size $(t+1) \times t$ whose entries are homogeneous polynomials of degree $m_{ij} = b_j - a_i$, similar to Definition 2.1. The property of (2.1) being *minimal* means that whenever $b_j - a_i = 0$, m_{ij} is required to be the polynomial 0.

The degree d_C and genus g_C of an ACM curve as in (2.1) are given by the formulae

$$d_C = \frac{1}{2}\left(\sum_{j=1}^{t} b_j^2 - \sum_{i=0}^{t} a_i^2\right), \qquad g_C = 1 + \frac{1}{6}\left(\sum_{j=1}^{t} b_j^3 - \sum_{i=0}^{t} a_i^3\right) - 2d_C. \tag{2.2}$$

Given the matrix S_+, the $t \times t$ minors of S_+ are a set of *minimal generators* for $\mathcal{I}_C$. Conversely, each minimal set of generators of $\mathcal{I}_C$ is the set of $(t+1) \times t$ minors of a matrix S_+ with $coker(S_+) \cong \mathcal{I}_C$.

Now suppose that we are given a homogeneous polynomial F of degree d defining a surface $X \subset \mathbb{P}^3$ containing C. Then F can be chosen as a minimal generator of $\mathcal{I}_C$ if and only if F is not in the image of the multiplication map

$$\mu_d : H^0(\mathbb{P}^3, \mathcal{I}_C(d-1)) \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathbb{P}^3, \mathcal{I}_C(d)). \tag{2.3}$$

We are interested in this because, whenever F is a minimal generator of $\mathcal{I}_C$, it is a minor of some matrix S_+. By definition, the surface X defined by such an F is determinantal of some suitable type. In order to state this relation precisely, consider two types of curves.

Definition 2.6. Let (a, b) be an admissible pair of length t and degree d. By replacing (a, b) with an equivalent pair if necessary, suppose that $a_1 > d$ and set $a_0 := d$. Given these conditions, denote by C_+ any curve with a minimal free resolution as in (2.1).

On the other hand, denote by C_- any curve with a minimal free resolution of the form

$$0 \to \bigoplus_{j=1}^{t-1} \mathcal{O}_{\mathbb{P}^3}(-b_j + b_t - d) \xrightarrow{S_-} \bigoplus_{i=1}^{t} \mathcal{O}_{\mathbb{P}^3}(-a_i + b_t - d) \to \mathcal{I}_{C_-} \to 0. \tag{2.4}$$

These numbers might seem arbitrary, but the idea is that the matrix S_+ defining C_+ is obtained from a matrix S as in Definition 2.1 by adding a row $R = (m_{00}, m_{01}, \ldots, m_{0t})$ of homogeneous polynomials with $deg(m_{0j}) = b_j - a_0 = b_j - d$; and the matrix S_- defining C_- is obtained from S by removing the last column.

We are ready to state a criterion for a surface to be determinantal. See [LLV24, Proposition 1.8].

Proposition 2.7 (Determinantal criterion). *Let (a, b) be an admissible pair of degree d and length t with $a_1 > d$, and let $X \subset \mathbb{P}^3$ be a surface defined by a homogeneous polynomial F of degree d. The following properties hold*

(i) If X contains a curve C_+ given as in (2.1), or a curve C_- given as in (2.4), then X is determinantal of type (a, b).

(ii) If $X \in det(a, b)$ is general, then it contains a curve C_+ and a curve C_-.

Proof.

(i) If X contains a curve C_+ then X will be determinantal of type (a, b) as long as F is not in the image of the multiplication map μ_d in (2.3). However, the hypothesis $a_1 > d$ implies that $H^0(\mathbb{P}^3, \mathcal{I}_{C_+}(d-1)) = 0$. Thus this condition is satisfied trivially.

Now suppose that X contains a curve C_-. Let $S_- = (m_{ij})$ be the $t \times (t-1)$ matrix defining C_-, and let F_i denote the $(t-1) \times (t-1)$ minor of S_- obtained by removing the i-th row, for $i = 1, \ldots, t$. Then $\{F_i : 1 \leq i \leq t\}$ is a set of minimal generators for $\mathcal{I}_{C_-}$. Since $C_- \subset X$, this implies that F has an expression

$$F = \sum_{i=1}^{t-1} (-1)^{t-1} m_{it} F_i,$$

where m_{it} is a homogeneous polynomial of degree

$$deg(m_{it}) = d - deg(F_i) = d - (a_i - b_t + d) = b_t - a_i.$$

In other words, $F = det(S)$ where $S = (m_{ij})$ is the $t \times t$ square matrix obtained from S_- by adding a column with entries m_{it}. In particular, $X \in det(a, b)$.

(ii) If $X \in det(a, b)$ is general, so is the matrix S such that $det(S) = F$; and so is the matrix S_+ obtained from S by adding a row R. Therefore, S_+ will define a curve $C_+ \subset X$ as required.

On the other hand, if $X \in det(a, b)$ is general, so is the matrix S_- obtained from S by removing the last column, which in turn defines a curve $C_- \subset X$ as required.

□

Remark 2.8. The hypothesis that $X \in det(a,b)$ is general is necessary for part (ii) of the previous proposition. For example, consider the pair $a = (3,3), b = (4,4)$, and the matrix

$$S = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

The double plane $X = \{det(S) = 0\}$ is an element of $det(a,b)$, but it does not contain a curve C_+. In this case, C_+ must have minimal free resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-6)^2 \xrightarrow{S_+} \mathcal{O}_{\mathbb{P}^3}(-5)^2 \oplus \mathcal{O}_{\mathbb{P}^3}(-2)$$

where

$$S_+ = \begin{pmatrix} F_1 & F_2 \\ x & 0 \\ 0 & x \end{pmatrix}$$

has entries F_1, F_2 of degree 4. This would mean that the generators of $\mathcal{I}_{C_+}$ are x^2, xF_1 and xF_2. Therefore, the plane $\{x = 0\}$ would be contained in C_+, which is a contradiction.

2.3 ACM curves

This section gathers various properties of ACM curves that will be useful afterwards.

Theorem 2.9. *[PS74, Theorem 6.2] Let C be a curve defined by (2.1). If $a_i < b_j$ for all $0 \leq i \leq t$ and $1 \leq j \leq t$, and S_+ is general, then C is smooth and irreducible.*

The following result is about an ACM curve as a point in the Hilbert scheme $\mathrm{Hilb}^{\mathbb{P}^3}$. Before stating it, let us fix some notation.

Definition 2.10. Let (a,b) be an admissible pair of degree $a_0 = d < a_1$. Denote by $\mathcal{H}_{a,b} \subset \mathrm{Hilb}^{\mathbb{P}^3}$ the locus of curves with minimal free resolution (2.1).

Theorem 2.11. *[Ell75, Theorem 2 (ii)] The family $\mathcal{H}_{a,b}$ is a smooth and irreducible open set of dimension*

$$\dim \mathcal{H}_{a,b} = 1 + \sum_{i=0}^{t}\sum_{j=1}^{t}\binom{b_j - a_i + 3}{3} - \sum_{i=0}^{t}\sum_{j=0}^{t}\binom{a_i - a_j + 3}{3} - \sum_{i=1}^{t}\sum_{j=1}^{t}\binom{b_i - b_j + 3}{3}.$$

2.4 Picard group

This section describes the Picard group of a general surface $X \in det(a,b)$. This description is crucial for Lemma 3.2; needed in the proof of Theorem 3.1; and in the proof of Theorem 4.18. Given a curve $C \subset X$, we will also denote by C its class in $Pic(X)$. We will denote the hyperplane class of X by H, so that $\mathcal{O}_X(H) \cong \mathcal{O}_X(1)$.

Theorem 2.12. *[Lop91, Corollary II.3.8] Let $C \subset \mathbb{P}^3$ be a smooth and irreducible curve and let $m(C)$ be the maximum degree among a minimal set of generators of $\mathcal{I}_C$. Let*

$$d \geq \max\{4, m(C)+1\}.$$

Then the general surface of degree d containing C has Picard group isomorphic to $\mathbb{Z}^2$, with generators H and C.

As an application of the previous theorem, we obtain a description of the Picard group of a general determinantal surface.

Corollary 2.13. *[LLV24, Corollary 1.10] Let (a, b) be an admissible pair of degree $d \geq 4$. Then the general $X \in det(a, b)$ has Picard group isomorphic to $\mathbb{Z}^2$ generated by H and C_-.*

Proof. Consider a general curve C_- as in (2.4), which is smooth and irreducible according to Theorem 2.9. The degrees of the minimal generators of $\mathcal{I}_{C_-}$ are $d-(b_t-a_i) < d$ for $1 \leq i \leq t$, which implies $m(C_-) < d$. Therefore, the general degree d surface containing C_- satisfies the hypothesis of Theorem 2.12. Since the general $X \in det(a, b)$ is obtained this way, the conclusion follows. □

It turns out that C_+ can be used as a generator of the Picard group of X too. This is what we prove next.

Proposition 2.14. *Let (a, b) be an admissible pair of degree d, and let $X \in det(a, b)$ be general. Then in $Pic(X)$*

$$C_+ = C_- + (b_t - d)H.$$

Proof. Let $S_+ = (m_{ij})$ be the matrix defining C_+. For any $u \in \mathbb{C}$ let

$$S_u := \begin{pmatrix} u \cdot m_{01} & u \cdot m_{02} & \dots & u \cdot m_{0,t-1} & m_{0t} \\ m_{11} & m_{12} & \dots & m_{1,t-1} & m_{1t} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{t1} & m_{t2} & \dots & m_{t,t-1} & m_{tt} \end{pmatrix}$$

and let $C_u \subset \mathbb{P}^3$ be the scheme defined by the $t \times t$ minors of S_u.

Clearly $C_u \subset X$ for any u, because X is defined by the minor of M_u obtained by removing the first row. Moreover, $C_1 = C_+$, while C_0 is the union of C_- and the intersection $D := X \cap \{m_{0t} = 0\}$, which has class $D = (b_t - d)H$ in $Pic(X)$.

