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PRESENTA:
GERARDO MARTIN FRANCO CORDOVA

DIRECTOR Y CODIRECTOR DE LA TESIS:
DR. MIGUEL ARTURO BALLESTEROS MONTERO,
INSTITUTO DE INVESTIGACIONES EN MATEMÁTICAS APLICADAS Y EN SISTEMAS, UNAM

DR. HERMANN SCHULZ-BALDES,
DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD FRIEDRICH-ALEXANDER DE ERLANGEN-NÚREMBERG

MIEMBROS DEL COMITE TUTOR:
DR. LUIS OCTAVIO SILVA PEREYRA, IIMAS, UNAM.
DRA. JUDITH CAMPOS CORDERO, DEPTO. MATEMATICAS, F. CIENCIAS, UNAM

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Resumen

En este trabajo desarrollamos teoría de dispersión directa para operadores de Schrödinger unidimensionales, discretos y con valores matriciales. Estos operadores, entendidos como operadores de Jacobi con valores matriciales, son análogos discretos de los operadores de Sturm-Liouville y describen interacciones entre vecinos más cercanos en una línea discreta. Al permitir que los coeficientes sean matrices autoadjuntas, generalizan los operadores de Jacobi escalares clásicos y capturan dinámicas más complejas relevantes para sistemas cuánticos multicomponentes.

Un objetivo central de nuestro trabajo es el análisis detallado de la matriz de dispersión asociada a tales operadores, con especial énfasis en sus propiedades analíticas y sus implicaciones espectrales. Para lograr esto, utilizamos el enfoque estacionario de la teoría de dispersión, en el cual la matriz de dispersión se construye mediante los valores a la frontera de los operadores resolventes correspondientes al operador perturbado y al no perturbado. Este enfoque nos proporciona una representación explícita y analítica de la matriz de dispersión en términos de funciones propias generalizadas.

La construcción comienza con una expansión completa en funciones propias del operador libre H_0, basadas en funciones propias generalizadas acotadas.

Luego, las funciones de onda para el operador perturbado pueden obtenerse a través de una ecuación de Lippmann–Schwinger que involucra los valores límite del resolvente. Bajo supuestos adecuados de decaimiento del potencial, la ecuación de Lippmann–Schwinger se reduce a una ecuación integral de tipo Volterra, que puede resolverse mediante aproximaciones sucesivas. Las soluciones resultantes se conocen como soluciones de Jost.

Estas soluciones de Jost constituyen el núcleo de nuestro análisis. Nos permiten definir y estudiar la matriz de dispersión en energías dentro del espectro absolutamente continuo del Hamiltoniano completo. Además, las propiedades analíticas de su continuación analítica en el parámetro espectral complejo nos proporcionan información sobre los estados acotados y los estados semi-acotados del Hamiltoniano completo.

También presentamos una representación alternativa de las soluciones de Jost como una expansión en una serie que involucra las soluciones de Jost libres. Esta formulación se conoce como operador de transformación y desempeña un papel clave en el análisis de problemas inversos de dispersión. En nuestro caso, utilizamos esta expresión para derivar estimaciones asintóticas de las soluciones de Jost en el límite de alta energía. Estas estimaciones asintóticas, junto con las propiedades analíticas de la matriz de dispersión, concluyen en la formulación y demostración de una versión discreta y matricial del teorema de Levinson, que relaciona el cambio de fase total de la matriz de dispersión con el número de estados acotados y semi-acotados.

Finalmente, establecemos rigurosamente la equivalencia entre las formulaciones estacionaria y dependiente del tiempo de la teoría de dispersión en este entorno discreto con valores matriciales.

En el Capítulo 4 se presentan resultados que no están incluidos en ningún artículo de investigación y que son producto de mi colaboracion con Volker Bach, Miguel Ballesteros, Jonathan Giovanni Gil, Iván Naumkin y Diego Terrazas. En particular, el capitulo incluye contribuciones de Jonathan Giovanni Gil Juárez y Diego Martin Terrazas Hernandez.
Este capítulo se incluye en la presente tesis porque constituye un trabajo en coautoría conmigo, tomando en cuenta las precisiones señaladas en este párrafo. Cabe aclarar que no representa la parte principal de mi tesis doctoral: mi contribución más importante se presenta en los demás capítulos, cuyos resultados fueron publicados en tres artículos de investigación [8, 9, 10], en coautoría con Miguel Ballesteros, Iván Naumkin y Hermann Schulz-Baldes.
Los Capítulos 4.1.1 y 4.2 contienen el estudio del operador resolvente y el principio del límite de absorción, que serán incluidos en los artículos y tesis referidos anteriormente. Asimismo, las Secciones 4.4 y 4.3 formarán parte de artículos posteriores. En estas últimas se construyen los operadores de Fourier generalizados y la fibración del operador de dispersión.

Zusammenfassung

Wir entwickeln eine direkte Streutheorie für eindimensionale, diskrete, matrixwertige Schrödinger-Operatoren. Diese Operatoren, dargestellt als matrixwertige Jacobi-Operatoren, sind diskrete Analoga zu Sturm-Liouville-Operatoren und beschreiben Wechselwirkungen zwischen nächsten Nachbarn auf einer diskreten Linie. Durch das Erlauben matixwertiger, selbstadjungierter Koeffizienten verallgemeinern sie klassische skalare Jacobi-Operatoren und beschreiben komplexere Dynamiken, wie sie in Mehrkomponenten-Quantensystemen auftreten.
Ein Hauptziel dieser Arbeit ist das detaillierte Studium der Streumatrix solcher Operatoren, insbesondere im Hinblick auf ihre analytischen Eigenschaften und Konsequenzen für das Spektrum des Hamiltonoperators. Zu diesem Zweck verwenden wir den stationären Zugang zur Streutheorie, bei dem die Streumatrix über die Randwerte der Resolventen der gestörten und ungestörten Operatoren konstruiert wird. Dieser Zugang ermöglicht eine explizite analytische Darstellung der Streumatrix in Form verallgemeinerter Eigenfunktionen.
Ausgangspunkt der Konstruktion ist eine vollständigen Eigenfunktionsentwicklung des freien Operators H_0, basierend auf beschränkten verallgemeinerten Eigenfunktionen.
Die Wellenfunktionen des gestörten Operators erhält man durch eine Lippmann–Schwinger Gleichung, die die Randwerte der Resolvente beinhaltet. Unter geeigneten Abfallbedingungen an das Potential vereinfacht sich diese Gleichung zu einer Volterra-Typ-Integralgleichung, die durch sukzessive Approximationen gelöst werden kann. Die entstehenden Lösungen sind als Jost-Lösungen bekannt.
Diese Jost-Lösungen bilden den Kern unserer Analyse. Sie ermöglichen die Definition und das Studium der Streumatrix für Energien im absolutstetigen Spektrum des vollständigen Hamiltonoperators. Darüber hinaus liefert die analytische Fortsetzung der Streumatrix im komplexen Spektralparameter Informationen über gebundene und halbgebundene Zustände des Hamiltonoperators.
Wir betachten zudem eine alternative Darstellung der Jost-Lösungen als Reihenentwicklung, die auf den freien Jost-Lösungen basiert. Diese Darstellung ist als Transformationsoperator bekannt und spielt eine zentrale Rolle in der inverse Streutheorie. Wir verwenden sie, um asymptotische Abschätzungen für die Jost-Lösungen im Hochenergiegrenzfall abzuleiten. Diese Abschätzungen, kombiniert mit den analytischen Eigenschaften der Streumatrix, führen zur Formulierung und zum Beweis einer diskreten matrixwertigen Version des Levinson-Theorems, welches die Phasenverschiebung der Streumatrix mit der Anzahl der gebundenen und halbgebundenen Zustände verknüpft.
Abschließend zeigen wir die Äquivalenz des stationären und des zeitabhängigen Zugangs zur Streutheorie im betachteten diskreten matrixwertigen Kontext.

Im Kapitel 4 werden Ergebnisse präsentiert, die in keinem Forschungsartikel enthalten sind und die aus meiner Zusammenarbeit mit Volker Bach, Miguel Ballesteros, Jonathan Giovanni Gil, Iván Naumkin und Diego Terrazas hervorgegangen sind. Insbesondere enthält dieses Kapitel Beiträge von Jonathan Giovanni Gil Juárez und Diego Martín Terrazas Hernández, die Teil ihrer jeweiligen Dissertationen sind.

Dieses Kapitel wird in die vorliegende Dissertation aufgenommen, da es eine gemeinsame Arbeit mit mir darstellt, unter Berücksichtigung der in diesem Absatz genannten Präzisierungen. Es ist jedoch hervorzuheben, dass es nicht den Hauptteil meiner Doktorarbeit darstellt; mein bedeutendster Beitrag wird in den übrigen Kapiteln entwickelt, deren Ergebnisse in drei Forschungsartikeln [8, 9, 10] veröffentlicht wurden, in Zusammenarbeit mit Miguel Ballesteros, Iván Naumkin und Hermann Schulz-Baldes.

Die Kapitel 4.1.1 und 4.2 enthalten die Untersuchung des Resolventenoperators und des Prinzips des Absorptionslimits, die in den zuvor erwähnten Artikeln und Dissertationen enthalten sein werden. Ebenso werden die Abschnitte 4.4 und 4.3 Teil zukünftiger Artikel und Dissertationen sein, in denen die verallgemeinerten Fourier-Operatoren konstruiert und die Faserung des Streuoperators analysiert wird.

Abstract

We develop direct scattering theory for one dimensional discrete matrix-valued Schrödinger operators. These operators, realized as matrix-valued Jacobi operators, are discrete analogues of Sturm-Liouville operators and describe nearest-neighbor interactions on a discrete line. By allowing the coefficients to be self-adjoint matrices, they generalize classical scalar Jacobi operators and capture more complex dynamics relevant to multi-component quantum systems.
A central goal of our work is the detailed analysis of the scattering matrix associated with such operators, with particular focus on its analytic properties and spectral implications. To achieve this, we use the stationary approach to scattering theory, in which the scattering matrix is constructed via the boundary values of resolvent operators corresponding to the perturbed and unperturbed operator. This approach gives us an explicit and analytically representation of the scattering matrix in terms of generalized eigenfunctions.
The construction begins with a complete eigenfunction expansion of the free operator H_0 based on bounded generalized eigenfunctions.
Then wave functions for the pertubated operator can be obtain through a Lippmann–Schwinger equation that involves the resolvent's boundary values. Under suitable decay assumptions on the potential, the Lippmann–Schwinger equation reduces to a Volterra-type integral equation, that can be solved via successive approximations. The resulting solutions are known as Jost solutions.
These Jost solutions form the core of our analysis. They allow us to define and study the scattering matrix at energies inside the absolutely continuous spectrum of the full Hamiltonian. Moreover, the analytic properties of its analytic continuation in the complex spectral parameter gives us information about the bound state and half-bound states of the full Hamiltonian. We also present an alternative representation of the Jost solutions as a series expansion involving the free Jost solutions. This formulation is known as Transformation operator and plays a key role in the analysis of inverse scattering problems, and we employ it to derive asymptotic estimates for the Jost solutions in the high-energy limit. These asymptotics, together with the analytic properties of the scattering matrix, conclude in the formulation and proof of a discrete matrix-valued version of Levinson's theorem, which relates the total phase shift of the scattering matrix to the number of bound and half-bound states.
Finally, we rigorously establish the equivalence between the stationary and time-dependent formulations of scattering theory in this discrete matrix-valued setting.

Chapter 4 presents results that are not included in any research article and arise from my collaboration with Volker Bach, Miguel Ballesteros, Jonathan Giovanni Gil, Iván Naumkin, and Diego Terrazas. In particular, this chapter includes contributions from Jonathan Giovanni Gil Juárez and Diego Martín Terrazas Hernández.

This chapter is included in the present dissertation because it constitutes joint work with me, taking into account the clarifications mentioned in this paragraph. It should be noted, however, that it does not represent the main part of my doctoral thesis; my most significant contributions are presented in the other chapters, whose results were published in three research articles [8, 9, 10], in collaboration with Miguel Ballesteros, Iván Naumkin, and Hermann Schulz-Baldes.

Chapters 4.1.1 and 4.2 contain the study of the resolvent operator and the limiting absorption principle, which will be included in the previously mentioned articles and thesis. Likewise, Sections 4.4 and 4.3 will be part of future articles, where the generalized Fourier operators are constructed and the fiber decomposition of the scattering operator is analyzed.

Contents

1 Introduction

The aim of this work is to present a study of the direct scattering problem for discrete matrix-valued Schrödinger operators. These operators are discrete analogues of classical Sturm-Liouville operators, generalized to act on sequences of vector-valued or matrix-valued functions. Specifically, we consider operators of the form

$$(H\psi)(n) = (W_n + \mathbf{1})\psi(n+1) + (W_{n-1} + \mathbf{1})\psi(n-1) + V_n\psi(n),$$

where $\psi(n) \in \mathbb{C}^L$, and $W_n, V_n \in \mathbb{C}^{L\times L}$ is a selfadjoint matrix-valued potential that decays suitably as $|n| \to \infty$.

The spectral and scattering theory of such operators plays a crucial role in understanding discrete quantum systems with internal degrees of freedom, such as multi-component lattice models. Moreover, discrete Schrödinger operators are intimately related to orthogonal matrix polynomials, which satisfy three-term recurrence relations.

Via the spectral theorem, such operators correspond to matrix-valued measures on the real line. The direct scattering problem involves determining the scattering data, such as the reflection and transmission coefficients, as well as the discrete spectrum, from a given potential W_n, V_n. This data forms the foundation for the inverse scattering problem, that conserns in reconstructing the potential and solving certain classes of nonlinear integrable equations.

The central objects in this theory are the Jost solutions, which are solutions of the generalized eigenvalue equation asociated to H constructed to have prescribed asymptotic behavior at infinity. These solutions encode the interaction between the potential and the free evolution and are used to define the scattering matrix, whose analytic and asymptotic properties are the main focus of our analysis.

Finally, we bring together the analysis of the scattering matrix in the formulation and proof of Levinson's theorem (see Theorem 1.15 below). This fundamental result relates the total phase shift of the scattering matrix on the absolutely continuous spectrum to the number of bound states and half-bound states of the operator H. Levinson's theorem relates the analytic properties of the scattering matrix and the spectral content of the operator.

In the final chapter, we study the equivalence between two formulations of the scattering matrix: the stationary approach developed in the main part of this work, and the more general time-dependent scattering theory. While this connection is well known in the context of one-dimensional scalar Schrödinger operators (see [28], Chapters 4 and 5), we include it here in the discrete matrix-valued setting for the sake of clarity and completeness.

Let us give a brief description of related works on scattering for one-dimensional discrete Schrödinger operators. Foundations and inverse scattering theory for the scalar case are laid out in [12, 16, 17, 18, 5, 27]. Levinson's theorem for one-dimensional discrete operators in the scalar case is proved in [19] and more recently in [13, 20, 23]. Scattering in a periodic background is treated in [15]. Works on the scattering theory for the matrix-valued case are scarce [26, 6, 7], but the latter two also construct Jost solutions and a scattering matrix under a moment condition similar as is done below. What is missing in [7], however, is the fine analysis of the analytic behavior of the Jost solutions and the scattering matrix at the band edges so that the authors could not conclude that there is a finite number of bound states nor analyze half-bound states nor prove a Levinson theorem. For scattering theory for continuous one-dimensional Schrödinger operators with a matrix-valued potential, there is also abundant literature, most of which is cited in the recent monograph by Aktosun and Weder [4]. A Levinson theorem in that framework is proved in [3, 2], and an index-theoretic perspective is given in [21] (in the scalar case, but this readily transposes to the matrix-valued case, *e.g.* [11, 20]). Complementary references for scattering theory on matrix Schrödinger operators and inverse scattering can be found in [22, 4].

The content of this work is structured as follows. We begin with an introduction where we briefly describe the main objects of study and summarize the key results obtained. In Section 1.2, we provide a brief introduction to matrix-valued Schrödinger operators, along with the definition and properties of the transfer matrix, which plays an important role in technical computations throughout the work. Then, in Section 2, we present some basic properties of the free Hamiltonian (Discrete Laplacian). Although elementary, these results are included to fix notation and to highlight several important facts relevant to the study of discrete Schrödinger operators.

In Section 3, we present a detailed analysis of the direct scattering problem. In particular, we construct the Jost solutions (see Section 1.3), and then study the properties of the scattering matrix (see Section 3.2). In Section 3.4 we establish a relation between the eigenvalues of the Hamiltonian and the zeros of the scattering matrix.

Section 3.6 is devoted to the analysis of the scattering matrix at the spectral band edges. We show how the behavior of the scattering matrix at the band edges is related to the existence of bounded solutions (half-bound states) of the generalized eigenvalue equation.

In Section 3.7, we construct the transformation operator, which plays a crucial role in the study of the inverse scattering problem. This construction is then used in Section 3.8 to analyze the high-energy asymptotics of the scattering matrix.

Section 3.10 is dedicated to the proof of Levinson's theorem, which relates the number of bound and half-bound states of the operator to the phase shift of the scattering matrix.

In Section 4, we study the equivalence between the stationary and time-dependent formulations of scattering theory. We begin by analyzing the resolvent operator and its boundary behavior, the so called limiting absorption principle (Section 4.1). We then construct a unitary diagonalization of the Hamiltonian, known as the generalized Fourier transform, which provides the key connection between the Jost solutions and the wave operators (Section 4.2). In Section 4.3, we prove the existence and completeness of the wave operators and derive their explicit relationship with the generalized Fourier transform. Finally, we show the equivalence between the scattering matrix defined in the stationary framework (in terms of Jost solutions) and the one obtained from the general time-dependent framework via wave operators.

To provide the reader with a general context regarding the content of this work, we begin with a brief description of the general framework of scattering theory.

1.1 Wave operators and scattering operator

Mathematical scattering theory is accurately described by some authors (e.g., [28]) as a branch of perturbation theory. Namely, one starts with a “simple” self-adjoint operator H_0, which is usually referred to as the free operator, and for which one has knowledge of the spectral properties that pertain to it. That is, one knows for example, its spectrum, absolutely continuous spectrum, essential spectrum, resolvent operators and their boundary values, etc. Then, after a suitable (in some sense “small”) self-adjoint perturbation V, commonly referred to as the potential, one obtains an operator $H := H_0 + V$, for which the aim is to study how the spectral properties are altered. In particular, scattering theory is mainly concerned with changes in the absolutely continuous part of the spectrum.

Let us illustrate the problem of scattering theory with one of its classical results:

Theorem (Kato-Rosenblum)**.** *If H and A are self-adjoint operators, and A is a trace class operator, then the absolutely continuous parts of $H + A$ and H are unitarily equivalent.*

In particular, this theorem implies that

$$\sigma_{\mathrm{ac}}(H) = \sigma_{\mathrm{ac}}(H + A).$$

One can observe an analogy between this result and Weyl's theorem concerning the essential spectrum.

Although, stated in this form it does not explicitly reference scattering theoretic objects, the Kato–Rosenblum theorem is in fact deeply connected to scattering theory. The proof of the theorem involves the construction of a unitary operator W between the absolutely continuous subspaces of the two self-adjoint operators H and $H + A$, satisfying the intertwining relation

$$WH = (H + A)W.$$

This unitary operator W is known as the wave operator, which is a central object in scattering theory.

Let us clarify the above. For the sake of simplicity, let us assume that H and H_0 are defined on the same Hilbert space $\mathcal{H}$. The unitary equivalence between the absolutely continuous parts of H and H_0 is closely related to the large-time asymptotic behavior of the solutions to the associated Schrödinger equations.

More specifically, consider the Schrödinger equation associated with the free operator:

$$i\frac{d}{dt}u_0(t) = H_0 u_0(t), \quad u_0(0) = h_0 \in \mathcal{H}, \quad u_0(t) = e^{-itH_0}h_0. \tag{1.1}$$

and the Schrödinger equation associated with the perturbed operator:

$$i\frac{d}{dt}u(t) = Hu(t), \quad u(0) = h \in \mathcal{H}, \quad u(t) = e^{-itH}h. \tag{1.2}$$

If for every $h_0 \in \mathcal{H}^0_{\mathrm{ac}}$, the absolutely continuous subspace of H_0, there exist vectors $h_\pm \in \mathcal{H}_{\mathrm{ac}}$, the absolutely continuous subspace of H, such that

$$\lim_{t\to\pm\infty} \|u(t) - u_0(t)\| = \lim_{t\to\pm\infty} \|e^{-itH}h_\pm - e^{-itH_0}h_0\| = \lim_{t\to\pm\infty} \|h_\pm - e^{itH}e^{-itH_0}h_0\| = 0, \tag{1.3}$$

then one can define the following operators, known as the wave operators (or Möller operators, first introduced by Christian Möller): $W_{\pm}(H, H_0) : \mathcal{H} \to \mathcal{H}$,

$$W_{\pm}(H, H_0)h_0 := \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} P^0_{\mathrm{ac}} h_0, \tag{1.4}$$

where P^0_{ac} denotes the projection onto the absolutely continuous subspace of H_0. It turns out that $W_{\pm}$ is a partial isometry with initial space $\mathcal{H}^0_{\mathrm{ac}}$ and final space $\mathcal{H}_{\mathrm{ac}}$ and satisfies the following relation, known as 'Intertwinting property'

$$W_{\pm} H_0 = H W_{\pm}. \tag{1.5}$$

Therefore, if the range of the wave operators $W_{\pm} \equiv W_{\pm}(H, H_0)$ equals the absolutely continuous subspace $\mathcal{H}_{\mathrm{ac}}$, that is,

$$\mathrm{Ran}(W_{\pm}) = \mathcal{H}_{\mathrm{ac}},$$

a property known as completeness, then the wave operators implement a unitary equivalence between the absolutely continuous parts of H and H_0.

The main objective of general mathematical scattering theory is to determine conditions on the perturbation $H - H_0$ that ensure the existence and completeness of the wave operators $W_{\pm}(H, H_0)$. In this context, one can reinterpret the Kato–Rosenblum theorem as a fundamental result in scattering theory.

Theorem (Kato–Rosenblum)**.** *Let H and H_0 be self-adjoint operators such that $H - H_0$ is trace class. Then the wave operators $W_{\pm}(H, H_0)$ exist and are complete.*

When the wave operators exist and are complete, one defines the scattering operator by

$$S(H, H_0) = W_{+}(H, H_0)^{*} W_{-}(H, H_0). \tag{1.6}$$

For each $h \in \mathcal{H}_{\mathrm{ac}}$, let u be the solution to the Schrödinger equation (1.2) with initial condition h. The scattering operator relates the initial condition $h_0^- \in \mathcal{H}^0_{\mathrm{ac}}$ of the solution u_0^- to the free Schrödinger equation (1.1) which behaves asymptotically as $t \to -\infty$ like u, to the initial condition $h_0^+ \in \mathcal{H}^0_{\mathrm{ac}}$ of the solution u_0^+ to (1.1) that behaves asymptotically as $t \to +\infty$ like u. The scattering operator is unitary and commutes with the free Hamiltonian,

$$H_0 S(H, H_0) = S(H, H_0) H_0.$$

The central object of interest in our analysis is the scattering matrix. This arises when one considers a spectral representation (or diagonalization) of the absolutely continuous part of the free Hamiltonian.

By the spectral theorem, there exists a measurable family of Hilbert spaces $\mathcal{H}(E)$, defined for almost every $E \in \sigma_{\mathrm{ac}}(H_0)$, and a unitary operator

$$\Phi_0 : \mathcal{H}^0_{\mathrm{ac}} \to \int_{\sigma}^{\oplus} \mathcal{H}(E)\, dE$$

such that

$$(\Phi_0 H_0^{\mathrm{ac}} h)(E) = E(\Phi_0 h)(E), \quad \text{for a.e. } E \in \sigma. \tag{1.7}$$

Since the scattering operator $S(H, H_0)$ commutes with H_0, it follows that $S(H, H_0)$ is diagonalized by Φ_0. That is, for almost every $E \in \sigma$, there exists a unitary operator $S(E) \in \mathcal{B}(\mathcal{H}(E))$ such that

$$(\Phi_0 S(H, H_0)h)(E) = S(E)(\Phi_0 h)(E). \tag{1.8}$$

The family $\{S(E)\}_{E\in\sigma}$ is known as the scattering matrix. It is important to note that the scattering matrix depends on the particular choice of the diagonalization $(\Phi_0, \int_\sigma^\oplus \mathcal{H}(E)\, dE)$, but it is unique up to unitary equivalence of the fibers $\mathcal{H}(E)$ for almost every $E \in \sigma_{\mathrm{ac}}(H_0)$. In the context of one-dimensional Schrödinger operators, it is possible to give an explicit construction of the unitary map Φ_0 and of the direct integral space $\int_\sigma^\oplus \mathcal{H}(E)\, dE$. Moreover, this can be done in such a way that $\mathcal{H}(E) = \tilde{\mathcal{H}}$ for almost every $E \in \sigma_{\mathrm{ac}}(H_0)$, where $\tilde{\mathcal{H}}$ is a fixed finite-dimensional Hilbert space. In particular, we can take $\tilde{\mathcal{H}} \cong \mathbb{C}^L$ for some $L \in \mathbb{N}$. Therefore, in this representation, the scattering matrix becomes an operator-valued (actually matrix-valued) function

$$S : \sigma_{\mathrm{ac}}(H_0) \to \mathcal{B}(\tilde{\mathcal{H}}) \cong \mathbb{C}^{L\times L}.$$

In this work, we are interested in studying the analytic properties of this matrix-valued function. In particular, we aim to relate these analytic properties to the spectral characteristics of the Hamiltonian H.

1.2 Matrix valued discrete Schrödinger operators

Now, we present the framework in which we study direct scattering theory, specifically, the case of matrix-valued Jacobi operators. We denote by $L^2(\mathbb{Z}, \mathbb{C}^L)$ the space of square summable sequence, namely $u : \mathbb{Z} \to \mathbb{C}^L$ such that

$$\|u\|_{L^2}^2 := \sum_{n\in\mathbb{Z}} \|u(n)\|_{\mathbb{C}^L}^2 < \infty.$$

Endowed with the usual norm $\|\cdot\|_{L^2}$.

Definition 1.1. *We consider matrix-valued Jacobi operators*

$$H : L^2(\mathbb{Z}, \mathbb{C}^L) \to L^2(\mathbb{Z}, \mathbb{C}^L),$$

defined by

$$(Hu)(n) = A_n u(n-1) + B_n u(n) + A_{n+1} u(n+1), \tag{1.9}$$

where $A, B : \mathbb{Z} \to \mathbb{C}^{L\times L}$ are sequences of self-adjoint matrices:

$$A_n^* = A_n, \quad B_n^* = B_n,$$

and each A_n is invertible.

Definition 1.2. *A particular case of the Jacobi operator arises when $A_n = I_{L\times L}$ and $B_n = 0$. This yields the discrete Laplacian operator, denoted by H_0:*

$$H_0 : L^2(\mathbb{Z}, \mathbb{C}^L) \to L^2(\mathbb{Z}, \mathbb{C}^L), \quad (H_0 u)(n) = u(n+1) + u(n-1).$$

Definition 1.3. *Given a Jacobi operator H, we define its potential by*

$$V = H - H_0.$$

In particular, the potential V is the operator given by

$$(Vu)(n) = (A_n - I)u(n-1) + B_n u(n) + (A_{n+1} - I)u(n+1).$$

We denote

$$W_n := A_n - I, \quad V_n := B_n,$$

so that the potential operator can be written as

$$(\tau_V u)(n) = W_n u(n-1) + V_n u(n) + W_{n+1} u(n+1). \tag{1.10}$$

The goal of this work is to study scattering theory for the pair (H, H_0). To this end, we impose a decay condition on the potential $V = H - H_0$. The assumptions we consider are as follows:

Definition 1.4. *We say that the Jacobi operator V, as defined in Definition 1.1, has finite p-moment if*

$$\sum_{n\in\mathbb{Z}} |n|^p \left(\|W_n\| + \|V_n\|\right) < \infty.$$

To study direct scattering theory, namely, the existence and completeness of the wave operators and the scattering matrix, it suffices to assume finite zero moment ($p = 0$) for the potential coefficients. However, for analyzing the analytic properties of the scattering matrix, particularly near the edges of the continuous spectrum, a first moment condition ($p = 1$) is required. Throughout the text, we explicitly indicate which hypothesis is needed in each context.

In some parts of the text, we interpret the right-hand side of Equation (1.9) more generally, applying it to arbitrary vector-valued sequences $u \in (\mathbb{C}^L)^{\mathbb{Z}}$, or even matrix-valued sequences $u \in (\mathbb{C}^{L\times L})^{\mathbb{Z}}$, not just those that are square-summable. To avoid ambiguity, we introduce the following notation:

Definition 1.5. *Let u be a vector-valued (resp. matrix-valued) sequence. We define the sequence $\tau_H u$ by*

$$(\tau_H u)(n) := A_n u(n-1) + B_n u(n) + A_{n+1} u(n+1). \tag{1.11}$$

Remark 1.6. *Throughout the text, we use the subscript $_0$ to denote quantities associated with the free operator H_0. In particular, we write $\tau_{H_0} \equiv \tau_0$, where*

$$(\tau_0 u)(n) = u(n-1) + u(n+1). \tag{1.12}$$

1.3 Stationary aproach and Jost solutions

As we mention in Section 1.1, our main aim in this text is to study the analytic properties of the scattering matrix. To this end, we seek more explicit formulas for the scattering matrix. This can be achieved through the stationary approach to scattering theory.

The stationary approach to scattering theory consists of representing the wave operators (as well as the scattering operator and scattering matrix) in terms of boundary values of the resolvent operators associated with H and H_0 (see, e.g., [29], Chapter 5, for a more general treatment of this approach). Thus, the description of the wave operators is reduced to the study of the boundary values of the resolvents. For Schrödinger operators, the stationary approach is closely related to the eigenfunction expansion, which we use to obtain a representation of the scattering matrix in terms of solutions to the generalized eigenvalue equation. Next, we explain this relation.

In Section 2.2, we see that for almost every $E \in \sigma_{\mathrm{ac}}(H_0)$, there exist solutions to the generalized eigenvalue problem

$$\tau_0 \psi^E_{0,(j,a)} = E \psi^E_{0,(j,a)} \tag{1.13}$$

with $\psi^E_{0,(j,a)} \in L^\infty(\mathbb{Z}, \mathbb{C}^L)$, where $j \in \{1, \dots, L\}$ and $a \in \{1, -1\}$, such that the set $\{\psi_{0,(j,a)}(n) : j \in \{1, \dots, L\}, a \in \{1, -1\}, n \in \mathbb{Z}\}$ is complete.

By "complete," we mean that there exists a unitary operator

$$\Phi_0 : L^2(\mathbb{Z}, \mathbb{C}^L)^0_{\mathrm{ac}} \to L^2(\sigma_{\mathrm{ac}}(H_0), L^2(\{(j,a)\}, \mathbb{C})) \tag{1.14}$$

such that for every $u \in L^1(\mathbb{Z}, \mathbb{C}^L) \subset L^2(\mathbb{Z}, \mathbb{C}^L)$, we have

$$(\Phi_0 u)(E)((j,a)) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \langle u(n), \psi^E_{0,(j,a)}(n) \rangle_{\mathbb{C}^L} = \langle u, \psi^E_{0,(j,a)} \rangle. \tag{1.15}$$

This is called an eigenfunction expansion of the operator H_0^{ac}; see Section 2.2 for details on the construction of Φ_0.

Formally, we could define the wave functions as,

$$\psi^E_{+,(j,a)} := W_+(H, H_0)\psi^E_{0,(j,a)}, \quad \psi^E_{-,(j,a)} := W_-(H, H_0)\psi^E_{0,(j,a)}, \tag{1.16}$$

in Section 4.3, see Def. 4.21, we give the precise definition of the wave function. Then, Eq. (1.13) together with the intertwining property (Eq. (1.5)) implies that $\psi^E_{\pm,(j,a)}$ are solutions to the generalized eigenvalue problem:

$$\tau_H \psi^E_{\pm,(j,a)} = E \psi^E_{\pm,(j,a)}.$$

Moreover, Eq. (1.15) implies that $\Phi_0 W^*_\pm$ provides an eigenfunction expansion of the absolutely continuous part of the operator H, denoted by H^{ac}.

Define now the operator

$$S(E) : L^2(\{(j,a)\}, \mathbb{C}) \to L^2(\{(j,a)\}, \mathbb{C}),$$

such that

$$\langle e_i, \psi^E_{-,\cdot}(n) \rangle_{\mathbb{C}^L} \mapsto \langle e_i, \psi^E_{+,\cdot}(n) \rangle_{\mathbb{C}^L}. \tag{1.17}$$

Then, using Eq. (1.15), we obtain

$$S(E)(\Phi_0 W_-(H,H_0)^* u)(E) = (\Phi_0 W_+(H,H_0)^* u)(E),$$

or equivalently,

$$S(E)(\Phi_0 u)(E) = (\Phi_0 W_+(H,H_0)^* W_-(H,H_0)u)(E) = (\Phi_0 S(H,H_0)u)(E). \tag{1.18}$$

Comparing Eq. (1.18) with Eq. (1.8), we conclude that $S(E)$ is the scattering matrix.
In Section 4, we formalize the above ideas by constructing an eigenfunction expansion of H^{ac} (see Section 4.2) and proving its relation with the wave operators (see Proposition 4.36). From this point, we adopt the representation of the scattering matrix given by Eq. (1.17). To proceed, we must find the wave functions given formally by Eq. (1.16), which are solutions to the Lippmann-Schwinger equation:

$$\psi^E_{\pm,(j,a)} := \psi_{0,(j,a)} + R(E \pm i0)\, V\, \psi^E_{\pm,(j,a)}, \tag{1.19}$$

where $R(E \pm i0)$ denotes the boundary values of the resolvent.
Since we can express the boundary values of the resolvent as an integral operator, Eq. (1.19), after appropriate normalization to ensure that the solution behaves like the free solution at $\pm\infty$, leads to an integral equation of Volterra type. Such equations can be solved by successive approximations, the resulting solutions are known as Jost solutions. Based on Eq. (1.17), we define the scattering matrix in terms of the Jost solutions and analyze its properties using the integral equation.
To make the above more concrete, in Section 2 we see that the discrete Laplacian H_0 is absolutely continuous and that its absolutely continuous spectrum is $\sigma_{\mathrm{ac}}(H_0) = [-2,2]$. Moreover, if the free wave functions $\psi_{0,(j,a)}$ are normalized to equal one at $n = 0$, they are called Jost free solutions, and take the form

$$u^E_{0,(j,a)}(n) = \lambda(E)^{a\cdot n} e_j, \quad a \in \{1,-1\}, \quad j \in \{1,\dots,L\}, \tag{1.20}$$

where $\lambda(E) \in \mathbb{C} \setminus \{0\}$ solves the equation $\lambda(E) + \lambda(E)^{-1} = E$ (see Definition 2.4).
For notational convenience, we group the solutions $u^E_{0,(j,\pm 1)}(n)$ as columns of the matrix-valued solutions:

$$u^E_{0,+}(n) = \lambda(E)^n \mathbf{1}, \qquad u^E_{0,-}(n) = \lambda(E)^{-n} \mathbf{1}, \tag{1.21}$$

where $\mathbf{1} \in \mathbb{C}^{L\times L}$ denotes the identity matrix. In Section 2.1, we show that the boundary values of the resolvent are given by

$$R_0(E \pm i0) : L^1(\mathbb{Z}) \to L^\infty(\mathbb{Z}),$$

$$(R_0(E \pm i0)u)(n) = \sum_{m\in\mathbb{Z}} G_0^{E\pm i0}(n,m)\, u(m), \tag{1.22}$$

where

$$G_0^{E\pm i0}(n,m) = \nu^E \lambda(E)^{\pm|n-m|}, \qquad \nu^E := \frac{1}{\lambda(E) - \lambda(E)^{-1}}. \tag{1.23}$$

Using this, the Lippmann-Schwinger equation becomes

$$
\begin{aligned}
u_{+,up}^E(n) &= u_{+,0}^E(n)N + (R_0(E+i0)Vu_{+,up}^E)(n) \\
&= \lambda(E)^n N + \nu^E \sum_{m=n+1}^{\infty} \lambda(E)^{m-n}(Vu_{+,up}^E)(m) + \nu^E \sum_{m=-\infty}^{n} \lambda(E)^{n-m}(Vu_{+,up}^E)(m),
\end{aligned}
\tag{1.24}
$$

where $N \in \mathbb{C}^{L\times L}$ is chosen so that

$$
\lim_{n\to+\infty} \lambda(E)^{-n}\left(u_{+,up}^E(n) - u_{+,0}^E(n)\right) = 0. \tag{1.25}
$$

Assume $V : L^\infty(\mathbb{Z}) \to L^1(\mathbb{Z})$. Multiplying Eq. (1.24) by $\lambda(E)^{-n}$ and taking the limit as $n \to +\infty$, we obtain (using Eq. (1.25))

$$
\mathbf{1} = \lim_{n\to+\infty} \lambda(E)^{-n} u_{+,up}^E(n) = N + \nu^E \sum_{m\in\mathbb{Z}} \lambda(E)^{-m}(Vu_{+,up}^E)(m).
$$

Solving for N and substituting into Eq. (1.24), we obtain a Volterra-type integral equation:

$$
u_{+,up}^E(n) = \lambda(E)^n + \sum_{m=n+1}^{\infty} \nu^E \left(\lambda(E)^{m-n} - \lambda(E)^{m-n}\right)(Vu_{+,up}^E)(m). \tag{1.26}
$$

This equation can be extended analytically for $z \in \mathbb{C}$, providing solutions with the asymptotic behavior described in Eq. (1.25). These are the Jost solutions, we introduce them below. Their existence is established in Section 3.1, and they serve as the fundamental tool in our analysis of the scattering matrix.

Definition 1.7 (Jost Solutions). *For every $z \in \mathbb{C}$, let u_+^z and u_-^z denote the $\mathbb{C}^{L\times L}$-valued solutions to the generalized eigenvalue problem:*

$$
\tau_H u_+^z = z u_+^z, \qquad \tau_H u_-^z = z u_-^z, \tag{1.27}
$$

satisfying the asymptotics

$$
u_+^z(n) = \lambda(z)^n(\mathbf{1} + o(1)) \quad \text{as } n \to +\infty, \qquad u_-^z(n) = \lambda(z)^{-n}(\mathbf{1} + o(1)) \quad \text{as } n \to -\infty. \tag{1.28}
$$

For $z = 2$, we denote by $v_\pm$ the solutions of

$$
\tau_H v_\pm = 2v_\pm,
$$

with asymptotic behavior

$$
v_\pm(n) = n(\mathbf{1} + o(1)) \quad \text{as } n \to \pm\infty. \tag{1.29}
$$

Definition 1.8. *For $z \in \mathbb{C} \setminus [-2,2]$, let w_+^z and w_-^z be the solutions of*

$$
\tau_H w_+^z = z w_+^z, \qquad \tau_H w_-^z = z w_-^z,
$$

satisfying

$$
w_+^z(n) = \lambda(z)^{-n}(\mathbf{1} + o(1)) \quad \text{as } n \to +\infty, \qquad w_-^z(n) = \lambda(z)^n(\mathbf{1} + o(1)) \quad \text{as } n \to -\infty. \tag{1.30}
$$

Remark 1.9. *For $z \in \mathbb{C} \setminus [-2,2]$, the asymptotic behaviors in Eqs. (1.28) and (1.30) imply that the columns of $(u_+^z \,|\, w_+^z)$ are linearly independent and span the solution space. That is, any solution u of $\tau_H u = zu$ can be written as*

$$u = u_+^z \alpha + w_+^z \beta,$$

for some $\alpha, \beta \in \mathbb{C}^{L\times L}$. A similar statement holds for $(u_-^z \,|\, w_-^z)$.

Definition 1.10. *For $E \in [-2,2]$, we define*

$$u_{\pm,\mathrm{out}}^E := u_\pm^z\big|_{z=E}.$$

Moreover, the following limit exists:

$$u_{\pm,\mathrm{in}}^E(n) := \lim_{\substack{z\to E\\ \Im(z)<0}} u_\pm^z(n).$$

These satisfy the asymptotic behaviors:

$$u_{+,\mathrm{out}}^E(n) = \lambda(E)^n(\mathbf{1}+o(1)) \quad \text{as } n \to +\infty, \tag{1.31}$$
$$u_{-,\mathrm{out}}^E(n) = \lambda(E)^{-n}(\mathbf{1}+o(1)) \quad \text{as } n \to -\infty, \tag{1.32}$$

$$u_{+,\mathrm{in}}^E(n) = \lambda(E)^{-n}(\mathbf{1}+o(1)) \quad \text{as } n \to +\infty, \tag{1.33}$$
$$u_{-,\mathrm{in}}^E(n) = \lambda(E)^{n}(\mathbf{1}+o(1)) \quad \text{as } n \to -\infty. \tag{1.34}$$

Remark 1.11. *For $E \in [-2,2]$, the asymptotic behavior implies that the columns of the matrix $(u_{+,\mathrm{out}}^E \; u_{+,\mathrm{in}}^E)$ are linearly independent and form a basis of the solution space. Thus, any solution u of $\tau_H u = Eu$ can be written as*

$$u = u_{+,\mathrm{out}}^E \alpha + u_{+,\mathrm{in}}^E \beta,$$

for some $\alpha, \beta \in \mathbb{C}^{L\times L}$. A similar statement holds for $(u_{-,\mathrm{out}}^E \,|\, u_{-,\mathrm{in}}^E)$.

1.4 Scattering matrix

Due to the asymptotic behavior of the Jost solutions (see Equation (1.28)), the columns of the matrices $(u_{+,out}^E, u_{+,in}^E)$ and $(u_{-,in}^E, u_{-,out}^E)$ are linearly independent for $E \in (-2,2)$ and, therefore, they form a basis of solutions. The same holds true for $(u_\pm^2, v_\pm)$. This implies that there are matrices $M_\pm^E, N_\pm^E \in \mathbb{C}^{L\times L}$ such that

$$u_{+,out}^E = u_{-,in}^E M_+^E + u_{-,out}^E N_+^E, \qquad u_{-,out}^E = u_{+,out}^E N_-^E + u_{+,in}^E M_-^E. \tag{1.35}$$

Moreover, it will be proved that the matrices $M_\pm^E$ have a meromorphic continuation to $\mathbb{C} \setminus [-2,2]$ (see Equation (3.54)). Assuming that $M_\pm^E$ are invertible, we can rewrite these equations as

$$u_{+,out}^E T_+^E = u_{-,in}^E + u_{-,out}^E R_+^E, \qquad u_{-,out}^E T_-^E = u_{+,in}^E + u_{+,out}^E R_-^E, \tag{1.36}$$

where

$$T_{\pm}^{E} = (M_{\pm}^{E})^{-1}, \qquad\qquad R_{\pm}^{E} = N_{\pm}^{E}(M_{\pm}^{E})^{-1}, \tag{1.37}$$

are the transmission and reflection coefficients, respectively. The interpretation of (1.36) in the case that $\lambda(E) \in \mathbb{S}^1 \setminus \{-1, 1\}$, corresponding a wave traveling to the right, is the following (we only describe the first equation in (1.36)): the incoming wave $u_{-,in}^{E}$ produces the outgoing wave $u_{+,out}^{E} T_{+}^{E}$ traveling to the right (*i.e.*, a transmitted wave) and the outgoing wave $u_{-,out}^{E} R_{+}^{E}$ traveling to the left (*i.e.*, a reflected wave). The relation between transmitted and reflected waves is described by the scattering matrix.

Definition 1.12. *For any $E \in (-2, 2)$, the scattering matrix $\mathcal{S}^E \in \mathcal{M}_{2L\times 2L}$ is defined by*

$$\mathcal{S}^{E} = \begin{pmatrix} T_{+}^{E} & R_{-}^{E} \\ R_{+}^{E} & T_{-}^{E} \end{pmatrix} = \begin{pmatrix} (M_{+}^{E})^{-1} & N_{-}^{E}(M_{-}^{E})^{-1} \\ N_{+}^{E}(M_{+}^{E})^{-1} & (M_{-}^{E})^{-1} \end{pmatrix}.$$

Notice that matrices $M_{\pm}^{E}$ are indeed invertible, see Proposition 3.16. In the case that $\lambda(E) \in \mathbb{S}^1 \setminus \{-1, 1\}$ represents a wave traveling to the right, then $u_{\pm,in}^{E}$ are incoming and $u_{\pm,out}^{E}$ are outgoing. In this case, the scattering matrix expresses the incoming Jost solutions $u_{\pm,in}^{E}$ in terms of the outgoing ones $u_{\pm,out}^{E}$:

$$\begin{pmatrix} u_{-,in}^{E} & -u_{+,in}^{E} \end{pmatrix} = \begin{pmatrix} u_{+,out}^{E} & -u_{-,out}^{E} \end{pmatrix} \mathcal{S}^{E} . \tag{1.38}$$

1.5 Results

We now briefly present the main results established in this work. As a preliminary observation, we note that the map $E \mapsto M_{\pm}^{E}$ can be analytically extended to a function $z \mapsto M_{\pm}^{z}$ defined on $\mathbb{C} \setminus [-2, 2]$, which is continuous up to the boundary $\overline{\mathbb{C}^{+}}$, see Section 3.2. In Section 3.4, we analyze the zeros of this extension and relate them to the eigenvalues of the Hamiltonian H. The main result of that section is the following:

Proposition 1.13. *The set of eigenvalues of H is a finite subset of $(-\infty, -2) \cup (2, \infty)$. Moreover, suppose that $E \in \mathbb{R} \setminus [-2, 2]$ is an eigenvalue of H. Define $n_E = \dim(\mathrm{Ker}(M_{-}^{E})) = \dim(\mathrm{Ker}(H - E))$. Then there exists a nonzero complex constant $c_E \in \mathbb{C} \setminus \{0\}$ such that*

$$\det(M_{-}^{z}) = (z - E)^{n_E} \left(c_E + \mathcal{O}(|z - E|)\right), \qquad z \to E.$$

In Section 3.6, we study the limit of $\mathcal{S}^{E}$ as $E \to [-2, 2]$. The main result of this section is the following, see Theorem 3.51

Theorem 1.14. *The limits*

$$T_{\pm}^{2} := \lim_{z \to 2} T_{\pm}^{2} \tag{1.39}$$

exist, where the limits are taken in the complex $z \in \mathbb{C}$, and they admit explicit expressions (see Equation (3.155)). The kernels and images of $T_{\pm}^{2}$ can be computed explicitly (see Equation (3.156)), and are closely related to the half-bound states.

Similarly, the limits

$$R^2_\pm := \lim_{\substack{E\to 2\\ E\in(-2,2)}} R^E_\pm \tag{1.40}$$

of the reflection coefficients exist, and also have explicit expressions (see Equation (3.158)). The kernels and images of $\mathbf{1} - R^1_\pm$ *can be explicitly determined (see Equation (3.159)).*

Section 3.10 is dedicated to the proof of the Levinson Theorem, which we state as follows:

Theorem 1.15 (Levinson Theorem)**.** *Let* H *be a Jacobi operator as defined in Definition 1.1. Assume that* $H - H_0$ *has finite first moment. Then the Hamiltonian* H *admits only a finite number* J_b *of eigenvalues* $E_1, \dots, E_{J_b} \in \mathbb{R}$ *(counted with multiplicity), all lying outside the interval* $[-2, 2]$*. Additionally, at the thresholds* $E = \pm 2$*, there exist* $J^\pm_h \leq L$ *linearly independent bounded solutions of* $Hu = \pm 2u$*, referred to as half-bound states. Setting* $J_h = J^-_h + J^+_h$*, the following identity holds:*

$$\frac{1}{2\pi i} \lim_{\varepsilon\downarrow 0} \int_{-2+\varepsilon}^{2-\varepsilon} \mathrm{Tr}\left((\mathcal{S}^E)^* \frac{d}{dE} \mathcal{S}^E \right) dE = L - \left(J_b + \frac{1}{2} J_h \right).$$

2 Basic spectral properties of the free Hamiltonian

Just as the Laplacian operator defined in $\mathbb{R}$, a standard method for studying the discrete Laplacian H_0 is to express it as a multiplication operator via a unitary transform. As in the continuous case, this can be achieved using the Fourier transform. However, an important distinction between the continuous and discrete Laplacians lies in the structure of the momentum space. In the continouos case the momentum space corresponds to the real numbers $\mathbb{R}$, while in the discrete case, the momentum space corresponds to the one-dimensional torus $\mathbb{T}^1$.