We claim that the family $\{C_u\}$ is flat at $u = 0$. To see this, observe that the flat limit of $\{C_u\}$ at $u = 0$ is contained in $C_- \cup D$ scheme-theoretically. Since X is general, C_- and D are smooth, irreducible curves intersecting transversally at $C_- \cdot D = deg(C_-) \cdot deg(m_{0t})$ different points. Therefore, it will suffice to prove that C_+ and $C_- \cup D$ have the same degree and arithmetic genus.

Let us compute the degree and arithmetic genus of $C_- \cup D$ and compare them to (2.2). Denote $e := deg(m_{0t}) = b_t - d$. Since D is the complete intersection of X and the surface $\{m_{0t} = 0\}$, then $deg(D) = de$. From (2.4) we obtain

$$\begin{aligned}
2 \cdot deg(C_-) &= \sum_{j=1}^{t-1}(b_j - e)^2 - \sum_{i=1}^{t}(a_i - e)^2 \\
&= \sum_{j=1}^{t-1}(b_j^2 - 2eb_j) - \sum_{i=1}^{t}(a_i^2 - 2ea_i) - e^2 \\
&= \left(\sum_{j=1}^{t-1} b_j^2 - \sum_{i=1}^{t} a_i^2\right) - 2e\left(\sum_{j=1}^{t-1} b_j - \sum_{i=1}^{t} a_i\right) - e^2 \\
&= \left(2 \cdot deg(C_+) - b_t^2 + d^2\right) - 2e(-e) - e^2 \\
&= 2 \cdot deg(C_+) - 2 \cdot deg(D).
\end{aligned}$$

Thus $deg(C_+) = deg(C_-) + deg(D) = deg(C_- \cup D)$. Similarly, the genus of D is given by the formula

$$g(D) = \frac{de(d+e-4)}{2} + 1 = \frac{de(b_t - 4)}{2} + 1.$$

From (2.4) we obtain

$$\begin{aligned}
6 \cdot g(C_-) &= 6 + \left(\sum_{j=1}^{t-1}(b_j - e)^3 - \sum_{i=1}^{t}(a_i - e)^3\right) - 12 \cdot deg(C_-) \\
&= 6 + \sum_{j=1}^{t-1}(b_j^3 - 3eb_j^2 + 3e^2 b_j) - \sum_{i=1}^{t}(a_i^3 - 3ea_i^2 + 3e^2 a_i) + e^3 - 12 \cdot deg(C_-) \\
&= \left(6 + \sum_{j=1}^{t-1} b_j^3 - \sum_{i=1}^{t} a_i^3\right) - 3e\left(\sum_{j=1}^{t-1} b_j^2 - \sum_{i=1}^{t} a_i^2\right) \\
&\qquad + 3e^2\left(\sum_{j=1}^{t-1} b_j - \sum_{i=1}^{t} a_i\right) + e^3 - 12 \cdot deg(C_-) \\
&= (6g(C_+) - b_t^3 + d^3 + 12 \cdot deg(C_+)) \\
&\qquad - 3e(2 \cdot deg(C_+) - b_t^2 + d^2) + 3e^2(-e) + e^3 - 12 \cdot deg(C_-) \\
&= 6g(C_+) + 12(deg(C_+) - deg(C_-)) - 6e \cdot deg(C_+) + 3de(e - d) \\
&= 6g(C_+) + 12(deg(C_+) - deg(C_-)) - 6e \cdot (deg(C_-) + de) + 3de(e - d) \\
&= 6g(C_+) + 12de - 6 \cdot C_- \cdot D - 3de(d + e) \\
&= 6(1 + g(C_+) - g(D) - C_- \cdot D).
\end{aligned}$$

Therefore, the arithmetic genus of $C_- \cup D$ is

$$p_a(C_- \cup D) = g(C_-) + g(D) + C_- \cdot D - 1 = g(C_+)$$

as claimed. This proves that $\{C_u\}$ is flat at $u = 0$. In other words, C_+ and $C_- \cup D$ are rationally equivalent. Since $H^1(X, \mathcal{O}_X) = 0$, rational and linear equivalence coincide in X, yielding $C_+ = C_- + D = C_- + (b_t - d)H \in Pic(X)$. □

Corollary 2.15. *[LLV24, Proposition 1.11] Let (a, b) be an admissible pair of degree $d \geq 4$, and let $X \in det(a, b)$ be general. Then $Pic(X)$ is generated by H and C_+.*

Chapter 3

Dimension of $det(a, b)$

This chapter is devoted to computing the dimension of $det(a, b)$ explicitly in terms of the pair (a, b). There is a small gap in [LLV24], and this chapter also shows how to fix it.

From now on, X denotes a surface in $\mathbb{P}^3$; C_+ a curve with minimal free resolution (2.1); d_+ and g_+ the degree and genus of C_+; and $\mathrm{Hilb}^{\mathbb{P}^3}, \mathrm{Hilb}^X$ the Hilbert schemes of closed subschemes inside $\mathbb{P}^3$ and X, respectively.

The main result proved in this chapter is the following.

Theorem 3.1. *[LLV24, Theorem 1] Let (a, b) be an admissible pair of degree d such that $a_1 > d$. Then $det(a, b)$ is an irreducible variety of dimension*

$$\dim det(a, b) = 1 + \dim \mathcal{H}_{a,b} - \binom{d-1}{3} - g_+ + (d-4)d_+, \tag{3.1}$$

where d_+, g_+ are given by (2.2).

The idea of the proof is very simple: since $d = a_0 < a_i$ for all $1 \leq i \leq t$, a curve C_+ with resolution (2.1) is contained in a unique degree d surface X, which is determinantal of type (a, b) by Proposition 2.7. This yields a morphism

$$\begin{aligned} \nu : \mathcal{H}_{a,b} &\to det(a, b) \subset |\mathcal{O}_{\mathbb{P}^3}(d)| \\ C_+ &\mapsto X, \end{aligned}$$

which is dominant, again by Proposition 2.7. To conclude the proof we compute the dimension of the fiber $\mathcal{H}^X_{a,b} := \nu^{-1}(X)$ over a general $X \in det(a, b)$. This is done in two steps: the first one is to compute the dimension of $|\mathcal{O}_X(C_+)|$; the second is to prove that $\dim |\mathcal{O}_X(C_+)| = \dim \mathcal{H}^X_{a,b}$.

Observe that, according to Proposition 2.14, the class $C_+ - K_X = C_- + (b_t - 2d + 4)H \in Pic(X)$ is ample for a sufficiently large b_t. Therefore, $H^i(X, C_+) = 0$ for $i > 0$ if we replace (a, b) for a sufficiently positive equivalent pair $(a + k, b + k)$. The following lemma provides an explicit and uniform bound for this to happen.

Lemma 3.2. *Let (a,b) be a degree d admissible pair with*

$$b_t > 2d - 4 + \max\left\{0, -\frac{C_-^2}{deg(C_-)}\right\}.$$

Then for any smooth $X \in det(a,b)$ and any (not necessarily smooth) $C_+ \in \mathcal{H}^X_{a,b}$, $H^i(X, \mathcal{O}_X(C_+)) = 0$ for $i > 0$.

Proof. We apply Moishezon's criterion to

$$L := C_+ - K_X = C_- + (b_t - 2d + 4)H.$$

Let $C \subset X$ be a reduced and irreducible curve. Since X is general, C_- is smooth, so if $C_- \cdot C < 0$ then necessarily $C = C_-$, and

$$L \cdot C = C_-^2 + (b_t - 2d + 4)deg(C_-) > 0.$$

Otherwise, we have

$$L \cdot C \geq (b_t - 2d + 4)deg(C) > 0.$$

We conclude that L is ample. By Kodaira's vanishing theorem

$$H^i(X, \mathcal{O}_X(C_+)) = H^i(X, \mathcal{O}_X(L + K_X)) = 0 \qquad \text{for } i > 0.$$

□

Corollary 3.3. *Let (a,b) be an admissible pair satisfying the hypothesis of Lemma 3.2. For any (not necessarily smooth) $C_+ \in \mathcal{H}_{a,b}$ and any degree d smooth surface X containing C_+,*

$$\dim|\mathcal{O}_X(C_+)| = \binom{d-1}{3} + g_+ - (d-4)d_+ - 1.$$

Proof. The adjunction formula implies that $C_+^2 = 2g_+ - 2 - C_+ \cdot K_X$. Since $H^i(X, \mathcal{O}_X(C)) = 0$ for $i = 1, 2$, the Riemann-Roch formula yields

$$\begin{aligned}
h^0(X, C_+) &= \chi(X, C_+) \\
&= \chi(\mathcal{O}_X) + \frac{1}{2}C_+ \cdot (C_+ - K_X) \\
&= 1 + \binom{d-1}{3} + \frac{1}{2}(C_+^2 - C_+ \cdot K_X) \\
&= 1 + \binom{d-1}{3} + g_+ - 1 - C_+ \cdot K_X \\
&= \binom{d-1}{3} + g_+ - (d-4)d_+.
\end{aligned}$$

□

Now we analyze the local nature of $\mathcal{H}^X_{a,b}$.

Lemma 3.4. *If $X \in det(a,b)$ is smooth then $\mathcal{H}^X_{a,b}$ is also smooth, and its dimension at a point C_+ is $\dim_{C_+} \mathcal{H}^X_{a,b} = \dim |\mathcal{O}_X(C_+)|$.*

Proof. Given $C_+ \in \mathcal{H}^X_{a,b}$, we first compute the dimension of the tangent space of Hilb^X at C_+. Consider the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C_+) \to N_{C_+/X} \to 0.$$

The long exact sequence in cohomology, together with $H^1(X, \mathcal{O}_X) = 0$, proves that the restriction map

$$H^0(X, \mathcal{O}_X(C_+)) \to H^0(N_{C_+/X})$$

is surjective and has kernel of dimension $h^0(X, \mathcal{O}_X) = 1$. This means that the tangent space to the Hilbert scheme Hilb^X at C_+ has dimension $\dim |\mathcal{O}_X(C_+)|$.

So far this proves that $\dim_{C_+} \mathcal{H}_{a,b} \leq \dim_{C_+} \mathrm{Hilb}^X \leq \dim |\mathcal{O}_X(C)|$, and $\mathcal{H}^X_{a,b}$ is smooth at C_+ if these inequalities are equalities.

To see that the previous inequalities are equalities, remember that the family $\mathcal{H}_{a,b}$ of curves with minimal free resolution (2.1) is a smooth open set of an irreducible component of $\mathrm{Hilb}^{\mathbb{P}^3}$ by Theorem 2.11. With respect to the natural inclusions $\mathcal{H}^X_{a,b} \subset \mathrm{Hilb}^X \subset \mathrm{Hilb}^{\mathbb{P}^3}$, we have that $\mathcal{H}^X_{a,b} = \mathcal{H}_{a,b} \cap \mathrm{Hilb}^X$ is open in $Hilb^X$. In particular, $|\mathcal{O}_X(C_+)| \cap \mathcal{H}^X_{a,b}$ is a non-empty open set of $|\mathcal{O}_X(C_+)|$ (it contains C_+). Therefore $\dim |\mathcal{O}_X(C_+)| = \dim_{C_+} |\mathcal{O}_X(C_+)| \cap \mathcal{H}^X_{a,b} \leq \dim_{C_+} \mathcal{H}^X_{a,b}$, and the result follows. □

Remark 3.5. In [LLV24, Lemma 2.1], instead of Lemma 3.2, it was proved that $H^i(X, C_+) = 0$ for $i > 0$ without restrictions on the admissible pair (a,b), but with the additional hypothesis that C_+ be smooth.

If C_+ is obtained from the matrix S such that $F = det(S)$ by adding a row of general homogeneous polynomials, then C_+ is always smooth as long as X is general. However, $\mathcal{H}^X_{a,b}$ is in general not irreducible, and it could be the case that the general element in another component of $\mathcal{H}^X_{a,b}$ (corresponding to a different, non general determinantal representation of X) is not smooth. If this is the case, [LLV24, Lemma 2.1] and Lemma 3.4 are not enough to guarantee that $\mathcal{H}^X_{a,b}$ is equidimensional.

Lemma 3.2 fills this gap at the cost of replacing (a,b) with a sufficiently positive equivalent admissible pair $(a+k, b+k)$. The proof of Theorem 3.1 below shows that (3.1) remains invariant after replacing (a,b) with $(a+k, b+k)$; this implies a posteriori that (3.1) can be computed using an admissible pair which does not satisfy the hypothesis of Lemma 3.2, as claimed originally in [LLV24].

Proof of Theorem 3.1. If (a,b) satisfies the hypothesis of Lemma 3.2 then the fiber $\mathcal{H}^X_{a,b}$ of

$$\nu : \mathcal{H}_{a,b} \to det(a,b)$$

at a general $X \in det(a,b)$ is equidimensional of dimension as in Corollary 3.3. Therefore

$$\dim det(a,b) = \dim \mathcal{H}_{a,b} - \dim \mathcal{H}^X_{a,b}$$

coincides with (3.1). To finish the proof, it remains to show that formula (3.1) does not change when we replace (a,b) with an equivalent pair, as long as $a_1 > d$.

Let (a,b) be an admissible pair with $a_1 > d$, and let $(a',b') = (a+1, b+1)$. Denote

$$A := \bigoplus_{i=0}^{t} \mathcal{O}_{\mathbb{P}^3}(-a_i), \qquad B := \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}^3}(-b_j).$$

We will further write $hom(B,A) := \dim Hom_{\mathcal{O}_{\mathbb{P}^3}}(B,A)$, and so on. With this notation, Theorem 2.11 becomes

$$\dim \mathcal{H}_{a,b} = 1 + hom(B,A) - hom(A,A) - hom(B,B).$$

Finally, denote by $d'_+, g'_+, hom(A',B')$, etc. the corresponding numbers in (3.1) for (a',b'). First, observe that

$$\begin{aligned} hom(B',A') - hom(B,A) &= \sum_{j=1}^{t} hom(\mathcal{O}_{\mathbb{P}^3}(-b_j-1), \mathcal{O}_{\mathbb{P}^3}(-d)) - hom(\mathcal{O}_{\mathbb{P}^3}(-b_j), \mathcal{O}_{\mathbb{P}^3}(-d)) \\ &= \sum_{j=1}^{t} \binom{b_j - d + 4}{3} - \binom{b_j - d + 3}{3} \\ &= \sum_{j=1}^{t} \binom{b_j - d + 3}{2} \end{aligned}$$

as $hom(\mathcal{O}_{\mathbb{P}^3}(-b'_j), \mathcal{O}_{\mathbb{P}^3}(-a'_i)) = hom(\mathcal{O}_{\mathbb{P}^3}(-b_j), \mathcal{O}_{\mathbb{P}^3}(-a_i))$ whenever $i > 0$. Likewise:

$$hom(A',A') - hom(A,A) = \sum_{i=1}^{t} \binom{a_i - d + 3}{2}$$

given that $d < a_i$ whenever $i > 0$. And lastly

$$hom(B',B') - hom(B,B) = 0.$$

Now using (2.2) we get:

$$\begin{aligned} d'_+ - d_+ &= \frac{1}{2}\left(\sum_{j=1}^{t} (b_j+1)^2 - b_j - \sum_{i=1}^{t} (a_i+1)^2 - a_i\right) \\ &= \sum_{i=1}^{t} b_i - a_i \\ &= d. \end{aligned}$$

Similarly:

$$\begin{aligned} g'_+ - g_+ &= \frac{1}{6}\left(\sum_{j=1}^{t}(b_j+1)^3 - b_j^3 - \sum_{i=1}^{t}(a_i+1)^3 - a_i^3\right) - 2(d'_C - d_C) \\ &= \frac{1}{6}\left(3\sum_{i=1}^{t}(b_i^2 - a_i^2) + (b_i - a_i)\right) - 2d \\ &= \frac{1}{2}\sum_{i=1}^{t}(b_i^2 - a_i^2) - \frac{3}{2}d. \end{aligned}$$

Putting all this together, the difference of formula (3.1) for (a', b') and (a, b) is equal to:

$$\begin{aligned} &\sum_{i=1}^{t}\binom{b_i - d + 3}{2} - \binom{a_i - d + 3}{2} - \frac{1}{2}\sum_{i=1}^{t}(b_i^2 - a_i^2) + \frac{3}{2}d + (d-4)d \\ &= \frac{1}{2}\sum_{i=1}^{t}\big((b_i^2 - a_i^2) - (2d-5)(b_i - a_i)\big) - \frac{1}{2}\sum_{i=1}^{t}(b_i^2 - a_i^2) + \frac{3}{2}d + (d-4)d \\ &= -\frac{2d-5}{2}d + \frac{3}{2}d + (d-4)d \\ &= 0. \end{aligned}$$

□

In Appendix A we include code for computing the dimension of a family $det(a, b)$ using Theorem 3.1.

Chapter 4

Noether-Lefschetz theory

This chapter analyzes first order deformations of determinantal surfaces in order to relate the families $det(a, b)$ with the *Noether-Lefschetz locus* $NL(d)$.

4.1 The Noether-Lefschetz locus

In [Lef24] Solomon Lefschetz proved the celebrated Noether-Lefschetz theorem, following a claim made by Max Noether in [Noe82].

Theorem 4.1 (Noether-Lefschetz Theorem)**.** *Let $d \geq 4$ be an integer. If $X \subset \mathbb{P}^3$ is a very general surface of degree d, then the restriction map*

$$Pic(\mathbb{P}^3) \to Pic(X)$$

is an isomorphism.

The term *very general* refers to the fact that the conclusion of Theorem 4.1 is satisfied outside a countable union of proper Zariski closed sets in $|\mathcal{O}_{\mathbb{P}^3}(d)|$. This complement, called the *Noether-Lefschetz locus*, is the main object of study throughout this chapter.

Definition 4.2. For any $d \geq 4$, the *Noether-Lefschetz locus* $NL(d) \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ is defined as the locus parameterizing smooth surfaces X such that $Pic(X) \neq \langle H \rangle$.

Lefschetz investigated extensively the topology of algebraic varieties. Another well-known result of his is the following.

Theorem 4.3 (Lefschetz hyperplane theorem)**.** *Let Y be a a complex irreducible projective variety of dimension n, and let $X \subset Y$ be a hyperplane section such that $Y \setminus X$ is smooth. Then the map*

$$i_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$$

induced by the inclusion $i : X \to Y$ is an isomorphism for $k < n - 1$, and injective for $k = n - 1$.

The following lemma is an application of Theorem 4.3. See [Laz17, Example 3.1.18].

Lemma 4.4. *Let $X \subset \mathbb{P}^3$ be a smooth surface. Then the restriction map*

$$i^* : H^2(\mathbb{P}^3, \mathbb{Z}) \to H^2(X, \mathbb{Z})$$

is injective, and its cokernel is torsion-free.

Proof. The injectivity is a direct consequence of Theorem 4.3, but can be seen directly as follows: $H^2(\mathbb{P}^3, \mathbb{Z})$ is generated by the hyperplane class, whose image $H \in H^2(X, \mathbb{Z})$ satisfies $(kH)^2 = k^2 \cdot deg(X) \neq 0$. Therefore, no multiple kH with $k \neq 0$ can be zero in $Pic(X)$.