There are several equivalent ways to represent the one-dimensional torus. The standard representation is

$$\mathbb{T}^1 := \mathbb{R}/2\pi\mathbb{Z},$$

which is equivalent to the unit circle in the complex plane,

$$\mathbb{S}^1 := \{w \in \mathbb{C} : |w| = 1\}.$$

An explicit diffeomorphism between these two descriptions is given by

$$\phi : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{S}^1, \quad \phi(k) = e^{ik},$$

where we write $k \equiv [k] \in \mathbb{R}/2\pi\mathbb{Z}$.

In studying the analytic properties of the scattering matrix, it becomes important to extend certain formulas from the real line to the complex plane in order to use not only the differential structure but the complex one. For this reason, it is helpful to view the torus as a real-analytic manifold embedded in a complex-analytic one. To that end, we also consider

$$\mathbb{C}/2\pi\mathbb{Z} \equiv \mathbb{T}^1 + i\mathbb{R},$$

which can be visualized as a cylinder containing $\mathbb{T}^1$ as its "central slice." The diffeomorphism ϕ extends naturally to an isomorphism between complex-analytic manifolds:

$$\phi : \mathbb{C}/2\pi\mathbb{Z} \to \mathbb{C} \setminus \{0\}.$$

In this extended setting, the unit circle $\mathbb{S}^1 \subset \mathbb{C} \setminus \{0\}$ corresponds to the real torus, while the interior of the circle (i.e., $0 < |z| < 1$) is mapped to the upper part of the cylinder, and the exterior (i.e., $|z| > 1$) to the lower part.

For practical purposes in this work, we adopt the representation $\mathbb{T}^1 \equiv \mathbb{S}^1 \subset \mathbb{C} \setminus \{0\} \equiv \mathbb{T}^1 \times i\mathbb{R}$. Note that $\mathbb{S}^1$ carries a natural measure μ, which is the pushforward of the Lebesgue measure on $\mathbb{R}$ under the composition $\phi \circ q$, where

$$q : [0, 2\pi] \to \mathbb{R}/2\pi\mathbb{Z}, \quad q(k) = [k].$$

We denote by $\mathcal{F} : L^2(\mathbb{Z}, \mathbb{C}^L) \to L^2(\mathbb{S}^1, \mathbb{C}^L)$ the Fourier transform, which is the operator defined on sequence with finite support by the formula

$$(\mathcal{F}u)(w) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} w^n u(n), \tag{2.1}$$

and then extended it by unitarity. It inverse $\mathcal{F}^{-1}$ is given by

$$(\mathcal{F}^{-1}f)(n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{S}^1} w^{-n} f(w)\; d\mu(w). \tag{2.2}$$

A direct computation shows that if $u \in (\mathbb{C}^L)^{\mathbb{Z}}$ has finite support then

$$(\mathcal{F}H_0 u)(w) = (w + 1/w)(\mathcal{F}u)(w),\ w \in \mathbb{S}^1. \tag{2.3}$$

Then, the Fourier transform allows us to write the discrete Laplacian as a multiplicacion operator by the function

$$\mathbb{S}^1 \ni w \mapsto w + 1/w.$$

This function can be naturally extend it to the whole plane $\mathbb{C} \setminus \{0\}$. We adopt the following notation,

Notation 2.1. *We denote by* $\mathcal{J} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, *the function given by*

$$\mathcal{J}(w) = w + 1/w.$$

With this notation, Eq. (2.3) together with the fact that finite support sequence are dense in L^2 give us

$$\mathcal{F}H_0\mathcal{F}^* = \mathcal{J},$$

here $\mathcal{J}$ denotes the multiplication operator by the function $\mathcal{J}$. As the Fourier transform is unitary, then the spectrum of H_0 is $\sigma(H_0) = [-2, 2]$ and it is purely absolutely continuous. The essential spectrum of H is thus $[-2, 2]$ (see Section XIII.4 in [24]).

Proposition 2.2. *Spectrum of* H_0 *is essential and absolutely continuous is given by* $[-2, 2]$.

2.1 Boundary values of free resolvent

For the subsequent analysis of the operator H and H_0, it is important to study the level sets of the function $\mathcal{J}\big|_{\mathbb{S}^1}$, instead we study the level sets of the extensions $\mathcal{J}$. For $z \in \mathbb{C} \setminus \{2, -2\}$ the equation $\mathcal{J}(w) = z$ has two different solutions. If w is a solution, the other one is $1/w$. Consider,

$$w_\pm(z) := \frac{1}{2}(z \pm \sqrt{z-2}\sqrt{z+2}), \tag{2.4}$$

here we are taking the branch of the square root $\sqrt{z} = \sqrt{|z|}e^{i\theta/2}$, $-\pi < \theta = \arg(z) \leq \pi$. One can verify that $w_+(z)w_-(z) = 1$, and $w_+(z) + w_-(z) = z$, then $w_\pm(z)$ are the solutions of $J(w) = z$. By definition, $w_\pm$ is analytic on $\mathbb{C} \setminus (-\infty, 2]$. However, if one takes the branch of square root $sqr(z) = \sqrt{|z|}e^{i\theta/2}$, $0 \leq \theta = \arg(z) < 2\pi$, one can verify that for $z \in \mathbb{C}$ with $\Re(z) < -2$,

$$w_\pm(z) = \frac{1}{2}(z \pm sqr(z-2)sqr(z+2)),$$

which proves that $w_\pm$ has analytic extension on $\mathbb{C} \setminus [-2, 2]$. If $w_\pm(z)$ is unitary, then $\overline{w_+(z)} = w_+(z)^{-1} = w_-(z)$ which using Eq. (2.4) implies that $z \in \mathbb{R}$, and together with $|w_+(z)| = 1$

implies $z \in [-2,2]$. Coversely, if $z \in [-2,2]$ then $|w_{\pm}(z)| = 1$. Then $w_{\pm}(z)$ is unitary if and only if $z \in [-2,2]$. Moreover, for $z = 2$, $w_{\pm}(2) = 1$ and for $z = -2$, $w_{\pm}(-2) = -1$. It follows then that for $z \in \mathbb{C} \setminus [-2,2]$ one of the solutions has modulus strictly bigger than one and the other one strictly less than one. For real $z \notin [-2,2]$ it is straightforward that $|w_-(z)| < 1$ and $|w_+(z)| > 1$. Since both w_+ and w_- are analytic in $\mathbb{C} \setminus [-2,2]$. Then a connexity argument implies that $|w_-(z)| < 1$ for every $z \in \mathbb{C} \setminus [-2,2]$. It follows that $\mathcal{J}|_{\mathbb{D}\setminus\{0\}} : \mathbb{D}\setminus\{0\} \to \mathbb{C}\setminus[-2,2]$, is a bijection. We denote its inverse, by $\lambda : \mathbb{C} \setminus [-2,2] \to \mathbb{D} \setminus \{0\}$

$$\lambda(z) = w_-(z). \tag{2.5}$$

Remark 2.3. *By definition (see Not. 2.1 and Eq. (2.5)), the function* $\lambda : \mathbb{C} \setminus [-2,2] \to \mathbb{D} \setminus \{0\}$ *is analytic. Moreover, if we take the branch of the square root given by* $sqr_+(z) := \sqrt{|z|}e^{i\theta/2}$, $0 \leq \theta = \arg(z) < 2\pi$. *Then, Eqs. (2.5) and (2.4) imply that*

$$\lambda(z) = \frac{1}{2}(z - sqr_+(z-2)\sqrt{z+2}), \ \Im(z) > 0. \tag{2.6}$$

Since the R.H.S. of Eq. (2.6) is analytic on $\mathbb{C} \setminus (-\infty, -2] \cup [2, \infty]$, *this proves that for* $E \in (-2,2)$ *the following limit exists*

$$\lambda(E + i0) := \lim_{\substack{z \to E \\ \Im(z) > 0}} \lambda(z) = \frac{1}{2}(E - i\sqrt{4 - E^2}). \tag{2.7}$$

In the same manner, taking now $sqr_-(z) = \sqrt{|z|}e^{i\theta/2}$, $-2\pi < \theta = \arg(z) \leq 0$, *we obtain that*

$$\lambda(E - i0) := \lim_{\substack{z \to E \\ \Im(z) < 0}} \lambda(z) = \frac{1}{2}(E + i\sqrt{4 - E^2}). \tag{2.8}$$

Moreover, using the general fact $\lim_{z \to 0} \sqrt{z} = 0$, *we obtain*

$$\lim_{z \to \pm 2} \lambda(z) = \pm 1. \tag{2.9}$$

Let $z \in \mathbb{C} \setminus [-2,2]$. *By defintion we obtain*

$$\overline{z} = \overline{\lambda(z)} + \frac{1}{\overline{\lambda(z)}}.$$

In particular $\{\overline{\lambda(z)}, 1/\overline{\lambda(z)}\}$ *are the solution of* $\mathcal{J}(w) = \overline{z}$. *Since* $|\overline{\lambda(z)}| = |\lambda(z)| < 1$ *then by definition we obtain*

$$\lambda(\overline{z}) = \overline{\lambda(z)}, \ z \in \mathbb{C} \setminus [-2,2]. \tag{2.10}$$

Definition 2.4. *From now on, we extend the function* λ *to* $[-2,2]$ *by the upper limit, see Rem. 2.3 and Eq. (2.7). Then,* $\lambda : \mathbb{C} \to \mathbb{D} \setminus \{0\} \cup \mathbb{S}^1_-$ *is analytic on* $\mathbb{C} \setminus [-2,2]$, *with analytic extension around* $E \in (-2,2)$, *and satisfies Eq. (2.9) and*

$$\lim_{\substack{z \to E \\ \Im(z) < 0}} \lambda(z) = \lambda(E)^{-1}. \tag{2.11}$$

Taking derivative in the equation $\lambda(z)+\lambda(z)^{-1} = z$, we obtain that $\lambda'(z)$ satisfies the following equation

$$\lambda'(z) = \frac{\lambda(z)^2}{\lambda(z)^2-1}, z \in \mathbb{C}. \tag{2.12}$$

Through the text we use the following notation,

$$\nu^z := \frac{\lambda'(z)}{\lambda(z)} = \frac{\lambda(z)}{\lambda(z)^2-1} = \frac{1}{\lambda(z)-\lambda(z)^{-1}}, \ z \in \mathbb{C} \setminus \{-2, 2\}. \tag{2.13}$$

Now we can obtain an explicit form of the Green function in terms of the function λ. We have the following lemma that relates the integration with respect the measure μ and the complex integration.

Proposition 2.5. *Let $f : U \subset \mathbb{C} \to \mathbb{C}^L$ be a continuous function, with U an open set such that $\mathbb{S}^1 \subset U$. Consider $f| : \mathbb{S}^1 \to \mathbb{C}^L$ the restriction of f to S^1 then*

$$\int_{\mathbb{S}^1} f| \; d\mu = \frac{1}{i}\int_{\mathbb{S}^1} \frac{f(z)}{z} dz.$$

Proof. Let us denote by $l : [0, 2\pi] \to \mathbb{S}^1, \ l(\theta) = e^{i\theta}$. By definition and change of measure we have

$$\int_{\mathbb{S}^1} f| \; d\mu = \int_0^{2\pi} (f| \circ l)(\theta) d\theta = \frac{1}{i}\int_0^{2\pi} (f| \circ l)(\theta) e^{-i\theta} l'(\theta) \; dx = \frac{1}{i}\int_{\mathbb{S}^1} \frac{f(z)}{z} \; dz.$$

□

Definition 2.6. *For $z \notin [-2, 2]$, we denote by $R_0(z)$ the resolvent operator of H_0, namely*

$$R_0(z) := (H_0 - z)^{-1}.$$

Definition 2.7. *For $z \notin [-2, 2]$ and $n, m \in \mathbb{Z}$ we define the Green function, $G_0^z(n, m) \in \mathbb{C}^{L\times L}$ as*

$$G_0^E(n, m)_{i,j} := \langle R_0(E)\delta_m e_j, \delta_n e_i\rangle.$$

Where $\delta_n : \mathbb{Z} \to \mathbb{C}^{L\times L}$ denotes the Knronecker's delta and $\{e_j\}$ the standar basis of $\mathbb{C}^L$.

Lemma 2.8. *Let $z \notin [-2, 2]$, the Green function $G_0^z(n, m)$ has the following explicit form*

$$G_0^z(n, m) = \nu^z \lambda(z)^{|n-m|}\mathbf{1}.$$

Proof. Using unitarity of $\mathcal{F}$ we obtain,

$$G_0^z(n, m)_{i,j} = \langle R_0(z)\delta_m e_j, \delta_n e_i\rangle_{L^2} = \langle \mathcal{F} R_0(z)\mathcal{F}^{-1}\mathcal{F}\delta_m e_j, \mathcal{F}\delta_n e_i\rangle_{L^2} = \langle M_g \mathcal{F}\delta_m e_j, \mathcal{F}\delta_n e_i\rangle_{L^2}.$$

Where $(M_g f)(w) = \frac{1}{w+1/w-z} f(w)$, and $(\mathcal{F}\delta_n e_i)(w) = \frac{1}{\sqrt{2\pi}} w^n e_i$. Then we have,

$$G_0^z(n, m)_{i,j} = \frac{\delta_{i,j}}{2\pi}\int_{S^1} \frac{1}{w+1/w-z} w^{n-m} \; d\mu(z) = \frac{\delta_{i,j}}{2\pi i}\int_{S^1} \frac{1}{w+1/w-z} w^{n-m-1} dw. \tag{2.14}$$

Suppose $n \geq m+1$. Note that the R.H.S. is the residue of a meromorphic function with pole at $\lambda(z) \in \mathbb{D}$. If we write $z = \lambda(z) + \frac{1}{\lambda(z)}$ we can compute the residue and obtain,

$$G_0^z(n,m)_{i,j} = \delta_{i,j} \nu^z \lambda(z)^{n-m-1}.$$

If $n = m$, then it also has a residue at $w = 0$, computing this residue explicitly we obtain

$$G_0^z(n,m)_{i,j} = \delta_{i,j} \nu^z.$$

If $m > n$, we use the previous computation and that $G_0^z(n,m) = G_0^{\bar{z}}(m,n)^*$. $\square$

From the above Lemma it is clear that the Green function has limit up to the boundary. Namely, for all $E \in (-2,2)$ we have

$$G_0^{E \pm i0}(n,m) := \lim_{\substack{z \to E \\ \pm \Im(z) > 0}} G_0^z(n,m) = \pm \nu^E \lambda(E)^{\pm |n-m|}. \tag{2.15}$$

Definition 2.9. *For $E \in (-2,2)$, we define the integral operators*

$$R_0(E \pm i0) : L^1(\mathbb{Z}) \to L^\infty(\mathbb{Z}),\ (R_0(E \pm i0)u)(n) := \sum_{n \in \mathbb{Z}} G_0^{E \pm i0}(n,m) u(m).$$

Proposition 2.10. *The operator valued function $R_0 : \mathbb{C} \setminus [-2,2] \to \mathcal{B}(L^1(\mathbb{Z}), L^\infty(\mathbb{Z}))$, has limit up to the boundary from above and below in the weak operator topology. Moreover, its limit is given by $R_0(E \pm i0)$, respectively. Namely, for all $u \in L^1$ and for all $\rho \in (L^\infty)^*$ one has*

$$\lim_{\substack{z \to 0 \\ \pm \Im(z) > 0}} \rho(R_0(z)u - R(E \pm i0)u) \to 0. \tag{2.16}$$

Proof. We make the proof for the sign '+', the sign '-' is analogues. Taking the representation of $R_0(z)$ and $R(E + i0)$ as integral operators we obtain

$$(R_0(z) - R(E \pm i0)u)(n) = \sum_{m \in \mathbb{Z}} (\nu^z \lambda(z)^{|n-m|} - \nu^E \lambda(E)^{|n-m|}) u(n). \tag{2.17}$$

Equation above together with the fact that $|\lambda(z)| \leq 1$ and Lebesgue convergence Theorem implies that $\delta_{n,j}(R_0(z)u - R(E \pm i0)u) \to 0$, where $\delta_{n,j}(u) = \langle u(n), e_j \rangle_{\mathbb{C}^L}$. Also note that for a compact neighborhood of E, $K \subset \mathbb{C} \setminus \{2, -2\}$ one has

$$\|(R_0(z) - R(E + i0))u\|_{L^\infty} \leq C \|u\|_{L^1}.$$

Using this above and the fact that $\delta_{n,j}$ is dense in $(L^\infty)^*$ we conclude the proof. $\square$

2.2 Eigenfunction expansion

In this section we contruct the unitary operator Φ_0 describe it in Section 1.3, see Eqs. (1.14) and (1.15). This operator is really important because it gives the conection beetwen the Jost solutions and the wave operators, we roughly explain this connection in Section 1.3 and we treat we all the details in Section 4. Note that the function $\lambda : (-2, 2) \to \mathbb{S}^1$, (see Def. 2.4) allows us to obtain a parametrization of the level sets of $\mathcal{J}$. We consider the function $\theta : (-2, 2) \times \{-1, 1\} \to \mathbb{S}^1$,

$$\theta(E, j) = \lambda(E)^j, \ j \in \{-1, 1\}. \tag{2.18}$$

Note that in particular $\theta(E, j) \in \mathcal{J}^{-1}(E)$.

Definition 2.11. *Let $f \in C(\mathbb{S}^1, \mathbb{C}^L)$ continuous function. We define $\Gamma_0 f : (-2, 2) \to L^2(\{-1, 1\}, \mathbb{C}^L)$, the following function*

$$(\Gamma_0 f)(E)(j) = f(\theta(E, j)) = f(\lambda(E)^j), \ j \in \{-1, 1\}.$$

Note that $(\Gamma_0 f)(E)$ is in some sense the restriction of the function f to the level set $\mathcal{J}^{-1}(E)$. For the sake of notation it is more convinient to think the space $L^2(\{-1, 1\}, \mathbb{C}^L)$ as $\mathbb{C}^{2L}$.

Remark 2.12. *The space $L^2(\{-1, 1\}, \mathbb{C}^L)$ can be naturally identified wiht $\mathbb{C}^{2L}$. One just need to consider the isomorphism $L^2(\{-1, 1\}, \mathbb{C}^L) \ni f \mapsto (f(-1), f(1)) \in \mathbb{C}^{2L}$. From now on, we identify $L^2(\{-1, 1\}, \mathbb{C}^L) \equiv \mathbb{C}^{2L}$. In this way, $\Gamma_0 f \in \mathbb{C}^{2L}$ is given by,*

$$(\Gamma_0 f)(E) = \begin{pmatrix} f(\lambda(E)^{-1}) \\ f(\lambda(E)) \end{pmatrix}. \tag{2.19}$$

The aim now is to extend $\Gamma_0 f$, to all $f \in L^2(\mathbb{S}^1, \mathbb{C}^L)$. For this, one should consider a normalization. Let $f \in C_0(\mathbb{S}^1 \setminus \{1, -1\}, \mathbb{C}^L)$ be a continuous function with compact support, let us say $supp(f) \subset \lambda([-t, t]) \cup \lambda^{-1}([-t, t])$, for some $t \in (0, 2)$. Then, $\lambda : (-t, t) \to \lambda([-t, t]) \subset \mathbb{S}^1 \cap \mathbb{C}^-$ and $\lambda^{-1} : (-t, t) \to \mathbb{S}^1 \cap \mathbb{C}^+$ are parametrization of the correspondent arcs, and notice that λ^{-1} change the orientation. Taking this above in account, together with Prop. 2.5 we compute

$$\begin{aligned} \int_{\mathbb{S}^1} \|f(k)\|^2 \, d\mu(k) &= \frac{1}{i} \int_{S^1} \frac{\|f(z)\|^2}{z} dz = \frac{1}{i} \int_{S^1_-} \frac{\|f(z)\|^2}{z} dz + \frac{1}{i} \int_{S^1_+} \frac{\|f(z)\|^2}{z} dz \\ &= \frac{1}{i} \int_{-2}^{2} \|f(\lambda(E))\|^2 \frac{\lambda(E)}{\lambda(E)^2 - 1} dE + \frac{1}{i} \int_{-2}^{2} \|f(\lambda(E)^{-1})\|^2 \frac{\lambda(E)}{\lambda(E)^2 - 1} dE \\ &= \frac{1}{i} \int_{-2}^{2} \nu^E \|(\Gamma_0 f)(E)\|^2_{\mathbb{C}^{2L}} dE. \end{aligned} \tag{2.20}$$

Moreover, Eq. (2.7) imply that for $t \in (0, 2)$

$$\int_{-t}^{t} |\nu^E| \, dE = \int_{-t}^{t} \frac{1}{\sqrt{4 - E^2}} \, dE \leq \pi.$$

Then the function $\nu^E \in L^1([-2, 2])$ and then Eq. (2.20) can be extended for all $f \in C(\mathbb{S}^1, \mathbb{C}^L)$ this leds us the following definition.

Definition 2.13. *For all $f \in C(\mathbb{S}^1, \mathbb{C}^L)$ continuous function. We define $\mathcal{F}_0 f : (-2,2) \to \mathbb{C}^{2L}$, as the following function*

$$(\mathcal{F}_0 f)(E) = \sqrt{-i\nu^E}(\Gamma_0 f)(E). \tag{2.21}$$

Def. 2.13 and Eq. (2.20) imply that for $f \in C(\mathbb{S}^1, \mathbb{C}^L)$ continuous function one has,

$$\|\mathcal{F}_0 f\|_{L^2((-2,2),\mathbb{C}^{2L})} = \|f\|_{L^2(\mathbb{S}^1,\mathbb{C}^L)}. \tag{2.22}$$

Therefore, the linear function $\mathcal{F}_0 : C(\mathbb{S}^1, \mathbb{C}^L) \subset L^2(\mathbb{S}^1) \to L^2((-2,2))$ can be extended to a unitary operator

$$\mathcal{F}_0 : L^2(\mathbb{S}^1, \mathbb{C}^L) \to L^2((-2,2), \mathbb{C}^{2L}).$$

Definition 2.14. *We denote by $\Phi_0 : L^2(\mathbb{Z}, \mathbb{C}^L) \to L^2((-2,2), \mathbb{C}^{2L})$, the unitary operator given by the composition*

$$\Phi_0 = \mathcal{F}_0 \mathcal{F}. \tag{2.23}$$

Explicitly, Φ_0 is given for $u \in L^1(\mathbb{Z}, \mathbb{C}^L)$ by the following formula, (see (2.1), (2.21) and (2.19))

$$(\Phi_0 u)(E) = \sqrt{\frac{-i\nu^E}{2\pi}} \sum_{n\in\mathbb{Z}} \begin{pmatrix} \lambda(E)^{-n} u(n) \\ \lambda(E)^{n} u(n) \end{pmatrix}. \tag{2.24}$$

By definition, Φ_0 is a unitary operator, and satisfies

$$\Phi_0 H_0 \Phi_0^* = M_E, \tag{2.25}$$

where $M_E : L^2((-2,2), \mathbb{C}^{2L}) \to L^2((-2,2), \mathbb{C}^{2L})$ denotes the multiplication operator, namely $(M_E f)(E) = E f(E)$.

Definition 2.15 (Free wave functions)**.** *For all $E \in (-2,2)$ we denote by $\psi^E_{0,\pm} \in (\mathbb{C}^{L\times L})^{\mathbb{Z}}$ the free wave functions as*

$$\psi^E_{0,\pm}(n) = \sqrt{-i\nu^E}\lambda(E)^{\pm n}. \tag{2.26}$$

Note that the columns of the matrix $\psi^E_{0,\pm}$ constitutes an eigenfunction expantion for the operator H_0. Solutions $\psi^E_{0,\pm}$ normalized to be $\mathbf{1}$ at $n = 0$ are called Jost free solutions. Moreover, this solutions admits an extension to $z \in \mathbb{C}$.

Definition 2.16 (Free Jost solutions)**.** *For all $z \in \mathbb{C}$, Free Jost solutions are given by*

$$u^z_{\pm,0}(n) = \lambda(z)^{\pm n}.$$

2.3 Transfer matrix

The difference expression (1.11) has associated a difference equation, the so called generalized eigenvalue equation,

$$\tau_H u = zu, \ \ z \in \mathbb{C}. \tag{2.27}$$

Note that Eq. (2.27) allows solutions $u \in (\mathbb{C}^L)^{\mathbb{Z}}$ and not only square summable ones. Generalized eigenvalue equation implies a second order difference equation given by

$$A_n u(n-1) + B_n u(n) + A_{n+1} u(n+1) = zu(n), \tag{2.28}$$

and due to the invertibility of A_n, Eq. (2.28) allow us to obtain a recurrence relation, namely

$$u(n+1) = (A_{n+1})^{-1}((z - B_n)u(n) - A_n u(n-1)), \ \ n \in \mathbb{Z}. \tag{2.29}$$

$$u(n-1) = (A_n)^{-1}((z - B_n)u(n) - A_{n+1} u(n+1)), \ \ n \in \mathbb{Z}. \tag{2.30}$$

The best way to use the recurrence equation is by using the transfer matrix.

Definition 2.17. *For $n \in \mathbb{Z}$ and $z \in \mathbb{C}$, we denote by $\mathcal{T}^z(n) \in \mathcal{M}_{2L\times 2L}(\mathbb{C})$ the transfer matrix, as the block matrix*

$$\mathcal{T}^z(n) = \begin{pmatrix} (z - B_n)(A_n)^{-1} & -A_n \\ (A_n)^{-1} & 0 \end{pmatrix} \tag{2.31}$$

Here $0 \in \mathcal{M}_{L\times L}(\mathbb{C})$ denotes the matrix with all entries 0.

Definition 2.17 and Eq. (2.29) imply that if $u \in (\mathbb{C}^L)^{\mathbb{Z}}$ or $u \in \mathcal{M}_{L\times L}(\mathbb{C})^{\mathbb{Z}}$ and u satisfies Eq. (2.27) then

$$\mathcal{T}^z(n) \begin{pmatrix} A_n u(n) \\ u(n-1) \end{pmatrix} = \begin{pmatrix} A_{n+1} u(n+1) \\ u(n) \end{pmatrix}. \tag{2.32}$$

Proposition 2.18. *Given two vectors (or matrices) a_0, b_0 and $n \in \mathbb{Z}$, there exists an unique solution of the equation (2.27), such that $u(n) = a_0$, $u(n+1) = b_0$. In particular, the vectorial subspace of $(\mathbb{C}^L)^{\mathbb{Z}}$ which consints in solutions of Eq. (2.27) has dimension $2L$.*

Remark 2.19. *Therefore, by definition for $n, m \in \mathbb{Z}$,*

$$\mathcal{T}^z(n,m) \begin{pmatrix} A_m u(m) \\ u(m-1) \end{pmatrix} = \begin{pmatrix} A_n u(n) \\ u(n-1) \end{pmatrix} \tag{2.33}$$

previous equations show, that any solution of Eq. (2.27) is determinated by two consecutive values.

Now we describe some properties satisfied by the transfer matrix. That are usefull for us later in the study of the operator.

Definition 2.20. *We denote by $\mathcal{J} \in \mathcal{M}_{2L\times 2L}(\mathbb{C})$ the unitary matrix*

$$\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \quad \mathcal{J}^{-1} = \mathcal{J}^* = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{2.34}$$

Here $I, 0$ denotes the identity and zero matrices respectively.

Remark 2.21. *If*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

conjugation with $\mathcal{J}$ *follows the next rule*

$$\mathcal{J} M \mathcal{J}^* = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}. \tag{2.35}$$

One important propriety satisfied by the Transfer matrix is the following

Proposition 2.22. *For all* $z \in \mathbb{C}$ *and* $n \in \mathbb{Z}$*, transfer matrix* $\mathcal{T}^z(n)$ *is invertible. Moreover,*

$$\mathcal{T}^z(n)^{-1} = \mathcal{J} \mathcal{T}^{\bar{z}}(n)^* \mathcal{J}^{-1} \tag{2.36}$$

Proof. One can easily see from Def. 2.17 that (recall that A_n and B_n are self-adjoint matrices)

$$\mathcal{T}^{\bar{z}}(n)^* = \begin{pmatrix} (A_n)^{-1}(z - B_n) & A_n^{-1} \\ -A_n & 0 \end{pmatrix}$$

Therefore, using conjugation rule 2.35

$$\mathcal{J} \mathcal{T}^{\bar{z}}(n)^* \mathcal{J}^{-1} = \begin{pmatrix} 0 & A_n \\ -A_n^{-1} & (A_n)^{-1}(z - B_n) \end{pmatrix}$$

Using eq. above and Eq. (2.31) one verifies that $\mathcal{T}^z(n)(\mathcal{J} \mathcal{T}^{\bar{z}}(n)^* \mathcal{J}^{-1}) = I$. ☐

Proposition 2.23. *For all* $n \in \mathbb{Z}$ *and* $z \in \mathbb{C}$, $\det(\mathcal{T}^z(n)) = 1$.

Proof. Note that

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} (z - B_n)(A_n)^{-1} & -A_n \\ A_n^{-1} & 0 \end{pmatrix} = \begin{pmatrix} A_n^{-1} & 0 \\ (z - B_n)(A_n)^{-1} & -A_n \end{pmatrix}$$

Using Schur complement formula for determinant we obtain

$$\det \begin{pmatrix} A_n^{-1} & 0 \\ (z - B_n)(A_n)^{-1} & -A_n \end{pmatrix} = \det(-A_n)\det(A_n^{-1}) = (-1)^L. \tag{2.37}$$

Using that

$$\det \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = (-1)^L$$

we obtain the result. ☐

Definition 2.24. *For* $n > m \in \mathbb{Z}$ *we define,*

$$\mathcal{T}^z(n, m) = \mathcal{T}^z(n-1) \dots \mathcal{T}^z(m).$$

And

$$\mathcal{T}^z(n, n) = I$$

$$\mathcal{T}^z(m, n) = \mathcal{T}^z(n, m)^{-1}.$$

Remark 2.25. *Note that Eq. (2.36) is also valid for $\mathcal{T}^z(n,m)$, because*

$$\begin{aligned}\mathcal{J}\mathcal{T}^{\overline{z}}(n,m)^*\mathcal{J}^* &= \mathcal{J}\mathcal{T}^{\overline{z}}(m)^*\mathcal{J}\mathcal{J}^*\mathcal{T}^{\overline{z}}(m-1)^*\mathcal{J}\dots\mathcal{J}^*\mathcal{T}^{\overline{z}}(n-1)^*\mathcal{J}^*\\ &= \mathcal{T}^z(m)^{-1}\dots\mathcal{T}^z(n-1)^{-1} = \mathcal{T}^z(n,m)^{-1}.\end{aligned} \tag{2.38}$$

Definition 2.26. *Let $u,v\in(\mathbb{C}^{L\times L})^{\mathbb{Z}}$, the wronskian $\mathcal{W}(u,v):\mathbb{Z}\to\mathbb{C}^{L\times L}$ is defined as*

$$\mathcal{W}(u,v)(n) = u(n+1)^*A_{n+1}v(n) - u(n)^*A_{n+1}v(n+1).$$

Remark 2.27. *One can see from definition that the wronskian satisfies the following for $u,v,w\in(\mathbb{C}^{L\times L})^{\mathbb{Z}}$ and $M\in\mathbb{C}^{L\times L}$ we have*

$$\begin{aligned}\mathcal{W}((u+w)M,v) &= M^*(\mathcal{W}(u,v)+\mathcal{W}(w,v)),\\ \mathcal{W}(u,(v+w)M) &= (\mathcal{W}(u,v)+\mathcal{W}(u,w))M.\\ \mathcal{W}(u,v) &= -\mathcal{W}(v,u)^*.\end{aligned}$$

Definition 2.28. *Let $u,v\in\mathbb{C}^{L\times L}$, we denote by $\Phi(u,v):\mathbb{Z}\to\mathbb{C}^{2L\times 2L}$ the sequence given by*

$$\Phi(u,v)(n) = \begin{pmatrix} A_n u(n) & A_n v(n)\\ u(n-1) & v(n-1)\end{pmatrix}$$

Remark 2.29. *Note that if $u,v\in\mathcal{M}_{L\times L}(\mathbb{C})$ are solutions of equation (**??**), Rem. 2.19 implies that*

$$T^z(n,m)\Phi(u,v)(m) = \Phi(u,v)(n). \tag{2.39}$$

Proposition 2.30. *Let u,v solutions of Eq. (2.27) , and $\tilde{u},\tilde{v}$ solution of $\tau(\overline{z})$. Then, we have that*

$$\mathcal{J}\Phi(\tilde{u},\tilde{v})(n)^*\mathcal{J}^*\Phi(u,v)(n) = \mathcal{J}\Phi(\tilde{u},\tilde{v})(m)^*\mathcal{J}^*\Phi(u,v)(m), \quad \textit{for all } n,m\in\mathbb{Z}.$$

Proof.

$$\begin{aligned}&\mathcal{J}\Phi(\tilde{u},\tilde{v})(n)^*\mathcal{J}^*\Phi(u,v)(n) = \mathcal{J}(\mathcal{T}^{\overline{z}}(n,m)\Phi(\tilde{u},\tilde{v})(m))^*\mathcal{J}^*\Phi(u,v)(n)\\ &= \mathcal{J}\Phi(\tilde{u},\tilde{v})(m)^*\mathcal{T}^{\overline{z}}(n,m)^*\mathcal{J}^*\Phi(u,v)(n) = \mathcal{J}\Phi(\tilde{u},\tilde{v})(m)^*\mathcal{J}^*\mathcal{J}\mathcal{T}^{\overline{z}}(n,m)^*\mathcal{J}^*\Phi(u,v)(n)\\ &= \mathcal{J}\Phi(\tilde{u},\tilde{v})(m)^*\mathcal{J}^*\mathcal{T}^z(n,m)^{-1}\Phi(u,v)(n) = \mathcal{J}\Phi(\tilde{u},\tilde{v})(m)^*\mathcal{J}^*\Phi(u,v)(m).\end{aligned} \tag{2.40}$$

□

Now we use the explict form of $\Phi(\tilde{u},\tilde{v})$ (see Def. 2.28) and conjugation rule (2.35) to compute explicitly $\mathcal{J}\Phi(\tilde{u},\tilde{v})(n+1)^*\mathcal{J}^*\Phi(u,v)(n+1)$ to obtain (see Def. 2.26).

$$\mathcal{J}\Phi(\tilde{u},\tilde{v})(n+1)^*\mathcal{J}^*\Phi(u,v)(n+1) = \begin{pmatrix} -\mathcal{W}(\tilde{v},u)(n) & -\mathcal{W}(\tilde{v},v)(n)\\ \mathcal{W}(\tilde{u},u)(n) & \mathcal{W}(\tilde{u},v)(n)\end{pmatrix} \tag{2.41}$$

In particular, using Eq. (2.41) and Prop. 2.30 we obtain the next proposition.

Proposition 2.31. *Let $u,v\in\mathcal{M}_{L\times L}(\mathbb{C})^{\mathbb{Z}}$ such that u is solution of $\tau(z)$ and v solution of $\tau(\overline{z})$ then*

$$\mathcal{W}(v,u)(n) = \mathcal{W}(v,u)(m), \;\; n,m\in\mathbb{Z}. \tag{2.42}$$

From now on, if u,v satisfies hypothesis of Prop. 2.31 we denote

$$\mathcal{W}(v,u)\equiv\mathcal{W}(v,u)(n).$$

3 Direct scattering theory

In the one-dimensional case, the scattering matrix can be constructed directly. This construction relies on the fact that we can obtain solutions to the eigenvalue equation associated with the perturbated Hamiltonian that behave like plane waves at infinity, these are known as Jost solutions. This solutions are obtained by solving a Volterra tyoe equation. Furthermore, we can analyze the analytic properties of these solutions and consequently transfer these properties to the scattering matrix. The expression of the scattering matrix in terms of the Jost solutions allows us to connect its analytic properties with the spectral properties of the perturbated Hamiltonian.

There are numerous results concerning the direct approach to scattering theory in the one-dimensional continuous setting. For instance, Chapters 5 and 6 of [28] provide a comprehensive exposition of the scalar case for the continuous Laplacian. Similarly, [4] offers a detailed treatment of the matrix-valued continuous case. In the discrete case, there are many results as well. For example, Chapter 10 of [27] offers a overview for Scattering theory on scalar valued Jacobi operators, in [15] construction of scattering matrix for scalar valued Jacobi operators with periodic coefficients and some analytic properties are proved, as well some inverse scattering theory. The results presented are based on the work did it by the author of this text and its collaborators, that can be found in [8, 9, 10].

In this section, we present the direct approach to scattering theory for the pair (H_0, H) where H_0 is the discrete Laplacian, and H is a matrix-valued Jacobi operator given by Eq. (1.1) and satisfying first moment condition, see Eq.(1.4). We begin by constructing the Jost solutions in Section 3.1.1 and analyzing some of their key properties. In Section 3.2, we define and study the scattering matrix.

Sections 3.4 and 3.5 explore how certain analytic properties of the scattering matrix correspond to spectral properties of the Hamiltonian. In Section 3.7, we derive expressions for the Jost solutions that allow us to study their asymptotic behavior, which is further detailed in Section 3.8. Finally, in Section 3.10, we bring together all these results in a formula that relates the analytic properties of the scattering matrix to the spectral characteristics of the perturbated Hamiltonian, this is known as Levinson's Theorem.

3.1 Solutions

In this section we study solutions of the generalized eigenvalue equation associated with H and H_0. Namely, we describe some solution of the equation (see Eq. (1.11))

$$\tau_H u = zu, \ \ z \in \mathbb{C}. \tag{3.1}$$

and (see Eq. (1.12))

$$\tau_0 u = zu, \ \ z \in \mathbb{C}. \tag{3.2}$$

Here we are considering matrix valued or vector valued sequence u. More relevant are the Jost solutions for the construction of the scattering matrix and solutions given in Lemma 3.4, that help us to analyze the asympthotic behavior of the scattering matrix at the thresholds. We start introducing the following solutions that are very relevant in many computations.

Definition 3.1. *For every $z \in \mathbb{C}$ and $n \in \mathbb{Z}$, we denote by s_n^z and c_n^z the solutions $s_n^z, c_n^z \in (\mathbb{C}^{L\times L})^{\mathbb{Z}}$ of the eigenvalue Eq. (3.2) with the initial conditions $s_n^z(n) = 0$, $s_n^z(n+1) = \mathbf{1}$ and $c_n^z(n) = \mathbf{1} = c_n^z(n+1)$. In the special case when $n = 0$ we omit the subscript $s^z \equiv s_0^z$, $c^z \equiv c_0^z$.*

Remark 3.2. *Since, s_n^z c_n^z are solutions to Eq. (3.2), they satisfy a recurence equation. Then, one can see inductively that the maps $\mathbb{C} \ni z \mapsto s^z(n), c^z(n)$ are analytic, for all $n \in \mathbb{Z}$. Moreover, explicitly one can verify that (see Def. 2.4)*

$$s_n^z(m) = \begin{cases} \nu^z(\lambda(z)^{m-n} - \lambda(z)^{n-m})\mathbf{1}. & z^2 \neq 4 \\ (\pm 1)^{m-n+1}(m-n)\mathbf{1}, & z = \pm 2 \end{cases} \tag{3.3}$$

and (for $z \neq -2$)

$$c_n^z(m) = \frac{\lambda(z)^{m-n} + \lambda(z)^{n-m+1}}{\lambda(z)+1}\mathbf{1}. \tag{3.4}$$

Indeed, R.H.S. of Eqs. (3.3) and (3.4) are solutions of Eq. (3.2) (recall $\lambda(z) + \lambda(z)^{-1} = z$) and have the same initial conditions that s_n^z and c_n^z respectively.

Eq. (3.3) and the fact that $|\lambda(z)| \leq 1$ gives us directly that

$$\|s_n^z(m)\| \leq C^z |\lambda(z)|^{-|m-n|}, z \in \mathbb{C} \setminus \{-2, 2\}. \tag{3.5}$$

Where C^z does not depend on n, m and is uniformly on compact sets $z \in \mathbb{C} \setminus \{-2, 2\}$.

Proposition 3.3. *We have the following estimates on s_n^z and c_n^z*

$$\|s_n^z(m)\| \leq |\lambda(z)|^{-|n-m|}|n-m|, \;\; n, m \in \mathbb{Z}, \;\; z \in \mathbb{C}. \tag{3.6}$$

$$\|c_n^z(m)\| \leq C|\lambda(z)|^{-|n-m|}, \;\; n, m \in \mathbb{Z}, \;\; z \in D(2; 1/2). \tag{3.7}$$

Where we denote

$$D(2; 1/2) := \{z \in \mathbb{C} : \|z - 2\| < 1/2\},$$

and C is a constant that does not depend on $n, m \in \mathbb{Z}$ and $z \in D(2; 1/2)$. Also we have,

$$\left\| \frac{d}{dz} s_n^z(m) \right\| \leq C^z |\lambda(z)|^{-|n-m|}|m-n|, \;\; z \in \mathbb{C} \setminus \{-2, 2\}, \tag{3.8}$$

where C^z does not depend on $n, m \in \mathbb{Z}$ and is uniformly bounded on compact sets $K \subset \mathbb{C} \setminus \{-2, 2\}$.

Proof. Using Eq. (3.3) we can write $s_n^z(m)$ as

$$s_n^z(m) = \operatorname{sgn}(m-n)\lambda(z)^{-|n-m|+1} \sum_{j=0}^{|m-n|-1} \lambda(z)^{2j}\mathbf{1}. \tag{3.9}$$

Then, we obtain (recall $|\lambda(z)| \leq 1$) Eq. (3.6). Eqs. (3.7) follows directly from Eq. (3.4). Using Eq. (3.3) one can obtain (see Eq. (2.12)) that for $z \in \mathbb{C} \setminus \{-2, 2\}$,

$$\frac{d}{dz} s_n^z(m) = \frac{d\nu^z}{dz}(\lambda(z)^{n-m} - \lambda(z)^{m-n})\mathbf{1} + (\nu^z)^2(m-n)(\lambda(z)^{m-n} + \lambda(z)^{n-m})\mathbf{1}. \tag{3.10}$$

Eq. (3.10) implies Eq. (3.8) □

3.1.1 Jost solutions

Lemma 3.4 (Jost solutions)**.** *Suppose that the potencial V has finite zero moment (see Def. 1.4). Then, for all $z \in \mathbb{C} \setminus \{-2,2\}$ the jost solutions u_+^z, u_-^z as defined in Definition 1.7 exist. Moreover, for every n, the functions $u_+^z(n), u_-^z(n)$ are holomorphic on $\mathbb{C} \setminus [-2,2]$ and continuous on $\overline{\mathbb{C}^+} \setminus \{-2,2\}$. The following Volterra equations are satisfied:*

$$A_{n+1}u_+^z(n) = u_{+,0}^z(n) + \sum_{j=n+1}^{\infty} (\tau_V s_\cdot^z(n)^*)(j)^* u_+^z(j), \qquad n \in \mathbb{Z}, \tag{3.11}$$

$$A_n u_-^z(n) = u_{-,0}^z(n) - \sum_{j=-\infty}^{n-1} (\tau_V s_\cdot^z(n)^*)(j)^* u_-^z(j), \qquad n \in \mathbb{Z}, \tag{3.12}$$

Here, we are using $s_\cdot^z(n)$ to denote the sequence $m \mapsto s_m^z(n)$. Then, (recall that A_j and B_j are self-adjoint)

$$(\tau_V s_\cdot^z(n)^*)(j)^* = s_{j-1}^z(n)W_j + s_j^z(n)V_j + s_{j+1}^z(n)W_{j+1}. \tag{3.13}$$

Further, if V has finite first moment, the jost solutions $u_+^{\pm 2}$, $u_-^{\pm 2}$ exist. They satisfies Eq. (3.11) and (3.12), and for all $n \in \mathbb{Z}$

$$\lim_{z \to \pm 2} u_\sigma^z(n) = u_\sigma^{\pm 2}(n), \ \sigma \in \{\pm\}. \tag{3.14}$$

Proof. We use Theorem 5.1. Suppose that $H - H_0$ has finite 0-moment. For $z \notin \{-2,2\}$, we take $g(n) = A_{n+1}^{-1}$, $K^z(n,j) = A_{n+1}^{-1}\lambda(z)^{j-n}(\tau_V s_\cdot^z(n)^*)(j)^*$, and (see Eq. (2.13))

$$M^z(j) = 2|\lambda(z)^{-1}\nu^z|\|A_{n+1}^{-1}\|(\|A_j - 1\| + \|B_j\| + \|A_{j+1} - 1\|).$$

Note that Eq. (3.3) imply that for $j > n \geq 0$

$$|K^z(n,j)| \leq M^z(j),$$

and cero moment assumption implies that for compact sets $K \subset \mathbb{C}\backslash\{-2,2\}$, $\{\sup_{z \in K} M^z(j)\} \in L^1(\mathbb{N})$. Using Theorem 5.1 we obtain a solution $\tilde{u}_+^z \in l^\infty(\mathbb{N}, \mathbb{C}^{L\times L})$ to the equation

$$A_{n+1}\tilde{u}_+^z(n) = \mathbf{1} + \sum_{j=n+1}^{\infty} (\tau_V s_\cdot^z(n)^*)(j)^* \lambda(z)^{j-n} \tilde{u}_+^z(j), \ n \geq 0. \tag{3.15}$$

We define $u_+^z(n) := \lambda(z)^n \tilde{u}_+^z(n)$, $n \geq 0$, and we define recursively,

$$u_+^z(n-1) := A_n^{-1}((z - B_n)u_+^z(n) - A_{n+1}u_+^z(n+1)), \ n \leq 0. \tag{3.16}$$

By construction, u_+^z is analytic on $\mathbb{C} \setminus [-2,2]$, continuous on $\overline{\mathbb{C}^+} \setminus \{-2,2\}$, satisfies Eq. (3.11) (for $n \geq 0$) and Eq. 1.28. It just remain to see that it is a solution of the eigenvalue equation. Proposition 5.3 implies that for $k > n \geq 0$

$$\sum_{j=n+1}^{k} (\tau_V s_\cdot^z(n)^*)^*(j) u_+^z(j) = \sum_{m=n+1}^{k} s_j^z(n)(\tau_V u_+^z)(j) + \mathcal{W}_V(s_\cdot^z(n)^*, u_+^z)(k) - \mathcal{W}_V(s_\cdot^z(n)^*, u_+^z)(n). \tag{3.17}$$

Defintion of u_+^z and Eq. (3.6) imply that $\mathcal{W}_V(s_\cdot^z(n)^*, u_+^z)(k) \to 0, k \to +\infty$. Direct computation using Def. 2.26 shows that $\mathcal{W}_V(s_\cdot^z(n)^*, u_+^z)(n) = (\mathbf{1} - A_{n+1})u_+^z(n)$. Using this above, taking limit in Eq. (3.17) and substituting in Eq. (3.11) we obtain that u_+^z satisfies the following equation (which is equivalent to (3.11)),

$$u_+^z(n) = u_{+,0}^z(n) + \sum_{j=n+1}^{\infty} s_j^z(n)(\tau_V u_+^z)(j), \ n \geq 0. \tag{3.18}$$

For all $n \in \mathbb{Z}$, let us denote the second summand in the R.H.S. of Eq. (3.18) by $S(n)$. An explicit computation, using that s_j^z is solution of Eq. (3.2) and that $s_{n-1}^z(n) = -\mathbf{1}$, shows that

$$((\tau_0 - z)S)(n) = -(\tau_V u_+^z)(n), \ n \in \mathbb{Z}. \tag{3.19}$$

Using Eqs. (3.19) and (3.18) (recall $\tau_H = \tau_0 + \tau_V$), we obtain that u_+^z satisfies Eq. (3.1) for $n \geq 1$, for $n \leq 0$ it satisfies Eq. (3.1) by definition. Finally, let us see that u_+^z satisfies Eq. (3.18) for $n < 0$. Note that $u_+^z - S$ is a solution of (3.2). Indeed, Eq. (3.19) implies $(\tau_0 - z)(u_+^z - S) = (\tau - z)u_+^z = 0$. For $n \geq 1$, we already know that $(u_+^z - S)(n) = u_{+,0}^z(n)$, since both are solutions of (3.2), then they are equal.