For the second part, the long exact sequence in cohomology of the pair $(\mathbb{P}^3, X)$

$$\ldots \to H^2(\mathbb{P}^3, \mathbb{Z}) \xrightarrow{i^*} H^2(X, \mathbb{Z}) \to H^3(\mathbb{P}^3, X; \mathbb{Z}) \to \ldots$$

shows that the cokernel of i^* is a subgroup of the relative cohomology group $H^3(\mathbb{P}^3, X; \mathbb{Z})$. The quotient space $\mathbb{P}^3/X$, obtained by identifying X to a point, is a CW-complex. Therefore, the theorem of universal coefficients implies that the torsion of

$$H^3(\mathbb{P}^3, X; \mathbb{Z}) \cong H^3(\mathbb{P}^3/X, \mathbb{Z})$$

is the same as the torsion of $H_2(\mathbb{P}^3/X, \mathbb{Z}) \cong H_2(\mathbb{P}^3, X; \mathbb{Z})$. Finally, consider the long exact sequence (this time for homology) of the pair $(\mathbb{P}^3, X)$:

$$\ldots \to H_2(X, \mathbb{Z}) \xrightarrow{i_*} H_2(\mathbb{P}^3, \mathbb{Z}) \to H_2(\mathbb{P}^3, X; \mathbb{Z}) \to H_1(X, \mathbb{Z}) \xrightarrow{i_*} H_1(\mathbb{P}^3, \mathbb{Z}) \to \ldots$$

Theorem 4.3 implies that the arrows marked i_* are isomorphisms. Therefore $H_2(\mathbb{P}^3, X; \mathbb{Z}) = 0$ has no torsion. □

It follows from Lemma 4.4 that a smooth degree d surface $X \subset \mathbb{P}^3$ belongs to the Noether-Lefschetz locus $NL(d)$ if and only if $Pic(X)$ is a free abelian group of rank at least 2. In this case, the hyperplane class H can always be taken as a basis element of $Pic(X)$.

The Noether-Lefschetz locus $NL(d)$ is a countable union of irreducible algebraic sets, called *components of the Noether-Lefschetz locus*.

Definition 4.5. A *component of the Noether-Lefschetz locus* $NL(d)$, or simply a *Noether-Lefschetz component*, is a maximal irreducible subset $\Sigma \subset NL(d)$.

4.2 Hodge loci

Denote by $U_d \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ the open set parameterizing smooth surfaces of degree d. The Noether-Lefschetz components, as in Definition 4.5, turn out to be algebraic closed sets inside U_d. However, there is an alternative way of describing a Noether-Lefschetz component as a *Hodge locus*. This construction has the advantage of endowing it with a natural scheme structure, as explained in this section.

Let us denote by $U_d \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ the locus parameterizing smooth surfaces, and by $\pi : \mathcal{X} \to U_d$ the universal family. This means that, given a smooth surface $X \in U_d$, the fiber $\mathcal{X}_X$ is isomorphic to X.

By Ehresmann's theorem [Ehr50], the projection $\pi : \mathcal{X} \to U_d$ is topologically a locally-trivial fibration. This implies that

$$\mathcal{H} := R^2\pi_*\underline{\mathbb{Z}}$$

is a locally constant sheaf of abelian groups with stalk over $X \in U_d$ equal to $H^2(X, \mathbb{Z})$. Fix a surface $X \in NL(d)$, and a cohomology class $\gamma \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ which is not a multiple of the hyperplane class H. Let $U \subset U_d$ be a disc, in the analytic topology, around X. Then γ defines a section $\overline{\gamma} \in H^0(U, \mathcal{H})$.

Definition 4.6. The *Hodge locus* defined by γ in U is the locus

$$NL(\gamma) := \{Y \in U : \overline{\gamma}_Y \in H^2(Y, \mathbb{Z}) \cap H^{1,1}(Y)\}.$$

The closure $\overline{NL(\gamma)} \subset U_d$ in the Zariski topology is simply known as the *Hodge locus* defined by γ.

Remark 4.7. Geometrically, $NL(\gamma)$ is the locus of surfaces Y for which the cohomology class γ is represented by an algebraic cycle in Y. This definition only makes sense locally around X: if $\{X_t\} \subset U_d$ is a non-contractible path starting and ending at $X_0 = X_1 = X$, and $\gamma_t \in H^2(X_t, \mathbb{Z})$ is the continuation of γ along $\{X_t\}$, then γ_0 and γ_1 might be different. In other words, the monodromy action of the fundamental group $\pi_1(U_d, X)$ acting on $H^2(X, \mathbb{Z})$ is not trivial in general.

The Hodge locus $\overline{NL(\gamma)}$ has a natural scheme structure defined locally by the section $\overline{\gamma}$. In general, $\overline{NL(\gamma)}$ need not be reduced. This scheme structure is hard to describe explicitly (e.g.: to give equations in $|\mathcal{O}_{\mathbb{P}^3}(d)|$ generating its ideal sheaf), but there is an explicit description of its Zariski tangent space at the point X, explained in what remains of this section.

Definition 4.8. Let X be a smooth projective variety. The *deformation map* of X is the map

$$\cdot : H^1(X, T_X) \times H^1(X, \Omega^1_X) \to H^2(X, \mathcal{O}_X)$$

obtained by composing the cup product

$$H^1(X, T_X) \times H^1(X, \Omega^1_X) \xrightarrow{\cup} H^2(X, T_X \otimes \Omega^1_X)$$

with the map in cohomology induced by the duality map

$$T_X \otimes \Omega^1_X \to \mathcal{O}_X.$$

The previous map is named *deformation map* because it relates first-order deformations of X with cohomology classes in $H^1(X, \Omega^1_X) = H^{1,1}(X)$ as explained below. Denote by $\mathrm{Def}^1(X)$ the space of first-order deformations of X; and by

$$\kappa : \mathrm{Def}^1(X) \to H^1(X, T_X)$$

the Kodaira-Spencer map.

Theorem 4.9. *[Ser07, Theorem 3.3.11(iii)] Let $\eta \in \mathrm{Def}^1(X)$ be a first-order deformation of X, and let $\mathcal{L}$ be an invertible sheaf on X. Then there is a first-order deformation of $\mathcal{L}$ along η if and only if*

$$\kappa(\eta) \cdot c_1(\mathcal{L}) = 0,$$

where $\cdot$ denotes the deformation map of X.

Now consider the exact sequence

$$0 \to T_X \to T_{\mathbb{P}^3}|_X \to N_{X/\mathbb{P}^3} \to 0$$

and the connection map

$$\delta : H^0(X, N_{X/\mathbb{P}^3}) \to H^1(X, T_X). \tag{4.1}$$

Geometrically, sections of the normal bundle $N_{X/\mathbb{P}^3}$ correspond to embedded first-order deformations of $X \subset \mathbb{P}^3$; elements in $H^1(X, T_X)$ to abstract first-order deformations of X (via the Kodaira-Spencer map); and δ is just the forgetful map. Moreover

$$H^0(X, N_{X/\mathbb{P}^3}) \cong T_{U_d, X}.$$

Theorem 4.10. *[Voi03, Lemma 5.16] Let $X \in U_d$ and let $\gamma \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ be different from a multiple of the hyperplane class. The tangent space to $\overline{NL(\gamma)}$ at the point X is equal to the kernel of the map*

$$\begin{aligned} \overline{\nabla}(\overline{\gamma}) : T_{U_d,X} &\to H^2(X, \mathcal{O}_X) \\ v &\mapsto \delta(v) \cdot \gamma, \end{aligned}$$

where $\delta : T_{U_d,X} \to H^1(X, \mathcal{O}_X)$ is the forgetful map (4.1) and $\cdot$ denotes the deformation map in Definition 4.8.

In other words, if $\gamma \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ is the cohomology class of a divisor $C \subset X$, then the tangent space of $\overline{NL(d)}$ at X is the space of embedded first-order deformations of $X \subset \mathbb{P}^3$ that extend to a deformation of the line bundle $\mathcal{O}_X(C)$.

4.3 Noether-Lefschetz theory

The study of the geometry and structure of Noether-Lefschetz components is known as *Noether-Lefschetz theory*. Let us briefly discuss some results in the area. For a detailed exposition on this topic, see [BN14].

We begin with an application of Theorem 4.10.

Proposition 4.11. *If Σ is an irreducible component of $NL(d)$, then*

$$\operatorname{codim} \Sigma \leq \binom{d-1}{3}$$

where the codimension is taken with respect to the ambient space U_d.

Proof. Suppose that $\Sigma = \overline{NL(\gamma)}$ for some class $\gamma \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$. Denote by $T_{\Sigma,X}$ the tangent space to Σ at X, where Σ is endowed with the scheme structure as Hodge locus. Then Theorem 4.10 yields an exact sequence

$$0 \to T_{\Sigma,X} \to T_{U_d,X} \xrightarrow{\overline{\nabla}(\overline{\gamma})} H^2(X,\mathcal{O}_X).$$

The inequality follows from the identity:

$$\dim H^2(X,\mathcal{O}_X) = \binom{d-1}{3}.$$

□

The previous upper bound on the codimension of Σ is sharp and relatively easy to prove. The corresponding lower bound is much harder, and it is due to Mark L. Green [Gre84].

Theorem 4.12 (Explicit Noether-Lefschetz theorem)**.** *Every component Σ of the Noether-Lefschetz locus satisfies*

$$d-3 \leq \operatorname{codim}\, \Sigma \leq \binom{d-1}{3}.$$

In [Gre88], Green simplified the proof of this theorem and further showed, for $d \geq 5$, that the only component Σ for which $\operatorname{codim}\,\Sigma = d-3$ is the one formed by surfaces containing a line. Independently that same year, Claire Voisin proved the uniqueness of the component with $\operatorname{codim}\,\Sigma = d-3$ in [Voi88]. Later on, she improved this result.

Theorem 4.13. *[Voi89] For $d \geq 5$, each Noether-Lefschetz component $\Sigma \subset NL(d)$ has codimension* $\operatorname{codim}\,\Sigma > 2d-7$*, with two exceptions:*

(i) The component of surfaces containing a line satisfies $\operatorname{codim}\,\Sigma = d-3$.

(ii) The component of surfaces containing a conic satisfies $\operatorname{codim}\,\Sigma = 2d-7$.

Heuristically, most components have maximal $\operatorname{codim}\,\Sigma$, as the standard terminology suggests.

Definition 4.14. The components with maximal codimension according to Theorem 4.12 are called *general*. The remaining components are called *special*.

The following result, proved by Ciro Ciliberto, Joe Harris and Rick Miranda in [CHM88], supports this idea.

Theorem 4.15 (Density Theorem)**.** *For $d \geq 4$, the union of the general components of $NL(d)$ is dense in $|\mathcal{O}_{\mathbb{P}^3}(d)|$ with respect to the analytic topology.*

Joe Harris conjectured (see [Gre89]) that the opposite was true for special components, namely that only finitely many of them existed. This was shown to be false by Claire Voisin in [Voi91] for $d \geq 168$ a multiple of 4. However, Gregorio Baldi, Bruno Klingler and Emmanuel Ullmo proved the following result in contrast to Theorem 4.15.

d	#special	#general	**#total**
4	0	5	**5**
5	2	6	**8**
6	7	9	**16**
7	14	11	**25**
8	30	12	**44**
9	51	17	**68**
10	88	24	**112**
11	146	24	**170**
12	239	29	**268**
13	367	33	**400**
14	571	38	**609**
15	852	43	**895**
16	1279	49	**1328**

Table 4.1: Distribution of special and general components formed by determinantal surfaces.

Theorem 4.16. *[BKU24, Theorem 6] For $d \geq 5$, the union of the special components of $NL(d)$ is not Zariski dense in U_d.*

The last theorem of this section aims at describing the distribution of the codimensions codim Σ in the interval described by Theorem 4.12.

Theorem 4.17. *[CL91, Theorem 1.1] For any $d \geq 8$ there exists a component of the Noether-Lefschetz locus $NL(d)$ of codimension c for every integer c such that*

$$\min\left\{\frac{3}{4}d^2 - \frac{17}{4}d + \frac{19}{3}, \frac{9}{2}d^{\frac{3}{2}}\right\} \leq c \leq \binom{d-1}{3}.$$

Theorems 4.13 and 4.17 illustrate the fact that most Noether-Lefschetz components have big codimensions, while those with smaller codimension are scarcer. In the following section we prove that the families $det(a, b)$ form Noether-Lefschetz components. It turns out that most of them are special, and many of them have codimension smaller than the lower bound in Theorem 4.17.

This fact is, in our opinion, what constitutes the main contribution of our work to the Noether-Lefschetz theory. Very little is known about the behavior and geometry of Noether-Lefschetz components with small codimensions, and determinantal surfaces provide several explicit examples of them.

Table 4.1, which was produced with the help of `Macaulay2` (see Appendix A), illustrates the amount of general and special Noether-Lefschetz components formed by determinantal surfaces of degree $d \leq 16$.

4.4 Proof of Theorem 4.18

This section is completely devoted to proving the following Theorem.

Theorem 4.18. *[LLV24, Corollary 1] Every family $det(a,b)$ contains, as a dense open set, an irreducible and generically reduced component of $NL(d)$.*

Fix an admissible pair (a,b) of degree $d \geq 4$, and let $X \in det(a,b)$ be a general element. Observe that, since X has Picard rank 2 (Corollary 2.13), the space of embedded deformations of $X \subset \mathbb{P}^3$ that keep a class $\gamma \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ of type $(1,1)$ does not depend on γ, as long as it is not a multiple of the hyperplane class H. This can be seen from Theorem 4.10: if $\gamma' = aH + b\gamma \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$, with $b \neq 0$, then for any $v \in T_{U_d,X}$

$$\begin{aligned}\overline{\nabla}(\overline{\gamma'})(v) &= \delta(v) \cdot (aH + b\gamma) \\ &= b\delta(v) \cdot \gamma \\ &= b\overline{\nabla}(\overline{\gamma}).\end{aligned}$$

The second equality uses the fact that any embedded deformation $\delta(v)$ of X, by definition, preserves H as a $(1,1)$ class, thus $\delta(v) \cdot H = 0$.

Two things follow from the previous observation. The first is that $\Sigma := \overline{NL(\gamma)}$ is a Noether-Lefschetz component, where γ is the class of a curve $C_+ \subset X$: otherwise, Σ must be contained in a larger Hodge locus $\overline{NL(\gamma')}$ for some cohomology class $\gamma' \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ which is not a multiple of the hyperplane class. This is a contradiction, as $\overline{NL(\gamma)} = \overline{NL(\gamma')}$.

The second is that

$$det(a,b) \cap U_d \subset \Sigma.$$

Indeed, Corollary 2.15 implies that $det(a,b) \cap U_d \subset NL(d)$. Since $det(a,b) \cap U_d$ is irreducible and Σ is the only Noether-Lefschetz component containing X, it must contain $det(a,b) \cap U_d$.

The next step in the proof of Theorem 4.18 is to show that this inclusion is an equality. In order to do this, let us assume that (a,b) satisfies the hypotheses of Lemma 3.2, implying that $H^1(X, C_+) = 0$. In this situation, the following result can be applied.

Proposition 4.19. *[Ser07, Corollary 3.3.15 (ii)] If $H^1(X,C) = 0$, then any first order deformation of X preserving $\mathcal{O}_X(C)$ admits a deformation of $C \subset X$ along it.*

To use this result, consider a (not necessarily complete) smooth curve $\{X_t\} \subset \Sigma$ passing through $X_0 = X$. Such a family can be constructed, for instance, intersecting Σ with a general linear space of complimentary dimension.

By Proposition 4.19 there exists a deformation $\{C_t \subset X_t\}$ with $C_+ = C_0$. Theorem 2.11 implies that C_t is a smooth ACM curve with the same Betti table as C_+ for all but a finite number of points t. Finally, for every t such that C_t is ACM, Proposition 2.7 says that $X_t \in det(a,b)$.

Since the union of the smooth curves passing through X is dense in Σ, this argument implies that the general element in Σ belongs to $det(a,b)$. Therefore, $det(a,b) \cap U_d = \Sigma$ as claimed.

To finish the proof of Theorem 4.18, it remains to see that Σ is a generically reduced Noether-Lefschetz component.

Lemma 4.20. *Suppose that*

$$b_t > 2d + \max\left\{0, -\frac{C_-^2}{deg(C_-)}\right\}.$$

Then the restriction map

$$H^0(X, N_{X/\mathbb{P}^3}) \to H^0(C_+, N_{X/\mathbb{P}^3}|_{C_+})$$

is an isomorphism.

Proof. Since $N_{X/\mathbb{P}^3} \cong \mathcal{O}_X(d)$, there is a short exact sequence

$$0 \to \mathcal{O}_X(dH - C_+) \to N_{X/\mathbb{P}^3} \to N_{X/\mathbb{P}^3}|_{C_+} \to 0.$$

From the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(d) \to \mathcal{O}_X(d) \to 0$$

and the fact that $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0$, it follows that every section of $\mathcal{O}_X(dH - C_+)$ is represented by a degree d homogeneous polynomial vanishing along C_+. The only such polynomial is, up to a scalar multiple, the one defining X itself. This polynomial defines the zero section of $\mathcal{O}_X(d)$, thus $H^0(X, \mathcal{O}_X(dH - C_+)) = 0$.

On the other hand, the same argument in the proof of Lemma 3.2 shows that, under the hypothesis on b_t,

$$H^1(X, \mathcal{O}_X(dH - C_+)) \cong H^1(X, \mathcal{O}_X(C_+ - 4H)) = 0.$$

The result follows. □

Proposition 4.21. *[LLV24, Proposition 2.5] In the current situation, there is a natural short exact sequence*

$$0 \to H^0(C_+, N_{C_+/X}) \to H^0(C_+, N_{C_+/\mathbb{P}^3}) \to T_{\Sigma,X} \to 0.$$

Proof. The normal bundle sequence $0 \to N_{C_+/X} \to N_{C_+/\mathbb{P}^3} \to N_{X/\mathbb{P}^3}|_{C_+} \to 0$ yields the following exact sequence in cohomology:

$$0 \to H^0(C_+, N_{C_+/X}) \to H^0(C_+, N_{C_+/\mathbb{P}^3}) \xrightarrow{\pi} H^0(C_+, N_{X/\mathbb{P}^3}|_{C_+}).$$

By Lemma 4.20, there is a natural isomorphism

$$H^0(C_+, N_{X/\mathbb{P}^3}|_{C_+}) \cong H^0(X, N_{X/\mathbb{P}^3}) \cong T_{U_d,X}.$$

Under this identification, π sends a first-order deformation of $C_+ \subset \mathbb{P}^3$ to the unique first-order deformation of X containing it. It follows that $\mathrm{Im}(\pi) \subset T_{U_d,X}$ is the space of first-order deformations of X containing a first-order deformation of C_+, which coincides with $T_{\Sigma,X}$ by Theorem 4.10. □

Corollary 4.22. *The component $det(a,b) \subset NL(d)$ is reduced at a general X.*

Proof. From the previous proposition

$$\begin{aligned}\dim T_{\Sigma,X} &= \dim H^0(C_+, N_{C_+/\mathbb{P}^3}) - \dim H^0(C_+, N_{C_+/X}) \\ &= \dim \mathcal{H}_{a,b} - \dim |\mathcal{O}_X(C_+)|.\end{aligned}$$

This coincides with $\dim det(a,b) = \dim \Sigma$ by Theorem 3.1. □

Chapter 5

The degree of $det(a, b)$

This chapter centers around the following question:

Question 1. *What is the degree of the family $det(a, b) \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$?*

One expects a combinatorial answer to Question 1 in terms of the pair (a, b), similar to the formula in Theorem 3.1. However, at the moment of writing this work there is no general answer to it. The main theorem proved in this chapter, which answers Question 1 in degree 4, is the following.

Theorem 5.1. *[LLV24, Theorem 2] For $d = 4$, there are five different families formed by determinantal surfaces, and they have degrees 320112, 136512, 38475, 2508 and 320 respectively.*

Section 5.1 describes the five families of quartic formed by determinantal surfaces; Section 5.2 explains some algebraic preliminaries concerning bilinear products in lattices; the proof of Theorem 5.1 is an application of the Gromov-Witten theory for K3 surfaces, as explained in [MP13]. Sections 5.3 and 5.4 present the necessary results from [MP13]; finally, Section 5.5 contains the proof of Theorem 5.1.

5.1 The five families

The goal of this section is to describe the five different families formed by degree 4 determinantal surfaces.

The following table lists the admissible pairs defining them; the degree d_- and genus g_- of the corresponding curve C_-; as well as the discriminant and coset of the Picard lattice $(Pic(X), H)$ of a general member X of the family (see Definition 5.5).

a	b	d_-	g_-	Δ	δ
$(5,5,5,5)$	$(6,6,6,6)$	6	3	20	2
$(5,5,5)$	$(6,6,7)$	3	0	17	1
$(5,5)$	$(7,7)$	4	1	16	0
$(5,6)$	$(7,8)$	2	0	12	2
$(5,5)$	$(6,8)$	1	0	9	1

(5.1)

The curves C_- have very simple geometric descriptions, which in turn give a handy characterization of the surfaces they parametrize, as described in the following lemma.