If one assumes that H has finite first moment, then the proof is similar, in this case we take $g(n) = A_{n+1}^{-1}, K^z(n,j) = A_{n+1}^{-1}\lambda(z)^{j-n}(\tau_V s_\cdot^z(n)^*)(j)^*$, and (see Eq. (3.13))

$$M^z(j) = |\lambda(z)^{-1}||j|\|A_{n+1}^{-1}\|(\|A_j - 1\| + \|B_j\| + \|A_{j+1} - 1\|).$$

Using Eq. (3.6) one obtains that $\|K^z(n,j)\| \leq M^z(j)$, and by the finite first moment assumption $\{\sup_{z\in C} M^z(j)\} \in L^1$. The solution $u_{-,0}^z$ is obtained in a similar manner. □

Remark 3.5. *In the special case, when $E \in [-2,2]$ we denote*

$$u_{\pm,out}^E := u_+^E. \tag{3.20}$$

Proposition 3.6. *Suppose that V has finite zero momment. Then, for all $E \in (-2,2)$, the following limits exist*

$$u_{\pm,in}^E(n) := \lim_{\substack{z\to E\\ \Im(z)<0}} u_\pm^z(n).$$

They have the following asympthotics

$$\begin{aligned} u_{+,in}^E(n) &= \lambda(E)^{-n}(\mathbf{1} + o(1)), \ n \to +\infty. \\ u_{-,in}^E(n) &= \lambda(E)^{n}(\mathbf{1} + o(1)), \ n \to -\infty. \end{aligned} \tag{3.21}$$

They satisfies Eq. (3.1). The following Volterra equation is satisfied,

$$A_{n+1}u_{+,in}^E(n) = u_{-,0}^E(n) + \sum_{j=n+1}^{\infty} (\tau_V s_\cdot^E(n)^*)(j)^* u_{+,in}^E(j) \tag{3.22}$$

$$A_n u_{-,in}^E(n) = u_{+,0}^E(n) - \sum_{j=-\infty}^{n-1} (\tau_V s_\cdot^E(n)^*)(j)^* u_{-,in}^E(j) \tag{3.23}$$

Proof. We proceed in the same manner as in Lemma 3.4, we take the same g, M^z, and K^z, $z \notin [-2,2]$, and for $E \in (-2,2)$ we change $K^E(n,m) = A_{n+1}^{-1}\lambda(E)^{n-j}(\tau_V s_\cdot^E(n)^*)(j)^*$. Note that know, K^z is continuous on $\overline{\mathbb{C}^-}$ (see Eq. (2.11)). We obtain, using Theorem 5.1, a solution $\tilde{v}_+^z$ (which by unicity coincides with the one obtained in Lemma 3.4, $\tilde{u}_+^z$, for $z \notin [-2,2]$) and such that is continuous on $\overline{\mathbb{C}^-}$. The same in the proof of Lemma 3.4 remains true, with exception that we define for $E \in (-2,2)$, $u_{+,in}^E(n) = \lambda(E)^{-n}\tilde{v}_+^E(n)$, $n \geq 0$. Then $u_+^z(n) = \lambda(z)^n\tilde{u}_+^z(n) = \lambda(z)^n\tilde{v}_+^z(n) \to u_{+,in}^E(n)$, $z \to E$, $\Im(z) < 0$. The rest is analogues to the proof of Lemma 3.4. □

Remark 3.7. *Prop. 3.6 proves that the function $\mathbb{C}^- \ni z \mapsto u_\pm^z$ has continuous extension to $\overline{\mathbb{C}^-}$. Namely, $u_\pm^{E-i0}(n) = u_{\pm,in}^E(n)$.*

Lemma 3.8. *Suppose that V has finite first moment. Then, the solutions $v_\pm$ introduced in Definition 1.7 exist.*

Proof. We give the construction of v_+. The assumption of finite first moment (see Def. 1.4) implies that there is $N \in \mathbb{N}$, enough large such that

$$\sum_{j=N+1}^{\infty} |j+1|(\|A_j - \mathbf{1}\| + \|B_j\| + \|A_{j+1} - \mathbf{1}\|) < 1/2. \tag{3.24}$$

Consider the Volterra-type equation (where we have the convention $\sum_{j=N+1}^{N} a_j = 0$.)

$$X(n) = n\mathbf{1} + \sum_{j=N+1}^{n} j(\tau_V X)(j) + n\sum_{j=n+1}^{\infty}(\tau_V X)(j),\ n \geq N. \tag{3.25}$$

We proceed to find a solution of Eq. (3.25). Consider the following vectorial space,

$$L^{\infty,-1}([N,\infty),\mathbb{C}^{L\times L}) := \{u \in (\mathbb{C}^{L\times L})^{[N,\infty)} : \{n^{-1}u(n)\} \in L^\infty([N,\infty),\mathbb{C}^{L\times L})\},$$

endowed with the norm

$$\|u\|_{\infty,-1} = \|\{n^{-1}u(n)\}\|_\infty = \sup_{n\geq N}\|n^{-1}u(n)\|,$$

which is a Banach space. Let us define the operator $T : L^{\infty,-1} \to L^{\infty,-1}$, by

$$(Tu)(n) = \sum_{j=N+1}^{n} j(\tau_V u)(j) + n\sum_{j=n+1}^{\infty}(\tau_V u)(j). \tag{3.26}$$

Note that for $u \in L^{\infty,-1}$, one has for $j \geq N+1$ that

$$\|(\tau_V u)(j)\| \leq |j+1|(\|A_j - \mathbf{1}\| + \|B_j\| + \|A_{j+1} - \mathbf{1}\|)\|u\|_{\infty,-1},$$

this together with Eq. (3.24) implies that the R.H.S. of Eq. (3.52) is indeed a convergent series, besides

$$\|n^{-1}(Tu)(n)\| \leq \|u\|_{\infty,-1}\sum_{j=N+1}^{\infty}|j+1|(\|A_j - \mathbf{1}\| + \|B_j\| + \|A_{j+1} - \mathbf{1}\|) \leq 1/2\|u\|_{\infty,-1},$$

having thus that T is well defined. Moreover, T is bounded and $\|T\| \leq 1/2$, which implies that $I-T$ is invertible. Let $X = (I-T)^{-1}g$, where $g(n) = n$. By definition, X is a solution of Eq. (3.25) such that $\{n^{-1}X(n)\}$ is bounded in $[N, \infty)$. Let us define $v_+(n) = X(n),\ n \geq N$, and recursively we define for $n \leq N$

$$v_+(n-1) = A_n^{-1}((2-B_n)v_+(n) - A_{n+1}v_+(n+1)).$$

An explicit computation using (3.52), shows that for $u \in L^{\infty,-1}$ one has

$$(Tu)(n+1) - (Tu)(n) = \sum_{j=n+1}^{\infty} (\tau_V u)(j),\ n \geq N. \tag{3.27}$$

Using Eq. (3.27) and the finite first moment assumption one obtains that $(Tu)(n+1) - (Tu)(n) \to 0, n \to \infty$, or equivalently (taking the mean sequence) $n^{-1}(Tu)(n) \to 0,\ n \to \infty$. This above, together with the fact that v_+ satisfies Eq. (3.25) imply that v_+ satisfies Eq. (1.29). It remains to prove that v_+ is a solution of Eq. (3.1). Since $v_+(n) = n + (Tv_+)(n),\ n \geq N$ and using Eq. (3.27) we obtain

$$\begin{aligned}((\tau_0 - 2)v_+)(n) &= (Tv_+)(n+1) + (Tv_+)(n-1) - 2(Tv_+)(n) \\ &= (Tv_+)(n+1) - (Tv_+)(n) + (Tv_+)(n-1) - (Tv_+)(n) \\ &= -(\tau_V v_+)(n),\ n \geq N+1,\end{aligned}$$

which together with the fact that $\tau_0 + \tau_V = \tau_H$ imply that v_+ is a solution of Eq. (3.1) for $n \geq N+1$. The fact that v_+ is solution of Eq. (3.1) for $n \leq N$ is given by definition. The construction of v_- is equivalent. ☐

Lemma 3.9. *Suppose that H has finite zero moment. Then, for every $z \in \mathbb{C} \setminus [-2, 2]$, there exist solutions $w_\pm^z \in (\mathbb{C}^{L\times L})^{\mathbb{Z}}$ to Eq. (3.1), that satisfy the following asympthotic behavior*

$$w_+^z(n) = \lambda(z)^{-n}(\mathbf{1} + o(1)), n \to +\infty, \quad w_-^z(n) = \lambda(z)^{n}(\mathbf{1} + o(1)), n \to -\infty. \tag{3.28}$$

Proof. We fix $z \in \mathbb{C} \setminus [-2, 2]$. In particular, note that $\lambda(z)^2 \neq 1$. To symplify the notation let us denote $\theta(z) = \lambda(z)^{-1} - \lambda(z)$, which is a constant that depends on z. Take $N \in \mathbb{N}$ enough large such that

$$\sum_{j=N+1}^{\infty} \|A_j - \mathbf{1}\| + \|B_j\| + \|A_{j+1} - \mathbf{1}\| < \frac{1}{2}\left|\theta(z)\lambda(z)\right|. \tag{3.29}$$

Consider the Volterra-type equation (recall the convention $\sum_{j=N+1}^{N} a_j = 0$),

$$X(n) = \lambda(z)^{-n}\mathbf{1} + \frac{1}{\theta(z)}\sum_{j=n+1}^{\infty} \lambda(z)^{j-n}(\tau_V X)(j) + \frac{1}{\theta(z)}\sum_{j=N+1}^{n} \lambda(z)^{n-j}(\tau_V X)(j),\ n \geq N. \tag{3.30}$$

We proceed to find a solution X to Eq. (3.30). Consider the following vectorial space,

$$\mathcal{B} := \{u \in (\mathbb{C}^{L\times L})^{[N,\infty)} : \{\lambda(z)^n u(n)\} \in L^\infty([N, \infty), \mathbb{C}^{L\times L})\},$$

endowed with the norm

$$\|u\|_{\mathcal{B}} = \|\{\lambda(z)^n u(n)\}\|_\infty = \sup_{n\geq N} \|\lambda(z)^n u(n)\|,$$

which is a Banach space. Next consider the operator $T : \mathcal{B} \to \mathcal{B}$, defined by

$$(Tu)(n) = \theta(z)^{-1} \sum_{j=n+1}^{\infty} \lambda(z)^{j-n}(\tau_V u)(j) + \theta(z)^{-1} \sum_{j=N+1}^{n} \lambda(z)^{n-j}(\tau_V u)(j). \tag{3.31}$$

For $u \in \mathcal{B}$, one has for $j \geq N+1$

$$\|\lambda(z)^j(\tau_V u)(j)\| \leq \|u\|_{\mathcal{B}} |\lambda(z)|^{-1}(\|A_j - \mathbf{1}\| + \|B_j\| + \|A_{j+1} - \mathbf{1}\|) \tag{3.32}$$

Eq. (3.32) together with zero moment assumption imply that the sum in R.H.S. of Eq. (3.31) is convergent. Moreover, Eq. (3.32) and Eq. (3.29) imply that

$$\|\lambda(z)^n(Tu)(n)\| \leq |\theta(z)\lambda(z)|^{-1} \sum_{j=N+1}^{\infty} (\|A_j - \mathbf{1}\| + \|B_j\| + \|A_{j+1} - \mathbf{1}\|)\|u\|_{\mathcal{B}} < \frac{1}{2}\|u\|_{\mathcal{B}}.$$

Previous equation shows that T is well defined. Moreover, T is bounded and $\|T\| \leq \frac{1}{2}$. Eq. (3.30) takes the form of $X(n) = \lambda(z)^{-n}\mathbf{1} + (TX)(n)$. Therefore, its solution is $X = (I-T)^{-1}g$, where $g(n) = \lambda(z)^{-n}\mathbf{1} \in \mathcal{B}$, which exists because $\|T\| \leq \frac{1}{2}$. Now, we define for $n \geq N$, $w_+^z(n) := X(n)$, and recursively for $n \leq N$,

$$w_+^z(n-1) := A_n^{-1}((z - B_n)w_+^z(n) - A_{n+1}w_+^z(n+1))$$

Therefore, from its definition w_+^z satisfies $\lambda(z)^n w_+^z(n) = O(1),\ n \to \infty$, also it satisfies (3.30), then its easy to see that it satisfies (3.28). It remains to see that w_+^z satisfies (3.1) for $n \geq N+1$. An explicit computation using Eq. (3.31) shows that for $u \in \mathcal{B}$ one has

$$(Tu)(n+1) + (Tu)(n-1) = z(Tu)(n) - (\tau_V u)(n),\ n \geq N+1. \tag{3.33}$$

Since $w_+^z(n) = g(n) + (Tw_+^z)(n),\ n \geq N$, one obtains using Eq. (3.33) that for $n \geq N+1$ (note that g satisfies $(\tau_0 - z)g = 0$),

$$((\tau_0 - z)w_+^z)(n) = (Tw_+^z)(n+1) + (Tw_+^z)(n-1) - z(Tw_+^z)(n-1) = -(\tau_V w_+^z)(n).$$

Which together with the fact that $\tau_0 + \tau_V = \tau_H$ complete the proof. $\square$

3.1.2 Solutions with prescribed data on 0 and 1

One of the key point of this text is the study of the wronskian of Jost solutions when the spectral parameter z tends to 2. It turns out to be more accessible to control the behavior of solutions with prescribed data on 0 and 1 as $z \to 2$ in a detailed manner. Via Wronskian identities this ultimately allows to deal with Jost solutions and the behavior of the scattering matrix as $z \to 2$.

We recall the solutions s_n^z, c_n^z to Eq. (3.2) given in Def. 3.1.

Lemma 3.10. *Let a, $b \in \mathbb{C}^{L\times L}$. For every $z \in \mathbb{C}$, the solution of the eigenvalue problem (3.1), Ψ^z with initial conditions $\Psi^z(0) = a$, $\Psi^z(1) = b$ satisfies the following equations: for every $n \in \mathbb{Z}^+$,*

$$\Psi^z(n) = s^z(n)(b-a) + c^z(n)a - \sum_{j=1}^{n-1} s^z_j(n)(\tau_V\Psi^z)(j), \tag{3.34}$$

and for every $n \in \mathbb{Z}^- \cup \{0\}$

$$\Psi^z(n) = s^z(n)(b-a) + c^z(n)a + \sum_{j=n+1}^{0} s^z_j(n)(\tau_V\Psi^z)(j). \tag{3.35}$$

Moreover, for every fixed n, $\Psi^z(n)$ is holomorphic on $\mathbb{C}$.

Proof. The result follows from Lemma 5.2, taking $A = \mathbf{1}, B(n) = -z\mathbf{1}$ and $F = -\tau_V$. Note that $s^z_m = s_m$ and $U = s^z(n)(b-a) + c^z(n)a$. The analyticity follows from the analyticity of $s^z_j(n)$ and $c^z_j(n)$ (see Rem. 3.2). ☐

Remark 3.11. *For some computations, we need to write Eqs. (3.34), and (3.35) as recurrence equations where the linear map τ_V is applied to s^z_j instead of Ψ^z. Using Prop. 5.3 we obtain*

$$\sum_{j=1}^{n-1} s^z_j(n)(\tau_V\Psi^z)(j) = \sum_{j=1}^{n-1}(\tau_V s^z_\cdot(n)^*)(j)^*\Psi^z(j) + \mathcal{W}_V(s^z_\cdot(n)^*,\Psi^z)(0) - \mathcal{W}_V(s^z_\cdot(n)^*,\Psi^z)(n-1). \tag{3.36}$$

By definition (see Def. 2.26) we obtain

$$\mathcal{W}_V(s^z_\cdot(n)^*,\Psi^z)(0) = s^z_1(n)W_1\Psi^z(0) - s^z_0(n)W_1\Psi^z(1),$$

$$\mathcal{W}_V(s^z_\cdot(n)^*,\Psi^z)(n-1) = -s^z_{n-1}(n)W_n\Psi^z(n) = (\mathbf{1} - A_n)\Psi^z(n).$$

Combining this with Eq. (3.36) we obtain

$$\sum_{j=1}^{n-1} s^z_j(n)(\tau_V\Psi^z)(j) = \sum_{j=0}^{n-1}(\tau^+_V s^z_\cdot(n)^*)(j)^*\Psi^z(j) + (A_n - \mathbf{1})\Psi^z(n). \tag{3.37}$$

where

$$(\tau^+_V u)(n) = \begin{cases} (\tau_V u)(n),\ n \geq 2 \\ V_1u(1) + W_2u(2),\ n = 1 \\ W_1u(1),\ n = 0. \end{cases} \tag{3.38}$$

In the same manner we obtain,

$$\sum_{j=n+1}^{0} s^z_j(n)(\tau_V\Psi^z)(j) = \sum_{j=n+1}^{1}(\tau^-_V s^z_\cdot(n)^*)(j)^*\Psi^z(j) + (\mathbf{1} - A_{n+1})\Psi^z(n), \tag{3.39}$$

where

$$(\tau_V^- u)(n) = \begin{cases} (\tau_V u)(n), \ n \leq -1 \\ W_0 u(-1) + V_0 u(0), \ n = 0 \\ W_1 u(0), \ n = 1. \end{cases} \tag{3.40}$$

Using Eqs. (3.34) and (3.35) together with Eqs. (3.37) and (3.39), we obtain that Ψ^z *satisfies the following equations: for every* $n \in \mathbb{Z}^+$

$$A_n \Psi^z(n) = s^z(n)(b-a) + c^z(n)a - \sum_{j=0}^{n-1} (\tau_V^+ s^z_\cdot(n)^*)(j)^* \Psi^z(j), \tag{3.41}$$

and for every $n \in \mathbb{Z}^- \cup \{0\}$

$$A_{n+1} \Psi^z(n) = s^z(n)(b-a) + c^z(n)a + \sum_{j=n+1}^{1} (\tau_V^- s^z_\cdot(n)^*)(j)^* \Psi^z(j). \tag{3.42}$$

Lemma 3.12. *Let* Ψ^z *be as in Lemma 3.10. Suppose that the potencial* V *has finite first moment (see Def. 1.3). Then, for all* $z \in D(2; 1/2)$*, the following estimate holds*

$$\|\Psi^z(n)\| \ \leq C|n||\lambda(z)|^{-|n|}, \ n \in \mathbb{Z}. \tag{3.43}$$

where C *depends on* a *and* b*, but not on* z *and* n*.*

Proof. Let $n \in \mathbb{Z}^+$, it follows from Eqs. (3.41), (3.6) and (3.7) that

$$\begin{aligned} &\|\Psi^z(n)\| |\lambda(z)|^n \frac{1}{n} && (3.44) \\ &= \|A_n\|^{-1} |\lambda(z)|^n \frac{1}{n} \Big\| s^z(n)(b-a) + c^z(n)a - \sum_{j=1}^{n-1} (\tau_V^+ s^z_\cdot(n)^*)(j)^* \Psi^z(j) \Big\| && (3.45) \\ &\leq C\Big(1 + \sum_{j=1}^{n-1} j(\|W_j\| + \|V_j\| + \|W_{j+1}\|) \|\Psi^z(j)\| \ |\lambda(z)|^j \frac{1}{j}\Big), \end{aligned}$$

and, therefore, Gronwall's Lemma (see Lemma 5.4) combined with finite first moment assumption yields (3.43). For negative n, the argument is similar, using (3.42). □

3.2 Scattering matrix

For $z \in \mathbb{C} \setminus [-2, 2]$, the Wronskians of the Jost solutions (see Def. 1.7 and Lemma 3.4) can be evaluated using that they are independent of n, Eq. (1.28) and Eq. (2.10) : taking the limits, either $n \to +\infty$ or $n \to -\infty$,

$$\mathcal{W}(u_+^{\bar{z}}, u_+^z) = 0 = \mathcal{W}(u_-^{\bar{z}}, u_-^z). \tag{3.46}$$

In the same manner, using Eq. (3.28) one obtains

$$\mathcal{W}(u_-^{\overline{z}}, w_-^z) = -\nu^z, \quad \mathcal{W}(u_+^{\overline{z}}, w_+^z) = \nu^z, \ z \in \mathbb{C} \setminus [-2,2]. \tag{3.47}$$

Moreover, taking limit $z \to E$, $\Im(z) > 0$, in Eq. (3.46) one obtains (recall $u_\pm^z$ is continuous on $\overline{\mathbb{C}^+}$ and see Prop. 3.6)

$$\mathcal{W}(u_{+,in}^E, u_{+,out}^E) = 0 = \mathcal{W}(u_{-,in}^E, u_{-,out}^E). \tag{3.48}$$

If, additionally, $E \in (-2,2)$ one obtains (see Rem. 3.5 and Prop. 3.6 and recall $|\lambda(E)| = 1$)

$$\mathcal{W}(u_{\pm,out}^E, u_{\pm,out}^E) = \mp(\nu^E)^{-1}\mathbf{1}, \quad \mathcal{W}(u_{\pm,in}^E, u_{\pm,in}^E) = \pm(\nu^E)^{-1}\mathbf{1}\ , \tag{3.49}$$

where

$$\nu^E \ = \ \frac{1}{\lambda(E) - \lambda(E)^{-1}}\ . \tag{3.50}$$

Definition 3.13. *For $z \in \mathbb{C} \setminus [-2,2]$ we denote by $M_\pm^z$, $N_\pm^z \in \mathbb{C}^{L \times L}$ the coefficients such that (see Rem. 1.9)*

$$u_+^z = u_-^z N_+^z + w_-^z M_+^z, \quad u_-^z = u_+^z N_-^z + w_+^z M_-^z. \tag{3.51}$$

We recall that in the case $E \in (-2,2)$, it is possible to decompose the states $u_{+,out}^E$ and $u_{-,out}^E$ on the basis $(u_{-,out}^E, u_{-,in}^E)$ and $(u_{+,in}^E, u_{+,out}^E)$, respectively, and that the matrices $M_\pm^E$ and $N_\pm^E$ are defined by

$$u_{+,out}^E = u_{-,in}^E M_+^E + u_{-,out}^E N_+^E, \qquad u_{-,out}^E = u_{+,out}^E N_-^E + u_{+,in}^E M_-^E. \tag{3.52}$$

In the same manner, one can decompose the states $u_{+,out}^E$, $u_{-,out}^E$ on the basis $(u_{-,out}^E, u_{-,in}^E)$ and $(u_{+,in}^E, u_{+,out}^E)$, respectively,

$$u_{+,in}^E = u_{-,in}^E L_+^E + u_{-,out}^E P_+^E, \qquad u_{-,in}^E = u_{+,out}^E P_-^E + u_{+,in}^E L_-^E. \tag{3.53}$$

We can express this coefficients in terms of the wronsian of the jost solutions.

Proposition 3.14. *Coefficients defined in Eqs. (3.52), (3.53) hold the following relations.*

$$\begin{array}{rclrcl} (P_-^E)^* = M_+^E & = & -\nu^E\,\mathcal{W}(u_{-,in}^E, u_{+,out}^E)\ , & M_-^E & = & (P_+^E)^* = \nu^E\,\mathcal{W}(u_{+,in}^E, u_{-,out}^E), \\ N_+^E & = & \nu^E\,\mathcal{W}(u_{-,out}^E, u_{+,out}^E)\ , & L_+^E & = & -\nu^E\,\mathcal{W}(u_{-,in}^E, u_{+,in}^E) \\ N_-^E & = & -\nu^E\,\mathcal{W}(u_{+,out}^E, u_{-,out}^E),\ , & L_-^E & = & \nu^E\,\mathcal{W}(u_{+,in}^E, u_{-,in}^E). \end{array} \tag{3.54}$$

Proof. We prove one of the identities, the others can be prove in the same manner. Using Eq. (3.52), linearity of wronskian (see Rem. 2.27) and Eqs. (3.48), (3.49) we obtain

$$\mathcal{W}(u_{-,in}^E, u_{+,out}^E) = \mathcal{W}(u_{-,in}^E, u_{-,in}^E) M_+^E = -\nu^E M_+^E. \tag{3.55}$$

We recall that $\nu^E \neq 0$ for $E \notin \{-2,2\}$ and satisfies $\overline{\nu^E} = -\nu^E$. □

In the case $z \in \mathbb{C} \setminus [-2,2]$ one can use Eq. (3.47) and Eq. (3.46) to see that

$$M_+^z := -\nu^z \mathcal{W}(u_-^{\overline{z}}, u_+^z), \quad M_-^z := \nu^z \mathcal{W}(u_+^{\overline{z}}, u_-^z),\ z \in \mathbb{C} \setminus [-2,2]. \tag{3.56}$$

In particular, note that the map $z \mapsto M_\pm^z$ is analytic on $\mathbb{C} \setminus [-2,2]$ and continuous on $\overline{\mathbb{C}^\pm}$. It satisfies,

$$\lim_{\substack{z \to E \\ \Im(z) > 0}} M_\pm^z = M_\pm^E, \qquad \lim_{\substack{z \to E \\ \Im(z) < 0}} M_\pm^z = P_\pm^E. \tag{3.57}$$

Equations (3.54) and (3.56) imply that

$$(N_+^E)^* \ = -N_-^E \ ,\ E \in (-2,2), \qquad (M_+^z)^* \ = \ M_-^{\overline{z}},\ z \in \mathbb{C} \setminus [-2,2].\ , \tag{3.58}$$

Lemma 3.15. *For every $E \in (-2,2)$, the following identities hold true:*

$$(M_-^E)^* M_-^E \ = \ \mathbf{1} \ + \ (N_-^E)^* N_-^E \ , \tag{3.59}$$

$$M_+^E N_-^E = -L_+^E M_-^E \ , \tag{3.60}$$

$$(M_+^E)^* M_+^E \ = \ \mathbf{1} \ + \ (N_+^E)^* N_+^E, \tag{3.61}$$

$$M_-^E N_+^E \ = \ -L_-^E M_+^E \ . \tag{3.62}$$

Proof. First we prove (3.61). It follows from Equations (3.49), (3.53), (3.52) that

$$-(\nu^E)^{-1}\mathbf{1} = W(u_{+,out}^E, u_{+,out}^E) = W(u_{-,in}^E M_+^E \ + \ u_{-,out}^E N_+^E, u_{-,in}^E M_+^E \ + \ u_{-,out}^E N_+^E). \tag{3.63}$$

Expanding the right hand side of (3.63) (see Rem. 2.27) and using Equations (3.46), (3.49) we get

$$-(\nu^E)^{-1}\mathbf{1} = -(\nu^E)^{-1}(M_+^E)^* M_+^E + (\nu^E)^{-1}(N_+^E)^* N_+^E,$$

This implies (3.61). Equation (3.59) is obtained in similar manner by expanding $\mathcal{W}(u_{-,out}^E, u_{-,out}^E)$. Now let us prove (3.62). It follows from Equations (3.48), (3.53) and (3.52) that

$$0 = \mathcal{W}(u_{+,out}^E, u_{+,in}^E) = \mathcal{W}(u_{-,in}^E M_+^E \ + \ u_{-,out}^E N_+^E, u_{-,in}^E L_+^E \ + \ u_{-,out}^E P_+^E). \tag{3.64}$$

Expanding the right hand side of (3.64) and using Equations (3.48), (3.49) and (3.54), we get

$$0 = -(\nu^E)^{-1}(M_+^E)^* L_+^E + (\nu^E)^{-1}(N_+^E)^* P_+^E = (\nu^E)^{-1}(M_+^E)^*(L_-^E)^* + (\nu^E)^{-1}(N_+^E)^*(M_-^E)^*.$$

Equation (3.60) is obtained in similar manner expanding $W(u_{-,in}^E, u_{-,out}^E)$. □

Proposition 3.16. *For $E \in (-2,2)$, $M_\pm^E$ is invertible and $\mathcal{S}^E$ is unitary.*

Proof. The invertibility of $M_\pm^E$ follows from Equations (3.59) and (3.61). Now we prove the unitarity. The off diagonal terms of $(\mathcal{S}^E)^* \mathcal{S}^E$ are (see Definition 1.12)

$$((M_+^E)^{-1})^* N_-^E (M_-^E)^{-1} + ((M_+^E)^{-1})^* (N_+^E)^* (M_-^E)^{-1}, \tag{3.65}$$

$$((M_-^E)^{-1})^* (N_-^E)^* (M_+^E)^{-1} + ((M_-^E)^{-1})^* N_+^E (M_+^E)^{-1} \tag{3.66}$$

and they vanish by (3.58). The diagonal terms are

$$((M_+^E)^{-1})^* (1 + (N_+^E)^* N_+^E)(M_+^E)^{-1}, \tag{3.67}$$

$$((M_-^E)^{-1})^* (1 + (N_-^E)^* N_-^E)(M_-^E)^{-1},$$

and they are both equal to $\mathbf{1}$, see (3.61) and (3.59). This proves the unitary of $\mathcal{S}^E$. □

3.3 Boundary limits of derivatives of Jost solutions

The previous section constructs the solutions $u^z_\pm$ and demonstrates their analyticity on $\mathbb{C} \setminus [-2,2]$ as well as their continuity on $\overline{\mathbb{C}^\pm}$. In this section, it is proved that the functions $\mathbb{C} \ni z \mapsto u^z_\pm(n)$ are continuously differentiable from above on $\overline{\mathbb{C}^+} \setminus \{-2,2\}$ in the sense of Def. 3.17. The proof is inspired by the continuous case that is presented in Deift and Trubowitz [14].

Definition 3.17. *Let U be an open set of $\mathbb{C}$, with $U \cap \mathbb{R} \neq \emptyset$. Consider $g : U \to \mathbb{C}$, such that g is analytic on $U \cap \mathbb{C}^+$. We say that g is differentiable from above at $E \in U \cap \mathbb{R}$ with differential (or derivative) $\frac{d}{dE} g(E)$ or $\dot{g}(E)$ if*

$$\lim_{\substack{z\to 0\\ \Im(z)\geq 0}} \left| \frac{g(E+z)-g(E)}{z} - \dot{g}(E) \right| = 0.$$

Then g is said to be differentiable from above on U if it is differentiable at every point of $U \cap \mathbb{R}$, and continuously differentiable from above on U if its derivative is a continuous (from above) function on $U \cap \overline{\mathbb{C}^+}$. Likewise, these concepts are defined for vector or matrix values functions.

For the proof of the differentiability from above of u^z_+, we will consider the sequence $\tilde{u}^z_+(n) = \lambda(z)^{-n} u^z_+(n)$. Due to (3.11), it satisfies

$$A_{n+1} \tilde{u}^z_+(n) = \mathbf{1} + \sum_{j=n+1}^{\infty} \lambda(z)^{j-n} (\tau_V s^z_\cdot(n)^*)(j)^* \tilde{u}^z_+(j), \qquad z \in \overline{\mathbb{D}}.$$

Hence let us set, for $z \in \overline{\mathbb{C}^+} \setminus \{-2,2\}$ and $j > n \in \mathbb{N}$,

$$H(z,n,j) = \lambda(z)^{j-n} (\tau_V s^z_\cdot(n)^*)(j)^*,$$

so that

$$A_{n+1} \tilde{u}^z_+(n) = \mathbf{1} + \sum_{j=n+1}^{\infty} H(z,n,j) \tilde{u}^z_+(j). \tag{3.68}$$

From the definition and Eq. (3.6) (also see Eq.(3.13)) one readily checks that

$$\|H(z,n,j)\| \leq (j-n)|\lambda(z)|^{-1} (\|W_j\| + \|V_j\| + \|W_{j+1}\|). \tag{3.69}$$

Note that $H(z,n,j)$ is contiuous differentiable from above and using Eqs. (3.8) and (3.5) one obtains that

$$\left\| \frac{d}{dz} H(z,n,j) \right\| \leq C^z (j-n)(\|W_j\| + \|V_j\| + \|W_{j+1}\|), \tag{3.70}$$

where C^z does not depend on n, m and is uniformly bounded on compact sets $K \subset \mathbb{C} \setminus \{-2,2\}$. Now formally deriving (3.68) w.r.t. z one obtains the equation

$$A_{n+1} \frac{d}{dz} \tilde{u}^z_+(n) = \sum_{j=n+1}^{\infty} \dot{H}(z,n,j) \tilde{u}^z_+(j) + \sum_{j=n+1}^{\infty} H(z,n,j) \frac{d}{dz} \tilde{u}^z_+(j), \tag{3.71}$$

for $n \in \mathbb{N}$. This gives us a Volterra type equation. Let us first verify that this equation has a bounded solution.

Lemma 3.18. *Assume that V has finite first moment (see Def. 1.4). For $z \in \mathbb{C} \setminus \{-2,2\}$, there exists a solution $h^z \in \ell^\infty(\mathbb{N}, \mathbb{C}^{L\times L})$ of the integral equation*

$$A_{n+1}h^z(n) = \sum_{j=n+1}^{\infty} \dot{H}(z,n,j)\tilde{u}_+^z(j) + \sum_{j=n+1}^{\infty} H(z,n,j)h^z(j),\ n \geq 0.$$

which is analytic on $\mathbb{C} \setminus [-2,2]$ and continuous on $\overline{\mathbb{C}^+} \setminus \{-2,2\}$.

Proof. By Lemma 3.4, $\tilde{u}_+^z \in \ell^\infty(\mathbb{N}, \mathbb{C}^{L\times L})$ with $\|\tilde{u}_+^z\|_\infty$ uniformly bounded on K, for every compact set $K \subset \mathbb{C}$. For $n \in \mathbb{N}$, (3.70) implies that

$$\sum_{j=n+1}^{\infty} \|\dot{H}(z,n,j)\tilde{u}_+^z(j)\| \leq C^z \sum_{j=n+1}^{\infty} j(\|V_j\| + \| + W_j\| + \|W_{j+1}\|),$$

where C^z is uniformly bounded on compact sets $K \subset \mathbb{C} \setminus \{-2,2\}$. This above together with the finite first moment assumption imply that the function

$$g^z(n) := \sum_{j=n+1}^{\infty} \dot{H}(z,n,j)\tilde{u}_+^z(j) \tag{3.72}$$

belongs to $\ell^\infty(\mathbb{N}, \mathbb{C}^{L\times L})$, is analytic in $z \in \mathbb{C} \setminus [-2,2]$, continuous in $z \in \overline{\mathbb{C}^+} \setminus \{2,2\}$ and uniformly bounded on compact sets $K \subset \mathbb{C} \setminus \{-2,2\}$.

Then, the result follows from the Theorem 5.1 with g^z defined as above and $K^z(n,j) = H(z,n,j)$, $M^z(j) = j|\lambda(z)|^{-1}(\|V_j\| + \|V_{j+1}\| + \|W_j\|)$. □

Proposition 3.19. *Assume that V has finite first moment (see Def. 1.4). For each $n \in \mathbb{N}$, the function $z \mapsto \tilde{u}_+^z(n)$ is continuously differentiable from above on $\mathbb{C} \setminus \{-2,2\}$. Moreover, the derivative $\frac{d}{dE}\tilde{u}_+^E \in \ell^\infty(\mathbb{N}, \mathbb{C}^{L\times L})$ and it satisfies (3.71).*

Proof. For $z \in \mathbb{C} \setminus \{-2,2\}$, let h^z the function defined in Lemma 3.18. Lemma 3.18 implies that $z \mapsto h^z(n)$ is continuous on $\overline{\mathbb{C}^+} \setminus \{2,-2\}$ and satisfies the following equation

$$A_{n+1}h^z(n) = \sum_{j=n+1}^{\infty} \dot{H}(z,n,j)\tilde{u}_+^z(j) + \sum_{j=n+1}^{\infty} H(z,n,j)h^z(j), \qquad n \in \mathbb{N}. \tag{3.73}$$

Take $n \in \mathbb{N}$. Since the map $\mathbb{C} \setminus [-2,2] \ni z \mapsto \tilde{u}_+^z(n)$ is analytic and $|\lambda(z)| < 1$, $\notin [-2,2]$ then the complex derivative $\frac{d}{dz}\tilde{u}_+^z(n)$ satisfies Eq. (3.73), and uniqueness implies $\frac{d}{dz}\tilde{u}_+^z = h(z)$, $z \notin [-2,2]$. Then, it is enough to prove that for $E \in (-2,2)$, $\frac{d}{dE}\tilde{u}_+^E(n) = h^E(n)$, then the result follows from the continuity of h^z. Take $z \in \overline{\mathbb{C}^+}$, using (3.68) and (3.73) one has that

$$A_{n+1}\left(\frac{\tilde{u}_+^{E+z}(n) - \tilde{u}_+^E(n)}{z} - h^E(n)\right) = G(z) + \sum_{j=n+1}^{\infty} H(E,n,j)\left(\frac{\tilde{u}_+^{E+z}(j) - \tilde{u}_+^E(j)}{z} - h^E(j)\right), \tag{3.74}$$

where

$$G(z)=\sum_{j=n+1}^{\infty}\left(\frac{H(E+z,n,j)-H(E,n,j)}{z}\tilde{u}_{+}^{E+z}(j)-\dot{H}(E,n,j)\tilde{u}_{+}^{E}(j)\right).$$

Eq. (3.70) implies that

$$\|\dot{H}(E,n,j)\tilde{u}_{+}^{E}(j)\|\leq Cj(\|V_j\|+\|W_j\|+\|V_{j+1}\|),$$

where C denotes a constant that does not depend on E,j,n. Then it is summable in j by the finite first moment assumption. On the other hand, the estimate

$$\left\|\frac{H(E+z,n,j)-H(E,n,j)}{z}\right\|\leq\int_0^1\|\dot{H}(E+tz,n,j)\|dt$$

and Eq. (3.70) leads to, for $0<|z|\leq 1$

$$\left\|\frac{H(E+z,n,j)-H(E,n,j)}{z}\tilde{u}_{+}^{E+z}(j)\right\|\leq Cj(\|V_j\|+\|W_j\|+\|V_{j+1}\|),$$

which is thus also summable in j. Therefore, the Lebesgue dominated convergence theorem implies that $G(z)\to 0$ as $z\to 0$, (recall $\tilde{u}_{+}^{z}(j)$ is continuous on $\overline{\mathbb{C}^{+}}$ and $H(z,n,j)$ is differentiable from above). Now by the Gronwall lemma applied to Eq. (3.74) (see, Lemma 5.4) and Eq. (3.69), one has

$$\left|\frac{\tilde{u}_{+}^{E+z}(n)-\tilde{u}_{+}^{E}(n)}{z}-h^{E}(n)\right|\leq C|G(z)|\exp\left(C\sum_{j=n+1}^{\infty}j(\|V_j\|+\|V_{j+1}\|+\|W_j\|)\right)\;\to\;0,$$

as $z\to 0$. This implies the desired result. ☐

Remark 3.20. *Recall that (see Proposition 3.4) by definition $u_{+}^{z}(n)=\lambda(z)^{n}\tilde{u}_{+}^{z}(n)$, so that Proposition 3.19 implies that the map $z\mapsto u_{+}^{z}(n)$ is continuously differentiable from above on $\mathbb{C}\setminus\{2,-2\}$ for $n\in\mathbb{N}$. Moreover, Eq. (3.1) implies that*

$$A_n u_{+}^{E}(n-1)=(E-V_n)u_{+}^{E}(n)-A_{n+1}u_{+}^{E}(n+1),$$

which along with the above allows to prove that $z\mapsto u_{+}^{z}(n)$ is continuously differentiable from above on $\mathbb{C}\setminus\{2,-2\}$ for all $n\in\mathbb{Z}$. In a similar way one proves that the map $z\mapsto u_{-}^{z}(n)$ is continuously differentiable from above on $\mathbb{C}\setminus\{-2,2\}$ for all $n\in\mathbb{Z}$. The above results imply that the map $(-2,2)\ni E\mapsto u_{\pm,out}^{E}(n)$ is continuous differentiable. In the same manner one can prove that the map $z\mapsto u_{\pm}^{z}(n)$ is continuous differentiable from below, and then conclude that the map $(-2,2)\ni E\mapsto u_{\pm,in}^{E}(n)$ is continuous differentiable.

3.4 Bound states

This section examines the behavior of the function $z \mapsto \det(M_\pm^z)$ at $z = E$, where $E \in \mathbb{R} \setminus [-2,2]$ is an eigenvalue of the operator H. The main result (see Proposition 3.24) establishes that the number of zeros of the analytic function $z \mapsto \det(M_\pm^z)$ on $\mathbb{C} \setminus [-2,2]$, counted with multiplicities, is equal to the number of eigenvalues of H, also counted with multiplicities.

Proposition 3.21. *For $z \in \mathbb{C} \setminus [-2,2]$, the following identity holds true:*

$$\dim(\mathrm{Ker}(H - z)) = \dim(\mathrm{Ker}(M_\pm^z)). \tag{3.75}$$

Moreover, N_-^z restricted to $\mathrm{Ker}(M_-^z)$ *is a bijection between* $\mathrm{Ker}(M_-^z)$ *and* $\mathrm{Ker}(M_+^z)$.

Proof. Let us prove (3.75) for M_+^z and the result for M_-^z is obtained in a similar fashion. Set

$$\mathcal{S}_+ = \{\phi \in \mathbb{C}^L : u_+^z \phi \in L^2(\mathbb{Z}, \mathbb{C}^L)\}, \qquad \mathcal{S}_- = \{\phi \in \mathbb{C}^L : u_-^z \phi \in L^2(\mathbb{Z}, \mathbb{C}^L)\},$$

then the function $T : \mathcal{S}_+ \to \mathrm{Ker}(H - E)$ defined by $T(\phi) = u_+^z \phi$ is linear and injective (because the columns of u_+^z are linearly independent by Remark 1.9 and these columns are precisely solutions to the eigenvalue problem). Let $u \in \mathrm{Ker}(H - E)$, then there exist $\phi \in \mathbb{C}^L$ such that $u(n) = u_+^z(n)\phi$ (indeed, write $u = u_+^z \phi + w_+^z \psi$ again by Remark 1.9 and notice that $w_+^z \psi \neq 0$ implies that $\lim_{n\to+\infty} \|w_+^z(n)\psi\| = \infty$ - see Eq. (3.28)). Thus T is surjective, and it is consequently an isomorphism.

Next we prove that $\mathcal{S}_+ = \mathrm{Ker}(M_+^z)$ which implies (3.75) (similarly, one proves that $\mathcal{S}_- = \mathrm{Ker}(M_-^z)$). Let us take $\phi \in \mathrm{Ker}(M_+^z)$ and multiply (3.51) by ϕ so that

$$u_+^z \phi = u_-^z N_+^z \phi. \tag{3.76}$$

This implies that $u_+^z \phi \in L^2(\mathbb{Z}, \mathbb{C}^L)$ and therefore $\phi \in \mathcal{S}_+$, which implies that $\mathrm{Ker}(M_+^z) \subset \mathcal{S}_+$. The other contention is proved by taking $\phi \in \mathcal{S}_+$ and multiplying (3.51) by ϕ. Then

$$u_+^z \phi = w_-^z M_+^z \phi + u_-^z N_+^z \phi.$$

Since $u_+^z \phi \in L^2(\mathbb{Z}, \mathbb{C}^L)$, it follows (using the asymptotic behavior of Jost solutions to compute the second term on the right of the next equation) that

$$\lim_{n\to-\infty} w_-^z(n) M_+^z \phi = \lim_{n\to-\infty} u_+^z(n)\phi - u_-^z(n) N_+^z \phi = 0.$$

The asymptotic behavior of Jost solutions implies that $M_+^z \phi = 0$ (since otherwise one would have $\lim_{n\to-\infty} \|w_-^z(n) M_+^z \phi\| = \infty$). The arguments above imply the first part of the statement.

Next, let us prove that $N_-^z\big|_{\mathrm{Ker}(M_-^z)}$ is a bijection between $\mathrm{Ker}(M_-^z)$ and $\mathrm{Ker}(M_+^z)$. Take $\phi \in \mathrm{Ker}(M_-^z)$. Eq. (3.51) implies that

$$u_-^z \phi = u_+^z N_-^z \phi,$$

and using the asymptotic behavior of Jost solutions (see Eq. (1.28)) one concludes that

$$u_+^z N_-^z \phi \in L^2(\mathbb{Z}, \mathbb{C}^L).$$

With help of Eq. (3.51) for u_+^z (*i.e.* $u_+^z N_-^z \phi = w_-^z M_+^z N_-^z \phi + u_-^z N_+^z N_-^z \phi$), one deduces as before (using a blow up argument) that $N_-^z \phi \in \operatorname{Ker}(M_+^z)$. This implies that N_-^z maps $\operatorname{Ker}(M_-^z)$ into $\operatorname{Ker}(M_+^z)$. Moreover, Eq. (3.51) and the above equations imply that

$$u_-^z \phi = u_+^z N_-^z \phi = u_-^z N_+^z N_-^z \phi.$$

Multiplying by $\lambda(z)^n$ and taking the limit $n \to -\infty$ in this identity (see also Eq. (1.28)), it follows that

$$\phi = N_+^z N_-^z \phi.$$

In a similar fashion, one proves that if $\phi \in \operatorname{Ker}(M_+^z)$ then $N_+^z \phi \in \operatorname{Ker}(M_-^z)$ and

$$\phi = N_-^z N_+^z \phi.$$

Then the restriction of N_+^z to $\operatorname{Ker}(M_+^z)$ is the inverse of $N_-^z\big|_{\operatorname{Ker}(M_-^z)}$, concluding the proof.□

Proposition 3.22. *The set of eigenvalues of H is a finite set contained in $(-\infty, -2) \cup (2, \infty)$.*

Proof. Since H is self-adjoint, its spectrum is contained in the real line. Let $E \in \mathbb{R}$ an eigenvalue of H and suppose that u is an eigenvector of H corresponding to E, *i.e.* $Hu = Eu$. Let us assume first that $E \in (-2, 2)$, Remark 1.11 implies that u can be written in the form

$$u = u_{+,out}^E \alpha + u_{+,in}^E \beta,$$

for some $\alpha, \beta \in \mathbb{C}^L$. As u is square integrable, one has

$$\lim_{n \to \infty} u(n) = 0.$$

Eqs. (1.31), (1.33) yield that

$$\lim_{n \to \infty} \lambda(E)^n \alpha + \lambda(E)^{-n} \beta = 0,$$

which is only possible when $\alpha = \beta = 0$ (recall $|\lambda(E)| = 1$, $\lambda(E)^2 \neq 1$) and hence $u = 0$. Consequently all eigenvalues must lie on $(-\infty, -2] \cup [2, \infty)$. It remains to rule out the points $\{-2, 2\}$. We analyze only $z = 2$, since the analysis for $z = -2$ is the same. The proof in this case is similar, but Remark 1.11 is not valid anymore because $u_{+,out}^2 = u_{+,in}^2$ (recall $\lambda(2) = 1$.) The columns of $u_{+,out}^2$ do not generate all solutions. Nevertheless, in Definition 1.7 in we introduce another solution v_+^1 such that the columns of $(u_{+,out}^2, \, v_+)$ generate all solutions. Now, following the line-of-argument for the case $E \in (-2, 2)$, one concludes that 2 is not an eigenvalue.

Proposition 3.21 implies that $E \in \mathbb{R} \setminus [-2, 2]$ is an eigenvalue of H if and only if the function $z \mapsto \det(M_\pm^z)$ has a zero at E. Since this function is analytic on $\mathbb{C} \setminus [-2, 2]$, we conclude that H has only a finite number of eigenvalues in $(-\infty, -2) \cup (2, \infty)$.

□

Proposition 3.21 claims that the number of zeros, counted without multiplicity, of the function $z \mapsto \det(M^z_\pm)$ on $\mathbb{C} \setminus [-2,2]$ is equal to the number of eigenvalues of H, counted without multiplicity. Proposition 3.24 below proves that they are also the same if counted with multiplicity. For the proof, the following technical statement is needed which is a discrete version of a result from [3] that was already used in [8].

Lemma 3.23. *Let $E \in \mathbb{R} \setminus [-2,2]$, an eigenvalue of H. Let $\alpha \in \mathrm{Ker}(M^E_-)$. The following equation holds true:*

$$(N^E_-\alpha)^* \frac{d}{dz} M^z_- \Big|_{z=E} \alpha = -\nu^E \|u^E_-\alpha\|^2 \tag{3.77}$$

Proof. Let $z \in \mathbb{C} \setminus [-2,2]$ and recall that the Jost solution u^z_+ satisfies the generalized eigenvalue equations $Hu^z_+ = zu^z_+$, namely

$$A_{n+1}u^z_+(n+1) + A_n u^z_+(n-1) + B_n u^z_+(n) = zu^z_+(n), \qquad \forall n \in \mathbb{Z}. \tag{3.78}$$

Taking derivative w.r.t. z in E, one obtains

$$A_{n+1}\dot{u}^E_+(n+1) + A_n \dot{u}^E_+(n-1) + B_n \dot{u}^E_+(n) = E\dot{u}^E_+(n) + u^E_+(n), \tag{3.79}$$

where $\dot{u}^r_+$ is given by $\dot{u}^r_+(n) = \frac{d}{dz}u^z_+(n)\Big|_{z=E}$. Taking adjoints and evaluating in $z = E$ in (3.78) leads to

$$u^E_+(n+1)^* A_{n+1} + u^E_+(n-1)^* A_n + u^E_+(n)^* B_n = Eu^E_+(n)^*. \tag{3.80}$$

Multiplying (3.79) on the left by $u^E_+(n)^*$ and (3.80) on the right by $\dot{u}^E_+(n)$ and subtracting the resulting equations, one obtains

$$\begin{aligned} u^E_+(n)^* A_{n+1}\dot{u}^E_+(n+1) - u^E_+(n+1)^* A_{n+1}\dot{u}^E_+(n) + u^E_+(n)^* A_n \dot{u}^E_+(n-1) - u^E_+(n-1)^* A_n \dot{u}^E_+(n) \\ = u^E_+(n)^* u^E_+(n). \end{aligned} \tag{3.81}$$

Recalling the definition of the Wronskian

$$\mathcal{W}(u^E_+, \dot{u}^E_+)(n) = u^E_+(n+1)^* A_{n+1}\dot{u}^E_+(n) - u^E_+(n)^* A_{n+1}\dot{u}^E_+(n+1),$$

one can rewrite the last equation as

$$\mathcal{W}(u^E_+, \dot{u}^E_+)(n-1) - \mathcal{W}(u^E_+, \dot{u}^E_+)(n) = u^E_+(n)^* u^E_+(n).$$

Multiplying this equation by $N^E_-\alpha$ on the right and by $(N^E_-\alpha)^*$ on the left implies that

$$(N^E_-\alpha)^*(\mathcal{W}(u^E_+, \dot{u}^E_+)(n-1) - \mathcal{W}(u^E_+, \dot{u}^E_+)(n))N^E_-\alpha = (u^E_-(n)\alpha)^* u^E_-(n)\alpha, \tag{3.82}$$

where Eq. (3.51) was used to exchange $u^E_+(n)N^E_-\alpha$ by $u^E_-(n)\alpha$ (recall that $\alpha \in \mathrm{Ker}(M^E_-)$). Since $\alpha \in \mathrm{Ker}(M^E_-)$, one has that $u^E_-\alpha \in L^2(\mathbb{Z}, \mathbb{C})$ (by using that $u^E_+(n)N^E_-\alpha = u^E_-(n)\alpha$ and the asymptotic properties of Jost solutions). Now take the sum in both sides of the Eq. (3.82) to get:

$$\sum_{n\in\mathbb{Z}} s(n-1) - s(n) = \sum_{n\in\mathbb{Z}} (u^E_-(n)\alpha)^* u^E_-(n)\alpha = \|u^E_-\alpha\|^2,$$

where $s(n) := (N_-^E\alpha)^*\mathcal{W}(u_+^E, \dot u_+^E)(n)N_-^E\alpha,\ n \in \mathbb{Z}$. Note that the l.h.s. of the equation is a telescoping series. Thus

$$\lim_{n\to-\infty} s(n) - \lim_{n\to+\infty} s(n) = \|u_-^E\alpha\|^2. \tag{3.83}$$

Computing $\dot u_+^E(n) = n\lambda(E)^{n-1}\tilde u_+^E(n) + \lambda(E)^n \frac{d}{dz}\tilde u_+^z(n)|_{z=E}$ and noticing that

$$\frac{d}{dz}\tilde u_+^z, \tilde u_+^z \in \ell^\infty(\mathbb{N}, \mathbb{C}^{L\times L}),$$

see Proposition 3.19, one obtains that

$$u_+^E(n), \qquad \dot u_+^E(n) \to 0, n \to +\infty. \tag{3.84}$$

Thus

$$s(n) \to 0 \quad \text{as } n \to +\infty.$$

Using the definition of the Wronskian (see Eq. (2.26)) and the general fact that $\frac{d}{dz} f(\overline{z})^*\Big|_{z=z_0} = \left(\frac{d}{dz} f(z)\Big|_{z=\overline{z_0}}\right)^*$, one obtains the following (for every $n \in \mathbb{Z}$):

$$\frac{d}{dz}\mathcal{W}(u_+^{\overline{z}}, u_-^z)\Big|_{z=E} = \mathcal{W}(\dot u_+^E, u_-^E)(n) + \mathcal{W}(u_+^E, \dot u_-^E)(n), \tag{3.85}$$

recall $\dot u_-^E$ is given by $\dot u_-^E(n) = \frac{d}{dz}u_-^z(n)\Big|_{z=E}$. The following computation now uses again Eq. (3.51) in order to replace $u_+^E(n)N_-^E\alpha$ by $u_-^E(n)\alpha$ (recall that $\alpha \in \mathrm{Ker}(M_-^E)$) and Eq. (3.85)

$$s(n) = (N_-^E\alpha)^*\mathcal{W}(u_+^E, \dot u_+^E)(n)N_-^E\alpha = \mathcal{W}(u_+^E N_-^E\alpha, \dot u_+^E)(n)N_-^E\alpha = \mathcal{W}(u_-^E\alpha, \dot u_+^E)(n)N_-^E\alpha$$

$$= \alpha^*\mathcal{W}(u_-^E, \dot u_+^E)(n)N_-^E\alpha = \alpha^*\left(\mathcal{W}(u_+^E, \dot u_-^E)(n) - \frac{d}{dz}\mathcal{W}(u_+^{\overline{z}}, u_-^z)\Big|_{z=E}\right)^* N_-^E\alpha =$$

$$= -\alpha^*\mathcal{W}(\dot u_-^E, u_-^E)(n)\alpha - \alpha^*(\frac{d}{dz}\mathcal{W}(u_+^{\overline{z}}, u_-^z)\Big|_{z=E})^*N_-^E\alpha,$$

Arguing as in (3.84), one gets $\dot u_-^E(n), u_-^E(n) \to 0,\ n \to -\infty$. Taking the limit $(n \to -\infty)$ in the last equation leads to

$$s(n)^* \to -(N_-^E\alpha)^*\frac{d}{dz}\mathcal{W}(u_+^{\overline{z}}, u_-^z)\Big|_{z=E}\alpha,\ n \to -\infty. \tag{3.86}$$

Eqs. (3.83) and (3.86) and the fact that $s(n) \to 0,\ n \to +\infty$, show that

$$-(N_-^E\alpha)^*\frac{d}{dz}\mathcal{W}(u_+^{\overline{z}}, u_-^z)\Big|_{z=E}\alpha = \|u_-^E\alpha\|^2. \tag{3.87}$$

Using Eq. (3.56) we obtain

$$\frac{d}{dz}M_-^z\Big|_{z=E} = \frac{d}{dz}\nu^z\Big|_{z=E}\mathcal{W}(u_+^E, u_-^E) + \nu^E\frac{d}{dz}\mathcal{W}(u_+^{\overline{z}}, u_-^z)\Big|_{z=E}. \tag{3.88}$$

Then, due to $\alpha \in \mathrm{Ker}(M_-^E) = \mathrm{Ker}\mathcal{W}(u_+^E, u_-^E)$,

$$(N_-^E\alpha)^* \frac{d}{dz} M_-^z\Big|_{z=E} \alpha = \nu^E (N_-^E\alpha)^* \frac{d}{dz} W(u_+^{\bar{z}}, u_-^z)\Big|_{z=E} \alpha. \tag{3.89}$$

Combining Eqs. (3.89) and (3.87), the result follows. $\square$

Proposition 3.24. *Suppose that $E \in \mathbb{R} \setminus [-2, 2]$ is an eigenvalue of H. Let us set $n_E = \dim(\mathrm{Ker}(M_-^E)) = \dim(\mathrm{Ker}(H - E))$. Then there exists a complex number $c_E \in \mathbb{C} \setminus \{0\}$ such that*

$$\det(M_-^z) = (z - E)^{n_E}(c_E + \mathcal{O}(|z - E|)), \qquad z \to E.$$

Proof. Let $\{u_1, ..., u_{n_E}\}$ be a basis of $\mathrm{Ker}(M_-^E)$. Since $N_-^E|_{\mathrm{Ker}(M_-^E)} : \mathrm{Ker}(M_-^E) \to \mathrm{Ker}(M_+^E)$ is an isomorphism (see Proposition (3.21)), it follows that $\{N_-^E u_1, ..., N_-^E u_{n_E}\}$ is a basis of $\mathrm{Ker}(M_+^E) = \mathrm{Ran}(M_-^E)^{\perp}$, the latter due to Eq. (3.58). Next let $\{v_{n_E+1}, ..., v_L\}$ be an orthonormal basis of $\mathrm{Ran}(M_-^E)$ and $\{u_{n_E+1}, ..., u_L\}$ such that $M_-^E u_i = v_i$. Then the sets $\{N_-^E u_1, ..., N_-^E u_{n_E}, v_{n_E+1}, ..., v_L\}$ and $\{u_1, ..., u_{n_E}, u_{n_E+1}, ..., u_L\}$ are basis of $\mathbb{C}^L$. We denote by U_1, V_1 and V_1, V_2 the matrices such that

$$U_1 = (u_1 ... u_L),\ U_2 = (u_1 ... u_{n_E}),\ \ V_1 = (N_-^E u_1 ... N_-^E u_{n_E}\ v_{n_E+1} ... v_L),\ V_2 = (N_-^E u_1 ... N_-^E u_{n_E}).$$

Then

$$V_1^* M_-^E U_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We let $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ denote the matrices satisfying

$$V_1^* \frac{d}{dz} M_-^z\Big|_{z=E} U_1 = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix},$$

where $\tilde{A} = V_2^* \frac{d}{dz} M_-^z\Big|_{z=E} U_2$. Lemma 3.23 shows that $\tilde{A}$ is positive, indeed if $\phi \in \mathbb{C}^{n_E} \setminus \{0\}$

$$\phi^* \tilde{A} \phi = (V_2\phi)^* \frac{d}{dz} M_-^z\Big|_{z=E} U_2\phi = (N_-^E U_2\phi)^* \frac{d}{dz} M_-^z\Big|_{z=E} U_2\phi = -\nu^E \|u_-^E U_2\phi\|^2 > 0.$$

Note that the last identity used $-\nu^E > 0$ for $E \in \mathbb{R} \setminus [-2, 2]$ and that the columns of U_2 are linearly independent, this implies that $U_2\phi \neq 0$. The fact that $\|u_-^E U_2\phi\|^2 \neq 0$ follows from Eq. (1.28), which implies that if $x \neq 0$ then $\lambda(E)^n u_-^E(n)x \to x \neq 0$ as $n \to -\infty$. With the help of Taylor's theorem and analyticity, it follows that

$$\begin{aligned} V_1^* M_-^z U_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + (z - E) V_1^* \frac{d}{dz} M_-^z\Big|_{z=E} U_1 + \mathcal{O}((z - E)^2) \\ &= \begin{pmatrix} (z - E)\tilde{A} & (z - E)\tilde{B} \\ (z - E)\tilde{C} & 1 + (z - E)\tilde{D} \end{pmatrix} + \mathcal{O}((z - E)^2) \quad \text{as } z \to E. \end{aligned} \tag{3.90}$$

Using the Schur formula (see Proposition 5.9) for the determinant in Eq. (3.90) one gets

$$\begin{aligned}\det(V_1^* M_-^z U_1) &= \det(1 + (z - E)\tilde{D} + \mathcal{O}((z - E)^2))\cdot \\ &\qquad \det((z - E)\tilde{A} + (z - E)^2 \tilde{B}(1 + o(1))\tilde{C} + \mathcal{O}((z - E)^2)) \\ &= \det(1 + (z - E)\tilde{D} + \mathcal{O}((z - E)^2)) \det((z - E)(\tilde{A} + \mathcal{O}(z - E))) \\ &= (z - E)^{n_E} g(z),\end{aligned}$$

where $g(z) = \det(1 + (z - E)D + \mathcal{O}(z - E)^2) \det(\tilde{A} + \mathcal{O}(z - E))$. From the last equation and the fact that $g(E) = \det(\tilde{A}) \neq 0$ the desired result follows. ☐

3.5 Half-bound states

This section analyzes the behavior of the function $z \mapsto \det(M_+^z)$ as $z \to \pm 2$. All results are presented for the case $z \to 2$, but they also hold for $z \to -2$; the corresponding proofs are essentially the same. Throughout this section, we denote

$$J_h^+ := \dim \operatorname{Ker}(\mathcal{W}(u_-^2, u_+^2)).$$

At first look, this definition may appear to differ from the one given in Theorem 1.15. However, as will be verified in Lemma 3.33 (see also Definition 3.28), these two definitions coincide.
This section provides a brief overview of the analysis, presenting only the key results necessary to describe the behavior of the determinant near the spectral edge. A more detailed treatment will be given in Section 3.6.
From the definition of u_+^2, we know that $u_+^2(j) \to \mathbf{1}$ as $j \to \infty$. Therefore, for sufficiently large j, the matrix $u_+^2(j)$ is invertible. To simplify the notation, we assume that $u_+^2(1)$ is already invertible. This assumption introduces no loss of generality, since one can always translate the origin accordingly.

Proposition 3.25. *There exist invertible matrices $P, Q \in \mathbb{C}^{L\times L}$ and matrix valued functions $A(z), B(z), C(z), D(z)$, for $z \in \mathbb{C} \setminus \{2\}$ sufficiently close to 2, such that*

$$P\ \mathcal{W}(u_-^{\overline{z}}, u_+^z) u_+^z(1)^{-1} u_+^2(1) Q = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \tag{3.91}$$

where

$$A(z) = (1-\lambda(z))\boldsymbol{A} + o(|1-\lambda(z)|), \quad B(z) = o(1), \quad C(z) = \mathcal{O}(|1-\lambda(z)|), \quad D(z) = \boldsymbol{D} + o(1). \tag{3.92}$$

In the previous equations, $\boldsymbol{A}$ is a matrix of size $J_h^+ \times J_h^+$ (and this determines the dimensions of the other matrices involved). Moreover, $\boldsymbol{A}$ and and $\boldsymbol{D}$ are invertible matrices. The invertibility of $u_+^z(1)$ for z close to 2 follows from the invertibility of $u_+^2(1)$ and Eq. (3.14).

In Section 3.6, we present a proof of Proposition 3.25 (see Proposition 3.50 and Equation (3.151)). This proposition is a key step to show that the limits

$$T_\pm^2 := \lim_{z\to 2} T_\pm^2$$

exist. It also implies the next proposition, which serves as a preparation for the proof of Levinson's theorem.

Proposition 3.26. *There is a constant* $c \in \mathbb{C} \setminus \{0\}$ *such that*

$$\det(M_+^z) = (\lambda(z) - 1)^{J_h^+ - L}(c + o(1)), \qquad z \to 2\,, \;\; z \in \mathbb{C}.$$

Proof. Using Eqs. (3.56) and (3.91) one has for $z \in \mathbb{C}$ that

$$\det(M_+^z) = (\nu^z)^L \det \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} (a + o(1)), \tag{3.93}$$

where $a = \det(-PQ)^{-1} \neq 0$, and Eq. (3.14) was used. Using Schur formula for the determinant (see Proposition 5.9) and Eq. (3.92), it follows that

$$\det \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} = \det(\boldsymbol{D} + o(1)) \det((1 - \lambda(z))\boldsymbol{A} + o(|1 - \lambda(z)|)) = (\lambda(z) - 1)^{J_h^+}(b + o(1)), \tag{3.94}$$

where b is a non-zero constant. Using Eqs. (3.93), (3.94) and (2.13), the required result follows. □

Remark 3.27. Using Eq. (3.58) and Proposition 3.26 one gets a similar result for M_-^z in a neighborhood of $z = 2$ in $\mathbb{C}$:

$$\det(M_-^z) = (\lambda(z) - 1)^{J_h^+ - L}(\bar{c} + o(1)). \tag{3.95}$$

The corresponding result in a neighborhood of $z = -2$ in $\mathbb{C}$ reads as:

$$\det(M_-^z) = (\lambda(z) + 1)^{J_h^- - L}(d + o(1)), \tag{3.96}$$

where $d \in \mathbb{C} \setminus \{0\}$. ◇

3.6 Band edge limit

The scattering matrix (see Definition 1.12) is well-defined for energies $E \in (-2, 2)$. However, at the band edges $E = \pm 2$, it exhibits singularities. These singularities depend on the decay properties of the potential $V = H - H_0$.

In particular, if the potential has a finite first moment, the limiting behavior of the scattering matrix at the band edges is related to the existence of bounded solutions to the generalized eigenvalue equation

$$(\tau_H u)(n) = E u(n), \quad E = \pm 2,$$

commonly referred to as half-bound states.

This behavior is reflected in Equation (3.56). If the matrix $\mathcal{W}(u_-^2, u_+^2)$ is invertible (generic case), then

$$(M_+^z)^{-1} \to 0 \quad \text{as} \quad z \to 2,$$

since $(\nu^z)^{-1} \to 0$ as $z \to 2$.

However, if the matrix $\mathcal{W}(u_-^2, u_+^2)$ is not invertible (exceptional case), then $(M_+^z)^{-1}$ develops a singularity at $z = 2$. This singularity is governed by the singular part of $\mathcal{W}(u_-^2, u_+^2)$, namely its kernel:

$$\operatorname{Ker}\left(\mathcal{W}(u_-^2, u_+^2)\right).$$

Its turns out that this kernel corresponds to the space of initial conditions of the half-bound states. In this case, one must use local estimates at $E = 2$ on particular solutions to the eigenvalue equation to extract the singularity induced by the singular part of $\mathcal{W}(u_-^2, u_+^2)$. The main difficulty in analyzing the limit at the band edge, as opposed to the case of bound states (Section 3.4), is that the function

$$z \mapsto M_\pm^z$$

is not differentiable at $z = 2$. Therefore, it is necessary to first derive local estimates for solutions with prescribed initial conditions, and then translate these estimates into information about the Wronskian:

$$\mathcal{W}(u_-^{\overline{z}}, u_+^z),$$

which ultimately governs the behavior of the scattering matrix near the band edge. We start with the description of the space $\mathrm{Ker}(\mathcal{W}(u_-^2, u_+^2)))$.

3.6.1 Half-bound states

In this section we study the singular part of the matrix $\mathcal{W}(u_-^2, u_+^2)$, namely $\mathrm{Ker}(\mathcal{W}(u_-^2, u_+^2))$. The main result (see Lemma 3.33) is that this space corresponds to the space of initial conditions of bounded solutions to the generalized eigenvalue equation. These solutions play a fundamental role in the limit of the scattering matrix as z tends to 2. Through all this section we assume that the potential V has finite first moment (see Def. 1.4).

Definition 3.28. *We introduce the notations*

$$\mathcal{N} := \mathrm{Ker}(\mathcal{W}(u_-^2, u_+^2)), \qquad \mathcal{L} := \mathrm{Ker}(\mathcal{W}(u_+^2, u_-^2)). \tag{3.97}$$

The generic case is referred as $\mathcal{N} = \{0\}$, otherwise one speaks of the exceptional case.

Lemma 3.29. *Let $u \in (\mathbb{C}^L)^{\mathbb{Z}}$ be a solution of the eigenvalue equation $\tau_H u = 2u$. The following items are equivalent:*

(i) u is $o(n)$ for $n \to +\infty$.

(ii) u is bounded as n tends to $+\infty$.

(iii) u converges as n tends to $+\infty$.

Moreover, u is $o(1)$ for $n \to +\infty$, if and only if $u = 0$.

Proof. It follows from Equations (1.28) and (1.29) that the columns of u_+^2 and v_+^2 form a basis of all solutions. Then, there are $\alpha, \beta \in \mathbb{C}^L$ such that

$$u = v_+^2\alpha + u_+^2\beta.$$

Due to the asymptotic behavior of v_+^2 at $+\infty$ each of the items is equivalent to $\alpha = 0$, then the asympotic behavior of u_+^2 yields the desired result. ☐

Remark 3.30. *The previous lemma remains valid if we replace* $+\infty$ *by* $-\infty$.

Lemma 3.31. *The following formula holds true*

$$\mathcal{W}(u_-^2, u_+^2) = -\sum_{j=-\infty}^{\infty} (\tau_V u_+^2)(j). \tag{3.98}$$

Proof. It follows from Lemma 3.4 that

$$\begin{aligned} u_+^2(n) &= \mathbf{1} + \sum_{j=n+1}^{\infty} (n-j)(\tau_V u_+^2)(j), \qquad n \in \mathbb{Z}, \\ u_-^2(n) &= \mathbf{1} - \sum_{j=-\infty}^{n-1} (n-j)(\tau_V u_-^2)(j), \qquad n \in \mathbb{Z}. \end{aligned} \tag{3.99}$$

By the first finite moment assumption $|jV_j|$ tends to zero as j tends to minus infinity, also there is a constant C such that $|u_+^2(j)| \leq C|j|$, for $j \leq 0$, (see Lemma 3.12), it follows that (recall $V_n = A_n - 1$)

$$\begin{aligned} &\mathcal{W}(u_-^2, u_+^2)(n) - \mathcal{W}_0(u_-^2, u_+^2)(n) = \\ &u_-^2(n+1)^*(nV_{n+1})\frac{u_+^2(n)}{n} - u_-^2(n)^*((n+1)V_{n+1})\frac{u_+^2(n+1)}{n+1} \to 0, \ n \to -\infty. \end{aligned} \tag{3.100}$$

Now for $n \in \mathbb{Z}$ we have

$$\begin{aligned} \mathcal{W}_0(u_-^2, u_+^2)(n) &= \left(\mathbf{1} - \sum_{j=-\infty}^{n} (n+1-j)(\tau_V u_-^2)(j)^*\right)\left(\mathbf{1} + \sum_{j=n+1}^{\infty} (n-j)(\tau_V u_+^2)(j)\right) \\ &\quad - \left(\mathbf{1} - \sum_{j=-\infty}^{n-1} (n-j)(\tau_V u_-^2)(j)^*\right)\left(\mathbf{1} + \sum_{j=n+2}^{\infty} (n+1-j)(\tau_V u_+^2)(j)\right) \\ &= -\sum_{j=n+1}^{\infty} (\tau_V u_+^2)(j) - \sum_{j=-\infty}^{n} (\tau_V u_+^2)(j)^* \sum_{j=n+1}^{\infty} (n-j)(\tau_V u_+^2)(j) + o(1), \end{aligned}$$

Here, we used that $u_\pm^2(\pm j)$ is bounded for $j \in \mathbb{N}$ and that $\lim_{m\to\infty} \sum_{j=-\infty}^{-m} |j|(\|V_j\| + \|W_j\|) + \sum_{j=m}^{\infty} |j|(\|V_j\| + \|W_j\|) = 0$. Taking the limit $n \to -\infty$ and using Eq. (3.100) we obtain the desired result. □

Lemma 3.32. *The next equations hold true:*

$$u_+^2(n) = -|n|(\mathcal{W}(u_-^2, u_+^2) + o(1)), \qquad n \to -\infty.$$

$$u_-^2(n) = n(\mathcal{W}(u_+^2, u_-^2) + o(1)), \qquad n \to +\infty.$$

Proof. It was proved in Lemma 3.31 that

$$\mathcal{W}(u_-^2, u_+^2) = -\sum_{j=-\infty}^{\infty} (\tau_V u_+^2)(j), \tag{3.101}$$

and, similarly we deduce that

$$\mathcal{W}(u_+^2, u_-^2) = \sum_{j=-\infty}^{\infty} (\tau_V u_-^2)(j). \tag{3.102}$$

Equation (3.11) implies that

$$\begin{aligned} u_+^2(n+1) - u_+^2(n) &= \sum_{j=n+2}^{\infty} (n+1-j)(\tau_V u_+^2)(j) - \sum_{j=n+1}^{\infty} (n-j)(\tau_V u_+^2)(j) \\ &= \sum_{j=n+1}^{\infty} (\tau_V u_+^2)(j) \\ &= -\mathcal{W}(u_-^2, u_+^2) - \sum_{j=-\infty}^{n} (\tau_V u_+^2)(j). \end{aligned}$$

Thus, $u_+^2(n+1) - u_+^2(n) \to -\mathcal{W}(u_-^2, u_+^2)$, as $n \to -\infty$ (recall that $|u_+^2(n)| \leq C|n|$ due to Equation (3.43)). This implies that

$$\frac{1}{|n|}(u_+^2(n) - u_+^2(0)) = \frac{1}{|n|} \sum_{j=n-1}^{0} u_+^2(j+1) - u_+^2(j) \to -\mathcal{W}(u_-^2, u_+^2), \;\; n \to -\infty.$$

and, therefore,

$$\frac{u_+^2(n)}{|n|} \;\to\; -\mathcal{W}(u_-^2, u_+^2), \qquad n \to -\infty.$$

This proves the first equality. The proof of the second is similar. □

The next result establishes a connection between the subspace $\mathcal{N}$ and $\mathcal{L}$ introduced in Definition 3.28 and the half-bound states.

Lemma 3.33 (Half-Bound States)**.** *The next equations hold true:*

$$\mathcal{N} = \{\xi \in \mathbb{C}^L : u_+^2\xi \textit{ is bounded}\}, \qquad \mathcal{L} = \{\chi \in \mathbb{C}^L : u_-^2\chi \textit{ is bounded}\}. \tag{3.103}$$

Moreover, since $\mathcal{W}(u_-^2, u_+^2)^* = -\mathcal{W}(u_+^2, u_-^2)$ *by definition, it follows that*

$$\mathcal{N} = (\mathcal{W}(u_+^2, u_-^2)\mathbb{C}^L)^\perp, \qquad \mathcal{L} = (\mathcal{W}(u_-^2, u_+^2)\mathbb{C}^L)^\perp, \tag{3.104}$$

and, therefore, $\dim(\mathcal{N}) = \dim(\mathcal{L})$.

Proof. Take $\xi \in \mathcal{N}$, then Lemmas 3.29 and 3.32 yield that $u_+^2\xi$ is bounded. Now taking $\xi \in \mathbb{C}^L$ such that $u_+^2\xi$ is bounded, Lemma 3.32 implies that

$$0 = \lim_{n\to-\infty} \frac{u_+^2(n)\xi}{n} = \mathcal{W}(u_-^2, u_+^2)\xi.$$

Therefore the first equality follows. The proof of the second equality is similar. □

Definition 3.34. *In Lemma 3.33, a characterization of $\mathcal{N}$ and $\mathcal{L}$ in terms of half-bound states is presented, in particular we prove that $(u_+^2(j)\xi)_{j\in\mathbb{Z}}$ is bounded, for every $\xi \in \mathcal{N}$ (i.e., it is a half-bound state). For $\xi \in \mathcal{N}$, let us define*

$$\Gamma\xi := \left(\xi - \sum_{j=-\infty}^{\infty} j(\tau_V u_+^2)(j)\xi\right). \tag{3.105}$$

Lemma 3.35. *For every $\xi \in \mathcal{N}$,*

$$\Gamma\xi = \lim_{n\to-\infty} u_+^2(n)\xi. \tag{3.106}$$

Proof. Let $\xi \in \mathcal{N} = \mathrm{Ker}(\mathcal{W}(u_-^2, u_+^2))$, we compute using Lemma 3.31 and Equation (3.18):

$$\begin{aligned}
\xi - u_+^2(n)\xi &= \sum_{j=n+1}^{\infty} (j-n)(\tau_V u_+^2)(j)\xi \\
&= \sum_{j=n+1}^{\infty} j(\tau_V u_+^2)(j)\xi - n\sum_{j=n+1}^{\infty} (\tau_V u_+^2)(j)\xi \\
&= \sum_{j=n+1}^{\infty} j(\tau_V u_+^2)(j)\xi + n\left(\mathcal{W}(u_-^2, u_+^2) + \sum_{j=-\infty}^{n} (\tau_V u_+^2)(j)\right)\xi \\
&= \sum_{j=n+1}^{\infty} j(\tau_V u_+^2)(j)\xi + n\sum_{j=-\infty}^{n} (\tau_V u_+^2)(j)\xi.
\end{aligned}$$

Notice that $u_+^2(j)\xi$ is bounded because $\xi \in \mathcal{N}$ (see Lemma 3.33). Then, the assumption of a finite first moment implies that the first term in the last equation converges, while the second term tends to zero as $n \to -\infty$. Using this above, taking limit in the last equation and using definition of Γ (see Definition 3.28), we complete the proof. □

Lemma 3.36. Γ *is a linear isomorphism from $\mathcal{N}$ to $\mathcal{L}$.*

Proof. Taking $\xi \in \mathcal{N}$ and $\chi = \Gamma\xi$, it follows from Lemma 3.35 and Equation (1.28) that

$$\lim_{n\to-\infty} u_+^2(n)\xi - u_-^2(n)\chi = 0.$$

Then, Remark 3.30 implies that

$$u_+^2(n)\xi = u_-^2(n)\chi, \ n \in \mathbb{Z}. \tag{3.107}$$

We deduce that $u_-^2\chi$ is bounded and, therefore, $\chi \in \mathcal{L}$ and it follows that $\Gamma\mathcal{N} \subset \mathcal{L}$. Let $\chi \in \mathcal{L}$ and $\xi = \lim_{n\to\infty} u_-^2(n)\chi$. As above, we obtain that $u_+^2(n)\xi = u_-^2(n)\chi$ for $n \in \mathbb{Z}$ and, therefore, $\Gamma\xi = \chi$. This proves the subjectivity and as $\mathcal{L}$ and $\mathcal{N}$ have the same dimension, that Γ is bijective. □

Remark 3.37. *It follows from (3.107) that*

$$\Gamma^{-1} = (u_+^2(n)^{-1}u_-^2(n))\big|_{\mathcal{L}}, \tag{3.108}$$

whenever $u_+^2(n)^{-1}$ exists.

3.6.2 Local estimates for solutions

This section is devoted to deriving technical estimates on the solutions of the eigenvalue equation close to $z = E$. This estimates will be instrumental in our subsequent analysis of the Wronskian. The main results of this section are the estimates obtained in Lemmas 3.41, 3.42, see also Def. 3.40. We recall the solutions s_n^z, c_n^z given in Def. 3.1, we also recall the solutions Ψ^z with particular initial condition at $n = 1, 0$, see Lemma 3.10. From now on, we consider a disc centered in 2 with radius 1/2,

$$D(2;1/2) := \{z \in \mathbb{C} : |z-2| \le 1/2\}.$$

Proposition 3.38. *For all $z \in D(2;1/2)$ the following holds true*

- *For all $n \in \mathbb{Z}$*

$$\frac{|s^z(n) - s^2(n)|}{|\lambda(z)-1|} \le C|n|^2|\lambda(z)|^{-|n|}, \qquad \frac{|c^z(n) - c^2(n)|}{|\lambda(z)-1|} \le C|n||\lambda(z)|^{-|n|}. \tag{3.109}$$

- *For all $n, m \in \mathbb{Z}$ with $m/n > 1$ we have*

$$\frac{|s_n^z(m) - s^z(m) - s_n^2(m) + s^2(m)|}{|\lambda(z)-1|} \le C|n\cdot m||\lambda(z)|^{-|m|}. \tag{3.110}$$

Where C denotes a constant that does not depend on n, m, z.

Proof. We start with the proof of the L.H.S of Eq. (3.109). Suppose that $n \ge 0$, using Eq. (3.9) we obtain (recall $|\lambda(z)| \le 1$)

$$\frac{s^z(n) - s^2(n)}{\lambda(z)-1} = \lambda(z)^{-n+1}\sum_{j=0}^{n-1}\frac{\lambda(z)^{2j}-1}{\lambda(z)-1} + n\frac{\lambda(z)^{-n+1}-1}{\lambda(z)-1} \tag{3.111}$$

Then we obtain

$$\left|\frac{s^z(n) - s^2(n)}{\lambda(z)-2}\right| \le |\lambda(z)|^{-n+1}|\lambda(z)+1|\sum_{j=0}^{n-1} j + n(n-1)|\lambda(z)+1||\lambda(z)|^{-n+1}. \tag{3.112}$$

The above equation implies the desired, for $n \leq 0$ is analogues. Now we prove the R.H.S. of Eq. (3.109). Suppose that $n \geq 0$. Taking Eq. (3.4), (note that $c^2(n) = 1$) we obtain

$$\frac{c^z(n) - c^2(n)}{\lambda(z) - 1} = \frac{1}{\lambda(z)+1}\left(\frac{\lambda(z)^n - 1}{\lambda(z)-1} + \frac{\lambda(z)^{-n+1}(1-\lambda(z)^n)}{\lambda(z)-1}\right). \tag{3.113}$$

Taking modulus in the above equation and using the general fact $a^n - b^n = (a-b)(\sum_{j=0}^{n-1} a^j b^{n-1-j})$ we obtain the desired. Now we prove Eq. (3.110). Suppose that $m \geq n \geq 0$, Eq. (3.3) implies that

$$s_n^z(m) = \lambda(z)^n s^z(m) - \lambda(z)^m s^z(n). \tag{3.114}$$

Then we have (recall $s_n^2(m) = m - n$)

$$\begin{aligned}\frac{s_n^z(m) - s^z(m) - s_n^2(m) + s^2(m)}{\lambda(z)-1} &= \frac{\lambda(z)^n s^z(m) - \lambda(z)^m s^z(n) - s^z(m) + n}{\lambda(z)-1}\\ &= \frac{s^z(m)(\lambda(z)^n - 1)}{\lambda(z)-1} + \frac{s^2(n) - \lambda(z)^m s^z(n)}{\lambda(z)-1}\\ = s^z(m)\frac{\lambda(z)^n-1}{\lambda(z)-1} + s^2(n)&\frac{1-\lambda(z)^m}{\lambda(z)-1} + \lambda(z)^m\frac{s^2(n)-s^z(n)}{\lambda(z)-1}\end{aligned} \tag{3.115}$$

Taking modulus in the previous equation and using Eqs. (3.6), (3.109) and the general fact $a^n - b^n = (a-b)(\sum_{j=0}^{n-1} a^j b^{n-1-j})$ we obtain the desired (recall $|\lambda(z)| \leq 1$ and $|m| \geq |n|$). □

Proposition 3.39. *Let Ψ^z solution of Eq. (3.1) with fixed initial conditions at 0 and 1, let us say $\Psi^z(0) = a$, $\Psi^z(1) = b$. Then one has for all $n \in \mathbb{Z}$,*

$$\frac{\Psi^z(n) - \Psi^2(n)}{\lambda(z) - 1} \to 0, \;\; z \to 2. \tag{3.116}$$

Proof. We note that (recall $\lambda(z) + 1/\lambda(z) = z$)

$$\frac{z-2}{\lambda(z)-1} = \frac{\lambda(z)-1}{\lambda(z)-1} + \frac{1/\lambda(z)-1}{\lambda(z)-1} \to 0. \tag{3.117}$$

Lemma 3.10 implies that the function $z \mapsto \Psi^z(n)$ is analytic. Then, the followin limit exists

$$\frac{d}{dz}\Psi^z(n)\Big|_{z=2} = \lim_{z\to 2}\frac{\Psi^z(n) - \Psi^2(n)}{z-2}\frac{\lambda(z)-1}{\lambda(z)-1} = \lim_{z\to 2}\frac{\Psi^z(n)-\Psi^2(n)}{\lambda(z)-1}\frac{\lambda(z)-1}{z-2} \tag{3.118}$$

Eqs. (3.117) and (3.118) imply the desired. □

Of particular importance for our analysis are the solutions with tinitial condition $a = u_+^2(0)$, $b = u_+^2(1)$.

Definition 3.40. *For all $z \in D(2; 1/2)$, the solution described in Lemma 3.10 with $a = u_+^2(0)$ and $b = u_+^2(1)$ is denoted by $\Theta^z \in (\mathbb{C}^{L\times L})^{\mathbb{Z}}$.*

Notice that $\Theta^2 = u_+^2$. In this case, we have that (see Eq. 3.18, recall $s_j(n) = n - j$ and $u_{+,0}^2 = 1$)

$$b - a = u_+^2(1) - u_+^2(0) = \sum_{j=1}^{\infty} (\tau_V u_+^2)(j) = \sum_{j=0}^{\infty} (\tau_V^+ u_{+,0}^2)(j)\Theta^2(j). \tag{3.119}$$

Let us set

$$d(n) := \sum_{j=n}^{\infty} (\tau_V^+ u_{+,0}^2)(j)\Theta^2(j).$$

It follows from Eqs. (3.41) and (3.42) that for every $n \in \mathbb{N}$,

$$\begin{aligned} A_n\Theta^z(n) = s^z(n)d(n) + c^z(n)a - \sum_{j=0}^{n-1} (\tau_V^+ s_\cdot^z(n)^*)(j)^*(\Theta^z(j) - \Theta^2(j)) \\ + \sum_{j=0}^{n-1} (\tau_V^+ (s^z(n) - s_\cdot^z(n))^*)(j)^*\Theta^2(j), \end{aligned} \tag{3.120}$$

and for every $n \in \mathbb{Z}^- \cup \{0\}$

$$A_{n+1}\Theta^z(n) = s^z(n)(b-a) + c^z(n)a + \sum_{j=n+1}^{1} (\tau_V^- s_\cdot^z(n)^*)(j)^*\Theta^z(j), \tag{3.121}$$

As usual we are denoting $s_\cdot^z(n)$ to the sequence $j \mapsto s_j^z(n)$, then (see Eq. (1.10))

$$\begin{aligned} &(\tau_V(s^z(n) - s_\cdot^z(n))^*)(j)^* \\ &= (s^z(n) - s_{j-1}^z(n))W_j + (s^z(n) - s_j^z(n))V_j + (s^z(n) - s_{j+1}^z(n))W_{j+1}. \end{aligned} \tag{3.122}$$

We provide some technical estimates for the solutions Θ^z for values of z close to 2. These estimates later allow us to derive bounds for the coefficients $M_\pm^z$ when z is near 2.

Lemma 3.41 (Regularity 1)**.** *Suppose that V has finite first moment. Then, there is a constant $C \in \mathbb{R}$ independent of n and z such that, for all $z \in D(2; 1/2)$,*

$$\left\| \frac{\Theta^z(n) - \Theta^2(n)}{\lambda(z) - 1} \right\| \le Cn|\lambda(z)|^{-n}, \qquad \forall n \in \mathbb{N}. \tag{3.123}$$

Proof. It follows from (3.120), that

$$A_n\Theta^z(n) - A_n\Theta^2(n) = (s^z(n) - s^2(n))d(n) + (c^z(n) - c^2(n))a \tag{3.124}$$

$$- \sum_{j=0}^{n-1} (\tau_V^+ s_\cdot^z(n)^*)(j)^*(\Theta^z(j) - \Theta^2(j))$$

$$+ \sum_{j=0}^{n-1} \tau_V^+ \Big(s^z(n)^* - s_\cdot^z(n)^* - s^2(n)^* + s_\cdot^2(n)^*\Big)(j)^*\Theta^2(j). \tag{3.125}$$

Equation (3.124), Prop. 3.38, and finite first moment assumption (which in particular imply that $nd(n)$ is uniformly bounded, see Def.(1.4)) give us

$$\frac{|\lambda(z)|^n}{n}\Big\|\frac{\Theta^z(n)-\Theta^2(n)}{\lambda(z)-1}\Big\| \leq C\Big[1+\sum_{j=0}^{n-1} j(\|V_j\|+\|W_j\|+\|W_{j+1}\|)\ \frac{|\lambda(z)|^j}{j}\ \Big\|\frac{\Theta^z(j)-\Theta^2(j)}{\lambda(z)-1}\Big\|\Big], \tag{3.126}$$

here we use that $\Theta^2(j)=u^2_+(j)$ is uniformly bounded for $j \geq 0$ (which is a consequence of the definition of the Jost solution at stake). Equation (3.208) and Gronwall's Lemma (see Lemma 5.4) combined with the finite first moment assumption imply (3.123). ☐

If the potential V has finite first moment. Lemma 3.12 (with Θ^2 playing the role of Ψ^2) implies that the series

$$\sum_{j=-\infty}^{\infty}(\tau_V\Theta^2)(j)=\sum_{j=-\infty}^{\infty}(\tau_V u^2_+)(j)=\sum_{j=-\infty}^{\infty}(V_j+W_j+W_{j+1})u^2_+(j), \tag{3.127}$$

converges to a matrix in $\mathbb{C}^{L\times L}$. This matrix plays an important role in the proofs as it is connected to the Wronskian of u^2_+ and u^2_-, see Lemma 3.31. In Lemma 3.33 (see Definition 3.28), we prove that for every vector ξ that belongs to its kernel, the sequence $(u^2_+(j)\xi)_{j\in\mathbb{Z}}$ is bounded. The corresponding states $(u^2_+(j)\xi)_{j\in\mathbb{Z}}$ are called half-bound states.

Lemma 3.42 (Regularity 2)**.** *Suppose that V has finite first moment. Let ξ belong to the kernel of (3.127). There is a constant $C\in\mathbb{R}$, independent of n and z, such that, for all $z\in D(2,1/2)$,*

$$\Big\|\frac{1}{\lambda(z)-1}(\Theta^z(n)-\Theta^2(n))\xi\Big\| \leq C|n||\lambda(z)|^n, \qquad \forall\, n\in\mathbb{Z}^-\cup\{0\}. \tag{3.128}$$

Proof. In this case, (see Equations (3.119), (3.38), (3.40) and recall that ξ belongs to the kernel of (3.127))

$$(b-a)\xi=\sum_{j=0}^{\infty}(\tau_V^+u^2_{+,0})(j)\Theta^2(j)\xi=-\sum_{j=-\infty}^{1}(\tau_V^-u^2_{+,0})(j)\Theta^2(j)\xi. \tag{3.129}$$

Hence let us set

$$d_-(n):=-\sum_{j=-\infty}^{n}(\tau_V^-u^2_{+,0})(j)\Theta^2(j).$$

Eq. (3.127) and Lemma 3.31 imply that $\xi\in\mathrm{Ker}(\mathcal{W}(u^2_-,u^2_+))=\mathcal{N}$ (see Def. 3.28), then Lemma 3.33 implies $u^2_+\xi=\Theta^2\xi$ is bounded, this together with finite first moment assumption on the potencial imply

$$|nd_-(n)| \leq \sum_{j=-\infty}^{n}|j|\,(\|V_j\|+\|W_j\|+\|W_{j+1}\|)\|\Theta^2(j)\xi\|<\infty. \tag{3.130}$$

It follows from Eq. (3.121) and (3.129) that for every $n \in \mathbb{Z}^- \cup \{0\}$

$$A_{n+1}\Theta^z(n)\xi = s^z(n)d_-(n)\xi + c^z(n)a\xi + \sum_{j=n+1}^{1} (\tau_V^- s_\cdot^z(n)^*)(j)V(j)(\Theta^z(j) - \Theta^2(j))\xi \tag{3.131}$$
$$+ \sum_{j=n+1}^{1} \big(\tau_V^-(s_\cdot^z(n)^* - s^z(n)^*)\big)(j)^*\Theta^2(j)\xi.$$

Then, we have that

$$\Big\|\frac{|\lambda(z)|^{-n}}{n}\frac{1}{\lambda(z)-1}(\Theta^z(n)\xi - \Theta^2(n)\xi)\Big\| \leq \frac{|\lambda(z)|^{-n}}{n}\frac{1}{|\lambda(z)-1|}\Big[\|(s^z(n) - s^2(n))d_-(n)\xi\|$$
$$+ \|(c^z(n) - c^2(n))a\xi\| + \sum_{j=n+1}^{1} \|(\tau_V^- s_\cdot^z(n)^*)(j)(\Theta^z(j) - \Theta^2(j))\xi\|$$
$$+ \sum_{j=n+1}^{1} \|\big(\tau_V^-(s_\cdot^z(n)^* - s^z(n)^* + s^2(n)^* - s_\cdot^2(n)^*)\big)(j)^*\Theta^2(j)\xi\|\Big]. \tag{3.132}$$

Bounding the first summand on the right is bounded using the second estimate from (3.109) and (3.130), the second summand with (3.109) and the fourth summand with (3.110) together with the finite first moment assumption (recall $\Theta^2\xi$ is bounded), one deduces

$$\frac{|\lambda(z)|^{-n}}{|n|}\Big\|\frac{\Theta^z(n) - \Theta^2(n)}{\lambda(z)-1}\xi\Big\| \tag{3.133}$$
$$\leq C\Big[1 + \sum_{j=n+1}^{1} |j|(\|W_j\| + \|W_{j-1}\| + \|V_j\|)\frac{|\lambda(z)|^{-j}}{|j|}\ \Big\|\frac{\Theta^z(j) - \Theta^2(j)}{\lambda(z)-1}\xi\Big\|\Big],$$

where we use also use Eq. (3.6). Equation (3.133) and Gronwall's inequality (see Lemma 5.4) imply (3.128). □

3.6.3 Analysis of the Wronskian

It follows from (3.54) and Definition 1.12 that the Wronskian is tightly connected to the scattering matrix. In particular, the invertibility of $M_\pm^z$ is essential for its definition. Since the purpose of this section is the analysis of the scattering matrix as z tends to 2, it is crucial to study the behavior of $\mathcal{W}(u_-^{\overline{z}}, u_+^z) = -(\nu^z)^{-1}M_+^z$ as z tends to 2 (the study of M_-^z is carried out using (3.58)). Regularity properties of u_+^z as z tends to 2 are thus relevant. As stated above, this will be deduced from the regularity results on Θ^z. Indeed, it turns out that these properties of Θ^z allow to identify lower order terms of $W(u_-^{\overline{z}}, u_+^z)$ with respect to $|\lambda(z) - 1|$. This holds because $\mathcal{W}(u_-^{\overline{z}}, u_+^z)$ can be written in terms of $\mathcal{W}(\Theta^{\overline{z}}, u_+^z)$ and $\mathcal{W}(u_-^{\overline{z}}, \Theta^z)$ (see Lemma 3.45). Then, most of this section is devoted to the study of $\mathcal{W}(\Theta^{\overline{z}}, u_+^z)$ and $\mathcal{W}(u_-^{\overline{z}}, \Theta^z)$. The main result of this section is Prop. 3.50, which describes the local behavior of the function $\mathcal{W}(u_-^{\overline{z}}, u_+^z)$ for $z \in \mathbb{C}$ close to $z = 2$.