Lemma 5.2. *The five different families of degree 4 determinantal surfaces parametrize surfaces that contain one of the following curves, listed in the same order as Table (5.1):*

(1) A degree 6 and genus 3 curve with minimal free resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4)^3 \to \mathcal{O}_{\mathbb{P}^3}(-3)^3.$$

(2) A twisted cubic curve.

(3) A complete intersection of two quadrics.

(4) A conic.

(5) A line.

Corollary 5.3 (of Theorem 3.1)**.** *All the families above have codimension 1 inside $|\mathcal{O}_{\mathbb{P}^3}(4)| \cong \mathbb{P}^{34}$.*

5.2 Classification of rank 2 lattices

Given a surface $X \in NL(4)$, we can consider its Picard group $Pic(X)$ together with the intersection product and the distinguished element $H \in Pic(X)$ satisfying $H^2 = 4$. An abelian group with this data will be called simply a *lattice* in this Chapter (See Definition 5.4).

In order to study the irreducible components Σ of $NL(4)$, we need to understand the lattices appearing as the Picard group of a generic element $X \in \Sigma$. This section discusses briefly the algebraic considerations that will be used in the rest of the Chapter.

Definition 5.4. Let $\mathbb{L}$ be a finitely generated free abelian group endowed with a bilinear product

$$\langle , \rangle : \mathbb{L} \times \mathbb{L} \to \mathbb{Z},$$

and let v be an element in $\mathbb{L}$. The pair $(\mathbb{L}, v)$ is called a *lattice of degree h* if $\mathbb{L}/\mathbb{Z}v$ is torsion-free; and $\langle v, v\rangle = h$. The *rank* of a lattice $(\mathbb{L}, v)$ is just the rank of $\mathbb{L}$ as an abelian group. Two lattices $(\mathbb{L}, v)$ and $(\mathbb{L}', v')$ are *isomorphic* if there exists an isomorphism of abelian groups $\varphi : \mathbb{L} \to \mathbb{L}'$ such that

(a) $\langle \varphi(v_1), \varphi(v_2)\rangle' = \langle v_1, v_2\rangle$ for all $v_1, v_2 \in \mathbb{L}$, and

(b) $\varphi(v) = v'$.

For the purposes of Theorem 5.1 we are only interested on rank 2 lattices. Any such lattice is given by a matrix

$$A = \begin{pmatrix} h & d \\ d & \chi \end{pmatrix} \tag{5.2}$$

by defining $\mathbb{L} := \mathbb{Z}^2$, $v := (1, 0)$ and $\langle v_1, v_2\rangle := v_1 A v_2^t$.

Definition 5.5. If $(\mathbb{L}, v)$ is the rank 2 lattice given by the matrix (5.2), the *discriminant* of $(\mathbb{L}, v)$ is defined as

$$\Delta := -det(A) = d^2 - h\chi;$$

and the *coset* of $(\mathbb{L}, v)$ is defined as

$$\delta := [d] \in (\mathbb{Z}/h\mathbb{Z})/\pm\,.$$

It is clear that Δ and δ only depend on the isomorphism class of $(\mathbb{L}, v)$, and not on the matrix A defining the lattice. Moreover, these invariants completely classify rank 2 lattices up to isomorphism.

Proposition 5.6. *Two rank 2 lattices of degree $h \neq 0$ are isomorphic if and only if they have the same discriminant and coset.*

Proof. Suppose that $(\mathbb{L}, v)$ has the same discriminant and coset as the ones in Definition 5.5, corresponding to a lattice defined by the matrix A in (5.2). It will be enough to find an element $w \in \mathbb{L}$ such that $\{v, w\}$ is a basis of $\mathbb{L}$; and such that, with respect to this basis, the intersection matrix of $\mathbb{L}$ is exactly A.

First, choose any w such that $\{v, w\}$ is a basis for $\mathbb{L}$. Then $\langle v, w\rangle \equiv \pm d \mod h$. By replacing w by $-w$ if needed, it can be assumed that $\langle v, w\rangle \equiv d \mod h$, which means that

$$\langle v, w\rangle = kh + d$$

for some $k \in \mathbb{Z}$. Observe that $\langle v, w - kv\rangle = d$, where $\{v, w - kv\}$ is again a basis for $\mathbb{L}$.

Replacing w with $w - kv$, it follows that

$$d^2 - h\chi = \Delta = d^2 - h \cdot \langle w, w\rangle.$$

Since $h \neq 0$, this implies that $\langle w, w\rangle = \chi$. That is, the intersection matrix with respect to $\{v, w\}$ is equal to A. □

The following sections will require to compare invariants of a lattice with those of its sublattices. For this purpose the following lemma will be useful.

Lemma 5.7. *Let $(\mathbb{L}, v)$ be a rank 2 lattice with discriminant Δ and let $\mathbb{L}' \subset \mathbb{L}$ be a subgroup containing v. Denote by Δ' the discriminant of the lattice $(\mathbb{L}', v)$ obtained by restricting the bilinear form $\langle\cdot,\cdot\rangle$ to $\mathbb{L}'$. Then Δ divides Δ' and Δ'/Δ is a perfect square.*

Proof. Pick a basis $\{v, w\}$ of $\mathbb{L}$, and a basis $\{v, w'\}$ for $\mathbb{L}'$. We can write

$$w' = av + bw$$

for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Replace w' by $w' - av \in \mathbb{L}'$ to assume that $a = 0$. Under these conditions, if (5.2) is the intersection matrix of $\mathbb{L}$ with respect to the basis $\{v, w\}$, then the intersection matrix of $\mathbb{L}'$ with respect to $\{v, w'\}$ is

$$\begin{pmatrix} h & bd \\ bd & b^2\chi \end{pmatrix}.$$

Therefore $\Delta' = b^2\Delta$. □

To finish this section, we present a simple necessary condition for a rank 2 lattice to be realized as the Picard lattice of a quartic surface $X \subset \mathbb{P}^3$.

Lemma 5.8. *Let $X \subset \mathbb{P}^3$ be a smooth quartic surface of Picard rank 2. Then the discriminant and coset of the lattice $(Pic(X), H)$ satisfy the following identities:*

$$\Delta > 0, \qquad \Delta \equiv \delta^2 \mod 8.$$

Proof. The fact that $\Delta > 0$ is a direct consequence of the Hodge index theorem. For the second part, pick a divisor class $D \in Pic(X)$ such that $\{H, D\}$ is a basis for $Pic(X)$. By replacing D with $D + nH$ for n sufficiently positive, we can assume that D is the class of a smooth curve $C \subset X$. With respect to this basis, the intersection matrix A in (5.2) satisfies

$$h = 4, \qquad \chi = D^2 = 2g(C) - 2.$$

Finally, we obtain

$$\Delta = d^2 - 4\chi = d^2 - 8 \cdot (g(C) - 1) \equiv \delta^2 \mod 8.$$

□

5.3 Noether-Lefschetz divisors

Let X be a K3 surface. A *quasi-polarization* on X of degree h is a choice of an element $H \in Pic(X)$ which is big and nef, and such that $H^2 = h$. An expository discussion on the construction and properties of the moduli space of quasi-polarized K3 surfaces can be found in [LSY15, Chapter 1]. For the purposes of this chapter, the following result is enough.

Theorem 5.9. *There exists a 19-dimensional projective moduli space $\mathcal{M}_h$ parameterizing quasi-polarized K3 surfaces (X, H) of degree h.*

There are two types of divisors in $\mathcal{M}_h$ that will be studied in this chapter.

Definition 5.10. Let Δ and δ be the discriminant and coset of a rank 2 lattice of degree $h > 0$. Then

$$\mathcal{P}_{\Delta,\delta} \subset \mathcal{M}_h$$

is defined as the closure of the elements $(X, H) \in \mathcal{M}_h$ such that $(Pic(X), H)$ is a rank 2 lattice of degree h.

Similarly, given integers d, g such that

$$\Delta_h(d, g) := d^2 - 2h(g-1) > 0,$$

denote by $\mu(d, g|\Delta, \delta)$ the number of elements w in a rank 2 lattice $(\mathbb{L}, v)$ of degree h with invariants (Δ, δ) such that

$$\langle v, w\rangle = d, \qquad \langle w, w\rangle = 2g-2.$$

Define the divisor

$$\mathcal{D}_{d,g} \subset \mathcal{M}_h$$

by the formula

$$\mathcal{D}_{d,g} := \sum_{\Delta,\delta} \mu(d, g|\Delta, \delta) \cdot \mathcal{P}_{\Delta,\delta}. \tag{5.3}$$

The formula (5.3) above makes sense because of the following lemma.

Lemma 5.11. *Let d, g be integers such that $\Delta_h(d, g) > 0$. Then*

$$0 \leq \mu(d, g|\Delta, \delta) \leq 2$$

for any pair (Δ, δ); and it is different from zero only for a finite number of such pairs.