From the definition of u_+^2, we know that $u_+^2(j)$ tends to $\mathbf{1}$ as j tends to infinity. Then, for large enough j, $u_+^2(j)$ is invertible. In order to simplify notations, we assume that $u_+^2(1)$ is already invertible. This does not imply any restriction because, translating the origin, we can always take it for granted.

Assumption 3.43. *We assume, without loss of generality, that $u_+^2(1)$ is invertible.*

Remark 3.44. *From now on, we consider the function $F(z) = \mathcal{W}(u_-^{\bar{z}}, u_+^z)$, defined on the complex plane, extended to $[-2,2]$ with the limit from above. Namely, for all $E \in [-2,2]$, see Prop. 3.5 and Rem. 3.6*

$$F(E) = \mathcal{W}(u_{-,in}^E, u_{+,out}^E). \tag{3.134}$$

Lemma 3.45. *For every $z \in \mathbb{C}$, it follows that*

$$\mathcal{W}(u_-^{\bar{z}}, u_+^z) = u_-^{\bar{z}}(1)^*(u_+^2(1)^*)^{-1}\mathcal{W}(\Theta^{\bar{z}}, u_+^z) + \mathcal{W}(u_-^{\bar{z}}, \Theta^z)u_+^2(1)^{-1}u_+^z(1). \tag{3.135}$$

Proof. The result follows from an expansion of the right hand side of (3.135) using Definition 3.40 of Θ^z and the definition of the Wronskians, evaluated on $n = 0$, and the identity (see (3.46))

$$\left(u_+^2(1)^*\right)^{-1}u_+^2(0)^*A_1 = A_1u_+^2(0)u_+^2(1)^{-1}.$$

Notice that, by definition, $\Theta^z(j) = u_+^2(j)$, for $j \in \{0,1\}$. ☐

Proposition 3.46. *The following formula holds true*

$$\mathcal{W}(\Theta^{\bar{z}}, u_+^z) = (1-\lambda(z))\mathbf{1} + o(|\lambda(z)-1|), \tag{3.136}$$

as z tends to 2 in $\mathbb{C}$.

Proof. Rem. 1.11 and the asympthotic behavior of w_+^z (see Eq. (1.30)) imply that $|\lambda(z)|^n|\Theta^z(n)|$ is bounded (with respect to n, for positive n). This implies that (see Eq. (1.28) and recall $A_{n+1} \to \mathbf{1}$)

$$\begin{aligned}\mathcal{W}(\Theta^{\bar{z}}, u_+^z) &= \lim_{n\to\infty}(\lambda(z)^n\Theta^{\bar{z}}(n+1)^* - \lambda(z)^{n+1}\Theta^{\bar{z}}(n)^*)A_{n+1}\\ &= (b-a)^* + (1-\lambda(z))a^* - \sum_{j=1}^{\infty}\lambda(z)^j(\tau_V\Theta^{\bar{z}})(j)^*,\end{aligned}$$

where in the second step we used (3.34) and Eqs. (3.3), (3.4). On the other hand,

$$\sum_{j=1}^{\infty}\lambda(z)^j(\tau_V(\Theta^{\bar{z}} - \Theta^2))(j)^* = o(|\lambda(z)-1|). \tag{3.137}$$

Indeed, Equations (3.123) and the first finite moment assumption imply that the series multiplied by $\frac{1}{\lambda(z)-1}$ is bounded by a summable function that does not depend on z, and (3.116) implies that each term of the sum multiplied by $\frac{1}{\lambda(z)-1}$ tends to zero. Hence interpreting the series as an integral with respect to a counting measure, Lebesgue's dominated convergence theorem shows (3.137). Furthermore, since Θ^2 is bounded for $j > 0$ (because $\Theta^2(j) = u_+^2(j)$, for $j > 1$) it follows that

$$\sum_{j=1}^{\infty}(\lambda(z)^j - 1 - j(\lambda(z)-1))(\tau_V\Theta^2)(j)^* = o(|\lambda(z)-1|). \tag{3.138}$$

Then, we obtain that

$$\mathcal{W}(\Theta^{\overline{z}}, u_+^z) = (b-a)^* + (1-\lambda(z))a^* - \sum_{j=1}^{\infty}(1 + j(\lambda(z)-1))(\tau_V\Theta^2)(j)^* + o(|\lambda(z)-1|).$$

This last equation and the fact that (see (3.119) and (3.18))

$$\begin{aligned}(b-a) &= \sum_{j=1}^{\infty}(\tau_V u_+^2)(j) = \sum_{j=1}^{\infty}(\tau_V\Theta^2)(j),\\ a &= \mathbf{1} - \sum_{j=1}^{\infty} j(\tau_V u_+^2)(j) = \mathbf{1} - \sum_{j=1}^{\infty} j(\tau_V\Theta^2)(j),\end{aligned} \tag{3.139}$$

imply the desired result. $\square$

In the following proposition, we relate the asymptotic behavior of the Wronskian to the isomorphism defined in Definition 3.34; see also Definition 3.28.

Proposition 3.47. *The following formula holds true*

$$\mathcal{W}(u_-^{\overline{z}}, \Theta^z) = \mathcal{W}(u_-^2, u_+^2) + o(1), \tag{3.140}$$

as z tends to 2 in $\mathbb{C}$. Moreover, if $\xi \in \mathcal{N}$, then

$$W(u_-^{\overline{z}}, \Theta^z)\xi = (1-\lambda(z))\Gamma\xi + o(|\lambda(z)-1|), \tag{3.141}$$

as z tends to 2 in $\mathbb{C}$.

Proof. Equation (3.140) follows from the continuity of the function $z \mapsto \Theta^z$, and the fact that $\Theta^2 = u_+^2$ also see Eq. (3.14). Now we prove (3.141). Rem. 1.11 and the asympthotic behavior of w_-^z (see Eq. (1.30)) imply that $|\lambda(z)|^{-n}|\Theta^z(n)|$ is bounded (with respect to n, for negative n). This implies that (see Equation (1.28) and (3.21), recall $\lambda(\overline{z})^* = \lambda(z)$, for $z \in \mathbb{C} \setminus [-2,2]$ and $\lambda(E)^* = \lambda(E)^{-1}$ for $E \in [-2,2]$, also see Rem. 3.44)

$$\begin{aligned}\mathcal{W}(u_-^{\overline{z}}, \Theta^z) &= \lim_{n\to-\infty}(\lambda(z)^{-n-1}\Theta^z(n) - \lambda(z)^{-n}\Theta^z(n+1))A_{n+1}\\ &= -(b-a) + \frac{1-\lambda(z)}{\lambda(z)}a - \sum_{j=-\infty}^{0}\lambda(z)^{-j}(\tau_V\Theta^z)(j),\end{aligned}$$

where (3.35) was used. Utilizing (3.116), (3.128) and the finite first moment assumption we deduce that (here we use again Lebesgue's dominated convergence theorem)

$$\sum_{j=-\infty}^{0} \lambda(z)^{-j}\tau_V(\Theta^z - \Theta^2)(j)\xi = o(|\lambda(z) - 1|). \tag{3.142}$$

Since $\Theta^2(j)\xi$ is bounded (see Lemma 3.33), it follows that for $j \leq 0$

$$\sum_{j=-\infty}^{0} (\lambda(z)^{-j} - 1 - j(1 - \lambda(z)))(\tau_V\Theta^2)(j)\xi = o(|\lambda(z) - 1|). \tag{3.143}$$

Then

$$\mathcal{W}(u_-^{\bar{z}}, \Theta^z)\xi = -(b-a)\xi + \frac{1-\lambda(z)}{\lambda(z)}a\xi - \sum_{j=-\infty}^{0}(1 + j(1-\lambda(z)))(\tau_V\Theta^2)(j)\xi + o(|\lambda(z) - 1|).$$

The desired result from this last equation and (3.139). Notice that we use that $\frac{1-\lambda(z)}{\lambda(z)} - (1 - \lambda(z)) = o(|1 - \lambda(z)|)$, and Eq. (3.105). Also, see that by assumption (see Lemma 3.31 and Definition 3.28) $\sum_{j=-\infty}^{\infty}(\tau_V\Theta^2)(j)\xi = 0$ and recall that $u_+^2 = \Theta^2$. □

The operator $u_-^{\bar{z}}(1)^*(u_+^2(1)^*)^{-1}$ that appears in (3.135) plays an important role because, when $z = 2$, it operates on half-bound states (see Remark 3.37) and it is present in the scattering matrix, in the limit when z tends to 2. For this reason, we also introduce a notation for this object.

Definition 3.48. *We denote*

$$\Omega := u_+^2(1)^{-1}u_-^2(1) \in \mathbb{C}^{L\times L}. \tag{3.144}$$

We recall the isomorphism $\Gamma : \mathcal{N} \to \mathcal{L}$ (see Def. 3.34). Notice that (see Rem. 3.37 and Lemma 3.36)

$$\Omega\big|_{\mathcal{L}} = \Gamma^{-1}. \tag{3.145}$$

Definition 3.49. *We denote by*

$$P_{\mathcal{N}}, \quad P_{\mathcal{L}}, \quad P_{\mathcal{N}^\perp}, \quad P_{\mathcal{L}^\perp} \tag{3.146}$$

the orthogonal projections onto $\mathcal{N}$, $\mathcal{L}$, $\mathcal{N}^\perp$ and $\mathcal{L}^\perp$ respectively.

Proposition 3.50. *There exist functions $X, Y : D(2; 1/2) \to \mathbb{C}^{L\times L}$ such that*

$$\begin{aligned}\mathcal{W}(u_-^{\bar{z}}, u_+^z) = \Big(&(1-\lambda(z))(\Omega^* + \Gamma)P_{\mathcal{N}} + \mathcal{W}(u_-^2, u_+^2)P_{\mathcal{N}^\perp} + X(z) + Y(z)P_{\mathcal{N}^\perp}\Big)\\ &\cdot u_+^2(1)^{-1}u_+^z(1),\end{aligned} \tag{3.147}$$

and $X(z) = o(|\lambda(z) - 1|)$ and $Y(z) = o(1)$, as z tends to 2 in $\mathbb{C}$.

Proof. The result follows from Eq. (3.14) (which in paricular implies $\mathcal{W}(u_-^{\bar{z}}, u_+^z) = \mathcal{W}(u_-^2, u_+^2) + o(1)$), Propositions 3.46, 3.47 and Lemma 3.45. □

3.6.4 Band edge limit of the scattering matrix

Let $\{e_1, \cdots, e_L\}$ be an orthonormal basis of $\mathbb{C}^L$ such that the first d vectors form a basis of $\mathcal{L}$ and the last vectors $L-d$ from a basis of $\mathcal{L}^\perp = \mathcal{W}(u_-^2, u_+^2)\mathbb{C}^L$, see Lemma 3.33. We take another orthonormal basis $\{v_1, \cdots, v_L\}$ of $\mathbb{C}^L$ such that the first d vectors form a basis of $\mathcal{N}$ and the last $L-d$ vectors form a basis of $\mathcal{N}^\perp$. Then define

$$P := \begin{pmatrix} e_1 & e_2 & \cdots & e_L \end{pmatrix}^*, \qquad Q := \begin{pmatrix} v_1 & v_2 & \cdots & v_L \end{pmatrix}. \tag{3.148}$$

We recall that $P_{\mathcal{L}}$ and $P_{\mathcal{L}^\perp}$ are the projections onto $\mathcal{L}$ and $\mathcal{L}^\perp$, respectively. Then

$$P_{\mathcal{L}}\Big((1-\lambda(z))(\Omega^* + \Gamma)P_{\mathcal{N}} + \mathcal{W}(u_-^2, u_+^2)P_{\mathcal{N}^\perp}\Big)P_{\mathcal{N}} = (1-\lambda(z))P_{\mathcal{L}}(\Omega^* + \Gamma)P_{\mathcal{N}} \tag{3.149}$$

and $P_{\mathcal{L}}(\Omega^* + \Gamma)P_{\mathcal{N}}$ defines a bijection between $\mathcal{N}$ and $\mathcal{L}$ (see Lemma 3.33): in view of Lemma 3.33 it is enough to prove that it is injective. This holds true because $\Gamma : \mathcal{N} \to \mathcal{L}$ is a bijection and, for every $\xi \in \mathcal{N}$, (see Equation (3.145))

$$\begin{aligned}\langle \Gamma\xi, P_{\mathcal{L}}(\Omega^* + \Gamma)\xi\rangle &= \langle P_{\mathcal{L}}\Gamma\xi, (\Omega^* + \Gamma)\xi\rangle = \langle \Gamma\xi, (\Omega^* + \Gamma)\xi\rangle \\ &= \langle \Omega\Gamma\xi, \xi\rangle + \|\Gamma\xi\|^2 = \|\xi\|^2 + \|\Gamma\xi\|^2.\end{aligned}$$

Moreover,

$$P_{\mathcal{L}^\perp}\Big((1-\lambda(z))(\Omega^* + \Gamma)P_{\mathcal{N}} + \mathcal{W}(u_-^2, u_+^2)P_{\mathcal{N}^\perp}\Big)P_{\mathcal{N}^\perp} = P_{\mathcal{L}^\perp}\mathcal{W}(u_-^2, u_+^2)P_{\mathcal{N}^\perp} \tag{3.150}$$

defines a bijection between $\mathcal{N}^\perp$ and $\mathcal{L}^\perp$ (see Lemma 3.33 and Definition 3.28). It follows that there are matrix-valued functions $A(z), B(z), C(z), D(z)$ such that

$$P\Big((1-\lambda(z))(\Omega^* + \Gamma)P_{\mathcal{N}} + \mathcal{W}(u_-^2, u_+^2)P_{\mathcal{N}^\perp} + X(z) + Y(z)P_{\mathcal{N}^\perp}\Big)Q = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \tag{3.151}$$

and

$$\begin{aligned} &A(z) = (1-\lambda(z))\boldsymbol{A} + o(|1-\lambda(z)|), \quad B(z) = o(1), \\ &C(z) = O(|1-\lambda(z)|), \quad D(z) = \boldsymbol{D} + o(1), \end{aligned} \tag{3.152}$$

where

$$\boldsymbol{A} := [P_{\mathcal{L}}(\Omega^* + \Gamma)P_{\mathcal{N}}]_\alpha^\beta, \qquad \boldsymbol{D} := [P_{\mathcal{L}^\perp}\mathcal{W}(u_-^2, u_+^2)P_{\mathcal{N}^\perp}]_\gamma^\delta, \tag{3.153}$$

are invertible where $[T]_\theta^\eta$ denotes the matrix representation of a linear transformation T in terms of the bases θ, η. Here $\alpha = \{v_1, ..., v_d\}$, which is a basis of $\mathcal{N}$, $\beta = \{e_1, ..., e_L\}$, which is a basis of $\mathcal{L}$, and $\gamma = \{v_{d+1}, ..., v_L\}$ and $\delta = \{e_{d+1}, ..., e_d\}$, which are bases of $\mathcal{N}^\perp$ and $\mathcal{L}^\perp$, respectively.

Theorem 3.51. *The transmission coefficients satisfy the following properties:*

$$T_+^z = T_+^2 + o(1), \qquad T_-^z = T_-^2 + o(1), \tag{3.154}$$

as z tends to 2 in $\mathbb{C}$, where

$$T_+^2 := Q \begin{pmatrix} 2\boldsymbol{A}^{-1} & 0 \\ 0 & 0 \end{pmatrix} P, \qquad T_-^2 := \left(Q \begin{pmatrix} 2\boldsymbol{A}^{-1} & 0 \\ 0 & 0 \end{pmatrix} P \right)^*. \tag{3.155}$$

Moreover,

$$\begin{aligned} T_+^2\mathbb{C}^L &= \mathcal{N}, \qquad & \operatorname{Ker}(T_+^2) &= \mathcal{W}(u_-^2, u_+^2)\mathbb{C}^L, \\ T_-^2\mathbb{C}^L &= \mathcal{L}, \qquad & \operatorname{Ker}(T_-^2) &= \mathcal{W}(u_+^2, u_-^2)\mathbb{C}^L. \end{aligned} \tag{3.156}$$

The reflection coefficients satisfy the following properties:

$$R_+^E = R_+^2 + o(1), \qquad R_-^E = R_-^2 + o(1), \tag{3.157}$$

as $E \in [-2, 2]$ tends to 2, where

$$R_+^2 := -\mathbf{1} + \Gamma T_+^2, \qquad R_-^2 := -\mathbf{1} + \Gamma^{-1} T_-^2. \tag{3.158}$$

Moreover,

$$\begin{aligned} \mathcal{L} &= (\mathbf{1} - R_+^2)\mathbb{C}^L, \qquad & \operatorname{Ker}(T_+^2) &= \operatorname{Ker}(\mathbf{1} - R_+^2), \\ \mathcal{N} &= (\mathbf{1} - R_-^2)\mathbb{C}^L, \qquad & \operatorname{Ker}(T_-^2) &= \operatorname{Ker}(\mathbf{1} - R_-^2). \end{aligned} \tag{3.159}$$

Proof. Equation (3.152) implies that the matrices $D(z)$ and $A(z) - B(z)D(z)^{-1}C(z)$ are invertible for $z \in \mathbb{C}$ in a neighborhood of 2, so using the Schur complement formula, it follows that (recall (2.13))

$$\begin{aligned} &(\nu^z)^{-1} \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}^{-1} \\ &= (\nu^z)^{-1} \begin{pmatrix} \mathbf{1} & 0 \\ -D(z)^{-1}C(z) & \mathbf{1} \end{pmatrix} \begin{pmatrix} \left(A(z) - B(z)D(z)^{-1}C(z)\right)^{-1} & 0 \\ 0 & D(z)^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & -B(z)D(z)^{-1} \\ 0 & \mathbf{1} \end{pmatrix} \\ &= \begin{pmatrix} 2\boldsymbol{A}^{-1} & 0 \\ 0 & 0 \end{pmatrix} + o(1), \end{aligned} \tag{3.160}$$

where (2.13) and (3.152) were used. The first equation in (3.155) follows from Equation (3.160), Proposition 3.50, Eqs. (3.14), (3.56) and Definition 1.12. The second equation in (3.155) is a consequence of the first equation, Definition 1.12 and (3.58).

Due to the definition (3.148) of P, the kernel of T_+^2 is generated by $\{e_{d+1}, \cdots, e_L\}$, and they are a basis of $W(u_-^2, u_+^2)\mathbb{C}^L$: since P is unitary, $P^* = P^{-1} = \left(e_1 \cdots e_L\right)$. Therefore, the kernel at stake equals the kernel of $\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \left(e_1 \cdots e_L\right)^{-1}$. Due to the definition (3.148) of

Q, the image of T_+^2 is generated by $\{v_1, \cdots, v_d\}$, and these vectors form a basis of $\mathcal{N}$. This proves the first line in (3.156). The second one follows from the fact that $T_-^2 = (T_+^2)^*$ (which can be deduced from (3.58) and (1.37)).

Next let us take small enough n such that $(u_-^2(n))^{-1}$ exists. Then, by continuity, $(u_{-,out}^E(n))^{-1}$ exists, for E in a neighborhood of 2. Using (1.36) leads to

$$\begin{aligned} R_+^E &= -(u_{-,out}^E(n))^{-1} u_{-,in}^E(n) + (u_{-,out}^E(n))^{-1} u_{+,out}^E(n) T_+^E \\ &\to -\mathbf{1} + (u_-^2(n))^{-1} u_+^2(n) T_+^2, \end{aligned} \tag{3.161}$$

as E tends to 2. Taking the limit $n \to -\infty$ in the right hand side of (3.161), we arrive at the first equation in (3.158) (see also Lemma 3.35 and (1.28)). The second equation is obtained similarly. Equations (3.159) follow from (3.156), (3.158) and the fact that Γ is a bijection from $\mathcal{N}$ onto $\mathcal{L}$. □

3.7 Transformation operator

We recall the Jost solutions constructed in Section 3 (see Lemma 3.4), where they are represented as solutions to a recurrence relation, which takes the form of a Volterra-type equation (see Eqs. (3.11), (3.12)). In this section, we present an alternative representation of the Jost solutions as a series involving the free Jost solutions. Specifically, we construct a sequence $(K(n,m))_{n,m\in\mathbb{Z}}$, with $K(n,m) \in \mathbb{C}^{L\times L}$, such that

$$u_+^z(n) = \sum_{m\in\mathbb{Z}} K(n,m) u_{+,0}^z(m), \tag{3.162}$$

where

$$K(n,m) = 0, \quad \text{for } m < n. \tag{3.163}$$

This representation, along with suitable estimates for the kernel K, plays a key role in the analysis of inverse scattering problems. In particular, we use it to derive asymptotic estimates for the Jost solutions u_+^z as $|z| \to \infty$. Equation (3.162) is commonly referred to as the transformation operator representation, as the kernel $K(n,m)$ defines an operator K in a suitable space (see 3.73) that "transforms" the free Jost solutions into the perturbed Jost solutions:

$$K u_{+,0}^z = u_+^z, \quad z \in \mathbb{C}.$$

The method we use to obtain K proceeds as follows. First, we formally derive a partial difference equation (see Eq. (3.166)) that K must satisfy.
Then, we derive an integral equation that is equivalent to the difference equation (see 3.170 and Eq. (3.181)). Finally, we solve the integral equation and show that the obtained solution possesses the desired properties (see Def. 3.65 and Rem. 3.66). Further, based on [27, 15] we obtain a recurrence relation that K satisfies (see Eq. (3.197)) that allows us to obtain some estimates on K (see Prop. 3.209). We begin by deriving the partial difference equation. The following computations are formal. Assume that u_+^z admits the representation given in

Eq. (3.162). Since u_+^z is a solution to the generalized eigenvalue problem Eq. (3.1), we obtain

$$\begin{aligned} Eu_+^z(n) &= (\tau_H u_+^z)(n) \\ &= \sum_{m\in\mathbb{Z}} (A_{n+1}K(n+1,m) + B_nK(n,m) + A_nK(n-1,m))u_{+,0}^E(m). \end{aligned} \tag{3.164}$$

On the other hand, $u_{+,0}^E$ is solution to the free eigenvalue equation (3.2) then

$$\begin{aligned} Eu_+^z(n) &= \sum_{m\in\mathbb{Z}} K(n,m)Eu_{+,0}^E(m) = \sum_{m\in\mathbb{Z}} K(n,m)(u_{+,0}^E(m+1) + u_{+,0}^E(m-1)) \\ &= \sum_{m\in\mathbb{Z}} (K(n,m-1) + K(n,m+1))u_{+,0}^E(m). \end{aligned} \tag{3.165}$$

By equating the coefficients in the series expansions (3.164) and (3.165), we obtain the following partial difference equation:

$$A_{n+1}K(n+1,m) + B_nK(n,m) + A_nK(n-1,m) = K(n,m-1) + K(n,m+1). \tag{3.166}$$

We note that Equation (3.166) can be seen as non-homogeneous problem of a discrete free operator in two dimension perturbated in one dimension, we recall the definition of $V = H - H_0$ (see Eq. (1.10)), then we can write Eq. (3.166) as

$$\begin{aligned} &K(n,m-1) + K(n,m+1) - K(n+1,m) - K(n-1,m) \\ &= W_{n+1}K(n+1,m) + V_nK(n,m) + W_nK(n-1,m). \end{aligned} \tag{3.167}$$

this above leds us to the following definition.

Definition 3.52. *We denote by* $\mathcal{L} : (\mathbb{C}^{L\times L})^{\mathbb{Z}^2} \to (\mathbb{C}^{L\times L})^{\mathbb{Z}^2}$, *the linear transform given by*

$$(\mathcal{L}u)(n,m) = u(n,m+1) + u(n,m-1) - u(n+1,m) - u(n-1,m).$$

Also, we denote by $V : (\mathbb{C}^{L\times L})^{\mathbb{Z}^2} \to (\mathbb{C}^{L\times L})^{\mathbb{Z}^2}$,

$$(Vu)(n,m) = W_{n+1}u(n+1,m) + V_nu(n,m) + W_nu(n-1,m). \tag{3.168}$$

Using this above we can write Eq. (3.167) as the equation

$$\mathcal{L}X = VX. \tag{3.169}$$

Later we will define an integral operator T such that Eq. (3.169) is equivalent to the following equation

$$X = u_0 + TVX, \tag{3.170}$$

for some u_0 solution of the homogeneous equation $\mathcal{L}u_0 = 0$. Moreover, we construct T in such a way that the solution X to Eq. (3.170) has the same asymphtotic behavior of u_0, namely

$$\lim_{n\to+\infty} \|X(n+m,n) - u_0(n+m,n)\| = 0,\ m \in \mathbb{Z}.$$

Before we give T, we need some definitions. We recall that for all $m, n \in \mathbb{Z}$, the function

$$E \mapsto s_m^E(n)$$

is a matrix valued polynomial with real coefficients, which can be constructed recursively using $s_m^E(m) = 0$, $s_m^E(m+1) = \mathbf{1}$ and $s_m^E(n+1) + s_m^E(n-1) = E s_m^E(n)$. Let us set a notation for this polynomial.

Definition 3.53. *We denote by* $P_{n,m} : \mathbb{C} \to \mathbb{C}^{L \times L}$*, the polynomial with real coefficients given by*

$$P_{n,m}(z) = s_m^z(n).$$

Remark 3.54. *The polynomials* $P_{n,m}$ *satisfy the following recurrence equation*

$$P_{n-1,m}(z) = z P_{n,m}(z) - P_{n+1,m}(z). \tag{3.171}$$

In particular, one can see (for example by induction over n*, with base induction* $P_{m-1,m}(z) = -\mathbf{1}$*) that for* $m > n$ *,* $P_{n,m}$ *is a polynomial of degree* $m - n - 1$*.*

For every $n, m \in \mathbb{Z}$ we can consider the operator

$$P_{n,m}(H_0) \in \mathcal{B}(L^2(\mathbb{Z}, \mathbb{C}^L)).$$

Note that the following relation is satisfied

$$P_{n+1,m}(H_0) + P_{n-1,m}(H_0) = H_0 P_{n,m}(H_0). \tag{3.172}$$

Definition 3.55. *Let us consider* $\{e_j : j = 1, \dots, L\}$ *the standard basis of* $\mathbb{C}^L$*. We denote by* $\Gamma(n, m, r, l) \in \mathbb{C}^{L \times L}$*, the matrix elements of the operator* $P_{n,r}(H_0)$*, i.e.,*

$$\Gamma(n, m, r, l)_{i,j} := \langle P_{n,r}(H_0) \delta_l e_j, \delta_m e_i \rangle_{L^2(\mathbb{Z}, \mathbb{C}^L)}.$$

Where $\delta_l : \mathbb{Z} \to \mathbb{C}^{L \times L}$*, denotes the Kronecker's delta.*

Remark 3.56. $\Gamma(n, m, r, l)$ *is the integration kernel of the operator* $P_{n,r}(H_0)$*. Namely, if* $u \in L^2(\mathbb{Z}, \mathbb{C}^L)$*, then (see Prop. 5.11)*

$$(P_{n,r}(H_0)u)(m) = \sum_{l \in \mathbb{Z}} \Gamma(n, m, r, l) u(l), \ m \in \mathbb{Z}. \tag{3.173}$$

Now we fix $(r, l) \in \mathbb{Z} \times \mathbb{Z}$, to obtain the sequence, $\mathbb{Z} \times \mathbb{Z} \ni (n, m) \mapsto \Gamma(n, m, r, l) \in \mathbb{C}^{L \times L}$. This sequence is a solution of the equation $\mathcal{L}u = 0$ with particular initials conditions.

Proposition 3.57. *For all* $r, l \in \mathbb{Z}$ *we have that*

$$\begin{aligned} &(\mathcal{L}\Gamma(\cdot, \cdot, r, l))(n, m) \\ &= \Gamma(n+1, m, r, l) + \Gamma(n-1, m, r, l) - \Gamma(n, m+1, r, l) - \Gamma(n, m-1, r, l) = 0. \end{aligned} \tag{3.174}$$

Moreover, for all $r, l \in \mathbb{Z}$ *one has*

$$\Gamma(r, m, r, l) = 0, \quad \Gamma(r+1, m, r, l) = \delta_l(m), \ m \in \mathbb{Z}. \tag{3.175}$$

Proof. Note that $\delta_m(n) = \delta_{m+1}(n+1)$, using this we obtain

$$(H_0\delta_m e_j)(n) = \delta_m(n+1)e_j + \delta_m(n-1)e_j = (\delta_{m-1} + \delta_{m+1})(n)e_j. \tag{3.176}$$

Then, $H_0\delta_m e_j = \delta_{m-1}e_j + \delta_{m+1}e_j$. Using this above, Eq. (3.172) and the fact that H_0 is a self-adjoint operator we obtain

$$\begin{aligned}\Gamma(n+1,m,r,l)_{i,j} + \Gamma(n-1,m,r,l)_{i,j} &= \langle (P_{n+1,r}(H_0) + P_{n-1,r}(H_0))\delta_l e_j, \delta_m e_i\rangle \\ &= \langle H_0 P_{n,r}(H_0)\delta_l e_j, \delta_m e_i\rangle = \langle \delta_l e_j, H_0\delta_m e_i\rangle \\ &= \langle \delta_l e_j, \delta_{m+1}e_i + \delta_{m-1}e_i\rangle \\ &= \Gamma(n,m+1,r,l)_{i,j} + \Gamma(n,m-1,r,l)_{i,j}.\end{aligned} \tag{3.177}$$

Using Eqs. (3.177) we obtain (3.174). The second part follows from an explicit computation using the fact that $P_{r,r}(H_0) = 0$ and $P_{r+1,r}(H_0) = I$. ☐

Eq. (3.6) imply that for all $E \in [-2,2]$, $\|P_{n,m}(E)\| = \|s_n^E(m)\| \le |n-m|$ (recall $|\lambda(E)| = 1$). Then, we have

$$\|P_{n,m}(H_0)\|_{\mathcal{B}} \le \|P_{n,m}\|_\infty \le |n-m|. \tag{3.178}$$

We recall that in the set of $L \times L$ matrices, $\mathbb{C}^{L\times L}$, one has the inner product

$$\langle M, N\rangle_{\mathbb{C}^{L\times L}} = MN^*, \tag{3.179}$$

where as always, N^* denotes the cojugate transpose of a matrix.

Lemma 3.58. *For all $n,m,r \in \mathbb{Z}^3$, the sequence $\mathbb{Z} \ni l \mapsto \Gamma(n,m,r,l)$ belongs to $L^2(\mathbb{Z},\mathbb{C}^{L\times L})$. Moreover,*

$$\|\Gamma(n,m,r,\cdot)\|_{L^2(\mathbb{Z},\mathbb{C}^{L\times L})} \le \sqrt{L}|r-n|.$$

Proof. Note that Definition 3.55 and self-adjointness of $P_{n,m}(H_0)$ implies that

$$\Gamma(n,m,r,l)_{i,j} = \sum_{k\in\mathbb{Z}} \langle \delta_l(k)e_j, (P_{n,r}(H_0)\delta_m e_i)(k)\rangle_{\mathbb{C}^L} = (P_{n,r}(H_0)\delta_m e_i)(l)_j.$$

Then, $\Gamma(n,m,r,\cdot)_i = P_{n,r}(H_0)\delta_m e_i \in L^2(\mathbb{Z},\mathbb{C}^L)$. Therefore, $\Gamma(n,m,r,\cdot) \in L^2(\mathbb{Z},\mathbb{C}^{L\times L})$. Moreover, Eq. (3.178) imply

$$\|\Gamma(n,m,r,\cdot)_i\|_{L^2} = \|P_{n,r}(H_0)\delta_m e_i\|_{L^2} \le \|P_{n,r}(H_0)\| \le |r-n|.$$

☐

Definition 3.59. *Let us take fix $N \in \mathbb{N}$. Consider $A : [N+1,\infty) \times \mathbb{Z} \to \mathbb{C}^{L\times L}$, such that for all $r \ge N+1$,*

$$A(r,\cdot) \in L^2(\mathbb{Z},\mathbb{C}^{L\times L})$$

and such that

$$\sum_{r=n+1}^{\infty} |r|\|A(r,\cdot)\|_{L^2} < \infty, \ \ n \in [N,\infty). \tag{3.180}$$

We define the sequence $TA : [N,\infty) \times \mathbb{Z} \to \mathbb{C}$, as follows

$$(TA)(n,m) = \sum_{r=n+1}^{\infty} \sum_{l\in\mathbb{Z}} \Gamma(n,m,r,l)A(r,l) = \sum_{r=n+1}^{\infty} \langle \Gamma(n,m,r,\cdot), A(r,\cdot)^*\rangle_{L^2(\mathbb{Z},\mathbb{C}^{L\times L})}. \tag{3.181}$$

Remark 3.60. *Cauchy-Schwartz Inequality, Lemma 3.58 and Hypothesis (3.180) imply that TA is well defined. Further, Eq. (3.173) implies*

$$(P_{n,r}(H_0)(A(r,\cdot)_i))(m) = \sum_{l\in\mathbb{Z}} \Gamma(n,m,r,l)A(r,l)_i,$$

where here we use M_i to denote the $i-$column of the matrix $M \in \mathbb{C}^{L\times L}$, $M_i = \langle M, e_i\rangle$. Therefore,

$$(TA)(n,m)_i = \sum_{r=n+1}^{\infty} (P_{n,r}(H_0)A(r,\cdot)_i)(m). \tag{3.182}$$

Proposition 3.61. *Let $A : [N+1,\infty)_{\mathbb{Z}} \times \mathbb{Z} \to \mathbb{C}$, such that satisfies the hypothesis of Definition 3.59, then for all $n \geq N+1,\ m \in \mathbb{Z}$, one has,*

$$(TA)(n,m+1) + (TA)(n,m-1) - (TA)(n+1,m) - (TA)(n-1,m) = A(n,m).$$

Proof. Let $n \geq N+1$ and $m \in \mathbb{Z}$, using Equation 3.181, we obtain

$$(TA)(n,m+1)+(TA)(n,m-1) = \sum_{r=n+1}^{\infty}\sum_{l\in\mathbb{Z}} A(r,l)\Gamma(n,m+1,r,l)+\Gamma(n,m-1,r,l). \tag{3.183}$$

Also, using Eq. (3.175) and $\Gamma(n-1,m,n,l) = -\delta_l(m)$, we obtain

$$\begin{aligned}(TA)(n-1,m)+(TA)(n+1,m) &= \sum_{r=n+1}^{\infty}\sum_{l\in\mathbb{Z}} A(r,l)(\Gamma(n+1,m,r,l)+\Gamma(n-1,m,r,l)) \\ &+ \sum_{l\in\mathbb{Z}} A(n,l)\Gamma(n-1,m,n,l) \\ &= \sum_{r=n+1}^{\infty}\sum_{l\in\mathbb{Z}} A(r,l)(\Gamma(n+1,m,r,l)+\Gamma(n-1,m,r,l)) - A(n,m).\end{aligned} \tag{3.184}$$

Taking the difference of the Eqs. (3.183) and (3.184) and using Proposition 3.57 we obtain the result. □

Using Prop. 3.61, one can see that one solution to Eq. (3.170) is also one solution to Eq. (3.169), indeed

$$\mathcal{L}X = \mathcal{L}u_0 + \mathcal{L}TVX = VX.$$

Moreover, Equation 3.170 can be solved finding an inverse of the the operator $I - TV$, in a proper space.

Definition 3.62. *Let us denote by $\mathcal{D}_N$ to the linear subspace of $(\mathbb{C}^{L\times L})^{[N,+\infty)\times\mathbb{Z}}$ given by the sequences $u : [N,+\infty)\times\mathbb{Z} \to \mathbb{C}^{L\times L}$ such that*

- *$u(n,\cdot) \in L^2(\mathbb{Z},\mathbb{C}^{L\times L}),\ n \geq N.$*
- *$\sup_{n\geq N} \|u(n,\cdot)\|_{L^2} < \infty$*

- $u(n,m) = 0,\ m < n,\ n \geq N$.

We endow $\mathcal{D}_N$ *with the norm* $\|u\|_{\mathcal{D}_N} = \sup_{n\geq N} \|u(n,\cdot)\|_{L^2}$.

One can see that $(\mathcal{D}_N, \|\cdot\|)$ is a Banach space (see Prop. 5.10).

Proposition 3.63. *Let* $u \in \mathcal{D}_N$*, then* $TVu \in \mathcal{D}_N$*, where for* $n \geq N$

$$(TVu)(n,m) = \sum_{r=n+1}^{\infty} \sum_{l\in\mathbb{Z}} \Gamma(n,m,r,l)(V_r u(r,l) + W_{r+1}u(r+1,l) + W_l u(r-1,l)).$$

Then, we can consider the operator $TV : \mathcal{D}_N \to \mathcal{D}_N$*. Moreover,* TV *is bounded and*

$$\|TV\|_{\mathcal{B}(\mathcal{D}_N)} \leq C \sum_{r=N+1}^{\infty} |r|(\|W_{r+1}\| + \|W_r\| + \|V_r\|). \tag{3.185}$$

Proof. Let $u \in \mathcal{D}_N$, and $r, l \in \mathbb{Z}$ with $r \geq N+1$ by definition (see Eq. (3.168))

$$(Vu)(r,l) = W_{r+1}u(r+1,l) + V_r u(r,l) + W_r u(r-1,l).$$

Then $(Vu)(r,\cdot) \in L^2(\mathbb{Z}, \mathbb{C}^{L\times L})$. Moreover,

$$\|(Vu)(r,\cdot)\|_{L^2} \leq (\|W_{r+1}\| + \|W_r\| + \|V_r\|)\|u\|_{\mathcal{D}_N}. \tag{3.186}$$

Therefore, Vu satisfies the hypothesis given in Definition 3.59, then TVu is well defined (see Rem. 3.60). Since $P_{n,r}(H_0)$ is a bounded operator $P_{n,r}(H_0)((Vu)(r,\cdot)_i) \in L^2(\mathbb{Z}, \mathbb{C}^L)$ and recall that $\|P_{n,r}(H_0)\| \leq C|r-n|$ (see Lemma 3.58) using this and (3.186) we have that

$$\sum_{r=n+1}^{\infty} \|P_{n,r}(H_0)((Vu)(r,\cdot)_i)\|_{L^2} \leq C\|u\|_{\mathcal{D}_N} \sum_{r=n+1}^{\infty} |r|(\|W_{r+1}\| + \|W_r\| + \|V_r\|) < \infty. \tag{3.187}$$

Therefore, Rem. 3.60 and Prop. 5.12 imply that $(TVu)(n,\cdot) \in L^2(\mathbb{Z}, \mathbb{C}^{L\times L})$ and

$$\|(TVu)(n,\cdot)_i\|_{L^2} \leq \sum_{r=n+1}^{\infty} \|P_{n,r}(H_0)((Vu)(r,\cdot)_i)\|_{L^2} < C\|u\|_{\mathcal{D}_N} \sum_{r=n+1}^{\infty} |r|(\|W_{r+1}\|+\|W_r\|+\|V_r\|).$$

Then,

$$\sup_{n\geq N} \|(TVu)(n,\cdot)\|_{L^2} < C\|u\|_{\mathcal{D}_N} \sum_{r=N+1}^{\infty} |r|(\|W_{r+1}\| + \|W_r\| + \|V_r\|) < \infty.$$

All that remains is to prove that $(TVu)(n,m) = 0$ for all $m < n$, this together with the above imply the result. By Rem. 3.54, for $r \geq n+1$, $P_{n,r}(E)$ is a polynomial of degree $r-n-1$, this implies that (see Proposition 5.13) we can write $P_{n,r}(H_0)((Vu)(r,\cdot)_i)$ as

$$(P_{n,r}(H_0)((Vu)(r,\cdot)_i))(m) = \sum_{j=m-(r-n-1)}^{m+r-n-1} C_j^{n,r}(Vu)(r,j)_i. \tag{3.188}$$

for some $C_j^{n,r} \in \mathbb{C}^{L\times L}$. If $m < n$ we have that $r-n+m-1 < r-1$. Note that in the sum of the RHS of Eq. (3.188), $j \leq m+r-n-1 < r-1$, then $u(r-1,j) = u(r,j) = u(r+1,j) = 0$ (recall that $u \in \mathcal{D}_N$). Hence, $(Vu)(r,j) = 0$ (see Def. 3.168). Therefore, $(P_{n,r}(H_0)((Vu)(r,\cdot)_i))(m) = 0,\ m < n$. This, together with Rem. 3.60, implies that $(TVu)(n,m) = 0,\ m < n$. □

Remark 3.64. *Equation (3.185) and finite first moment assumption, allow us to take* N *(enough large) such that* $\|TV\|_{\mathcal{B}(\mathcal{D}_N)} < 1$. *One can verify that the function*

$$u_0(n,m) = \delta_n(m)$$

(where δ_n *denotes the Kronecker's Delta) belongs* $\mathcal{D}_N$ *and it also satisfies*

$$u_0(n,m+1) + u_0(n,m-1) - u_0(n+1,m) - u_0(n-1,m) = 0.$$

That gives us the following definition.

Definition 3.65. *Let* $N \in \mathbb{N}$, *such that* $\|TV\|_{\mathcal{D}_N} < 1$. *Consider* $u_0 \in \mathcal{D}_N$ *given by* $u_0(n,m) = \delta_n(m)$ *(see Rem. 3.64). We define* $K : [N,\infty) \times \mathbb{Z} \to \mathbb{C}^{L\times L}$ *as*

$$K := (I - TV)^{-1}(u_0) \in \mathcal{D}_N.$$

We extend K *to all* $\mathbb{Z}\times\mathbb{Z}$, *defying it recursively for* $n \leq N$ *and* $m \in \mathbb{Z}$ *as*

$$\begin{aligned} &K(n-1,m) := \\ &(A_n)^{-1}(K(n,m-1) + K(n,m+1) - A_{n+1}K(n+1,m) - B_nK(n,m)). \end{aligned} \tag{3.189}$$

Remark 3.66. *By construction* K *satisfies the following integral equation*

$$K(n,m) = \delta_n(m) + \sum_{r=n+1}^{\infty}\sum_{l\in\mathbb{Z}} \Gamma(n,m,r,l)(V_rK(r,l) + W_{r+1}K(r+1,l) + W_rK(r-1,l)). \tag{3.190}$$

Further, Proposition 3.61 and Def. 3.189 imply that K *satisfies Eq. (3.166). By construction* $K(n,m) = 0,\ n \geq N, m < n$, *using Eq. (3.189) one can extend this property for all* $n \in \mathbb{Z}$, *namely*

$$K(n,m) = 0,\ n \in \mathbb{Z},\ m < n. \tag{3.191}$$

Further, Eq. (3.190) implies that

$$\|K(n,m) - \delta_n(m)\| \to 0,\ n \to +\infty. \tag{3.192}$$

Now we obtain a recurrence expression for K.

Definition 3.67. *Consider* M_j *a sequence in* $\mathbb{C}^{L\times L}$, *we use the following notation for the product*

$$\prod_{j=n}^{m} M_j = M_m M_{m-1} \cdots M_n.$$

Remark 3.68. *We denote by* $\|\cdot\|$ *the operator norm in* $\mathbb{C}^{L\times L}$. *Consider* M_j *a sequence in* $\mathbb{C}^{L\times L}$, *such that* $\|M_j\| \to 0$. *Note that*

$$\left\| \prod_{j=l+1}^{m} (\mathbf{1} + M_j) - \mathbf{1} \right\| \leq \prod_{j=l+1}^{m} (\mathbf{1} + \|M_j\|) - 1 \leq \exp\left(\sum_{j=l+1}^{\infty} \|M_j\| \right) - 1 \to 0,\ j \to \infty, \tag{3.193}$$

where in the first inequality, we expand the product, take the difference with $\mathbf{1}$*, use the property* $\|MN\| \leq \|M\|\|N\|$ *and rewrite the result as the second term. The above equation shows that the following limit exists,*

$$\lim_{m\to\infty}\prod_{j=n}^{m}(1+M_j)\in\mathbb{C}^{L\times L}. \tag{3.194}$$

Lemma 3.69. *Let* $K:\mathbb{Z}\times\mathbb{Z}\to\mathbb{C}^{L\times L}$ *the sequence given in Definition 3.65. Consider the following normalization (see Rem. 3.68, recall* $A_n = W_n + \mathbf{1}$ *see (Def. 1.3))*

$$\tilde{K}(n,m)=\prod_{j=n+1}^{\infty}A_j\cdot K(n,m). \tag{3.195}$$

Then, we have that $\tilde{K}$ *satisfies the following properties:*

$$\tilde{K}(n,n)=1,\ \ n\in\mathbb{Z}. \tag{3.196}$$

For every $j\geq 0$*, we have*

$$\tilde{K}(n,n+j+1)=\tilde{K}(n+1,n+j)+\sum_{m=n+1}^{\infty}(\mathbf{1}-F_{m+1})\tilde{K}(m+1,m+j)-E_m\tilde{K}(m,m+j), \tag{3.197}$$

where

$$E_m=\prod_{j=m+1}^{\infty}A_j\cdot V_m\left(\prod_{j=m+1}^{\infty}A_j\right)^{-1},\ F_m=\prod_{j=m+1}^{\infty}A_j\cdot(A_m)^2\left(\prod_{j=m+1}^{\infty}A_j\right)^{-1}. \tag{3.198}$$

Proof. Recall K satisfies Equation (3.166) (see Rem 3.68). If we take $m=n-1$ in Eq. (3.166), then we obtain that (recall $K(n,m)=0$ if $m<n$)

$$A_nK(n-1,n-1)=K(n,n). \tag{3.199}$$

Eq. (3.192) imply that $K(n,n)\to\mathbf{1}$, $n\to\infty$, taking n enough large we obtain $A_n = K(n,n)K(n-1,n-1)^{-1}$, then for all $M\geq n$

$$\prod_{j=n}^{M}A_j=A_MA_{M-1}\cdots A_n=K(M,M)K(n-1,n-1)^{-1}, \tag{3.200}$$

Taking limit $M\to+\infty$ (see Rem 3.68) and recall $K(M,M)\to\mathbf{1}$ we obtain Eq. (3.196) for n enough large. We use Eq. (3.199) to extend it for all $n\in\mathbb{Z}$. For the second part, substituing Eq. (3.195) in Eq. (3.166) we obtain

$$\tilde{K}(n-1,m)+E_n\tilde{K}(n,m)+F_{n+1}\tilde{K}(n+1,m)=\tilde{K}(n,m-1)+\tilde{K}(n,m+1) \tag{3.201}$$

If we make $m=n+j$, $j\geq 0$ we can write Eq. (3.201) as

$$\tilde{K}(n-1,n+j)-\tilde{K}(n,n+j+1)=\tilde{K}(n,n+j-1)-E_n\tilde{K}(n,n+j)-F_{n+1}\tilde{K}(n+1,n+j) \tag{3.202}$$

Taking the sum over n in Eq. (3.202) (note that LHS is telescopic) and using that

$$\lim_{n\to+\infty} K(n,n+j)=0,\ \ j>0,$$

we obtain Eq. (3.197). □

Now we obtain bounds for K. The following is based on [27] Chapter. 10.1.

Definition 3.70. *For $n\in\mathbb{Z}$, and $j\geq 1$ we denote*

$$C(n)=\sum_{m=n}^{\infty}\|V_m\|+\|A_{m+1}^2-1\|. \tag{3.203}$$

$$D(n,j)=\prod_{m=1}^{j-1}1+C(n+m). \tag{3.204}$$

Remark 3.71. *One can verify using Eq. (3.203) and Eq. (3.204) that*

$$C(n+1)\leq C(n),\quad D(n+1,j)\leq D(n,j),\quad D(n,j)\leq D(n,j+1). \tag{3.205}$$

Also one can see that,

$$D(n+1,j)+D(n+1,j)C(n+1)=D(n,j+1). \tag{3.206}$$

Moreover, since $C(n)\to 0,\ n\to\infty$ the limit $\lim j\to+\infty D(n,j)$, exists (see Rem 3.68), and

$$D(n,j)\leq \lim_{j\to+\infty} D(n,j):=D(n) \tag{3.207}$$

Further, the finite first moment assumption implies

$$\sum_{n\in\mathbb{Z}} C(n)<\infty. \tag{3.208}$$

Proposition 3.72. *For all $n\in\mathbb{Z}$ and $j\geq 1$ one has that*

$$\|\tilde{K}(n,n+j)\|\leq D(n,j)C(n+[j/2]+1). \tag{3.209}$$

Proof. The proof is by induction over j. Let us take $j=1$, using Eq. (3.197) (recall $K(n,m)=0,\ m<n$ and $\tilde{K}(n,n)=1$) we obtain that for all $n\in\mathbb{Z}$

$$\|\tilde{K}(n,n+1)\|\leq\sum_{m=n+1}^{\infty}\|V_m\|\leq D(n,1)C(n+1).$$

Assume that for all $n\in\mathbb{Z}$, and $r\leq j$ estimate (3.209) holds. Let $n\in\mathbb{Z}$, using Eq. (3.197) and induction hypothesis we obtain

$$\begin{aligned}\|\tilde{K}(n,n+j+1)\|\leq{}& D(n+1,j-1)C(n+2+[(j-1)/2])\\&+\sum_{m=n+1}^{\infty}\|A_{m+1}^2-1\|D(m+1,j-1)C(m+2+[(j-1)/2])\\&+\|V_m\|D(m,j)C(m+1+[j/2]).\end{aligned} \tag{3.210}$$

Using Eqs. (3.205) and (3.206) in Eq. (3.210) we obtain

$$\begin{aligned}\|\tilde{K}(n,n+j+1)\| &\leq C(n+[(j+1)/2]+1)(D(n+1,j)+D(n+1,j)C(n+1))\\ &= C(n+[(j+1)/2]+1)D(n,j+1).\end{aligned} \tag{3.211}$$

□

Using Eqs. (3.208), (3.207) and (3.209) one can see that for all $n \in \mathbb{Z}$,

$$\sum_{m=n+1}^{\infty} \|K(n,m)\| < D(n) \sum_{j=1}^{\infty} C([j/2]+n+1) < \infty. \tag{3.212}$$

Then, we obtain the following definition.

Definition 3.73 (Transformation operator)**.** *Consider $u : \mathbb{Z} \to \mathbb{C}^L$, such that $u(n) = O(1)$, $n \to +\infty$. We denote by $Ku : \mathbb{Z} \to \mathbb{C}^L$, the following sequence*

$$(Ku)(n) = \sum_{m=n}^{\infty} K(n,m)u(m). \tag{3.213}$$

Note that Eq. (3.212) implies that Ku is well defined.

We recall the jost free solution $u^z_{+,0}(n) = \lambda(z)^n$. In particular, note that for all $z \in \mathbb{C}$, $u^z_{+,0}(n) = O(1)$, $n \to +\infty$. Also, we recall the jost solutions u^z_+ (see Def. 1.7). The next proposition gives the relation beetwen them and the transformation operator.

Proposition 3.74. *For all $z \in \mathbb{C}$, one has that*

$$u^z_+(n) = (Ku^z_{+,0})(n) = \sum_{m=n}^{\infty} K(n,m)u^z_{+,0}(m). \tag{3.214}$$

Proof. Let $z \in \mathbb{C}$, let use denote u^z the R.H.S. of Eq. (3.214). First, we see that u^z is solution to the eigenvalue equation (3.1). Recall that K by construction satisfies Eq. (3.166), also $K(n,m) = 0$, $m < n$ (see Rem. 3.66) then

$$\begin{aligned}(\tau_H u^z)(n) &= \sum_{m\in\mathbb{Z}} (A_{n+1}K(n+1,m) + B_nK(n,m) + A_nK(n-1,m))u^z_{+,0}(m)\\ &= \sum_{m\in\mathbb{Z}} (K(n,m-1) + K(n,m+1))u^z_{+,0}(n)\\ &= \sum_{m\in\mathbb{Z}} K(n,m)(u^z_{+,0}(m+1) + u^z_{+,0}(m-1)) = zu^z(n).\end{aligned}$$

Now, Eqs. (3.195), (3.196) imply

$$\lambda(z)^{-n}u^z(n) = \left(\prod_{j=n+1}^{\infty} A_j\right)^{-1} + \sum_{m=n+1}^{\infty} K(n,m)\lambda(z)^{m-n} \to \mathbf{1},$$

where we are using Eq. (3.212) (note that for $n \in \mathbb{N}$, $D(n) \leq D(1)$). The fact that u^z_+ and u^z are both solutions to the eigenvalue equation with the same asympthotic behavior at $+\infty$ implies the result. □

Remark 3.75. *In a similar manner, one can prove that there exist coefficients* $K_-(n,m)$ *such that for all* $z \in \mathbb{C}$

$$u_-^z(n) = \sum_{m=-\infty}^{n} K_-(n,m) u_{-,0}^z(m).$$

Where, $K_-(n,m) = 0, \;\; m > n,$ *and*

$$K_-(n,n) = A_n^{-1} A_{n-1}^{-1} A_{n-2}^{-1} \cdots .$$

3.8 High energy limit

In this section we study the behavior of the scattering coefficient $M_z^{\pm}$ for $|z| \to \infty$. This behavior is important in the proof of Levinson theorem.

Proposition 3.76. *Jost solutions* $u_{\pm}^z$ *given in Def. 1.7 have the following behavior as* $|z| \to \infty$*. (see Def. 3.67)*

$$\lim_{|z|\to\infty} z^n u_+^z(n) = A_{n+1}^{-1} A_{n+2}^{-1} A_{n+3}^{-1} \cdots , \qquad \lim_{|z|\to\infty} z^{-n} u_-^z(n) = A_n^{-1} A_{n-1}^{-1} A_{n-2}^{-1} \cdots .$$

Proof. It follows from Eq. (3.214), recall that $K(n,\cdot) \in L^1(\mathbb{N})$, also that for $m > n$, if $\lambda(z)^{-n+m} \to 0, \; |z| \to \infty$, and use dominated convergence theorem, for $m = n$ recall that (see Eq. (3.195) and Def. 3.67)

$$K(n,n) = A_{n+1}^{-1} A_{n+2}^{-1} A_{n+3}^{-1} \cdots .$$

□

Proposition 3.77 (High Energy Limit)**.**

$$\lim_{|z|\to\infty} M_+^z = \lim_{N\to+\infty} A_{-N}^{-1} A_{-N+1}^{-1} \cdots A_0^{-1} \cdots A_{N-1}^{-1} A_N^{-1}.$$

$$\lim_{|z|\to\infty} M_-^z = \lim_{N\to+\infty} A_N^{-1} A_{N-1}^{-1} \cdots A_0^{-1} \cdots A_{-N+1}^{-1} A_{-N}^{-1}.$$

Proof. Using Eq. (3.56) and evaluating the wronskian at 0 we obtain (recall $\lambda(z) \to 0, \; |z| \to \infty$.)

$$\begin{aligned} \lim_{|z|\to\infty} M_+^z &= \lim_{|z|\to\infty} -\nu^z \mathcal{W}(u_-^{\overline{z}}, u_+^z)(0) \\ &= \lim_{|z|\to\infty} \lambda(z) u_-^{\overline{z}}(1)^* A_1 u_+^z(0) - \lambda(z) u_-^{\overline{z}}(0) A_1 u_+^z(1) \\ &= \lim_{|z|\to\infty} z\lambda(z)(\overline{z}^{-1} u_-^{\overline{z}}(1))^* A_1 u_+^z(0), \end{aligned} \tag{3.215}$$

where we use the fact that $z u_+^z(1)$ converge when $|z| \to \infty$, which in particular implies that $u_+^z(1) \to 0$. Note that the relation $\lambda(z)^2 + 1 = z\lambda(z)$, in particular implies that $z\lambda(z) \to 1$, using this and Prop. 3.76 (recall $A_n^* = A_n$) in Eq. (3.215) we obtain the result. The other limit can be computed using $M_-^{\overline{z}} = (M_+^z)^*$ see Eq. (3.58). □

3.9 Time delay

The (total) time delay is by definition the quantity

$$\mathrm{Tr}\Big((\mathcal{S}^E)^* \frac{d}{dE}\mathcal{S}^E\Big) \;=\; \det(\mathcal{S}^E)^{-1}\frac{d}{dE}\det(\mathcal{S}^E)$$

for $E \in (-2,2)$ (the above identity is referred to as Jacobi's formula). This section provides a formula for it in terms of the determinant of M_-^E and M_+^E.

Proposition 3.78. *Let $E \in (-2,2)$. The following identity holds true:*

$$\det(\mathcal{S}^E) = \det(M_-^E)^{-1}\det((M_+^E)^*)$$

Proof. Applying the Schur complement formula for the determinant (see Proposition 5.9) to the definition (1.12) of the scattering matrix leads to

$$\det(\mathcal{S}^E) = \det(M_-^E)^{-1}\det((M_+^E)^{-1} - N_-^E N_+^E (M_+^E)^{-1}). \tag{3.216}$$

Using Eqs. (3.58) and (3.61) one obtains that

$$\begin{aligned}\det((M_+^E)^{-1} - N_-^E N_+^E (M_+^E)^{-1}) &= \det((\mathbf{1} - N_-^E N_+^E)(M_+^E)^{-1}) \\ &= \det((\mathbf{1} + (N_+^E)^* N_+^E)(M_+^E)^{-1}) \\ &= \det((M_+^E)^*).\end{aligned} \tag{3.217}$$

Equations (3.216) and (3.217) imply the claim. □

Propositions 3.78 and Rem. 3.20 (also see Eq. (3.54)) imply that the function $(-2,2) \ni E \mapsto \det(\mathcal{S}^E)$ is continuous differentiable. This allows us to state the next result.

Corollary 3.79. *For every $E \in (-2,2)$, the following hold true:*

$$\det(\mathcal{S}^E)^{-1}\frac{d}{dE}\det(\mathcal{S}^E) = \det((M_+^E)^*)^{-1}\frac{d}{dE}\det((M_+^E)^*) - \det(M_-^E)^{-1}\frac{d}{dE}\det(M_-^E). \tag{3.218}$$

Proof. Using Proposition 3.78, an explicit computation gives

$$\frac{d}{dE}\det(\mathcal{S}^E) = \det(M_-^E)^{-1}\frac{d}{dE}\det((M_+^E)^*) - \det(M_-^E)^{-2}\frac{d}{dE}\det(M_-^E)\det((M_+^E)^*). \tag{3.219}$$

Multiplying Eq. (3.219) by $\det(\mathcal{S}^E)^{-1} = \det(M_-^E)\det((M_+^E)^*)^{-1}$ one gets the stated result. □

3.10 Levinson Theorem

This section is devoted to presenting a proof of Levinson's theorem, see Theo. 1.15. To provide a clear and detailed argument, we separate the technical components into lemmas. The main idea is presented at the final of this section. We begin with the following lemma.

Lemma 3.80. *For every $\eta > 0$, there exists $V_\pm \subset \mathbb{C}^+ \cup \mathbb{C}^-$ open set and $\varepsilon > 0$ that satisfies,*

a) $||\arg(\lambda(z) \mp 1)| - \pi/2| < \eta,\ z \in V_\pm$.

b) For every $E \in (2-\varepsilon, 2)$, there exists $\delta > 0$ such that

$$\{E + it :\ 0 < |t| < \delta\} \subset V_+.$$

Similarly, for every $E \in (-2, -2+\varepsilon)$ there exists $\delta > 0$ such that

$$\{E + it :\ 0 < |t| < \delta\} \subset V_-.$$

where $-\pi \leq \arg(z) < \pi$.

Proof. We make the proof for $'+'$, the other is similar. Take $\eta > 0$. Consider the following opet set,

$$U := \{z \in \mathbb{C} : ||\arg(z)| - \pi/2| < \eta\}. \tag{3.220}$$

Note that the following open set,

$$U_1 := \mathbb{D} \cap (U + 1), \tag{3.221}$$

is not empty ($\tan(\arg(e^{i\theta} - 1)) = \sin(\theta) \cdot (\cos(\theta) - 1)^{-1} \to \infty$). Moreover, since $\lambda : \mathbb{C} \backslash [-2, 2] \to \mathbb{D}$ is homeomorphism, the set

$$V := \lambda^{-1}(U_1), \tag{3.222}$$

is an open set. Consider the point in the intersection

$$\{z_0\} = \mathbb{S}^1 \cap \{z \in \mathbb{C} : \arg(z - 1) = \pi/2 + \eta\},$$

note that

$$\{\overline{z_0}\} = \mathbb{S}^1 \cap \{z \in \mathbb{C} : \arg(z - 1) = -\pi/2 + \eta\}.$$

By definition, $\lambda(\overline{z_0}) = \lambda(z_0) \in (-2, 2)$. Suppose that for some real $r \in (\lambda(z_0), 2)$ there exist a sequence $z_n \in \mathbb{C} \setminus (V \cup [-2, 2])$, with $z_n \to r$. Then since $\lambda : \mathbb{C} \setminus [-2, 2] \to \mathbb{D}$, is homeomorphism, there exists a sequence $\lambda(z_n) \in \mathbb{D} \setminus U_1$ such that $\lambda(z_n) \to r \in \mathbb{S}^1 \setminus \{2, \lambda(z_0)\}$ this is impossible. Then V and $\varepsilon = 2 - \lambda(z_0)$ are the desired. $\square$

Remark 3.81. *We recall that the number of zeros of the function $z \mapsto \det(M_-^z)$ is finite (see Props. 3.22 and 3.75). Then, we can take $r > 0$ and $R > 0$ such that, $[-2, 2] \subset B(0; r) \subset B(0; R)$ and all the zeros of the function $z \mapsto \det(M_-^z)^{-1}$ are inside of annulus the $B(0; R) \setminus B(0; r)$.*

Lemma 3.82. *Consider $r > 0$ as in Rem. 3.81. For all $\eta > 0$, there exists $\varepsilon > 0$ that satisfies the following property:*
For every $E \in (2-\varepsilon, 2)$, there exist $\delta > 0$ such that for all $0 < b < \delta$ and all rectificable path $\gamma : [0, 1] \to B(0; r) \setminus [-2, 2]$, with $\gamma(0) = E - ib = \overline{\gamma(1)}$ one has

$$\left| \int_\gamma \det(M_-^z)^{-1} \frac{d}{dz} \det(M_-^z)\ dz - (J_h^+ - L)i\pi \right| < \eta. \tag{3.223}$$

Proof. Eq. (3.95) implies that there exists $s > 0$ such that

$$\det(M_-^z) = (\lambda(z) - 1)^{J_h^+ - L} g(z),\ z \in \overline{B(2;s)} \subset B(0;r),$$

where g is analytic on $B(2;s) \setminus [-2,2]$, continuous on $\overline{B(2;s)}$ and $g(2) \neq 0$. An explicit computation using the above equation shows

$$\det(M_-^z)^{-1} \frac{d}{dz} \det(M_-^z) = (J_h^+ - L) \frac{\lambda'(z)}{\lambda(z) - 1} + \frac{g'(z)}{g(z)},\ z \in B(2;s) \setminus [-2,2]. \tag{3.224}$$

Take $\eta > 0$. Lemma 5.14 implies that there exists $0 < r_1 < s$, such that for all rectificable path $\gamma \subset B(2;r_1) \setminus [-2,2]$ one has

$$\left| \int_\gamma \frac{g'(z)}{g(z)}\ dz \right| < \eta/2. \tag{3.225}$$

If $J_h^+ - L = 0$, the result follows taking $\varepsilon = 2 - r_1$. Suppose $J_h^+ - L \neq 0$. Take V_+ and $0 < \varepsilon < 2 - r_1$ given in Lemma 3.80 corresponding to $\eta' = \eta(4(J_h^+ - L))^{-1}$. We check that this $\varepsilon > 0$ satisfies the desired property. Let $E \in (2 - \varepsilon, 2)$, in particular note $E \in B(2;r_1)$. By item $b)$ of Lemma 3.80, there exists $\delta > 0$ such that for all $0 < b < \delta$,

$$E + ib, E - ib \in V_+ \cap B(2;r_1).$$

Take $\gamma : [0,1] \to B(0;r) \setminus [-2,2]$, with $\gamma(0) = E - ib$, $\gamma(1) = E + ib$. Since the map $B(0;r) \setminus [-2,2] \ni z \mapsto \det(M_-^z)^{-1} \frac{d}{dz} \det(M_-^z)$ is analytic, by virtue of the Cauchy Theorem w.l.o.g. we can suppose $\gamma([0,1]) \subset B(2;r_1) \setminus [-2,2]$. Since $E - ib \in V_+$ by the item $a)$ of Lemma 3.80 (also recall $\Im(\lambda(E - ib)) > 0$) we know that

$$\lambda(\gamma(0)) - 1 = \lambda(E - ib) - 1 = re^{i\theta},\ 0 < \theta - \pi/2 < \eta', \tag{3.226}$$

and $\lambda(\gamma(1)) - 1 = re^{-i\theta}$. Now, recall that the map $\lambda : B(2;r_1) \setminus [-2,2] \mapsto \mathbb{D}$ is injective and differentiable. Then, we have

$$\int_\gamma \frac{\lambda'(z)}{\lambda(z) - 1} dz = \int_{\lambda(\gamma) - 1} \frac{1}{z}\ dz = 2i(\pi - \theta). \tag{3.227}$$

Where in the last equality we are using the fact that $\lambda \circ \gamma - 1$ is a path on $\{z \in \mathbb{C} : \Re(z) < 0\}$, such that $(\lambda \circ \gamma - 1)(0) = re^{i\theta}$, and $(\lambda \circ \gamma - 1)(1) = re^{-i\theta}$, (one could, for exaple, use Cauchy Theorem and change the path for this other $l(t) = re^{it}$, $t \in [\theta, 2\pi - \theta]$, and compute explicity the integral). Finally, using Eqs. (3.224), (3.225) and (3.227) together with (3.226) we obtain the result. □

Remark 3.83. *In the same manner, a result similar to Lemma 3.82 can be obtained. Namely, for $\eta > 0$, there exists $\varepsilon > 0$ such that for all for $E \in (-2, -2 + \varepsilon)$, the result of Lemma 3.82 holds by replacing J_h^+ with J_h^-, and taking $\gamma(0) = E + ib = \overline{\gamma(1)}$.*

Now we are ready to prove Levinson Theorem.

Proof of Theorem 1.15. Let $\eta > 0$. Consider $R, r > 0$ as in Rem. 3.81. Take $\varepsilon > 0$ described in Lemma 3.82 for $\eta/4$ (also see Rem. 3.83). Let $e \in (2-\varepsilon, 2)$, and consider $\delta > 0$ as in Lemma. 3.82. Identities of M_+^E and M_-^E in terms of the wronskian (see Eqs. (3.54)) together with Eq. (3.56) and Props. 3.4, 3.19 and Rem. 3.20 imply that the analytic function $\mathbb{C} \setminus [-2,2] \ni z \mapsto \det(M_-^z)$ and its derivative converge uniformly from above on compact sets $K \subset \overline{\mathbb{C}^+} \setminus \{-2,2\}$ to $\det(M_-^E)$ and its derivative (respectively). In particular, there exists $0 < b < \delta$ such that

$$\left| \int_{L_+^b} \det(M_-^z)^{-1} \frac{d}{dz} \det(M_-^z)\, dz - \int_{-e}^{e} \det(M_-^E)^{-1} \frac{d}{dE} \det(M_-^E)\, dE \right| < \frac{\eta}{4}, \tag{3.228}$$

where $L_b^+ \subset B(0;r) \setminus [-2,2]$ is a rectificable that goes from $-e+ib$ to $e+ib$. Also, using that $\mathbb{C} \setminus [-2,2] \ni z \mapsto (M_+^{\bar{z}})^*$, is an analytic function that converges uniformly over compacts from below to $(M_+^E)^*$ we obtain,

$$\left| - \int_{L_-^b} \det((M_+^{\bar{z}})^*)^{-1} \frac{d}{dz} \det((M_+^{\bar{z}})^*\, dz + \int_{-e}^{e} \det((M_+^E)^*)^{-1} \frac{d}{dE} \det(M_+^E)^*\, dE \right| < \frac{\eta}{4}, \tag{3.229}$$

where $L_b^- \subset B(0;r) \setminus [2,2]$ is a rectificable that goes from $-e-ib$ to $-e+ib$. Now we consider two rectificable paths, $C_\pm^b \subset B(0;r) \setminus [-2,2]$. Such that C_+^b goes from $e+ib$ to $e-ib$, and C_-^b goes from $-e-ib$ to $-e+ib$. We consider the path $C_e^b := L_+^b + C_+^b + (-L_-^b) + C_-^b$. The Argument principle and Rem. 3.81 imply (see Prop. 3.24)

$$\int_{C_R} + \int_{C_e^b} \det(M_-^z)^{-1} \frac{d}{dz} \det(M_-^z)\, dz = 2\pi i J_b. \tag{3.230}$$

Since the function $\det(M_-^z) \to c,\ c \neq 0, |z| \to \infty$, (see Prop. 3.77) we have that

$$\int_{C_R} \det(M_-^z)^{-1} \frac{d}{dz} \det(M_-^z)\, dz \to 0,\ R \to \infty.$$

Taking limit $R \to \infty$ in Eq. (3.230) we obtain that,

$$\int_{C_e^b} \det(M_-^z)^{-1} \frac{d}{dz} \det(M_-^z)\, dz = 2\pi i J_b. \tag{3.231}$$

Using Corollary 3.79 we obtain

$$\begin{aligned} &\left| \int_{-e}^{e} \det(\mathcal{S}^E)^{-1} \frac{d}{dE} \det(\mathcal{S}^E)\, dE + 2\pi i (J_b + \frac{1}{2} J_h - L) \right| \\ &= \left| \int_{-e}^{e} \det((M_+^E)^*)^{-1} \frac{d}{dE} \det((M_+^E)^*) - \int_{-e}^{e} \det(M_-^E)^{-1} \frac{d}{dE} \det(M_-^E) + 2\pi i (J_b + \frac{1}{2} J_h - L) \right| \end{aligned} \tag{3.232}$$

Using Eqs. (3.228) and (3.229), in Eq. (3.232) and the fact that $M_-^z = (M_+^{\overline{z}})^*$ together with Eq. (3.231), Eq. (3.223) and the election of $\varepsilon > 0$ (also recall $J_h = J_h^+ + J_h^-$), we obtain

$$\begin{aligned}
&\left| \int_{-e}^{e} \det(\mathcal{S}^E)^{-1} \frac{d}{dE} \det(\mathcal{S}^E)\ dE + 2\pi i (J_b + \frac{1}{2} J_h - L) \right| \\
&\leq \eta/2 + \left| \int_{L_-^b - L_+^b} \det(M_-^z)^{-1} \frac{d}{dz} \det(M_-^z)\ dz + 2\pi i (J_b + \frac{1}{2} J_h - L) \right| \\
&= \eta/2 + \left| - \int_{-C_-^b - C_+^b} \det(M_-^z)^{-1} \frac{d}{dz} \det(M_-^z)\ dz + i\pi (J_h - 2L) \right| < \eta.
\end{aligned} \tag{3.233}$$

□

4 Wave operator and Scattering operator

In the previous section, we defined the scattering matrix as the object that maps the outgoing waves to the incoming waves. This is commonly referred to as stationary scattering theory, where one can derive explicit formulas for the scattering matrix in terms of the boundary values of the resolvent.

On the other hand, in the abstract (time-dependent) framework of scattering theory, the scattering matrix is defined via the wave operators, see Eq. (1.6).

In this section, we demonstrate the equivalence between these two definitions for Jacobi operators as given in Definition 1.1. This equivalence relies on the fact that the boundary values of the resolvent can be expressed in terms of the Wronskian of the Jost solutions, introduced in 1.7. We begin this section by analyzing the resolvent operator of the Hamiltonian. The main ideas of this section are taken from [27] (Chapters 4 and 5), where the analysis of the one-dimensional Laplacian on the half-line and the full line is presented. This section is joint work with J. Gil, D. Terrazas, M. Ballesteros and I. Naumkin.

4.1 Limit absortion principle

The limiting absorption principle concerns the study of the limit values of the resolvent operator as it approaches to the spectrum of the operator. Naturally, such limits do not exist in the operator norm topology; however, they do exist when we consider a suitable weaker topology on the space of bounded operators, equivalently, when we weight the operator.

Analyzing the boundary values of the resolvent is fundamental in stationary scattering theory, as these values allow us to derive explicit expressions for the wave operators, and consequently for the scattering operator and the scattering matrix. In general, the study of these limits is quite intricate, and a variety of methods have been developed to approach them in the most general settings.

In our specific context, we are able to represent the resolvent operator as an integral operator whose kernel is given in terms of the Jost solutions (see Prop. 4.7). This representation enables us to derive explicit formulas for the boundary values of the kernel in terms of the Jost solutions (see Prop. 4.18). Then we can traslate the information of the kernel and obtain the boundary values of the resolvent in the spectrum (see Theorem 4.17). These, in turn, lead directly to expressions for the wave operators and the scattering operator.

We begin this section by analyzing the integral kernel of the resolvent operator.

4.1.1 Green's function

Our approach to studying the boundary behavior of the resolvent operator involves expressing it as an integral operator and analyzing its integral kernel, known as the Green's function. In particular, if the zero moment assumption (see Def. 1.4) is satisfied, i.e. it is possible to construct a basis of solution such as the ones describe in Def. 1.7, we can express the Green's function using this solutions. This allows the entire analysis of the Jost solutions to be directly transferred to the Green's function. We begin by introducing some notation.

Notation 4.1. *We denote by* $\{e_j : j \in \{1, \ldots, L\}\}$ *the standard basis of* $\mathbb{C}^L$. *For* $j \in \{1, \ldots, L\}$ *and* $m \in \mathbb{Z}$ *, we denote by* $\delta_m^j : \mathbb{Z} \to \mathbb{C}^L$ *to the vector valued sequence given by*

$$\delta_m^j(n) = \delta_{m,n} e_j.$$

Here δ *denotes the usual Kronecker delta function. Also, we denote by* $\delta_m : \mathbb{Z} \to \mathbb{C}^{L\times L}$*, the matrix valued function given by*

$$\delta_m(n) = \delta_{m,n}\mathbf{1}.$$

Remark 4.2. *In particular, note that* $\{\delta_m^j : m \in \mathbb{Z}, j \in \{1, \ldots, L\}\}$ *is an orthonormal basis of* $L^2(\mathbb{Z}, \mathbb{C}^L)$.

Definition 4.3. *For* $z \in \mathbb{C} \setminus \sigma(H)$*, we denote by* $R(z) : L^2(\mathbb{Z}, \mathbb{C}^L) \to L^2(\mathbb{Z}, \mathbb{C}^L)$ *the resolvent operator, namely*

$$R(z) = (H - z)^{-1}.$$

Definition 4.4. *For* $z \in \mathbb{C} \setminus \sigma(H)$*, we denote the Green's function by*

$$G^z : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}^{L\times L}.$$

This function is given as usual by the matrix elements of $R(z)$*, namely,*

$$G^z(n, m)_{i,j} = \langle R(z)\delta_m^j, \delta_n^i \rangle_{L^2}. \tag{4.1}$$

The green function plays an important role for the study of the resolvent. In particular, it is the integration kernel of the resolvent.

Proposition 4.5. *Let* $z \in \mathbb{C} \setminus \sigma(H_0)$. *For all* $u \in L^2(\mathbb{Z}, \mathbb{C}^L)$ *we have that*

$$(R(z)u)(n) = \sum_{m\in\mathbb{Z}} G^z(n, m)u(m), \; n \in \mathbb{Z}. \tag{4.2}$$

Proof. Note that by definition one has that $\langle u, \delta_n^j \rangle_{L^2} = \langle u(n), e_j \rangle_{\mathbb{C}^L} = u(n)_j$. This above implies

$$u(n) = \sum_{j=1}^{L} \langle u(n), e_j \rangle_{\mathbb{C}^L} e_j = \sum_{j=1}^{L} \langle u, \delta_n^j \rangle_{L^2} e_j. \tag{4.3}$$

As a consequence of Remark 4.2, we know that $u \in L^2(\mathbb{Z}, \mathbb{C}^L)$ can be written as

$$u = \sum_{m\in\mathbb{Z}} \sum_{i=1}^{L} \langle u, \delta_m^i \rangle_{L^2} \delta_m^i. \tag{4.4}$$

Applying Eq. (4.3) to $R(z)u$ and then Eq. (4.4) we obtain

$$\begin{aligned}
(R(z)u)(n) &= \sum_{j=1}^{L} \langle R(z)u, \delta_n^j \rangle_{L^2} e_j = \sum_{j=1}^{L} \left\langle R(z) \left(\sum_{m\in\mathbb{Z}} \sum_{i=1}^{L} \langle u, \delta_m^i \rangle_{L^2} \delta_m^i \right), \delta_n^j \right\rangle_{L^2} e_j \\
&= \sum_{j=1}^{L} \sum_{m\in\mathbb{Z}} \sum_{i=1}^{L} \left\langle R(z)\delta_m^i, \delta_n^j \right\rangle_{L^2} \langle u, \delta_m^i \rangle_{L^2} e_j = \sum_{m\in\mathbb{Z}} \sum_{j=1}^{L} \sum_{i=1}^{L} G^z(n, m)_{j,i} u(m)_i e_j \qquad (4.5) \\
&= \sum_{m\in\mathbb{Z}} \sum_{j=1}^{L} (G^z(n, m)u(n))_j e_j = \sum_{m\in\mathbb{Z}} G^z(n, m)u(m).
\end{aligned}$$

□

For matrix Jacobi operators just as in the scalar case, an explicit expression for the Green's function can be obtained. This expression is obtained by solving a difference equation. Let $m \in \mathbb{Z}$, Eq. (4.1) implies that the $j-column$ of the matrix $G^z(n,m)$ is given by

$$G^z(n,m)_j = (R(z)\delta^j_m)(n).$$

Then $G(\cdot,m)_j = R(z)\delta^j_m \in L^2(\mathbb{Z},\mathbb{C}^L)$, and satisfies the equation $(\tau - z)G_j(\cdot,m) = \delta^j_m$. Since the above is satisfied for each j, the matrix valued sequence $n \mapsto G^z(n,m)$ satisfies the equation

$$(\tau - z)G^z(\cdot,m) = \delta_m. \tag{4.6}$$

Expanding Eq. (4.6) we obtain the difference equation

$$A_nG^z(n-1,m) + B_nG^z(n,m) + A_{n+1}G^z(n+1,m) - zG^z(n,m) = \delta_m(n),\ n \in \mathbb{Z}. \tag{4.7}$$

Eq. (4.7) implies that $G^z(\cdot,m)$ satisfies Eq. (3.2) for $n \geq m+1$. Since the columns of the matrix $(u^z_+\ w^z_+)$ (see Rem. 1.9) form a basis of the space of solutions and $G^z(\cdot,m) \in l^2[m,\infty)$, we have that $G^z(\cdot,m)|_{[m+1,\infty)} = u^z_+|_{[m+1,\infty)}\alpha^z_m$, for some $\alpha^z_m \in \mathbb{C}^{L\times L}$. Using the difference equation Eq. (4.7) with $n = m+1$ we obtain,

$$G^z(n,m) = u^z_+(n)\alpha^z_m,\ n \geq m, \tag{4.8}$$

In the same manner, we obtain that

$$G^z(n,m) = u^z_-(n)\beta^z_m,\ n \leq m, \tag{4.9}$$

for some $\beta^E_m \in \mathbb{C}^{L\times L}$. In particular, we obtain that

$$u^z_-(m)\beta^z_m = G^z(m,m) = u^z_+(m)\alpha^z_m. \tag{4.10}$$

Now, taking $n = m$ in Eq. (4.7) and using Eqs. (4.8) and (4.9) we obtain

$$A_mu^z_-(m-1)\beta^z_m + B_mu^z_-(m)\beta^z_m + A_{m+1}u^z_+(m+1)\alpha^z_m - zu^z_-(m)\beta^z_m = \mathbf{1}.$$

Since u^z_- satisfies Eq. (3.1) we have

$$A_mu^z_-(m-1)\beta^z_m + B_mu^z_-(m)\beta^z_m - zu^z_-(m)\beta^z_m = -A_{m+1}u^z_-(m+1)\beta^z_m.$$

Using the two equations above we obtain

$$A_{m+1}u^z_+(m+1)\alpha^z_m - A_{m+1}u^z_-(m+1)\beta^z_m = \mathbf{1} \tag{4.11}$$

Therefore, Eqs. (4.10) and (4.11) give us the following system of equations for α^z_m, β^z_m,

$$\begin{pmatrix} A_{m+1}u^z_+(m+1) & A_{m+1}u^z_-(m+1) \\ u^z_+(m) & u^z_-(m) \end{pmatrix} \begin{pmatrix} \alpha^z_m \\ -\beta^z_m \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ 0 \end{pmatrix} \tag{4.12}$$

Now, it just let to obtain an inverse of the matrix that appears in the L.H.S. of Eq. (4.12). Note that, in fact this is the matrix $\Phi(u^z_+, u^z_-)(m+1)$ defined in Def. (2.28).

Proposition 4.6. *For all $z \in \mathbb{C}\setminus\sigma(H)$, $\mathcal{W}(u_-^{\bar z}, u_+^z)$ and $\mathcal{W}(u_+^{\bar z}, u_-^z)$ are invertible. Moreover, we have that*

$$\begin{aligned}\Phi(u_+^z, u_-^z)(m+1)^{-1} &= \begin{pmatrix} -\mathcal{W}(u_-^{\bar z}, u_+^z)^{-1} & 0 \\ 0 & \mathcal{W}(u_+^{\bar z}, u_-^z)^{-1} \end{pmatrix} \mathcal{J}\Phi(u_+^{\bar z}, u_-^{\bar z})(m+1)^*\mathcal{J}^* \\ &= \begin{pmatrix} -\mathcal{W}(u_-^{\bar z}, u_+^z)^{-1}u_-^{\bar z}(m)^* & \mathcal{W}(u_-^{\bar z}, u_+^z)^{-1}u_-^{\bar z}(m+1)^*A_{m+1} \\ -\mathcal{W}(u_+^{\bar z}, u_-^z)^{-1}u_+^{\bar z}(m)^* & \mathcal{W}(u_+^{\bar z}, u_-^z)^{-1}u_+^{\bar z}(m+1)^*A_{m+1} \end{pmatrix}\end{aligned} \tag{4.13}$$

Proof. We prove that $\mathcal{W}(u_-^{\bar z}, u_+^z)$ is invertible, the other can be done in the same manner. Let $\alpha, \beta \in \mathbb{C}^{L\times L}$ (see Rem. 1.9) such that

$$u_+^z = u_-^z\alpha + w_-^z\beta. \tag{4.14}$$

An explicit computation, using Eq. (3.28) and Eq. (1.28) and then taking the limit $n \to -\infty$, shows that $\mathcal{W}(u_-^{\bar z}, w_-^z) = \lambda(z)^{-1} - \lambda(z)$. Using this above and Eq. (3.46) one obtains that $\mathcal{W}(u_-^{\bar z}, u_+^z) = (\lambda(z)^{-1} - \lambda(z))\beta$. Let $\phi \in \mathrm{Ker}(\mathcal{W}(u_-^{\bar z}, u_+^z))$. Then $\phi \in \mathrm{Ker}(\beta)$, this and Eq. (4.14) imply that $u_+^z\phi = u_-^z\alpha\phi \in L^2$, since $Hu_+^z\phi = zu_+^z\phi$ and $z \notin \sigma(H)$ then $u_+^z\phi = 0$ which implies $\phi = 0$. Next we prove Eq. 4.13. Using Eq. (2.41) (with $\tilde v = u_-^{\bar z}$, $\tilde u = u_+^{\bar z}, u = u_+^z, v = u_-^z$) we obtain (recall that the wronskian does not depend on n)

$$\mathcal{J}\Phi(u_+^{\bar z}, u_-^{\bar z})(m+1)^*\mathcal{J}^*\Phi(u_+^z, u_-^z)(m+1) = \begin{pmatrix} -\mathcal{W}(u_-^{\bar z}, u_+^z) & 0 \\ 0 & \mathcal{W}(u_+^{\bar z}, u_-^z) \end{pmatrix}$$

where Eq. (3.46) was used. Using this above and the fact that the diagonal entries of the R.H.S. of the previous equation are invertible we obtain the result. The second equality in Eq. (4.13) follows by explicit computation using Def. 2.28 and Eq. (2.35). □

Using Eqs. (4.12) and Eq. (4.13) we obtain:

$$\begin{pmatrix} \alpha_m^z \\ -\beta_m^z \end{pmatrix} = \Phi(u_+^z, u_-^z)(m+1)^{-1}\begin{pmatrix} \mathbf{1} \\ 0 \end{pmatrix} = \begin{pmatrix} -\mathcal{W}(u_-^{\bar z}, u_+^z)^{-1}u_-^{\bar z}(m)^* \\ -\mathcal{W}(u_+^{\bar z}, u_-^z)^{-1}u_+^{\bar z}(m)^* \end{pmatrix} \tag{4.15}$$

Now we can obtain an expression for the Green's function.

Proposition 4.7. *For $z \in \mathbb{C}\setminus\sigma(H)$ and $n, m \in \mathbb{Z}$. The Green's functions has the following expression*

$$G^z(n,m) := \begin{cases} -u_+^z(n)\mathcal{W}(u_-^{\bar z}, u_+^z)^{-1}u_-^{\bar z}(m)^*, & n \geq m \\ u_-^z(n)\mathcal{W}(u_+^{\bar z}, u_-^z)^{-1}u_+^{\bar z}(m)^*, & n \leq m. \end{cases} \tag{4.16}$$

Proof. It follows using Eqs. (4.8), (4.9) and Eq. (4.15). □

Remark 4.8. *For $z \notin \sigma(H)$, we can use Eq. 4.16 to express the Green's function in terms of the coefficient $M_\pm^z$ (see Eq. (3.56)). Namely,*

$$G^z(n,m) := \begin{cases} \nu^z u_+^z(n)(M_+^z)^{-1}u_-^{\bar z}(m)^*, & n \geq m \\ \nu^z u_-^z(n)(M_-^z)^{-1}u_+^{\bar z}(m)^*, & n \leq m. \end{cases} \tag{4.17}$$

Since H is self-adjoint, we have that $\sigma(H) \subset \mathbb{R}$. In particular, if $z \in \mathbb{C}^+ \cup \mathbb{C}^-$, then Eq. (4.16) follows. Moreover, the Jost solutions have limit up to $(-2, 2)$ from above and below (see Rem. 3.5 and Prop. 3.6), and $M_\pm^E$ is invertible for $E \in (-2, 2)$ (see Prop. 3.16). Then, Eq. (4.16) implies that $G^z(n,m)$ also has limit up to the boundary.

4.1.2 Boundary values

Proposition 4.9 (Boundary values limit for the Green's function)**.** *For all $E \in (-2,2)$, and $n, m \in \mathbb{Z}$ the following limits exist*

$$G^{E+i0}(n,m) := \lim_{\substack{z\to E \\ \Im(z)>0}} G^z(n,m) = \begin{cases} -u^E_{+,out}(n)\mathcal{W}(u^E_{-,in}, u^E_{+,out})^{-1}u^E_{-,in}(m)^*, & n \geq m \\ u^E_{-,out}(n)\mathcal{W}(u^E_{+,in}, u^E_{-,out})^{-1}u^E_{+,in}(m)^*, & n \leq m. \end{cases} \tag{4.18}$$

$$G^{E-i0}(n,m) := \lim_{\substack{z\to E \\ \Im(z)<0}} G^z(n,m) = \begin{cases} -u^E_{+,in}(n)\mathcal{W}(u^E_{-,out}, u^E_{+,in})^{-1}u^E_{-,out}(m)^*, & n \geq m \\ u^E_{-,in}(n)\mathcal{W}(u^E_{+,out}, u^E_{-,in})^{-1}u^E_{+,out}(m)^*, & n \leq m. \end{cases} \tag{4.19}$$

Proof. It follows from Eq. (4.16), using the continuity from above of the Jost solutions and Prop. 3.6, also see Rem. 3.5. Note that Prop. 3.16 and Eq. (3.54) imply that $\mathcal{W}(u^E_{-,in}, u^E_{+,out})$ and $\mathcal{W}(u^E_{+,in}, u^E_{-,out})$ are invertible for $E \in (-2,2)$. ☐

Having obtained the boundary values of the Green's function, we are now in a position to define the operator $R(E \pm i0)$ for $E \in (-2,2)$ as an integral operator with kernel $G^{E\pm i0}$ (see Proposition 4.5). Naturally, the operator $R(E \pm i0)$ does not belong to $\mathcal{B}(L^2)$. Instead, we consider a larger operator space, in this case, we have

$$R(E \pm i0) \in \mathcal{B}(L^1(\mathbb{Z}, \mathbb{C}^L), L^\infty(\mathbb{Z}, \mathbb{C}^L)).$$

Since the Jost solutions $u^E_{\pm,\mathrm{out}}$ and $u^E_{\pm,\mathrm{in}}$ are bounded, Proposition 4.9 implies that

$$\sup_{n,m\in\mathbb{Z}} \left\|G^{E\pm i0}(n,m)\right\| < \infty. \tag{4.20}$$

Therefore, the operator

$$R(E \pm i0) : L^1(\mathbb{Z}, \mathbb{C}^L) \to L^\infty(\mathbb{Z}, \mathbb{C}^L),$$

defined by

$$(R(E \pm i0)u)(n) := \sum_{m\in\mathbb{Z}} G^{E\pm i0}(n,m)\, u(m), \tag{4.21}$$

is well-defined and bounded.
We now require the following uniform estimate on the Jost solutions.

Lemma 4.10. *Assume that $H - H_0$ has finite zero moment. Then, for all compact set $K \subset \mathbb{C} \setminus \{-2,2\}$, one has*

$$\sup_{n\in\mathbb{Z}, z\in K} \|\lambda(z)^{-n}u^z_+(n)\| \leq C_K, \tag{4.22}$$

where C_K denotes a constant that does not depend on $n \in \mathbb{Z}, z \in K$.

Proof. Let $n \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \{-2, 2\}$, Eq. (3.11) imply that

$$\|\lambda(z)^{-n} u_+^z(n)\| \leq C \left(\mathbf{1} + \sum_{j=n+1}^{\infty} \|\lambda(z)^{j-n} (\tau_V s_\cdot^z(n)^*(j))^*\| \|\lambda(z)^{-j} u_+^z(j)\| \right).$$

Using equation above and Lemma 5.4 we obtain that,

$$\|\lambda(z)^{-n} u_+^z(n)\| \leq C \exp\left(C \sum_{j=n+1}^{\infty} \|\lambda(z)^{j-n} (\tau_V s_\cdot^z(n)^*(j))^*\| \right). \tag{4.23}$$

Eq. (3.3) (also see Eq. (3.13)) implies that for $j > n$,

$$\sup_{z \in K} \|\lambda(z)^{j-n} (\tau_V s_\cdot^z(n)^*(j))^*\| \leq C_K (\|A_j - \mathbf{1}\| + \|A_{j+1} - \mathbf{1}\| + \|B_j\|), \tag{4.24}$$

where C_K denotes a constant that does not depend on $n \in \mathbb{Z}, z \in K$. Then, Eqs. (4.23), (4.24) and the finite zero moment assumption imply Eq. (4.22). □

Lemma 4.11. *Assume that $H - H_0$ has finite first moment. Let $E \in (-2, 2)$ and $K \subset \mathbb{C} \setminus \{-2, 2\}$ be a compact neihborhood of E. Then, for all $1 \geq \alpha > 0$, and $z \in K$, one has*

$$\sup_{n \in \mathbb{Z}} \|(1 + |n|)^{-\alpha} (\lambda(z)^{\mp n} u_\pm^z(n) - \lambda(E)^{\mp n} u_{\pm,out}^E(n))\| \leq C_K |\lambda(z) - \lambda(E)|^\alpha, \tag{4.25}$$

$$\sup_{n \in \mathbb{Z}} \|(1 + |n|)^{-\alpha} (\lambda(z)^{\mp n} u_\pm^z(n) - \lambda(E)^{\pm n} u_{\pm,in}^E(n))\| \leq C_K |\lambda(z) - \lambda(E)^{-1}|^\alpha, \tag{4.26}$$

where C_K is a constant that does not depend on $z \in K, n \in \mathbb{Z}$.