Proof. Let $(\mathbb{L}, v)$ be a rank 2 lattice of degree h with invariants (Δ, δ), and let $\{v, v'\}$ be a basis for $\mathbb{L}$. Then $w := av + a'v'$ satisfies $\langle v, w\rangle = d$ if and only if

$$a = \frac{d - a'\langle v, v'\rangle}{h}. \tag{5.4}$$

Assuming this equality, the condition that $\langle w, w\rangle = 2g - 2$ translates to

$$(a')^2 = \frac{\Delta_h(d, g)}{\Delta}. \tag{5.5}$$

Since a' determines a uniquely, $\mu(d, g|\Delta, \delta)$ is bounded by the number of solutions of (5.5), which is at most 2. Moreover, Δ must be a divisor of $\Delta_h(d, g)$ for $\mu(d, g|\Delta, \delta)$ to be different from zero, thus there are only a finite number of such pairs (Δ, δ). □

Remark 5.12. In the previous proof, given a value for a', the fact that a is an integer is equivalent to

$$d \equiv a'\langle v, v'\rangle \equiv \pm \left(\frac{\Delta_h(d, g)}{\Delta}\right) \delta \mod h.$$

Given d, g this can be used to compute the expression (5.3) for $\mathcal{D}_{d,g}$. In fact, the sum runs on the pairs (Δ, δ) such that $\Delta_h(d, g)/\Delta$ is a perfect square; and if $\delta = [\delta_0]$ is the class of an element $\delta_0 \in \mathbb{Z}/h\mathbb{Z}$ then

$$\mu(d, g|\Delta, \delta) = \#\left\{\epsilon \in \{1, -1\} : d \equiv \epsilon \left(\frac{\Delta_h(d, g)}{\Delta}\right) \delta_0 \mod h\right\}.$$

5.4 Noether-Lefschetz numbers

Let us explain the reason to pay attention to the divisors $\mathcal{P}_{\Delta,\delta}$. Consider the open set $U_4 \subset |\mathcal{O}_{\mathbb{P}^3}(4)|$ parameterizing smooth quartic surfaces. This open set carries a universal family $\pi : \mathcal{X} \to U_4$ and a line bundle $\mathcal{L}$ on $\mathcal{X}$, such that the restriction of $\mathcal{L}$ to a fiber $X = \mathcal{X}_u, u \in U_4$ is isomorphic to $\mathcal{O}_X(1)$. This means that $\pi : \mathcal{X} \to U_4$ together with the line bundle $\mathcal{L}$ is a family of principally polarized K3 surfaces of degree 4. Therefore, there is an induced morphism

$$\varphi : U_4 \to \mathcal{M}_4.$$

Now consider an irreducible component $\Sigma \subset NL(4) \subset U_4$, whose general element $X \in \Sigma$ has a Picard lattice $(Pic(X), H)$ of rank 2 and degree 4 with invariants (Δ, δ). It follows that $\varphi(\Sigma) \subset \mathcal{P}_{\Delta,\delta}$. It can be shown that any $X \in U_d$ with an isomorphic Picard lattice actually belongs to Σ. Therefore, one has the equality

$$\Sigma = \overline{\varphi^{-1}(\mathcal{P}_{\Delta,\delta})}.$$

We are interested in computing the degree of $\overline{\varphi^{-1}(\mathcal{P}_{\Delta,\delta})}$. It turns out that one gets a cleaner answer for the corresponding degrees using the divisors $\mathcal{D}_{d,g}$. The definition below is different from that in [MP13, Theorem 2].

Definition 5.13. Given two integers d, g, the *Noether-Lefschetz number* is defined as

$$NL_{d,g} := \deg \overline{\varphi^{-1}(\mathcal{D}_{d,g})}.$$

The numbers $NL_{d,g}$ are computed by Davesh Maulik and Rahul Pandharipande in [MP13]. To explain their result, we need some notation. Define the power series

$$A := \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{8}}, \qquad B := \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n^2}{8}}, \qquad \Psi := 108 \sum_{n>0} q^{n^2}.$$

Then Θ is defined by the relation

$$\begin{aligned} 2^{22}\Theta := {} & 3A^{21} - 81A^{19}B^2 - 627A^{18}B^3 - 14436A^{17}B^4 \\ & - 20007A^{16}B^5 - 169092A^{15}B^6 - 120636A^{14}B^7 \\ & - 621558A^{13}B^8 - 292796A^{12}B^9 - 1038366A^{11}B^{10} \\ & - 346122A^{10}B^{11} - 878388A^9B^{12} - 207186A^8B^{13} \\ & - 361908A^7B^{14} - 56364A^6B^{15} - 60021A^5B^{16} \\ & - 4812A^4B^{17} - 1881A^3B^{18} - 27A^2B^{19} + B^{21}. \end{aligned}$$

Theorem 5.14. *[MP13, Corollary 2] Let d, g be numbers such that $\Delta_4(d, g) > 0$. Then the number $NL_{d,g}$ is equal to the coefficient of $q^{\frac{\Delta_4(d,g)}{8}}$ in $\Theta - \Psi$.*

The coefficients of $\Theta - \Psi$ can be computed explicitly, although this process becomes increasingly more costly as $\Delta_4(d, g)$ grows bigger. Nonetheless, for the purposes of Theorem 5.1 the following partial expansion is enough. It displays all the terms $q^{\frac{\Delta}{8}}$ of $\Theta - \Psi$ for which $\Delta \leq 20$:

$$\Theta - \Psi = -1 + 320q^{9/8} + 5016q^{3/2} + 76950q^2 + 136512q^{17/8} + 640224q^{5/2} + \ldots \qquad (5.6)$$

In Appendix A we have included a computer function to determine the coefficients of $\Theta - \Psi$.

5.5 Proof of Theorem 5.1

This section carries out the computations necessary to apply Theorem 5.14 in order to prove Theorem 5.1. The objective is to use (5.3) to express $\mathcal{P}_{\Delta,\delta}$ as a linear combination of various $\mathcal{D}_{d,g}$, for each pair (Δ, δ) appearing in Table (5.1). During this section, the degree and genus of a class w in a rank 2 lattice $(\mathbb{L}, v)$ of degree 2 refer to the integers (d, g) such that

$$\langle v, w\rangle = d, \qquad \langle w, w\rangle = 2g - 2.$$

Remark 5.12 will be used repeatedly during this section. We will write $\delta = 0, 1$ or 2, meaning that δ is the class of $0, 1$ or $2 \in \mathbb{Z}/4\mathbb{Z}$, respectively. Furthermore, we will only consider pairs (Δ, δ) satisfying the conditions of Lemma 5.8. For such a pair, we will refer to $\mu(d, g|\Delta, \delta)$ as a *relevant coefficient*.

5.5.1 $(\Delta, \delta) = (20, 2)$

A general element in the family (1) according to Lemma 5.2 has Picard lattice generated by H and a class of degree $d_- = 6$ and genus $g_- = 3$. Let us follow the steps in Remark 5.12 for $\mathcal{D}_{6,3}$. We have the options $\Delta = 20$ or 5, yielding a single relevant coefficient:

$$\mu(6, 3|20, 2) = 2.$$

The identity (5.3) amounts to

$$\mathcal{P}_{20,2} = \frac{1}{2}\mathcal{D}_{6,3}. \tag{5.7}$$

5.5.2 $(\Delta, \delta) = (17, 1)$

Let us follow Remark 5.12 for $\mathcal{D}_{3,0}$ according to Lemma 5.2 (2). We have the only option $\Delta = 17$ and a single relevant coefficient:

$$\mu(3, 0|17, 1) = 1.$$

This yields the identity

$$\mathcal{P}_{17,1} = \mathcal{D}_{3,0}. \tag{5.8}$$

5.5.3 $(\Delta, \delta) = (16, 0)$

Although the steps are essentially the same, this is the only case where Remark 5.12 must be used more than once. First follow Remark 5.12 for $\mathcal{D}_{4,1}$ according to Lemma 5.2 (3). We have the options $\Delta = 16, 4, 1$, which yields the following relevant coefficients:

$$\mu(4, 1|16, 0) = 2, \qquad \mu(4, 1|4, 2) = 2, \qquad \mu(4, 1|1, 1) = 2.$$

This yields the identity

$$\mathcal{D}_{4,1} = 2\mathcal{P}_{16,0} + 2\mathcal{P}_{4,2} + 2\mathcal{P}_{1,1}. \tag{5.9}$$

We repeat the process for $\mathcal{P}_{4,2}$ and $\mathcal{P}_{1,1}$. For $\mathcal{P}_{4,2}$, we follow Remark 5.12 for $\mathcal{D}_{2,1}$. The options $\Delta = 4, 1$ result in the following relevant coefficients:

$$\mu(2,1|4,2) = 2, \qquad \mu(2,1|1,1) = 2.$$

This yields the identity

$$\mathcal{D}_{2,1} = 2\mathcal{P}_{4,2} + \mathcal{P}_{1,1}. \tag{5.10}$$

We need to repeat the process for $\mathcal{P}_{1,1}$, which appears both in (5.9) and (5.10). In order to do this, we follow Remark 5.12 for $\mathcal{D}_{1,1}$. The only option is $\Delta = 1$, which results in

$$\mu(1,1|1,1) = 1.$$

Therefore

$$\mathcal{D}_{1,1} = \mathcal{P}_{1,1}. \tag{5.11}$$

Combining (5.9), (5.10) and (5.11) we obtain the identity

$$\mathcal{P}_{16,0} = \frac{1}{2}\mathcal{D}_{4,1} - \frac{1}{2}\mathcal{D}_{2,1} - \frac{1}{2}\mathcal{D}_{1,1}. \tag{5.12}$$

5.5.4 $(\Delta, \delta) = (12, 2)$

Let us follow Remark 5.12 for $\mathcal{D}_{2,0}$ according to Lemma 5.2 (4). The options $\Delta = 12, 3$ result in the single relevant coefficient

$$\mu(2,0|12,2) = 2.$$

Therefore

$$\mathcal{P}_{12,2} = \frac{1}{2}\mathcal{D}_{2,0}. \tag{5.13}$$

5.5.5 $(\Delta, \delta) = (9, 1)$

Let us follow Remark 5.12 for $\mathcal{D}_{1,0}$ according to Lemma 5.2 (5). The options $\Delta = 9, 1$ result in the following relevant coefficients:

$$\mu(1,0|9,1) = 1, \qquad \mu(1,0|1,1) = 1.$$

This yields

$$\mathcal{D}_{1,0} = \mathcal{P}_{9,1} + \mathcal{P}_{1,1}. \tag{5.14}$$

Equations (5.14) and (5.11) imply the identity

$$\mathcal{P}_{9,1} = \mathcal{D}_{1,0} - \mathcal{D}_{1,1}. \tag{5.15}$$

5.5.6 Computing degrees

Finally, Theorem 5.14 applied to equations (5.7), (5.8), (5.12), (5.13) and (5.15) implies Theorem 5.1, as shown next. All the required degrees appear in (5.6).