Proof. We prove Eq. (4.25), Eq. (4.26) follows in the same manner. Let $E \in (-2, 2)$ and $n \in \mathbb{Z}$, Eq. (3.11) implies that

$$(1 + |n|)^{-\alpha} (\lambda(z)^{-n} u_+^z(n) - \lambda(E)^{-n} u_{+,out}^E(n)) = L(z, n) + R(z, n), \tag{4.27}$$

where

$$L(z, n) = A_{n+1}^{-1} \sum_{j=n+1}^{\infty} \frac{\lambda(z)^{j-n} (\tau_V s_\cdot^z(n)^*)(j)^* - \lambda(E)^{j-n} (\tau_V s_\cdot^E(n)^*)(j)^*}{(1 + |n|)^\alpha} \lambda(z)^{-j} u_+^z(j). \tag{4.28}$$

$$R(z, n) = A_{n+1}^{-1} \sum_{j=n+1}^{\infty} \frac{(1 + |j|)^\alpha}{(1 + |n|)^\alpha} \lambda(E)^{j-n} (\tau_V s_\cdot^E(n)^*)(j)^* \frac{\lambda(z)^{-j} u_+^z(j) - \lambda(E)^{-j} u_{+,out}^E(j)}{(1 + |j|)^\alpha}. \tag{4.29}$$

Let us denote for $j > n$,

$$r(z, j, n) := \lambda(z)^{j-n} (\tau_V s_\cdot^z(n)^*)(j)^* - \lambda(E)^{j-n} (\tau_V s_\cdot^E(n)^*)(j)^*.$$

Consider $K \subset \overline{\mathbb{C}^+} \setminus \{-2, 2\}$ a compact neighborhood of E. Using Equation (3.3) and Prop. 5.5 (also see Eq. (3.13)) we obtain that for $j \geq n + 1$ and $z \in K$,

$$\|r(z, j, n)\| \leq C_K |\lambda(z) - \lambda(E)|^\alpha (1 + 2(j - n))^\alpha (\|A_j - \mathbf{1}\| + \|A_{j+1} - \mathbf{1}\| + \|B_j\|), \tag{4.30}$$

using in Eq. (4.30) the general fact

$$\frac{1+2(j-n)}{1+|n|} \leq 2(1+|j|),\ j \geq n \in \mathbb{Z},$$

and that $\alpha \leq 1$, we obtain for $j > n \in \mathbb{Z}$ and $z \in K$ that

$$\frac{\|r(z,j,n)\|}{(1+|n|)^\alpha} \leq C_K|\lambda(z)-\lambda(E)|^\alpha(1+|j|)(\|A_j-\mathbf{1}\|+\|A_{j+1}-\mathbf{1}\|+\|B_j\|).$$

Using above equation, Eqs. (4.22), (4.28) and the first finite moment assumption we obtain for $z \in K$ that

$$\sup_{n\in\mathbb{Z}} \|L(z,n)\| \leq C_K|\lambda(z)-\lambda(E)|^\alpha. \tag{4.31}$$

Using Equation (3.3) (also see Eq. (3.13)) and that $1 \geq \alpha > 0$ we obtain for $j > n \in \mathbb{Z}$,

$$\frac{(1+|j|)^\alpha}{(1+|n|)^\alpha}\|\lambda(E)^{j-n}(\tau_V s_\cdot^E(n)^*)(j)^*\| \leq C_K(1+|j|)(\|A_j-\mathbf{1}\|+\|A_{j+1}-\mathbf{1}\|+\|B_j\|),$$

using above equation and Eq. (4.29) we obtain

$$\|R(z,n)\| \leq C_K \sum_{j=n+1}^{\infty} w_j\|(1+|j|)^{-\alpha}(\lambda(z)^{-j}u_+^z(j)-\lambda(E)^{-j}u_{+,out}^E(j))\|, \tag{4.32}$$

where $w_j = (1+|j|)(\|A_j-\mathbf{1}\|+\|A_{j+1}-\mathbf{1}\|+\|B_j\|)$. Taking norm in Eq. (4.27), using Eqs. (4.31), (4.32) and Gronwall inequality (see Prop. 5.4), together with the first moment assumption we obtain the result. □

Remark 4.12. *Taking the representation of the Green's function in terms of the Jost solutions (Eq. (4.16)), and applying Lemma 4.10, we obtain that for* $n \geq m$ *and* $K \subset \mathbb{C}\setminus\sigma(H)$ *compact set,*

$$\|G^z(n,m)\| \leq \|\lambda(z)^{n-m}\|\|\lambda(z)^{-n}u_+^z(n)\mathcal{W}(u_-^{\overline{z}},u_+^z)^{-1}\overline{\lambda(\overline{z})^m}u_-^{\overline{z}}(m)^*\| \leq C_K, \tag{4.33}$$

where C_K *is a constant that does not depend on* $z \in K, n, m \in \mathbb{Z}$*. One can obtain the same bound for* $n \leq m$*. Moreover, using Eq. (4.18) one can extend the uniform bound for compact sets* $K \subset (C\setminus\sigma(H))\cup(-2,2)$*.*

Remark 4.13. *Let* $E \in (-2,2)$ *and* $K \subset (\mathbb{C}\setminus\sigma(H))\cup(-2,2)$ *compact neighborhood of* E *such that* $|\lambda(E)-\lambda(z)| < 1,\ z \in K$*. Taking* $n = 1, 0$ *in Eq. (4.25) (with* $\alpha = 1$*) we obtain*

$$|u_+^z(1)-u_{+,out}^E(1)| \leq C_K|\lambda(z)-\lambda(E)|,\ |u_+^z(0)-u_{+,out}^E(0)| \leq C_K|\lambda(z)-\lambda(E)|,$$

using Eq. (4.26) with $n = 1, 0$*, we obtain* $|u_-^{\overline{z}}(0)-u_{-,in}^E(0)| \leq C_K|\lambda(\overline{z})-\lambda(E)^{-1}| = C_K|\lambda(z)-\lambda(E)|$ *(recall* $\overline{\lambda(\overline{z})} = \lambda(z)$, $\overline{\lambda(E)} = \lambda(E)^{-1}$*) and* $|u_-^{\overline{z}}(1)-u_{-,in}^E(1)| \leq C_K|\lambda(z)-\lambda(E)|$*. Then,*

$$\|\mathcal{W}(u_-^{\overline{z}},u_+^z)-\mathcal{W}(u_{-,in}^E,u_{+,out}^E)\| \leq C_K(\lambda(z)-\lambda(E)),\ z \in K. \tag{4.34}$$

Taking the representation of the Green's function in terms of the Jost solutions (Eq. (4.16) and Eq. (4.18)) , we obtain that for $n \geq m \in \mathbb{Z}$,

$$\begin{aligned} &G^z(n,m) - G^{E+i0}(n,m) \\ &= (\lambda(E)^{n-m} - \lambda(z)^{n-m})\lambda(E)^{-n}u^E_{+,out}(n)\mathcal{W}(u^E_{-,in}, u^E_{+,out})^{-1}\lambda(E)^m u^E_{-,in}(m)^* \\ &+ \lambda(z)^{n-m}(\lambda(E)^{-n}u^E_{+,out}(n) - \lambda(z)^{-n}u^z_+(n))\mathcal{W}(u^E_{-,in}, u^E_{+,out})^{-1}\lambda(E)^m u^E_{-,in}(m)^* \\ &+ \lambda(z)^{n-m}\lambda(z)^{-n}u^z_+(n)(\mathcal{W}(u^E_{-,in}, u^E_{+,out})^{-1} - \mathcal{W}(u^{\overline{z}}_-, u^z_+)^{-1})\lambda(E)^m u^E_{-,in}(m)^* \\ &+ \lambda(z)^{n-m}\lambda(z)^{-n}u^z_+(n)\mathcal{W}(u^{\overline{z}}_-, u^z_+)^{-1}(\lambda(E)^m u^E_{-,in}(m)^* - \lambda(z)^m u^z_-(m)^*). \end{aligned} \tag{4.35}$$

Applying Lemma 4.11 in Eq. (4.35), also using Eq. (4.22), Prop. 5.5 (with the general fact $(1+|n|)(1+|m|) \geq (1+|n-m|)$*) and Eq. (4.34) we obtain that for* $1 \geq \alpha > 0$

$$\frac{\|G^z(n,m) - G^{E+i0}(n,m)\|}{(1+|m|)^\alpha(1+|n|)^\alpha} \leq C_K|\lambda(E) - \lambda(z)|^\alpha,$$

and we can obtain the same for $n \leq m$*. In particular, for all* $z \in K$ *we have*

$$\sup_{n,m\in\mathbb{Z}} \frac{\|G^z(n,m) - G^{E+i0}(n,m)\|}{(1+|m|)^\alpha(1+|n|)^\alpha} \leq C_K|\lambda(E) - \lambda(z)|^\alpha. \tag{4.36}$$

Note that for $u \in L^1$, one has by definition $R(E \pm i0)u \in L^\infty$ then for all $v \in L^1$ the inner product

$$\langle R(E \pm i0)u, v\rangle = \sum_{n\in\mathbb{Z}} \langle (R(E \pm i0)u)(n), v(n)\rangle_{\mathbb{C}^L},$$

is well defined.

Proposition 4.14. *Suppose that* $H - H_0$ *has finite zero moment. Let* $u, v \in L^1 \subset L^2$*, then for all* $E \in (-2,2)$ *one has*

$$\lim_{\substack{z\to E \\ \pm\Im(z)>0}} \langle R(z)u, v\rangle = \langle R(E \pm i0)u, v\rangle. \tag{4.37}$$

Proof. We prove the limit from above. Using Prop. 4.5 and Def. (4.21), we obtain

$$\langle R(z)u, v\rangle - \langle R(E \pm i0)u, v\rangle = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \langle (G^z(n,m) - G^{E+i0}(n,m))u(m), v(n)\rangle_{\mathbb{C}^L}. \tag{4.38}$$

Using Cauchy-Schwarz inequality in Eq. (4.38) we obtain

$$|\langle R(z)u, v\rangle - \langle R(E \pm i0)u, v\rangle| \leq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \|v(n)\|\|G^z(n,m) - G^{E+i0}(n,m)\|\|u(m)\|. \tag{4.39}$$

Let $K \subset \overline{\mathbb{C}^+} \setminus \{-2,2\}$ be a compact neiborhood of E. Rem. 4.12 implies that $\|G^z(n,m) - G^{E+i0}(n,m)\| \leq C_K$, then Eq. (4.39), Dominated Lebesgue Theorem, and Prop. 4.9 give us the result. $\square$

Remark 4.15. *For $u, v \in L^1$, using Prop. 4.14 one can extend the function $\mathbb{C}^\pm \mapsto \langle R(z)u, v\rangle$ from above and below to $E \in (-2, 2)$. Moreover, in the same manner as in Prop. 4.14 one can prove that this extension is continuous on $\overline{\mathbb{C}^\pm} \setminus \{-2, 2\}$.*

The next proposition is the Limit Absorption Principle. For $\alpha \in \mathbb{R}^+$ we denote

$$\mathcal{M}_{-\alpha} : L^2(\mathbb{Z}, \mathbb{C}^L) \to L^2(\mathbb{Z}, \mathbb{C}^L), \ (\mathcal{M}_{-\alpha}u)(n) = (1 + |n|)^{-\alpha}u(n), \tag{4.40}$$

which is a bounded operator. For $\alpha \geq 1$, let us denote

$$C_{-\alpha} := \sum_{n\in\mathbb{Z}} \frac{1}{(1 + |n|)^{2\alpha}} \leq 1 + 2\sum_{n=1}^{\infty} \frac{1}{|n|^2} < \infty.$$

Remark 4.16. *Next, we see that if $\alpha \geq 1$ and $E \in (-2, 2)$, the operator (see Eq. (4.21))*

$$\mathcal{M}_{-\alpha}R(E \pm i0)\mathcal{M}_{-\alpha} \in \mathcal{B}(L^2).$$

First, if $u \in L^2$, Cauchy-schwarz inequality implies that

$$\sum_{m\in\mathbb{Z}}(1 + |m|)^{-\alpha}\|G^{E\pm i0}(n, m)u(m)\| \leq C_{-\alpha}^{1/2} \sup_{n,m\in\mathbb{Z}} \|G^{E\pm i0}(n, m)\|\|u\|_{L^2}.$$

Therefore,

$$\sum_{n\in\mathbb{Z}}(1 + |n|)^{-2\alpha} \left(\sum_{m\in\mathbb{Z}}(1 + |m|)^{-\alpha}\|G^{E\pm i0}(n, m)u(m)\|\right)^2 \leq C_{-\alpha}^2 \sup_{n,m\in\mathbb{Z}} \|G^{E\pm i0}(n, m)\|^2\|u\|_{L^2}^2.$$

The above equation implies that $\mathcal{M}_{-\alpha}R(E \pm i0)\mathcal{M}_{-\alpha} \in \mathcal{B}(L^2)$. Moreover,

$$\|\mathcal{M}_{-\alpha}R(E \pm i0)\mathcal{M}_{-\alpha}\| \leq C_{-\alpha} \sup_{n,m\in\mathbb{Z}} \|G^{E\pm i0}(n, m)\|.$$

Theorem 4.17 (Limit Absorption Principle)**.** *Let H be a matrix-valued self-adjoint Jacobi operator as in Def. 1.1. Further, assume that $H - H_0$ has finite first moment (see Def. 1.4). Take $\alpha \in \mathbb{R}$ with $2 \geq \alpha > 1$. Then, the operator valued function (see Eq. (4.40))*

$$\mathbb{C}^\pm \ni z \mapsto \mathcal{M}_{-\alpha}(H - z)^{-1}\mathcal{M}_{-\alpha} \in \mathcal{B}(L^2(\mathbb{Z}, \mathbb{C}^L)),$$

has limit up to $(-2, 2)$ (from above and below respectively). Moreover, for $E \in (-2, 2)$ and $K \subset \overline{\mathbb{C}^+}\setminus(\{2, -2\}\cup\sigma_d(H))$ compact neighboorhod of E. For all $z \in K$, with $|\lambda(E)-\lambda(z)| < 1$ one has (see Rem. 4.16),

$$\|\mathcal{M}_{-\alpha}R(z)\mathcal{M}_{-\alpha} - \mathcal{M}_{-\alpha}R(E + i0)\mathcal{M}_{-\alpha}\| \leq |\lambda(z) - \lambda(E)|^{\alpha-1}C_K. \tag{4.41}$$

Where C_K is a constant that does not depend on $z \in K$

Proof. Let $E \in (-2,2)$, and $K \subset \overline{\mathbb{C}^+} \setminus (\{2,-2\} \cup \sigma_d(H))$ compact neighboorhod of E with $|\lambda(z) - \lambda(E)| < 1$ for $z \in K$. Let $u \in L^2$, using Cauchy-Schwarz inequality and Eq. (4.36) we obtain that

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}} (1+|m|)^{-\alpha}(1+|n|)^{1-\alpha}\|(G^z(n,m) - G^{E+i0}(n,m))u(m)\| \\
&= \sum_{m\in\mathbb{Z}} (1+|m|)^{-1}\frac{\|(G^z(n,m) - G^{E+i0}(n,m)\|}{(1+|m|)^{\alpha-1}(1+|n|)^{\alpha-1}}\|u(m)\| \\
&\le C_K C_{-1}^{1/2}|\lambda(E) - \lambda(z)|^{\alpha-1}\|u\|_{L^2}.
\end{aligned} \tag{4.42}$$

Therefore, using Prop. 4.5 (also see Rem. 4.16) and Eq. (4.42) we have

$$\begin{aligned}
&\|(\mathcal{M}_{-\alpha}R(z)\mathcal{M}_{-\alpha} - \mathcal{M}_{-\alpha}R(E+i0)\mathcal{M}_{-\alpha})u\|^2 \\
&\le \sum_{n\in\mathbb{Z}} (1+|n|)^{2\alpha}\left(\sum_{m\in\mathbb{Z}} (1+|m|)^{-\alpha}\|(G^z(n,m) - G^{E+i0}(n,m))u(m)\|\right)^2 \\
&= \sum_{n\in\mathbb{Z}} (1+|n|)^{-2}\left(\sum_{m\in\mathbb{Z}} (1+|m|)^{-\alpha}(1+|n|)^{1-\alpha}\|(G^z(n,m) - G^{E+i0}(n,m))u(m)\|\right)^2 \\
&\le C_{-1}^2 C_K^2|\lambda(E) - \lambda(z)|^{2(\alpha-1)}\|u\|_{L^2}^2.
\end{aligned}$$

Equation above implies Eq. (4.41), which in particular implies the limit from above. Limit from below is proved in the same manner. □

4.2 Generalized Fourier transform

The wave operators can be expressed in terms of the Jost solutions via the generalized Fourier transforms. These transforms are unitary operators that diagonalize the absolutely continuous part of H. The idea behind constructing such unitary operators is to consider the family of functions

$$(-2,2) \ni E \mapsto U_n(E) \equiv \begin{pmatrix} u^E_{-,\mathrm{in}}(n)^* \\ u^E_{+,\mathrm{in}}(n)^* \end{pmatrix} \in \mathbb{C}^{2L\times L}, \quad n \in \mathbb{Z}.$$

It turns out that the columns of the matrices U_n, i.e., the set $\{(U_n)_j : n \in \mathbb{Z},\ j \in \{1,\dots,L\}\}$, after suitable normalization, form an eigenfunction expansion of the absolutely continuous part H^{ac} of the operator H. This construction leads to a partial isometry

$$\Phi_+ : L^2(\mathbb{Z},\mathbb{C}^L) \to L^2((-2,2),\mathbb{C}^{2L}),$$

which diagonalizes the operator H on its absolutely continuous subspace.
We begin by using the well-known Stone formula to express the spectral measure of H in terms of the boundary values of the resolvent.

Proposition 4.18 (Stone formula)**.** *Suppose that $H - H_0$ has finite zero moment. Let E_H the spectral measure of H. Then for all $[a,b] \subset (-2,2)$ and $u, v \in L^1(\mathbb{Z},\mathbb{C}^L)$ one has*

$$\langle E_H(a,b)u, v\rangle = \frac{1}{2\pi i}\int_a^b \langle (R(E+i0) - R(E-i0))u, v\rangle dE \tag{4.43}$$

Proof. By Prop. 3.22, there are no point spectrum on $[-2,2]$, namely $\sigma_p(H) \subset \mathbb{R} \setminus [-2,2]$. In particular, for all $a \in (-2,2)$, $E_H(\{a\}) = 0$. Using Prop. 5.6 we obtain

$$\begin{aligned}\langle E_H(a,b)u, v\rangle &= \frac{1}{2\pi i} \lim_{\varepsilon \downarrow 0} \left\langle \int_a^b (R(E+i\varepsilon) - R(E-i\varepsilon))u \; dE, v \right\rangle \\ &= \frac{1}{2\pi i} \lim_{\varepsilon \downarrow 0} \int_a^b \langle (R(E+i\varepsilon) - R(E-i\varepsilon))u, v\rangle \; dE \\ &= \frac{1}{2\pi i} \int_a^b \langle (R(E+i0) - R(E-i0))u, v\rangle \; dE,\end{aligned}$$

where in the last equality we use Prop. 4.14 (also see Rem. 4.15) and Lebesgue dominated theorem. ☐

Remark 4.19. *Using Eq. (4.18) with $n \geq m$, and Eq. (3.54) we obtain*

$$\begin{aligned}&(\nu^E)^{-1}(G^{E+i0}(n,m) - G^{E-i0}(n,m)) \\ &\qquad = u^E_{+,in}(n)((M^E_-)^*)^{-1}u^E_{-,out}(m)^* + u^E_{+,out}(n)(M^E_+)^{-1}u^E_{-,in}(m)^*.\end{aligned} \tag{4.44}$$

Now, we use Eq. (3.52) and we write in Eq .(4.44) $u^E_{\pm,in}$ in terms of $u^E_{\pm,out}$ (recall that M^E is invertible for $E \in (-2,2)$,) then we obtain

$$\begin{aligned}&(\nu^E)^{-1}(G^{E+i0}(n,m) - G^{E-i0}(n,m)) \\ &= (u^E_{-,out}(n)(M^E_-)^{-1} - u^E_{+,out}(n)N^E_-(M^E_-)^{-1})((M^E_-)^*)^{-1}u^E_{-,out}(m)^* \\ &+ u^E_{+,out}(n)(M^E_+)^{-1}(((M^E_+)^*)^{-1}u^E_{+,out}(m)^* - ((M^E_+)^*)^{-1}(N^E_+)^*u^E_{-,out}(m)^*).\end{aligned} \tag{4.45}$$

Now, we use that $\mathcal{S}^E$ (see Def. 1.12) is unitary, see Prop. 3.16. In particular, computing the bottom off diagonal term of $\mathcal{S}^E(\mathcal{S}^E)^*$ we obtain

$$N^E_+(M^E_+)^{-1}((M^E_+)^*)^{-1} = -(M^E_-)^{-1}((M^E_-)^*)^{-1}(N^E_-)^*. \tag{4.46}$$

Taking adjoint in Eq. (4.46) and using it in Eq. (4.45) we obtain,

$$\begin{aligned}G^{E+i0}(n,m) - G^{E-i0}(n,m) &= \nu^E u^E_{-,out}(n)(M^E_-)^{-1}((M^E_-)^*)^{-1}u^E_{-,out}(m)^* \\ &\quad + \nu^E u^E_{+,out}(n)(M^E_+)^{-1}((M^E_+)^*)^{-1}u^E_{+,out}(m)^*,\end{aligned} \tag{4.47}$$

one can obtain the same for $m \geq n$. In the same manner, one obtains

$$\begin{aligned}G^{E+i0}(n,m) - G^{E-i0}(n,m) &= \nu^E u^E_{+,in}(n)((M^E_-)^*)^{-1}(M^E_-)^{-1}u^E_{+,in}(m)^* \\ &\quad + \nu^E u^E_{-,in}(n)((M^E_+)^*)^{-1}(M^E_+)^{-1}u^E_{-,in}(m)^*.\end{aligned} \tag{4.48}$$

Rem. 4.19 tells us how we should normalize the Jost solution.

Remark 4.20. *For $E \in (-2,2)$ by definition (see Def. 2.4), $\lambda(E) \in \mathbb{S}^1_- \setminus \{1,-1\}$, then we have (see Eq. (2.13))*

$$-i\nu^E = \frac{1}{i(\lambda(E) - \lambda(E)^{-1})} = \frac{1}{-4\pi \Im(\lambda(E))} \in \mathbb{R}^+.$$

Now, we define solutions of the generalized eigenvalue equaitons, which correspond to the base of the eigenfunction expantion of H,

Definition 4.21 (Wave functions)**.** *For all $E \in (-2,2)$ we define the following generalized eigenvalues of H,*

$$\psi^E_{-,out}(n) = \sqrt{-i\nu^E} u^E_{-,out}(n)(M^E_-)^{-1} \quad \psi^E_{+,out}(n) = \sqrt{-i\nu^E} u^E_{+,out}(n)(M^E_+)^{-1}. \tag{4.49}$$

$$\psi^E_{-,in}(n) = \sqrt{-i\nu^E} u^E_{-,in}(n)((M^E_+)^*)^{-1}, \ \psi^E_{+,in}(n) = \sqrt{-i\nu^E} u^E_{+,in}(n)((M^E_-)^*)^{-1}. \tag{4.50}$$

Proposition 4.22. *Let $E \in (-2,2)$, for all $n, m \in \mathbb{Z}$ the following identity holds (see Def. 4.21)*

$$\begin{aligned} G^{E+i0}(n,m) - G^{E-i0}(n,m) &= i(\psi^E_{-,out}(n)\psi^E_{-,out}(m)^* + \psi^E_{+,out}(n)\psi^E_{+,out}(m)^*) \\ &= i(\psi^E_{-,in}(n)\psi^E_{-,in}(m)^* + \psi^E_{+,in}(n)\psi^E_{+,in}(m)^*). \end{aligned} \tag{4.51}$$

Proof. It follows from Eqs. (4.49), (4.50) and Rem. 4.19, see in particular Eqs. (4.48) and (4.47). ☐

Let us now introduce the generalized Fourier transforms.

Definition 4.23 (Generalized Fourier transform)**.** *For $u \in L^1(\mathbb{Z}, \mathbb{C}^L)$, we define $\Phi_\pm u : (-2,2) \to \mathbb{C}^{2L}$, the function given by (see Def. 4.21)*

$$(\Phi_+ u)(E) := \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \begin{pmatrix} \psi^E_{-,in}(n)^* u(n) \\ \psi^E_{+,in}(n)^* u(n) \end{pmatrix}, \tag{4.52}$$

$$(\Phi_- u)(E) := \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \begin{pmatrix} \psi^E_{+,out}(n)^* u(n) \\ \psi^E_{-,out}(n)^* u(n) \end{pmatrix}. \tag{4.53}$$

Next, we extend $\Phi_\pm$ to a unitary operator from the absolutely continuous space of H to $L^2((-2,2), \mathbb{C}^{2L})$.

Proposition 4.24. *For all $u, v \in L^1(\mathbb{Z}, \mathbb{C}^L)$ and $E \in (-2,2)$ one has*

$$\langle (\Phi_\pm u)(E), (\Phi_\pm v)(E) \rangle_{\mathbb{C}^{2L}} = \frac{1}{2\pi i} \langle (R(E+i0) - R(E-i0))u, v \rangle_{L^2}. \tag{4.54}$$

Proof. We prove the $'+'$ the other is similar. Using Eq. (4.52) we obtain

$$\begin{aligned} \langle (\Phi_+ u)(E), (\Phi_+ v)(E) \rangle &= \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \left\langle \begin{pmatrix} \psi^E_{-,in}(m)^* u(m) \\ \psi^E_{+,in}(m)^* u(m) \end{pmatrix}, \begin{pmatrix} \psi^E_{-,in}(n)^* v(n) \\ \psi^E_{+,in}(n)^* v(n) \end{pmatrix} \right\rangle \\ &= \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \langle \psi^E_{-,in}(m)^* u(m), \psi^E_{-,in}(n)^* v(n) \rangle + \langle \psi^E_{+,in}(m)^* u(m), \psi^E_{+,in}(n)^* v(n) \rangle \\ &= \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \langle (\psi^E_{-,in}(n)\psi^E_{-,in}(m)^* + \psi^E_{+,in}(n)\psi^E_{+,in}(m)^*) u(m), v(n) \rangle \\ &= \frac{1}{2\pi i} \sum_{n\in\mathbb{Z}} \left\langle \sum_{m\in\mathbb{Z}} (G^{E+i0}(n,m) - G^{E-i0}(n,m)) u(m), v(n) \right\rangle, \end{aligned} \tag{4.55}$$

where in the last equality we use Eq. (4.51). Eq. (4.55) imply the result. ☐

Remark 4.25. *Prop. 4.24 and Eq. (4.43) imply that for all $[a,b] \subset (-2,2)$ and $u,v \in L^1(\mathbb{Z},\mathbb{C}^L)$ one has*

$$\int_a^b \langle (\Phi_\pm u)(E), (\Phi_\pm v)(E)\rangle_{\mathbb{C}^{2L}} \; dE = \langle E_H(a,b)u, v\rangle_{L^2(\mathbb{Z})}. \tag{4.56}$$

In particular, taking $u = v \in L^1(\mathbb{Z},\mathbb{C}^L)$ in Eq. (4.56) one obtains

$$\int_a^b \|(\Phi_\pm u)(E)\|^2_{\mathbb{C}^{2L}} \; dE = \|E_H(a,b)u\|^2_{L^2(\mathbb{Z})}. \tag{4.57}$$

By Fatou's Lemma one obtains that

$$\int_{-2}^{2} \|(\Phi_\pm u)(E)\|^2_{\mathbb{C}^{2L}} \; dE \leq \liminf_{n\to\infty} \int_{-2+1/n}^{2+1/n} \|(\Phi_\pm u)(E)\|^2_{\mathbb{C}^{2L}} \; dE = \|E_H(-2,2)u\|^2_{L^2}.$$

Therefore, the function $E \mapsto \|(\Phi_\pm u)(E)\|^2_{\mathbb{C}^{2L}}$ belongs to $L^1((-2,2))$. Then, Lebesgue covergence theorem implies

$$\|\Phi_\pm u\|_{L^2((-2,2),\mathbb{C}^{2L})} = \|E_H(-2,2)u\|_{L^2(\mathbb{Z},\mathbb{C}^L)}. \tag{4.58}$$

Using the fact that $L^1(\mathbb{Z})$ is dense in $L^2(\mathbb{Z})$, Equation (4.58) allows us to extend the linear function $\Phi_\pm : L^1(\mathbb{Z},\mathbb{C}^L) \subset L^2(\mathbb{Z},\mathbb{C}^L) \to L^2((-2,2),\mathbb{C}^{2L})$, to a bounded operator.

Definition 4.26 (Generalized Fourier operator)**.** *(See Rem. 4.25) We denote by*

$$\Phi_\pm \in \mathcal{B}(L^2(\mathbb{Z},\mathbb{C}^L), L^2((-2,2),\mathbb{C}^{2L})),$$

the bounded operator defined by the continuous extension of the map (see Def. 4.23)

$$L^1(\mathbb{Z}) \ni u \mapsto \Phi_\pm u \in L^2((-2,2)).$$

Remark 4.27. *The relation given by the Eq. (4.57) can now be extended for all $u \in L^2(\mathbb{Z},\mathbb{C}^L)$. Indeed, take $u_n \in L^1(\mathbb{Z},\mathbb{C}^L)$ such that $\|u_n - u\|_{\mathbb{C}^L} \underset{n\to\infty}{\overset{L^2(\mathbb{Z},\mathbb{R})}{\to}} 0$. Continuity of $\Phi_\pm$, implies that $\|\Phi_\pm u_n - \Phi_\pm u\|_{\mathbb{C}^{2L}} \underset{n\to\infty}{\overset{L^2((-2,2),\mathbb{R})}{\to}} 0$. Then, there exists subsequence $\{u_{n_k}\}$ such that*

$$\|\Phi_\pm u_{n_k} - \Phi_\pm u\|_{\mathbb{C}^{2L}} \underset{n\to\infty}{\overset{a.e.}{\to}} 0.$$

By Lebesgue dominated theorem we obtain

$$\left|\|\Phi_\pm u_{n_k}\|_{L^2(a,b)} - \|\Phi_\pm u\|_{L^2(a,b)}\right| \leq \|\Phi_\pm u_{n_k} - \Phi_\pm u\|_{L^2(a,b)} \to 0. \tag{4.59}$$

Taking limit $k \to \infty$ in both size of Eq. (4.57) we obtain the desired. This above implies that for every U borel subset of $(-2,2)$, and $u \in L^2(\mathbb{Z},\mathbb{C}^L)$ one has,

$$\langle E_H(U)u, u\rangle_{L^2(\mathbb{Z})} = \int_{\mathbb{R}} \|(\Phi_\pm u)(E)\|^2_{\mathbb{C}^{2L}} \chi_U(E) \; dE. \tag{4.60}$$

Let us denote by $L^2(\mathbb{Z})_{ac}$ the absolutely continuous space related to H, and by P_{ac} the projection on this space. Next, prove that $E_H(-2,2)L^2(\mathbb{Z}) = L^2(\mathbb{Z})_{ac}$. If $u \in E_H(-2,2)L^2(\mathbb{Z},\mathbb{C}^L)$ (namely $E_H(-2,2)u = u$), using Eq. (4.60) one has

$$\langle E_H(U)u,u\rangle_{L^2(\mathbb{Z})} = \langle E_H(U\cap(-2,2))u,u\rangle_{L^2(\mathbb{Z})} = \int_{\mathbb{R}} \|(\Phi_\pm u)(E)\|^2_{\mathbb{C}^{2L}} \chi_{U\cap(-2,2)}(E)\ dE, \quad (4.61)$$

which implies that $u \in L^2(\mathbb{Z})_{ac}$. Conversely, suppose $u \in L^2(\mathbb{Z})_{ac}$ and let us denote f_u the Radon-Nikodym derivative of the measure $E_{H,u}$ with respect the Lebesgue measure. Props. 2.2 and 3.22 imply that the set $\sigma(H)\setminus(-2,2)$ has Lebesgue measure 0. Then,

$$\langle E_H(\sigma(H)\setminus(-2,2))u,u\rangle_{L^2(\mathbb{Z})} = \int_{\sigma(H)\setminus(-2,2)} f_u(x)\ dx = 0.$$

This implies that

$$\|u - E_H(-2,2)u\| = \langle u - E_H(-2,2)u, u - E_H(-2,2)u\rangle = \langle E_H(\sigma(H)\setminus(-2,2))u,u\rangle = 0.$$

Then $E_H(-2,2)u = u$, which implies the required. As a consequence we have

$$P_{ac} = E_H(-2,2). \quad (4.62)$$

Proposition 4.28. *For compact support continuous functions $f \equiv (f_1,f_2) \in C_0((-2,2),\mathbb{C}^{2L})$, the adjoint operator of $\Phi_\pm$, $(\Phi_\pm)^* : L^2((-2,2),\mathbb{C}^{2L}) \to L^2(\mathbb{Z},\mathbb{C}^L)$ is given by*

$$((\Phi_+)^*f)(n) := \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} \psi^E_{-,in}(n) f_1(E) + \psi^E_{+,in}(n) f_2(E)\ dE. \quad (4.63)$$

$$((\Phi_-)^*f)(n) := \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} \psi^E_{+,out}(n) f_1(E) + \psi^E_{-,out}(n) f_2(E)\ dE. \quad (4.64)$$

Moreover, the following identities hold

$$(\Phi_\pm)^*\Phi_\pm = E_H(-2,2) = P_{ac}, \quad \Phi_\pm H = M_E \Phi_\pm, \quad (4.65)$$

Here $P_{ac} : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})_{ac}$, denotes the projection on the absolutely continuous space related to H, and $M_E : L^2((-2,2)) \to L^2((-2,2))$ denotes the multiplication operator, $(M_E f)(E) = Ef(E)$.

Proof. Let us prove the statment for the sign $'+'$ the other is similar. Let $f \in C_0((-2,2),\mathbb{C}^{2L})$, let us denote by $A_f(n)$ the R.H.S. of Eq. (4.63). Let u finite supported sequence, an explicit computation using Eq. (4.52) shows

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \langle A_f(n), u(n)\rangle_{\mathbb{C}^L} &= \int_{-2}^{2} \left\langle \begin{pmatrix} f_1(E) \\ f_2(E) \end{pmatrix}, \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \begin{pmatrix} \psi^E_{-,in}(n)^* u(n) \\ \psi^E_{+,in}(n)^* u(n) \end{pmatrix} \right\rangle_{\mathbb{C}^{2L}} dE \\ &= \langle f, \Phi_+ u\rangle_{L^2((-2,2))} = \langle (\Phi_+)^* f, u\rangle_{L^2(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} \langle ((\Phi_+)^* f)(n), u(n)\rangle_{\mathbb{C}^L},\end{aligned}$$

where in the first equality we use that sum is finite, then it commutes with the integral. Since the above equation satisfies for all u of finite support, then $(\Phi_+)^* f = A_f$. Let us prove the second part. Using Eq.(4.58), one obtains that for all $u \in L^2(\mathbb{Z})$

$$\langle (\Phi_+^*\Phi_+ - E_H(-2,2))u, u\rangle_{L^2(\mathbb{Z})} = \|\Phi_+ u\|^2 - \|E_H(-2,2)u\|^2 = 0. \tag{4.66}$$

Since the operator $\Phi_+^*\Phi_+ - E_H(-2,2)$ is self-adjoint Eq. (4.78) implies that $\Phi_+^*\Phi_+ = E_H(-2,2)$, Eq. (4.62) implies the other equality. Finally, an explicit computation using Eq. (4.52) and the fact that $(\tau_H \psi_{\pm,in})(n)^* = E\psi_{\pm,in}(n)^*$, shows that for finite supported sequences $(\Phi_+ H u)(E) = E(\Phi_+ u)(E)$, then by continuity and density we obtain the R.H.S. of Eq. (4.65). □

Remark 4.29. *Equation (4.65) and Prop. 5.7 imply that*

$$\mathrm{Ran}(P_{ac}) = \mathrm{Ran}((\Phi_\pm)^*\Phi_\pm) = \mathrm{Ran}((\Phi_\pm)^*). \tag{4.67}$$

Then $(\Phi_\pm)^*\Phi_\pm = P_{ac}$ is the orthogonal projection on $\mathrm{Ran}((\Phi_\pm)^*)$. Next, we see that $\Phi_\pm(\Phi_\pm)^*$ is the orthogonal projection on $\mathrm{Ran}(\Phi_\pm)$. Take $g \in L^2((-2,2), \mathbb{C}^{2L})$, and write it as $g = f + h$ with $f \in \mathrm{Ran}(\Phi_\pm)$ and $h \in \mathrm{Ran}(\Phi_\pm)^\perp = \mathrm{Ker}((\Phi_\pm)^*)$. Also, write $f = \Phi_\pm u$ with $u \in \mathrm{Ran}((\Phi_\pm)^*) = \mathrm{Ran}(P_{ac})$ (recall $\mathrm{Ran}(\Phi_\pm) = \mathrm{Ran}(\Phi_\pm(\Phi_\pm)^*$ see Eq. 4.67). Then, using Eq. (4.65) we have

$$\Phi_\pm(\Phi_\pm)^* g = \Phi_\pm(\Phi_\pm)^* f = \Phi_\pm(\Phi_\pm)^*\Phi_\pm u = \Phi_\pm P_{ac} u = \Phi_\pm u = f,$$

which establishes the statement. Therefore, $\Phi_\pm$ is a partial isometry, that has intial space $\mathrm{Ker}(\Phi_\pm)^\perp = \mathrm{Ran}((\Phi_\pm)^*)$ and final space $\mathrm{Ran}(\Phi_\pm)$.

Proposition 4.30. *For all $(a,b) \subset (-2,2)$ and $u \in L^2(\mathbb{Z}, \mathbb{C}^L)$ we have*

$$(\Phi_\pm E_H(a,b)u)(E) = \chi_{(a,b)}(E)(\Phi_\pm u)(E), \text{ a.e.} \tag{4.68}$$

Proof. We prove it for $'+'$ the other is similar. Let $z \in \mathbb{C}^\pm$, Eq. (4.65) implies $\Phi_+(H - z) = (M_E - z)\Phi_+$. Since $z \notin \sigma(M_E), \sigma(H)$ then we obtain

$$\Phi_+ R_H(z) = \Phi_+(H-z)^{-1} = (M_E - z)^{-1}\Phi_+ = R_{M_E}(z)\Phi_+. \tag{4.69}$$

Now, we use the Stone's formula (see Prop. 5.6), we recall that H has no point spectrum in $(-2,2)$ which implies $E_H(\{a\}) = 0,\ a \in (-2,2)$,

$$\begin{aligned}
\Phi_+ E_H(a,b)u &= \Phi_+ \lim_{\varepsilon\downarrow 0} \int_a^b (R_H(x+i\varepsilon) - R_H(x-i\varepsilon))u \, dx \\
&= \lim_{\varepsilon\downarrow 0} \Phi_+ \int_a^b (R_H(x+i\varepsilon) - R_H(x-i\varepsilon))u \, dx \\
&= \lim_{\varepsilon\downarrow 0} \int_a^b (R_{M_E}(x+i\varepsilon) - R_{M_E}(x-i\varepsilon))\Phi_+ u \, dx \\
&= \frac{1}{2}(E_{M_E}(a,b) + E_{M_E}[a,b])\Phi_+ u = M_{\chi_{(a,b)}}\Phi_+ u.
\end{aligned} \tag{4.70}$$

where we use Eq. (4.69) in the third equality. In the last equality, we use that the multiplication operator $M_E \in \mathcal{B}(L^2(-2,2))$ has no point spectrum and its spectral measure is given by the multiplication operator by the characteristic function, $E_{M_E}(U) = M_{\chi_U}$. □

Proposition 4.31. *The following identity holds true*

$$\Phi_{\pm}(\Phi_{\pm})^* = I_{L^2((-2,2),\mathbb{C}^{2L})}. \tag{4.71}$$

Proof. Since $\Phi_{\pm}$ is a partial isometry (see 4.29), it is enough to prove that its final space is the whole space, namely $\mathrm{Ran}(\Phi_{\pm}) = L^2((-2,2),\mathbb{C}^{2L})$. We prove this for the sign $'+'$ the other is similar. Let $f = (f_1, f_2) \in \mathrm{Ran}(\Phi_+)^{\perp}$,, then for all $u \in L^2(\mathbb{Z},\mathbb{C}^L)$ we have

$$0 = \langle f, \Phi_+ u\rangle_{L^2((-2,2))}. \tag{4.72}$$

Take $u \in L^2(\mathbb{Z},\mathbb{C}^L)$ a sequence with compact support, and $(a,b) \subset [-2,2]$. Using Eq. (4.68) and Eq. (4.52) we obtain

$$(\Phi_+ E_H(a,b)u)(E) = \chi_{(a,b)}(E)(\Phi_+ u)(E) = \frac{\chi_{(a,b)}(E)}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \begin{pmatrix} \psi^E_{-,in}(n)^* u(n) \\ \psi^E_{+,in}(n)^* u(n) \end{pmatrix}. \tag{4.73}$$

Substituing $E_H(a,b)u$ in Eq. (4.72) and using Eq. (4.73) we obtain

$$\begin{aligned} 0 = \langle f, \Phi_+ E_H(a,b)u\rangle_{L^2((-2,2))} &= \int_{-2}^{2} \langle f(E), (\Phi_+ E_H(a,b)u)(E)\rangle_{\mathbb{C}^{2L}}\, dE \\ &= \int_{-2}^{2} \left\langle f(E), \frac{\chi_{(a,b)}(E)}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \begin{pmatrix} \psi^E_{-,in}(n)^* u(n) \\ \psi^E_{+,in}(n)^* u(n) \end{pmatrix} \right\rangle_{\mathbb{C}^{2L}} dE \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \int_a^b \left\langle \psi^E_{-,in}(n) f_1(E), u(n)\right\rangle_{\mathbb{C}^L} + \left\langle \psi^E_{+,in}(n) f_2(E), u(n)\right\rangle_{\mathbb{C}^L}\ dE. \end{aligned} \tag{4.74}$$

where we use that u has compact support, then the sum is finite and it comutes with the integral. Consider δ_n the Kronecker delta and $\{e_j\}$ the standar basis of $\mathbb{C}^L$. Taking $u = \delta_n e_j$ in Eq. (4.74) for each $n \in \mathbb{Z}$ and $j \in \{1,\dots,L\}$, one concludes that

$$0 = \int_a^b \psi^E_{-,in}(n) f_1(E) + \psi^E_{+,in}(n) f_2(E)\ dE, n \in \mathbb{Z}. \tag{4.75}$$

Since Eq. (4.75) is satisfied for all $(a,b) \subset [-2,2]$ we have that for all $n \in \mathbb{Z}$

$$H^E(n) := \psi^E_{-,in}(n) f_1(E) + \psi^E_{+,in}(n) f_2(E) = 0,\ a.e. \tag{4.76}$$

For all $n \in \mathbb{Z}$, let us denote

$$U_n := \{E \in (-2,2) : H^E(n) = \psi^E_{-,in}(n) f_1(E) + \psi^E_{+,in}(n) f_2(E) \neq 0\}.$$

We know that $U := \bigcup_{n\in\mathbb{Z}} U_n$ has Lebesgue measure zero. Take $E \in (-2,2) \setminus U$, then $H^E(n) = 0, n \in \mathbb{Z}$. An explicit computation using Eq. (4.50) and Eqs. (3.49), (3.54) shows that

$$0 = \mathcal{W}(u^E_{+,out}, H^E) = \sqrt{-i\nu^E}\mathcal{W}(u^E_{+,out}, u^E_{-,in})((M^E_+)^*)^{-1} f_1(E) = -\sqrt{-i\nu^E}\nu^E f_1(E). \tag{4.77}$$

Eq. (4.77) implies that $f_1(E) = 0$. Then, $f_1 = 0$, a.e. In the same manner, one proves that $f_2 = 0$, a.e., which implies that $f \equiv 0$. Therefore, $\mathrm{Ran}(\Phi_+) = L^2((-2,2),\mathbb{C}^{2L})$. □

In summary, the operator $\Phi_\pm$ is a partial isometry with initial space the absolutely continuous space asociated to H, and final space $L^2((-2,2),\mathbb{C}^{2L})$. Moreover, it satisfies

$$\Phi_\pm H(\Phi_\pm)^* = M_E. \tag{4.78}$$

In particular, we conclude

$$\sigma_{ac}(H) = [-2,2].$$

4.3 Wave operator

Let H be a Jacobi operator as defined in Definition 1.1. In this section, we prove that, under the assumption of a finite first moment, the wave operators associated with the pair (H,H_0) exist, where H_0 is the discrete Laplacian (see Definition 1.2). Moreover, we provide an explicit expression for these wave operators in terms of the unitary operators introduced in Section 4.2 (see Definition 4.26).
We begin with a brief review of the fundamental concepts of scattering theory.

Definition 4.32. *Let (H,H_0) be two self-adjoint operators defined on a Hilbert space. We say that the wave operators of the pair (H,H_0) exist if the following limit exist*

$$W_\pm(H,H_0) := s - \lim_{t\to\pm} e^{iHt}e^{-iH_0t}P^0_{ac}. \tag{4.79}$$

Where P^0_{ac} denotes the orthogonal projection on the absolutely continuous space $\mathcal{H}_0^{ac}$ related to H_0. In this case, we call $W_\pm(H,H_0)$ the wave operators.

Definition 4.33. *Let (H,H_0) be two self-adjoint operators defined on a Hilbert space $\mathcal{H}$. Assume that the wave operators exist. We say that the wave operators are complete if*

$$\operatorname{Ran}(W_\pm(H,H_0)) = \mathcal{H}_{ac},$$

where $\mathcal{H}_{ac}$ denotes the absolutely continuous space related to H.

Now we prove the main result of this section. We recall the partial isometries defined in Section 4.2 (see Def. 4.26)

$$\Phi_\pm \in \mathcal{B}(L^2(\mathbb{Z},\mathbb{C}^L), L^2((-2,2),\mathbb{C}^{2L})).$$

In the particular case $H - H_0 = 0$, using Eqs. (4.49), (4.50) ($M^E_\pm = 0$ if $V = 0$ see (3.54)). For all $u \in L^1(\mathbb{Z},\mathbb{C}^L)$ we have

$$(\Phi_{0,-}u)(E) = (\Phi_{0,+}u)(E) = \sqrt{\frac{-i\nu^E}{2\pi}}\sum_{n\in\mathbb{Z}}\begin{pmatrix}\lambda(E)^{-n}u(n)\\ \lambda(E)^n u(n)\end{pmatrix}, \tag{4.80}$$

which is the unitary operator given in Def. 2.14 (see Eq. (2.24)), $\Phi_0 = \Phi_{0,-} = \Phi_{0,+}$. We need the following lemmata.

Lemma 4.34. *Let $g \in C_0((-2,2), \mathbb{C}^L)$ be a continuous function with compact support. For all $t \in \mathbb{R}^+$, consider the following sequence*

$$I(t,n) = \int_{-2}^{2} \lambda(E)^{-n} e^{-iEt} g(E)\, dE, \;\; n \in \mathbb{Z}^+ \cup \{0\}.$$

Then,

$$\sum_{n=0}^{\infty} \|I(t,n)\|^2_{\mathbb{C}^L} \to 0, \;\; t \to +\infty.$$

We have the same result for $n \in \mathbb{Z}^-$ and the limit $t \to -\infty$.