$$\begin{aligned}(5.7): \deg \overline{\varphi^{-1}(\mathcal{P}_{20,2})} &= \frac{1}{2}\deg \overline{\varphi^{-1}(\mathcal{D}_{6,3})}\\ &= \frac{1}{2}640224\\ &= 320112.\end{aligned}$$

$$\begin{aligned}(5.8): \deg \overline{\varphi^{-1}(\mathcal{P}_{17,1})} &= \deg \overline{\varphi^{-1}(\mathcal{D}_{3,0})}\\ &= 136512.\end{aligned}$$

$$\begin{aligned}(5.12): \deg \overline{\varphi^{-1}(\mathcal{P}_{16,0})} &= \frac{1}{2}\deg \overline{\varphi^{-1}(\mathcal{D}_{4,1})} - \frac{1}{2}\deg \overline{\varphi^{-1}(\mathcal{D}_{2,1})} - \frac{1}{2}\deg \overline{\varphi^{-1}(\mathcal{D}_{1,1})}\\ &= \frac{1}{2}76950 - \frac{1}{2}0 - \frac{1}{2}0\\ &= 38475.\end{aligned}$$

$$\begin{aligned}(5.13): \deg \overline{\varphi^{-1}(\mathcal{P}_{12,2})} &= \frac{1}{2}\deg \overline{\varphi^{-1}(\mathcal{D}_{2,0})}\\ &= \frac{1}{2}5016\\ &= 2508.\end{aligned}$$

$$\begin{aligned}(5.15): \deg \overline{\varphi^{-1}(\mathcal{P}_{9,1})} &= \deg \overline{\varphi^{-1}(\mathcal{D}_{1,0})} - \deg \overline{\varphi^{-1}(\mathcal{D}_{1,1})}\\ &= 320 - 0\\ &= 320.\end{aligned}$$

These are the degrees of the five families in Lemma 5.2, as claimed in Theorem 5.1.

Appendix A

Computer code

This appendix includes computer code based on Definition 2.1, Theorem 3.1 and Theorem 5.1. This code has been written by the author of this thesis using the computer software `Macaulay2`. Most likely it can be improved and optimized in various ways, but it is included as a tool for anyone wishing to play around with the results of the thesis.

A.1 Determinantal surfaces

A.1.1 Overview

The following is a `Macaulay2` package. It is intended to be used by saving the code in a `.m2` file and loading it using the command `loadPackage`. It contains two functions, `dimDet` and `listAdmissiblePairs`. After loading the package, one can use the command `help DeterminantalSurfaces` to see the documentation. Nonetheless, below is an explanation of each function.

1) `dimDet` takes an admissible pair (a, b) as argument, and computes the dimension of the family $det(a, b)$ of determinantal surfaces using Theorem 3.1. The argument should be a pair of Lists, each containing the entries of a and b, respectively. The following is an example, featuring the second family in Table 5.1, as seen in a `Macaulay2` terminal:

```
i1 : a = {5, 5, 5};

i2 : b = {6, 6, 7};

i8 : dimDet(a, b)

o8 = 33

o8 : QQ
```

2) `listAdmissiblePairs` takes two integers (d, t) as argument, and computes the list of all admissible pairs of degree d and length t, up to redundancy. That is, up to equivalence according to Definition 2.1; and up to taking the transpose of the corresponding matrix. The following is an example of the expected behavior, as seen in a `Macaulay2` terminal:

```
i1 : listAdmissiblePairs(4, 2)

o9 = {({0, 0}, {1, 3}), ({0, 0}, {2, 2}), ({0, 1}, {2, 3})}

o9 : List
```

A.1.2 Code

```
newPackage(
    "DeterminantalSurfaces",
    Version => "1.4",
    Date => "May 1, 2025",
    Headline => "a package for computing families of determinantal surfaces",
    Authors => {{
        Name => "Manuel Leal",
        HomePage => "https://sites.google.com/view/manuel-leal"
    }}
)


export {"dimDet", "listAdmissiblePairs"}

dimDet = method()
dimDet (List, List) := (a, b) -> {
    t := #b;
    d := 0;
    for i in 0..(t-1) do d = d + b_i - a_i;
    auxK := d+1-a_0;
    dimC := 0;
    for i in 0..(t-1) do
        for j in 0..(t-1) do {
            dimC = dimC + binomial(b_j-a_i+3, 3);
            if(a_j >= a_i) then dimC = dimC - binomial(a_j-a_i+3, 3);
            if(b_j >= b_i) then dimC = dimC - binomial(b_j-b_i+3, 3);
            };
    for i in 0..(t-1) do
        dimC = dimC + binomial(b_i+auxK-d+3, 3) - binomial(a_i+auxK-d+3, 3);
    dC := -d^2;
```

```
    for i in 0..(t-1) do dC = dC + (b_i+auxK)^2-(a_i+auxK)^2;
    dC = dC/2;
    gC := -d^3;
    for i in 0..(t-1) do gC = gC + (b_i+auxK)^3-(a_i+auxK)^3;
    gC = (gC/6) + 1 - 2*dC;
    return dimC - binomial(d-1, 3) - gC + (d-4)*dC + 1;
}

listAdmissiblePairs = method()
listAdmissiblePairs (ZZ, ZZ) := (d, t) -> {
    if t == 1 then return {({0}, {d})};
    LA := listA(d, t);
    L := {};
    for a in LA do {
        k := d;
        for i in 0..(t-1) do k = k - (a_(t-1) + 1 - a_i);
        LB := listPartition(k, t);
        for b in LB do {
            bC := for i in 0..(t-1) list b_i + a_(t-1) + 1;
            q := true;
            for i in 0..(t-1) do {
                if(bC_(t-1) - a_i > bC_(t-1-i)-a_0) then break;
                if(bC_(t-1) - a_i < bC_(t-1-i)-a_0) then {q = false; break;}
            };
            if(q == true) then L = append(L, (a, bC));
        };
    };
    return L;
}

nextA = method()
nextA (List, ZZ) := (a, d) ->
{
    t := sub(#a, ZZ);
    if a_0 == 1 then return a;
    if a_1 == d-t then return for i in 1..t list 1;

    q := sub(t-1, ZZ);
    while a_q == d-t do q = q-1;
    return for i in 0..(t-1) list if i < q then sub(a_i, ZZ) else 1+sub(a_q, ZZ);
}

listA = method()
```

```
listA (ZZ, ZZ) := (d, t) -> {
    a := for i in 1..t list 0;
    L := {a};
    while(a_0 == 0) do {
        a = nextA(a, d);
        auxDiff := 0;
        for i in 0..(t-1) do auxDiff = auxDiff + a_(t-1) + 1 - a_i;
        if auxDiff <= d and sub(a_0, ZZ) == 0 then L = append(L, a);
    };
    return L;
}

nextPartition = method()
nextPartition (List, ZZ, ZZ) := (b, k, t) -> {
    if(b_(t-1) - b_0 <= 1) then return b;
    q := t-1;
    while (b_q >= b_(t-1) - 1) do q = q-1;
    b2 := for i in 0..(t-1) list if i < q then b_i else b_q + 1;
    auxK := 0;
    for i in 0..(t-2) do auxK = auxK + b2_i;
    return for i in 0..(t-1) list if i < t-1 then b2_i else k - auxK;
}

listPartition = method()
listPartition (ZZ, ZZ) := (k, t) -> {
    b := for i in 1..t list if i < t then 0 else k;
    L := {b};
    while(b_(t-1) - b_0 >= 2) do {
        b = nextPartition(b, k, t);
        L = append(L, b);
    };
    return L;
}

beginDocumentation()
document {
    Key => DeterminantalSurfaces,
    Headline => "a package for computing families of determinantal surfaces",
    EM "DeterminantalSurfaces",
    " is a complementary package to https://doi.org/10.1002/mana.202400132. ",
    "Compatible with version 1.24.11 of Macaulay2."
}
document {
```

```
    Key => {(dimDet, List, List), dimDet},
    Headline => "computes the dimension of a family of determinantal surfaces",
    Usage => "dimDet(a, b)",
    Inputs => {
        "a" => List,
        "b" => List => {"with the same length as ", TT "a"}
    },
    Outputs => {
        { "returns an integer, the dimension of the family ",
        "of determinantal surfaces determined by the pair (a, b)" }
    },
    EXAMPLE lines ///
        dimDet({0,0,0}, {1,1,1})
        dimDet({0,0,0,0}, {1,1,1,1})
    ///,
}
document {
    Key => {(listAdmissiblePairs, ZZ, ZZ), listAdmissiblePairs},
    Headline => "finds all the admissible pairs (up to equivalence and ",
    "transposition of the matrix) of a given degree and length",
    Usage => "listAdmissiblePairs(d, t)",
    Inputs => {
        "d" => ZZ,
        "t" => ZZ
    },
    Outputs => {
        { "a List, containing all the admissible pairs of degree d ",
        "and length t" }
    },
    EXAMPLE lines ///
        listAdmissiblePairs(3, 3)
        listAdmissiblePairs(4, 2)
    ///,
}
```

A.2 Modular form

A.2.1 Overview

Here we present a simple function to compute the coefficients of the modular form $\Theta - \Psi$ appearing in (5.6). It takes an integer Δ as argument, and returns the coefficient of $q^{\Delta/8}$ in $\Theta - \Psi$.

A.2.2 Code

```
modularFormMP = Delta -> {
    Delta = sub(Delta, ZZ);
    R := QQ[q];
    L := {3, 0, -81, -627, -14436, -20007, -169092, -120636, -621558,
    -292796, -1038366,-346122, -878388, -207186, -361908, -56364, -60021,
    -4812, -1881, -27, 0, 1};
    P := 0_R;
    r := floor(sqrt(Delta/8));
    if(Delta == 8*r^2 and Delta != 0) then P = 108*q^Delta;
    A := 1_R;
    B := 1_R;
    for n in 1..(floor(sqrt(Delta))) do {
        A = A + 2*q^(n^2);
        B = B + 2*(-1)^n * q^(n^2);
    };
    T := 0_R;
    for k in 0..21 do T = T + L_k * A^(21-k) * B^k;
    return coefficient(q^Delta, 2^(-22)*T-P);
}
```

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