Proof. An explicit computation using Equation (2.12) (also see Eq. (2.13)) implies

$$\begin{aligned} \frac{d}{dE}\lambda(E)^{-n}e^{-itE} &= \lambda(E)^{-n}e^{-itE}\left(-n\lambda(E)^{-1}\frac{d}{dE}\lambda(E) - it\right) \\ &= -i\lambda(E)^{-n}e^{-itE}(-ni\nu^E + t). \end{aligned} \tag{4.81}$$

Recall that for $E \in (-2,2)$, $-i\nu^E > 0$ (see Rem. 4.20), then for all $t \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$ we have $-ni\nu^E + t > 0$. Moreover, since g has compact support we can take

$$c := \inf\{-i\nu^E : E \in supp(g)\} > 0.$$

Using integration by parts and Eq. (4.81) we obtain we obtain

$$\begin{aligned} I(t,n) &= \int_{-2}^{2} \frac{d}{dE}\left(\lambda(E)^{-n}e^{-iEt}\right)\frac{ig(E)}{-in\nu^E + t}\, dE \\ &= -\int_{-2}^{2} \lambda(E)^{-n}e^{-iEt}\frac{d}{dE}\left(\frac{ig(E)}{-in\nu^E + t}\right)\, dE, \end{aligned} \tag{4.82}$$

For $t \geq 1$, we have $|-in\nu^E + t| = -in\nu^E + t \geq cn + t \geq c + 1$, $n \in \mathbb{Z}^+ \cup \{0\}$. Then, we obtain (recall g has compact support on $(-2,2)$ and $\nu^E \in C^\infty(-2,2)$)

$$\left\|\frac{d}{dE}\left(\frac{ig(E)}{-in\nu^E + t}\right)\right\| \leq C\frac{1}{|cn + t|}, \tag{4.83}$$

where C is a constant that does not depend on E, n, t. Using Eq. (4.83) in Eq. (4.82) (also recall that $|\lambda(E)| = 1$) we obtain

$$\|I(t,n)\|_{\mathbb{C}^L} \leq C\frac{1}{|cn + t|}. \tag{4.84}$$

Eq. (4.84) together dominated Lebesgue theorem imply the result. $\square$

Lemma 4.35. *Consider $f \in C([-2,2], \mathbb{C}^L)$ a continuous function, and $r : [-2,2] \times \mathbb{Z} \to \mathbb{C}^{L \times L}$, $K \in L^2(\mathbb{Z}, \mathbb{R})$ such that*

$$\|r(E,n)\| \leq K(n), \;\; E \in [-2,2].$$

Let,

$$I(t,n) := \int_{-2}^{2} e^{itE} r(E,n) f(E)\, dE. \tag{4.85}$$

Then,

$$\|I(t,n)\|_{L^2(\mathbb{Z})} \to 0, |t| \to \pm\infty.$$

Proof. For each $n \in \mathbb{Z}$, Riemann-Lebesgue lemma, see Prop. 5.8, implies $\|I(t,n)\| \to 0,\ |t| \to \infty$. Since $\|I(t,n)\|^2 \le K(n)^2 \in L^1(\mathbb{Z},\mathbb{R})$ then Lebesgue convergece theorem imply the result. □

Proposition 4.36. *Let H be a matrix valued self-adjoint Jacobi operator as in Def. 1.1. Assume that $V = H - H_0$ has finite first moment (see Def. 1.4). Then the wave operators $W_\pm(H,H_0)$ exist and are complete. Moreover, the wave operators are given by the expression (see Def. 2.14, and Def. 4.26)*

$$W_+(H,H_0) = \Phi_+^* \Phi_0, \quad W_-(H,H_0) = \Phi_-^* \Phi_0. \tag{4.86}$$

Proof. Since $\Phi_\pm$ is a partial isometry with inital space $L^2(\mathbb{Z},\mathbb{C}^L)$ and final space the absolutely continuous space related with H, and Φ_0 is unitary, it is enough to prove that the limit in Eq. (4.79) is given by the R.H.S. of Eq. (4.86). Namely, it is enough to prove that for all $u \in L^2(\mathbb{Z},\mathbb{C}^L)$ we have

$$\lim_{t\to\pm\infty} \|(e^{iHt}e^{-iH_0t}P^0_{ac} - (\Phi_\pm)^*\Phi_0)u\|_{L^2(\mathbb{Z})} = 0. \tag{4.87}$$

We prove Eq. (4.87) for the sign $'+'$, the other one is similar. Equation (4.78) implies that for $t \in \mathbb{R}$

$$\Phi_+ e^{-itH}(\Phi_+)^* = e^{-itM_E}. \tag{4.88}$$

Multiplying by $(\Phi_+)^*$ to the left in both sides of Eq. (4.88), using that $(\Phi_+)^*\Phi_+ = P_{ac}$ and $\mathrm{Ran}((\Phi_+)^*) = L^2(\mathbb{Z})_{ac}$ (see Rem. 4.29) and the fact that e^{-itH} let $L^2(\mathbb{Z})_{ac}$ invariant, we obtain

$$e^{-itH}(\Phi_+)^* = (\Phi_+)^* e^{-itM_E}. \tag{4.89}$$

Using Eq. (4.89), and the fact that e^{iHt} is unitary, also recall that H_0 is absolutely continuous (see Prop. 2.2) we obtain

$$\begin{aligned}\|e^{iHt}e^{-iH_0t}P^0_{ac}u - (\Phi_+)^*\Phi_0 u\|_{L^2(\mathbb{Z})} &= \|e^{-iH_0t}u - e^{-iHt}(\Phi_+)^*\Phi_0 u\|_{L^2(\mathbb{Z})} \\ &= \|e^{-iH_0t}u - (\Phi_+)^*e^{-iM_Et}\Phi_0 u\|_{L^2(\mathbb{Z})} \\ &= \|(\Phi_0)^*e^{-iM_Et}\Phi_0 u - (\Phi_+)^*e^{-iM_Et}\Phi_0 u\|_{L^2(\mathbb{Z})} \\ &= \|((\Phi_0)^* - (\Phi_+)^*)e^{-iM_Et}\Phi_0 u\|_{L^2(\mathbb{Z})}.\end{aligned} \tag{4.90}$$

Let us prove that Eq. (4.90) tends to 0 for $u \in \Phi_0^{-1}(C_0(-2,2))$, since this space is dense in $L^2(\mathbb{Z})$, this will imply the result.
Let $u \in \Phi_0^{-1}(C_0((-2,2),\mathbb{C}^{2L}))$, let us say $(f_1,f_2) = \Phi_0 u$, using Eq. (4.63) (also see Eq. (4.50)) we obtain

$$(((\Phi_0)^* - (\Phi_+)^*)e^{-iM_Et}\Phi_0 u)(n) = I_1(t,n) + I_2(t,n), \tag{4.91}$$

where

$$
\begin{aligned}
I_1(t,n) &= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} \sqrt{-i\nu^E}(\lambda(E)^n - u^E_{-,in}(n)((M^E_+)^*)^{-1})e^{-iEt} f_1(E)\, dE, \\
I_2(t,n) &= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} \sqrt{-i\nu^E}(\lambda(E)^{-n} - u^E_{+,in}(n)((M^E_-)^*)^{-1})e^{-iEt} f_2(E)\, dE.
\end{aligned}
\tag{4.92}
$$

Let $n \le 0$. Eq. (3.23) imply that

$$
u^E_{-,in}(n) = \lambda(E)^n + r^E(n). \tag{4.93}
$$

where

$$
r^E(n) = ((A_n)^{-1} - \mathbf{1})\lambda(E)^n - (A_n)^{-1} \sum_{j=-\infty}^{n-1} \tau_V(s^E_{\cdot}(n)^*)(j)^* u^E_{-,in}(j). \tag{4.94}
$$

Then, we can write $I_1(t,n)$ as

$$
\begin{aligned}
I_1(t,n) = \frac{1}{\sqrt{2\pi}} &\int_{-2}^{2} \lambda(E)^n e^{-iEt}(\mathbf{1} - ((M^E_+)^*)^{-1})\sqrt{-i\nu^E} f_1(E)\, dE \\
- \frac{1}{\sqrt{2\pi}} &\int_{-2}^{2} e^{-iEt} r^E(n)\sqrt{-i\nu^E}((M^E_-)^*)^{-1}) f_1(E)\, dE.
\end{aligned}
\tag{4.95}
$$

Eqs. (4.22) implies that $u^E_{-,in}(j)$ is uniformly bounded with respect $j \in \mathbb{Z}, E \in supp(f_1)$ (recall $|\lambda(E) = 1|$, $E \in (-2,2)$). Eq. (3.3) (also see Eq. (3.13)) imply that for $E \in supp(f_1)$ (recall $\nu^E \in C^\infty(-2,2)$) we have

$$
\|\tau_V(s^E_{\cdot}(n)^*)(j)^*\| \le C(\|A_j - 1\| + \|B_j\| + \|A_{j+1}\|).
$$

where C is a constant that does not depend on $E \in supp(f_1), n \in \mathbb{Z}, j \in \mathbb{Z}$. Then, applying this bounds to Eq. (4.94) we obtain

$$
\|r^E(n)\| \le C\left(\|(A_n)^{-1} - \mathbf{1}\| + \sum_{j=-\infty}^{n-1} \|A_j - 1\| + \|B_j\| + \|A_{j+1}\|\right) := K(n). \tag{4.96}
$$

where C is a constant that does not depend on $E \in supp(f_1), n \in \mathbb{Z}$. Note that the assumption og finite first moment (see Eq. (1.4)) implies that $K \in L^1(\mathbb{Z}^- \cup \{0\}, \mathbb{R}) \subset L^2(\mathbb{Z}^- \cup \{0\})$. Lemma 4.34 applied to the first summand of the R.H.S. of Eq. (4.95) and Lemma 4.35 applied to the second summand imply that

$$
\|I_1(t,\cdot)\|_{L^2(\mathbb{Z}^- \cup 0)} \to 0,\ t \to +\infty. \tag{4.97}
$$

Now, for $n > 0$ we use Eq. (3.53) and Eqs. (3.22), (3.11) to write

$$
\begin{aligned}
u^E_{-,in}(n)((M^E_+)^*)^{-1} &= u^E_{+,out}(n) + u^E_{+,in}(n)L^E((M^E_+)^*)^{-1} \\
&= \lambda(E)^n + \lambda(E)^{-n}L^E((M^E_+)^*)^{-1} + r^E(n).
\end{aligned}
\tag{4.98}
$$

where

$$\begin{aligned} r^E(n) &= (A_{n+1}^{-1} - \mathbf{1})(\lambda(E)^n + \lambda(E)^{-n} L^E((M_+^E)^*)^{-1}) \\ &\quad + A_{n+1}^{-1} \sum_{j=n+1}^{\infty} \tau_V(s_{\cdot}^E(n)^*)(j)^* (u_{+,out}^E(j) + u_{+,in}^E(j) L^E((M_+^E)^*)^{-1})). \end{aligned} \tag{4.99}$$

In the same manner as before, one can prove that $\|r^E(n)\| \leq CK(n),\ n \in \mathbb{Z}^+$, where $K \in L^2(\mathbb{Z}^+)$ and C is a constant that does not depend on $E \in supp(f_1),\ n \in \mathbb{Z}$. Using Eq. (4.98) we write $I_1(t,n)$ for $n > 0$ as

$$\begin{aligned} I_1(t,n) = &- \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} \lambda(E)^{-n} e^{-iEt} L^E((M_+^E)^*)^{-1} \sqrt{-i\nu^E} f_1(E)\, dE \\ &- \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} e^{-iEt} r^E(n) \sqrt{-i\nu^E} f_1(E)\, dE \end{aligned} \tag{4.100}$$

Lemma 4.34 applied to the first summand of the R.H.S. of Eq. (4.100) and Lemma 4.35 applied to the second summand imply that

$$\|I_1(t,\cdot)\|_{L^2(\mathbb{Z}^+)} \to 0,\ t \to +\infty. \tag{4.101}$$

Eqs. (4.97) and (4.101) imply $\|I_1(t,\cdot)\|_{L^2(\mathbb{Z})} \to 0,\ t \to +\infty$. In the same manner, one proves that $\|I_2(t,\cdot)\|_{L^2(\mathbb{Z})} \to 0,\ t \to +\infty$. This above, together with Eq. (4.91) and Eq. (4.90) imply the result. □

4.4 Scattering operator and scattering matrix

In this section, we prove that the scattering matrix (see Definition 1.12) defined via the stationary approach coincides with the one obtained through the general (time-dependent) approach.

Definition 4.37. *If the wave operators exist and are complete. We define the scattering operator as follows*

$$S(H, H_0) := W_+(H, H_0)^* W_-(H, H_0).$$

We recall the definition of the scattering matrix in terms of our incoming and outgoing solutions (see Def. 1.12). $\mathcal{S}^E \in \mathbb{C}^{2L \times 2L}$ is the unitary matrix that satisfies the equation

$$\begin{pmatrix} u_{-,in}^E & -u_{+,in}^E \end{pmatrix} = \begin{pmatrix} u_{+,out}^E & -u_{-,out}^E \end{pmatrix} \mathcal{S}^E\ . \tag{4.102}$$

Taking adjoint in Eq. (4.102) we obtain

$$\begin{pmatrix} u_{-,in}^E(n)^* \\ -u_{+,in}^E(n)^* \end{pmatrix} = (\mathcal{S}^E)^* \begin{pmatrix} u_{+,out}^E(n)^* \\ -u_{-,out}^E(n)^* \end{pmatrix}, n \in \mathbb{Z}. \tag{4.103}$$

Before proving the main result of this section we obtain some identites that we need later.

Proposition 4.38. *For all $E \in (-2,2)$ the following identity holds,*

$$\begin{pmatrix} M_+^E & 0 \\ 0 & M_-^E \end{pmatrix} \mathcal{S}^E \begin{pmatrix} ((M_+^E)^*)^{-1} & 0 \\ 0 & ((M_-^E)^*)^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\mathcal{S}^E)^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.104}$$

Proof. Let $E \in (-2,2)$. Since $\mathcal{S}^E$ is unitary (see Prop. 3.16), using Def. 1.12 one can compute the off diagonal terms of $\mathcal{S}^E(\mathcal{S}^E)^*$ (which are zero) and obtain

$$\begin{aligned} -(M_+^E)^{-1}((M_+^E)^*)^{-1}(N_+^E)^* &= N_-^E(M_-^E)^{-1}((M_-^E)^*)^{-1} \\ -N_+^E(M_+^E)^{-1}((M_+^E)^*)^{-1} &= (M_-^E)^{-1}((M_-^E)^*)^{-1}(N_-^E)^*. \end{aligned} \tag{4.105}$$

Using Def. 1.12 and Eq. (4.105) we obtain

$$\begin{aligned} \begin{pmatrix} M_+^E & 0 \\ 0 & M_-^E \end{pmatrix} \mathcal{S}^E &= \begin{pmatrix} \mathbf{1} & M_+^E N_-^E (M_-^E)^{-1} \\ M_-^E N_+^E (M_+^E)^{-1} & \mathbf{1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{1} & -((M_+^E)^*)^{-1}(N_+^E)^*(M_-^E)^* \\ -((M_-^E)^*)^{-1}(N_-^E)^*(M_+^E)^* & \mathbf{1} \end{pmatrix}. \end{aligned} \tag{4.106}$$

Multiplying Eq. (4.106) to the right by the diagonal matrix, $\operatorname{diag}(((M_+^E)^*)^{-1}, ((M_-^E)^*)^{-1})$ we obtain the result. □

Proposition 4.39. *The following relation between the generalized solutions defined in Def. 4.21 and Scattering matrix holds*

$$\mathcal{S}^E \begin{pmatrix} \psi_{+,out}^E(n)^* \\ \psi_{-,out}^E(n)^* \end{pmatrix} = \begin{pmatrix} \psi_{-,in}^E(n)^* \\ \psi_{+,in}^E(n)^* \end{pmatrix}, \; n \in \mathbb{Z}. \tag{4.107}$$

Proof. By definition (see Def. 4.21) we can write

$$\begin{pmatrix} \psi_{+,out}^E(n)^* \\ \psi_{-,out}^E(n)^* \end{pmatrix} = \sqrt{-i\nu^E} \begin{pmatrix} ((M_+^E)^*)^{-1} & 0 \\ 0 & ((M_-^E)^*)^{-1} \end{pmatrix} \begin{pmatrix} u_{+,out}^E(n)^* \\ u_{-,out}^E(n)^* \end{pmatrix}, \tag{4.108}$$

and

$$\begin{pmatrix} \psi_{-,in}^E(n)^* \\ \psi_{+,in}^E(n)^* \end{pmatrix} = \sqrt{-i\nu^E} \begin{pmatrix} (M_+^E)^{-1} & 0 \\ 0 & (M_-^E)^{-1} \end{pmatrix} \begin{pmatrix} u_{-,in}^E(n)^* \\ u_{+,in}^E(n)^* \end{pmatrix}. \tag{4.109}$$

Using Eqs. (4.108), (4.104) and Eq. (4.103) we obtain

$$\begin{aligned} \begin{pmatrix} M_+^E & 0 \\ 0 & M_-^E \end{pmatrix} \mathcal{S}^E \begin{pmatrix} \psi_{+,out}^E(n)^* \\ \psi_{-,out}^E(n)^* \end{pmatrix} &= \sqrt{-i\nu^E} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} (\mathcal{S}^E)^* \begin{pmatrix} u_{+,out}^E(n)^* \\ -u_{-,out}^E(n)^* \end{pmatrix} \\ &= \sqrt{-i\nu^E} \begin{pmatrix} u_{-,in}^E(n)^* \\ u_{+,in}^E(n)^* \end{pmatrix}, \end{aligned}$$

the last equation together with Eq. (4.109) give us the result. □

Now we consider the multiplication operator by $\mathcal{S}^E$.

Definition 4.40. *Let* $M_{\mathcal{S}^E} : L^2((-2,2),\mathbb{C}^{2L}) \to L^2((-2,2),\mathbb{C}^{2L})$*, the multiplition operator given by*

$$(M_{\mathcal{S}^E} f)(E) = \mathcal{S}^E f(E).$$

We recall the generalized Fourier operators defined in Def. 4.26.

Proposition 4.41. *The following relation holds true,*

$$M_{\mathcal{S}^E}\Phi_- = \Phi_+. \tag{4.110}$$

Proof. Let u compact supported sequence, using Def. 4.23 and Eq. (4.107) we obtain

$$\begin{aligned}(M_{\mathcal{S}^E}\Phi_- u)(E) = \mathcal{S}^E(\Phi_- u)(E) &= \frac{1}{\sqrt{2\pi}}\mathcal{S}^E \sum_{n\in\mathbb{Z}} \begin{pmatrix}\psi^E_{+,out}(n)^* u(n)\\ \psi^E_{-,out}(n)^* u(n)\end{pmatrix} \\ &= \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} \mathcal{S}^E \begin{pmatrix}\psi^E_{+,out}(n)^* u(n)\\ \psi^E_{-,out}(n)^* u(n)\end{pmatrix} = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} \begin{pmatrix}\psi^E_{-,in}(n)^* u(n)\\ \psi^E_{+,in}(n)^* u(n)\end{pmatrix} \\ &= (\Phi_+ u)(E).\end{aligned} \tag{4.111}$$

The above relation can be extended by density to all $L^2(\mathbb{Z},\mathbb{C}^L)$. □

Remark 4.42. *Let* H *be a matrix valued self-adjoint Jacobi operator as in Def. 1.1. Assume that* $V = H - H_0$ *has finite first moment (see Def. 1.4). By Prop. 4.36, the wave operators exist and are complete. Then we can consider the scattering operator (see Def. 4.37),*

$$S(H,H_0) := W_+(H,H_0)^* W_-(H,H_0).$$

Moreover, recall that the unitary operator Φ_0 *(see Def. 2.14) diagonalizes* H_0*. Then, it diagonalizes* $S(H,H_0)$ *as well. Namely, there exists* $\tilde{\mathcal{S}} : (-2,2) \to \mathbb{C}^{2L\times 2L}$*, measurable function such that*

$$\Phi_0 S(H,H_0)(\Phi_0)^* = M_{\tilde{\mathcal{S}}}. \tag{4.112}$$

The following proposition shows that the scattering matrix $\mathcal{S}^E$, defined as the object that map outgoing solutions to incoming ones (see Def. 1.12) is in fact the diagonalization of the scattering operator $\tilde{\mathcal{S}}$ (see Eq. (4.112)).

Proposition 4.43. *Consider the matrix valued functions* $\mathcal{S}^E$, $\tilde{\mathcal{S}}$ *(see Def. 1.12 and Eq. (4.112)), then*

$$\mathcal{S}^E = \tilde{\mathcal{S}}(E), \text{ a.e. } E \in (-2,2). \tag{4.113}$$

Proof. Using Eq.(4.110) and Eq. (4.71) we obtain $M_{\mathcal{S}^E} = \Phi_+(\Phi_-)^*$. Using this above together with Eq. (4.112) and Eq. (4.86) we obtain

$$M_{\tilde{\mathcal{S}}} = \Phi_0 S(H,H_0)(\Phi_0)^* = \Phi_0 W_+(H,H_0)^* W_-(H,H_0)(\Phi_0)^* = \Phi_+(\Phi_-)^* = M_{\mathcal{S}^E}.$$

This implies the result. □

5 Appendix

In this section, we present technical results. For the sake of completeness, we include their proofs. Some of them are referenced in the bibliography, where the reader can find a complete proof.

Theorem 5.1 (Volterra Equation)**.** *Let* $g \in L^\infty(\mathbb{N}, \mathbb{C}^{L\times L})$ *and* $K(n,m) \in \mathbb{C}^{L\times L}$ *for each* $m, n \in \mathbb{N}$. *Consider the Volterra sum equation*

$$f(n) = g(n) + \sum_{m=n+1}^{\infty} K(n,m) f(m), \ \ n \in \mathbb{N}. \tag{5.1}$$

and suppose there is a sequence $M \in L^1(\mathbb{N}, \mathbb{R})$ *such that* $\|K(n,m)\| \leq M(m)$ *for each* $m > n \in \mathbb{N}$. *Then, Equation* (5.1) *has a unique solution* $f \in L^\infty(\mathbb{N}, \mathbb{C}^{L\times L})$. *Moreover, if* $g^z(n)$ *and* $K^z(n,m)$ *depend continuously (resp. analytically) on a parameter* $z \in U$ *(open or close set) (for every* n*), and for every compact set* $C \subset U$*, one has* $\{\sup_{z\in C} |M^z(m)|\}_{m\in\mathbb{N}} \in L^1(\mathbb{N})$*, and* $g^z(n)$ *is uniformly bounded with respect to* $n \in \mathbb{N}$ *and* $z \in C$*, then* $f^z(n)$ *is continuous (resp. analytic) and uniformly bounded with respect* $n \in \mathbb{N}$ *and* $z \in C$.

Proof. This is a matrix valued version of Lemma 7.8 [27]. For each $k \in \mathbb{N}$, if one finds a solution $f \in l^\infty(\mathbb{N} \cap [k, \infty), \mathcal{M})$, then it can be extended to a solution in $l^\infty(\mathbb{N}, \mathcal{M})$ by defining recursively $f(n) = g(n) + \sum_{m=n+1}^\infty K(n,m)f(m)$ for each $n < k$. Since $M \in l^1(\mathbb{N}, \mathbb{R})$, there exists $k \in \mathbb{N}$ such that $\sum_{m=k+1}^\infty M(m) < 1/2$. Then, w.l.o.g., we can assume that $k = 0$, *i.e.*, $\sum_{m=1}^\infty M(m) < 1/2$. Then let us introduce the operator $T : l^\infty(\mathbb{N}, \mathcal{M}) \to l^\infty(\mathbb{N}, \mathcal{M})$ by

$$(Tf)(n) = \sum_{m=n+1}^{\infty} K(n,m) f(m)$$

which is well-defined because

$$\sum_{m=n+1}^{\infty} \|K(n,m)f(m)\| \leq \sum_{m=n+1}^{\infty} \|K(n,m)\| \|f(m)\| \leq 1/2 \|f\|_\infty.$$

Moreover, the last equation also implies that T is bounded and $\|T\| < 1/2$, therefore $I - T$ is invertible and $f := (I-T)^{-1} g$ is a solution to the equation on $l^\infty(\mathbb{N}, \mathbb{C}^{L\times L})$.

Now we assume that $g(n) \equiv g^z(n)$ and $K(n,m) = K^z(n,m)$ depend continuously (resp. holomorphically) on a parameter $z \in U$ (open or closed set) (for every n), for every compact set $C \subset U$, $\{\sup_{z\in C} |M^z(m)|\}_{m\in\mathbb{N}} \in L^1(\mathbb{N})$, and $g^z(n)$ is uniformly bounded with respect to n and z. Let us take a compact set $C \subset U$. Since the series $(Tg^z)(n) = \sum_{m=n+1}^\infty K^z(n,m) g^z(m)$ converges uniformly on C, $(Tg^z)(n)$ is then continuous (holomorphic) on C for each $n \in \mathbb{N}$ and it is uniformly bounded with respect to n and $z \in C$. Repeating the argument, one obtains that the same holds true for $T^j g^z$, for every natural number j. Using that $\|T^j g\| \leq (1/2)^j \sup_{n,z}\{\|g(n)\|\}$, it follows that the series $f^z(n) = \sum_{j=0}^\infty (T^j g)^z(n)$ converges uniformly on C and is uniformly bounded with respect $n \in \mathbb{N}$ and $z \in C$. This implies that the map $z \mapsto f^z(n)$ is continuous (holomorphic) on compact sets. Then it is continuous (holomorphic) on U. □

Lemma 5.2 (Variation of parameters)**.** *Let* $F : (\mathbb{C}^{L\times L})^{\mathbb{Z}} \to (\mathbb{C}^{L\times L})^{\mathbb{Z}}$ *be a function. Consider the following difference equation*

$$A_n X(n-1) + B_n X(n) + A_{n+1} X(n+1) = (F(X))(n), \tag{5.2}$$

where $A_n, B_n \in \mathbb{C}^{L\times L}$ *for each* $n \in \mathbb{Z}$ *with* A_n *invertible. Consider the free equation*

$$A_n X(n-1) + B_n X(n) + A_{n+1} X(n+1) = 0, \tag{5.3}$$

For all $n \in \mathbb{Z}$*, let us denote* s_m *the solution to Eq.* (5.3) *such that*

$$s_m(m) = 0, \quad s_m(m+1) = A_{m+1}^{-1}.$$

Fix $C, D \in \mathbb{C}^{L\times L}$*, consider* U *the solution to the free equation with intial conditions* $U(0) = C$*,* $U(1) = D$*. Then,* S *is a solution to Equation* (5.2)*, with initial conditions* $S(0) = C$*,* $S(1) = D$*, if and only if satisfies the sum equation*

$$S(n) = U(n) + \sum_{j=1}^{n-1} s_j(n)(F(S))(j), \; n \geq 1 \tag{5.4}$$

and for $n \in \mathbb{Z}^- \cup \{0\}$

$$S(n) = U(n) - \sum_{j=n+1}^{0} s_j(n)(F(S))(j), \; n \leq 0. \tag{5.5}$$

where we identify $\sum_{j=1}^{0} a_j \equiv 0$*.*

Proof. For $u \in (\mathbb{C}^{L\times L})^{\mathbb{Z}}$ we denote

$$(\tau u)(n) = A_n u(n-1) + B_n u(n) + A_{n+1} u(n+1).$$

We consider the linear function $L : (\mathbb{C}^{L\times L})^{\mathbb{Z}} \to (\mathbb{C}^{L\times L})^{\mathbb{Z}}$,

$$(Lu)(n) = \sum_{j=1}^{n-1} s_j(n) u(j), \; n \geq 1.$$

$$(Lu)(n) = -\sum_{j=n+1}^{0} s_j(n) u(j), \; n \leq 0.$$

In particular, note that $(Lu)(0) = (Lu)(1) = 0$. An explicit computation using that $s_n(n-1) = -A_n^{-1}$ and $\tau s_j = 0$ (making the cases $n \in \{-1, 0, 1\}$ separately) shows that

$$(\tau \circ Lu)(n) := A_n (Lu)(n-1) + B_n (Lu)(n) + A_{n+1} (Lu)(n+1) = u(n). \tag{5.6}$$

Suppose that S satisfies Eqs. (5.4), (5.5). In particular $S = U + L \circ F(S)$, then Eq. (5.6) implies (recall $\tau U = 0$) $\tau S = \tau U + \tau(L \circ F(S)) = \tau \circ L(F(S)) = F(S)$, then S satisfies Eq. (5.2). Conversely, suppose that S satisfies Eq. (5.2) and $S(0) = C$, $S(1) = D$. In particular,

$$\tau S = F(S). \tag{5.7}$$

Consider $U_0 := S - L(F(S))$, we note that U_0 is a solution of Eq. (5.3). Indeed, Eqs. (5.6) and (5.7) imply $\tau U_0 = \tau S - \tau L(F(S)) = F(S) - F(S) = 0$. Besides, $U_0(0) = S(0) = C = U(0)$ and $U_0(1) = S(1) = D = U(1)$. Since U and U_0 are both solutions of Eq. (5.3) with same initial conditions they are equal. Therefore, $S = U_0 + L(F(S)) = U + L(F(S))$. □

Proposition 5.3. *(Integration by parts) Consider τ the difference expression given by*

$$(\tau u)(n) = A_n u(n-1) + B_n u(n) + A_{n+1} u(n+1),$$

with A_n, B_n self-adjoint matrices. Recall that for matrix valued sequence we have

$$\mathcal{W}_\tau(u,v)(n) = u(n+1)^* A_{n+1} v(n) - u(n)^* A_{n+1} v(n+1).$$

The following identity holds true,

$$\sum_{m=n}^{k} ((\tau u)(m))^* v(m) = \sum_{m=n}^{k} u(m)^* (\tau v)(m) + \mathcal{W}(u,v)(k) - \mathcal{W}(u,v)(n-1),$$

Proof. Explicit computation shows,

$$(\tau u)^* v - u^*(\tau v) = d_- \mathcal{W}_\tau(u,v) \tag{5.8}$$

Recall that $(d_- v)(n) = v(n) - v(n-1)$. Finally, a telescopic argument implies the result. □

Lemma 5.4 (Gronwall lemma)**.** *Let α a real positive number and $(w_n)_{n\in\mathbb{N}}, (u_n)_{n\in\mathbb{N}}$ real positive sequences such that*

$$\sum_{j=1}^{\infty} w_j < \infty, \qquad u_n \le K,\ n \in \mathbb{N},$$

for some $K \in \mathbb{R}$ and

$$u_n \le \alpha + \sum_{j=n+1}^{\infty} w_j u_j. \tag{5.9}$$

Then for all $n \in \mathbb{N}$, it follows that

$$u_n \le \alpha \exp\left(\sum_{j=n+1}^{\infty} w_j\right).$$

Proof. Let us define the functions $W, U : \mathbb{R} \to [0,\infty)$ by setting

$$U|_{[-n,-n+1)} = u_n, \qquad W|_{[-n,-n+1)} = w_n, \qquad n \in \mathbb{N},$$

and both U and W vanish on $[0,\infty)$. For every $t \in [-n,-n+1)$, one has that

$$U(t) = u_n \le \alpha + \sum_{j=n+1}^{\infty} w_j u_j = \alpha + \int_{-\infty}^{-n} WU \le \alpha + \int_{-\infty}^{t} WU. \tag{5.10}$$

For the rest of the proof, one argues as in the proof of the Gronwall lemma for the continuous case. We provide a few lines with the key steps, for the convenience of the reader. Let us define $V(t) = e^{-\int_{-\infty}^{t} W} \int_{-\infty}^{t} WU$. It is clear that $\frac{d}{dt}V(t) = e^{-\int_{-\infty}^{t} W} W(t)[U(t) - \int_{-\infty}^{t} WU] \leq \alpha e^{-\int_{-\infty}^{t} W} W(t)$, for every $t \notin -\mathbb{N} \cup \{0\}$. Integrating, one gets

$$V(t) \leq \int_{-\infty}^{t} \alpha e^{-\int_{-\infty}^{s} W} W(s) = \alpha(1 - e^{-\int_{-\infty}^{t} W}).$$

This implies that

$$\int_{-\infty}^{t} WU \leq \alpha e^{\int_{-\infty}^{t} W} - \alpha,$$

which together with (5.10) implies $u_n = U(-n) \leq \alpha e^{\int_{-\infty}^{-n} W} = \alpha e^{\sum_{j=n+1}^{\infty} w_j}$. □

Proposition 5.5. *Let $w, y \in \mathbb{C}$ with $|w|, |y| \leq 1$. For $n \geq 0$, and $1 \geq \alpha > 0$ one has*

$$|w^n - y^n|^\alpha \leq 2|w - y|^\alpha |n|^\alpha. \tag{5.11}$$

Proof. Using that $|w^n - y^n| \leq |w|^n + |y|^n \leq 2$, we obtain

$$|w^n - y^n| = |w^n - y^n|^{1-\alpha}|w^n - y^n|^\alpha \leq 2^{1-\alpha}\left(|w - y| \sum_{j=0}^{n-1} |w|^{n-1-j}|y|^j\right)^\alpha \leq 2|w - y|^\alpha |n|^\alpha. \tag{5.12}$$

□

Proposition 5.6 (Stone's formula)**.** *Let $\mathcal{H}$ be a Hilbert space and $H : D(H) \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator. Consider E_H the spectral measure of H. Then for all $h \in D(H)$ and $(a, b) \subset \mathbb{R}$ one has*

$$\frac{1}{2}(E_H[a, b] + E_H(a, b))h = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_a^b (R(x + i\varepsilon) - R(x - i\varepsilon))h \, dx. \tag{5.13}$$

Here, for $z \notin \sigma(H)$, $R(z) = (H - z)^{-1}$ denotes the resolvent operator.

Proof. see VII.13 in [25] □

Proposition 5.7. *Let H_1,H_2 Hilbert spaces, and $T : H_1 \to H_2$ a bounded linear operator with $T^* : H_2 \to H_1$ its adjoint. Then* $\mathrm{Ran}(T^*T) = \mathrm{Ran}(T^*)$*.*

Proof. The contention $\subset$ is clear. Take $u \in \mathrm{Ran}(T^*)$, then $u = T^*x$ with $x \in H_2$. Since $H_2 = \mathrm{Ran}(T) \oplus \mathrm{Ran}(T)^\perp$, there exist $w \in \mathrm{Ran}(T)$, $y \in \mathrm{Ran}(T)^\perp = \mathrm{Ker}(T^*)$ with $x = w + y$. Then, $u = T^*x = T^*w + T^*y = T^*w \in \mathrm{Ran}(T^*T)$. □

Proposition 5.8 (Riemann-Lebesgue Lemma)**.** *For all $f \in L^1(\mathbb{R})$, one has*

$$\int_{\mathbb{R}} e^{itx} f(x) \, dx \to 0, \ \ |t| \to \pm\infty.$$

Proof. Let $(a,b) \subset \mathbb{R}$, and $\chi_{(a,b)}$ the charasteristic function asociated to (a,b). Then

$$\int_{\mathbb{R}} e^{itx} \chi_{(a,b)}(x)\ dx = \frac{e^{itb} - e^{ita}}{it} \to 0, |t| \to \pm\infty. \tag{5.14}$$

The above equation implies that for every simple function $s \in L^1(\mathbb{R})$, the limit holds. Take $f \in L^1(\mathbb{R})$, and let $\varepsilon > 0$. Consider $s \in L^1(\mathbb{R})$ simple function such that $\|f - s\|_{L_1} < \varepsilon/2$ and take T such that if $|t| > T$, $|\int_{\mathbb{R}} e^{itx} s(x)\ dx| < \varepsilon/2$. Then, for $|t| > T$ we have

$$\left| \int_{\mathbb{R}} e^{itx} f(x)\ dx \right| \leq \|f - s\|_{L^1} + \int_{\mathbb{R}} e^{itx} s(x)\ dx < \varepsilon.$$

□

Proposition 5.9 (Schur formula for the determinant). *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a block matrix with square matrices* A *and* D. *If* D *is invertible then*

$$\det(M) = \det(D) \det(A - BD^{-1}C).$$

Proposition 5.10. *Let* $\mathcal{H}$ *be a Hilbert space and consider* $\mathcal{D}$ *the linear subspace of* $\mathcal{H}^{\mathbb{Z}^2}$, *given by the functions* $u : \mathbb{Z}^2 \to \mathcal{H}$ *such that satisfy*

- $u(n, \cdot) \in L^2(\mathbb{Z}, \mathcal{H})$.
- $\sup_{n \in \mathbb{Z}} \|u(n, \cdot)\|_{L^2} < \infty$
- $u(n,m) = 0,\ m < n$.

Also, consider the function $\|.\| : \mathcal{D} \to \mathbb{R}$, *given by*

$$\|u\| = \sup_{n \in \mathbb{Z}} \|u(n, \cdot)\|_{L^2}.$$

Then $(\mathcal{D}, \|\cdot\|)$ *is a Banach space. Recall that here*

$$\|u\|_{L^2}^2 = \sum_{n \in \mathbb{Z}} \|u(n)\|_{\mathcal{H}}^2.$$

Proof. It is clear that $(\mathcal{D}, \|\cdot\|)$ is a normed linear space. Now we see that it is complete. Let $(u_r)_{r \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{D}$. Note that this implies that for each $n \in \mathbb{Z}$ the sequence $(u_r(n, \cdot))_{r \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{Z}, \mathcal{H})$, which is a Banach space. Then, for each $n \in \mathbb{Z}$ there is $v_n \in L^2(\mathbb{Z}, \mathcal{H})$ such that

$$\lim_{r \to \infty} \|u_r(n, \cdot) - v_n\|_{L^2} = 0. \tag{5.15}$$

Note that Eq. (5.15) and the fact that $u_r(n,m) = 0, m < n$ imply that if $m < n$ then

$$\|v_n(m)\|_{\mathcal{H}} = \|v_n(m) - u_r(n,m)\|_{\mathcal{H}} \leq \|u_r(n, \cdot) - v_n\|_{L^2} \to 0,\ r \to \infty.$$

Therefore,

$$v_n(m) = 0 \quad m < n. \tag{5.16}$$

Also, since (u_r) is a Cauchy sequence, there exist $N \in \mathbb{N}$, such that for all $r \geq N$

$$\|u_r(n, \cdot) - u_N(n, \cdot)\|_{L^2} \leq 1, \; n \in \mathbb{Z}. \tag{5.17}$$

Now, since $u_N \in \mathcal{D}$ then there is a constant C_N such that

$$\|u_N(n, \cdot)\|_{L^2} \leq C_N, \; n \in \mathbb{Z}. \tag{5.18}$$

Eqs. (5.17) and (5.18) imply that for all $r \geq N$

$$\|u_r(n, \cdot)\|_{L^2} \leq \|u_N(n, \cdot)\|_{L^2} + \|u_r(n, \cdot) - u_N(n, \cdot)\|_{L^2} \leq C_N + 1, \; n \in \mathbb{Z}.$$

Taking limit as $r \to +\infty$ in above equation, we obtain that

$$\|v_n\|_{L^2} \leq C_N + 1, \; n \in \mathbb{Z}. \tag{5.19}$$

Now we define $u(n, m) := v_n(m)$, then by definition and Eqs. (5.16), (5.19) we have that $u \in \mathcal{D}$. Finally, we prove that $u_r \to u, \; r \to +\infty$. Let $\epsilon > 0$, since u_r is a Cauchy sequence there are $N \in \mathbb{N}$ such that for all $r, s \geq N$

$$\|u_r(n, \cdot) - u_s(n, \cdot)\|_{L^2} < \epsilon/2, \; n \in \mathbb{Z}. \tag{5.20}$$

For $n \in \mathbb{Z}$ we know by definition and Eq. (5.15) that there are $L_n \geq N$ such that

$$\|u_{L_n}(n, \cdot) - u(n, \cdot)\|_{L^2} < \epsilon/2. \tag{5.21}$$

Then Eqs. (5.20) and (5.21) imply that for all $r \geq N$ and $n \in \mathbb{Z}$ we have that

$$\|u_r(n, \cdot) - u(n, \cdot)\|_{L^2} \leq \|u_r(n, \cdot) - u_{L_n}(n, \cdot)\|_{L^2} + \|u_{L_n}(n, .) - u(n, .)\|_{L^2} < \epsilon.$$

Since the above equation is true for all $n \in \mathbb{Z}$, we have that for all $r \geq N$

$$\|u_r - u\| = \sup_{n \in \mathbb{Z}} \|u_r(n, .) - u(n, .)\|_2 \leq \epsilon.$$

Therefore, the sequence u_r converge to $u \in \mathcal{D}$. □

Proposition 5.11. *Let $\{e_j : j = 1, \ldots, L\}$ be the standar basis of $\mathbb{C}^L$ and $\delta_j : \mathbb{Z} \to \mathbb{C}^{L \times L}$, the Kronecker's delta. Consider $T \in \mathcal{B}(L^2(\mathbb{Z}, \mathbb{C}^L))$ a bounded operator, and let*

$$G(n, m)_{i,j} = \langle T\delta_m e_j, \delta_n e_i \rangle_{L^2(\mathbb{Z}, \mathbb{C}^L)}.$$

Then, for all $u \in L^2(\mathbb{Z}, \mathbb{C}^L)$ we have that

$$(Tu)(n) = \sum_{m \in \mathbb{Z}} G(n, m)u(m), \; n \in \mathbb{Z}. \tag{5.22}$$

Proof. Note that by definition one has that $\langle u, \delta_n e_i \rangle_{L^2} = \langle u(n), e_i \rangle_{\mathbb{C}^L} = u(n)_i$. This above implies

$$u(n) = \sum_{i=1}^{L} \langle u(n), e_i \rangle_{\mathbb{C}^L} e_i = \sum_{j=1}^{L} \langle u, \delta_n^i \rangle_{L^2} e_i. \tag{5.23}$$

Since $\{\delta_m e_j : m \in \mathbb{Z},\ j \in 1, \dots, L\}$ is an orthonormal basis of $\mathbb{C}^{L \times L}$, then we have that $Tu \in L^2(\mathbb{Z}, \mathbb{C}^L)$ can be written as

$$u = \sum_{m \in \mathbb{Z}} \sum_{j=1}^{L} \langle u, \delta_m^j \rangle_{L^2} \delta_m^j, \tag{5.24}$$

where the limit is taken in L^2. Applying Eq. (5.23), Eq. (5.24) we obtain

$$\begin{aligned}
(Tu)(n) &= \sum_{i=1}^{L} \langle Tu, \delta_n^i \rangle_{L^2} e_i = \sum_{i=1}^{L} \left\langle T \left(\sum_{m \in \mathbb{Z}} \sum_{j=1}^{L} \langle u, \delta_m^j \rangle_{L^2} \delta_m^j \right), \delta_n^i \right\rangle_{L^2} e_i \\
&= \sum_{i=1}^{L} \sum_{m \in \mathbb{Z}} \sum_{j=1}^{L} \left\langle T\delta_m^j, \delta_n^i \right\rangle_{L^2} \langle u, \delta_m^j \rangle_{L^2} e_i = \sum_{m \in \mathbb{Z}} \sum_{i=1}^{L} \sum_{j=1}^{L} T(n,m)_{i,j} u(m)_j e_i \\
&= \sum_{m \in \mathbb{Z}} \sum_{i=1}^{L} (G(n,m)u(n))_i e_i = \sum_{m \in \mathbb{Z}} G(n,m) u(m).
\end{aligned} \tag{5.25}$$

□

Proposition 5.12. *Let $(u_r)_{r=1}^{\infty}$ be a sequence such that $u_r \in L^2(\mathbb{Z}, \mathbb{C}^L)$. Suppose that*

$$\sum_{r=1}^{\infty} \|u_r\|_{L^2} < \infty.$$

Then, for all $m \in \mathbb{Z}$, $\sum_{r=1}^{\infty} u_r(m)$ exist. Besides, there exist $u \in L^2(\mathbb{Z}, \mathbb{C}^2)$ such that

$$\lim_{M \to \infty} \left\| \sum_{r=1}^{M} u_r - u \right\|_{L^2},$$

and we have that

$$u(m) = \sum_{r=1}^{\infty} u_r(m).$$

Proof. Note that for all $m \in \mathbb{Z}$, $\|u_r(m)\|_{\mathbb{C}^L} \leq \|u_r\|_{L^2}$, then

$$\sum_{r=1}^{\infty} \|u_r(m)\|_{\mathbb{C}^L} \leq \sum_{r=1}^{\infty} \|u_r\|_{L^2} < \infty.$$

Therefore, $\sum_{r=1}^{\infty} u_r(m) \in \mathbb{C}^L$ exist. Now, since the sequence $(u_r)_{r=1}^{\infty}$ is norm-summable then is summable in L^2 so there exist $u \in L^2$ such that

$$\lim_{M\to\infty} \left\| \sum_{r=1}^{M} u_r - u \right\|_{L^2}.$$

Finally, note that

$$\left\| \sum_{r=1}^{M} u_r(m) - u(m) \right\|_{\mathbb{C}^L} \le \left\| \sum_{r=1}^{M} u_r - u \right\|_{L^2} \to 0,\ M \to \infty.$$

□

Proposition 5.13. *Consider the space of vector valued sequence* $(\mathbb{C}^L)^{\mathbb{Z}}$*. Let* $T : (\mathbb{C}^L)^{\mathbb{Z}} \to (\mathbb{C}^L)^{\mathbb{Z}}$ *be a second order difference linear transform, namely,*

$$(Tu)(n) = C_{n,n-1}u(n-1) + C_{n,n}u(n) + C_{n,n+1}u(n+1).$$

form some $C_{n,m} \in \mathbb{C}^{L\times L}$*. Let* $P : \mathbb{C} \to \mathbb{C}$ *be a polynomial of degree* d*, let us say*

$$P(z) = A_d z^d + \cdots + A_1 z + A_0. \tag{5.26}$$

Then the operator $P(T)$*, defined as*

$$P(T) = A_d T^d + \cdots + A_1 T + A_0,$$

is a difference linear transform of order at most $2d$*, namely*

$$(P(T)u)(n) = \sum_{j=n-d}^{n+d} C^P_{n,j} u(j),\ u \in (\mathbb{C}^L)^{\mathbb{Z}},$$

for some $C^P_{n,j} \in \mathbb{C}^{L\times L}$*.*

Proof. We prove the result for the polynomial $P(z) = z^d$ and the rest is clear. The proof is by induction over d. The result is clear for $d = 1$ (by definition of T). Now suppose that T^d is a difference operator of order $2d$, namely

$$(T^d u)(n) = \sum_{j=n-d}^{n+d} C^d_{n,j} u(j),\ u \in (\mathbb{C}^L)^{\mathbb{Z}}, \tag{5.27}$$

for some $C^d_{n,j} \in \mathbb{C}^{L\times L}$. Then, Eq. (5.27) and definition of T imply that

$$\begin{aligned} (T^{d+1}u)(n) = (T^d(Tu))(n) &= \sum_{j=n-d}^{n+d} C^d_{n,j}(Tu)(j) \\ &= \sum_{j=n-d}^{n+d} C^d_{n,j}(C_{j,j-1}u(j-1) + C_{j,j}u(j) + C_{j,j+1}u(j+1)). \end{aligned} \tag{5.28}$$

If we set

- $C^{d+1}_{n,n-d-1} := C^{d}_{n,n-d} C_{n-d,n-d-1}$.
- $C^{d+1}_{n,n-d} := C^{d}_{n,n-d+1} C_{n-d+1,n-d} + C^{d}_{n,n-d} C_{n-d,n-d}$.
- $C^{d+1}_{n,j} := C^{d}_{n,j+1} C_{j+1,j} + C^{d}_{n,j} C_{j,j} + C^{d}_{n,j-1} C_{j-1,j}, \quad n-d+1 \leq j \leq n+d-1.$
- $C^{d+1}_{n,n+d} := C^{d}_{n,n+d-1} C_{n+d-1,n+d} + C^{d}_{n,n+d} C_{n+d,n+d}$.
- $C^{d+1}_{n,n+d+1} := C^{d}_{n,n+d} C_{n+d,n+d+1}$.

Then by Eq. (5.28) we have that

$$(T^{d+1}u)(n) = \sum_{j=n-d-1}^{n+d+1} C^{d+1}_{n,j} u(j).$$

This finish the induction and then the proof. □

Lemma 5.14. *Let U be an open conected set with $\overline{U}$ compact. Consider a continuous function $g : \overline{U} \to \mathbb{C}$, analytic on U and such that $g(z_0) \neq 0$, for some $z_0 \in \overline{U}$. Then, for each $\eta > 0$, there exist $r > 0$, such that for all all $\gamma \subset B(z_0; r) \cap U$ rectifiable path, it follows that*

$$\left| \int_\gamma \frac{g'(z)}{g(z)} dz \right| < \eta.$$

Proof. Since $g(z_0) \neq 0$, we can assume w.l.o.g. that $\Re(g(z_0)) = c > 0$ (we multiply everything by a constant complex number). We set

$$\log : \mathbb{C} \setminus [-\infty, 0] \to \mathbb{C}$$

an analytic branch of logarithm. By the continuity of g, there exist D compact neighborhood of z_0 such that $g(D \cap \overline{U}) \subset \{z \in \mathbb{C} : \Re(z) \geq c/2\}$. Then, the function

$$\log \circ g : D \cap \overline{U} \to \mathbb{C}$$

is uniformly continuous, and analytic on $U \cap \overset{\circ}{D}$. Take $\eta > 0$, by the uniformly continuity there exists $r > 0$ with $B(z_0; r) \subset D$ and such that, $|\log \circ g(b) - \log \circ g(a)| < \eta$ for $a, b \in D \cap \overline{U}$ with $|a - b| < 2r$. Take $\gamma : [0, 1] \to B(z_0; r) \cap U$, rectificable path we compute

$$\int_\gamma \frac{g'(z)}{g(z)} dz = \int_\gamma (\log \circ g)' = \log \circ g(\gamma(1)) - \log \circ g(\gamma(0)).$$

The desired result is consequence from the fact that $|\gamma(1) - \gamma(0)| \leq |\gamma(1) - z_0| + |\gamma(0) - z_0| < 2r$, and the election of r. □

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